MASTER

A robust optimization approach to the capacitated lot sizing problem

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A Robust Optimization Approach to the Capacitated Lot Sizing Problem

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Contents

Abstract

In this thesis the Capacitated Lot Sizing Problem under demand uncertainty is considered. In this problem production has to be planned for a single resource over a finite horizon for a fixed number of products while being constraint by per period capacity restrictions. Costs are incurred for setting up production, holding inventory and back-orders. Based on a Mixed Integer Linear Optimization formulation we have derived a Stochastic Counterpart that assumes demand to be normally distributed as well as a Robust Counterpart that assumes demand to range in a specified interval. The realized service level when using the Stochastic Counterpart can be influenced by means of a β-service level constraint whereas for the Robust Counterpart we can set the back-order cost and the size of the demand interval.

Both methods were compared in a simulation study in a rolling horizon setting whereby the per period demands for each product are drawn from the Normal distribution. First, we fitted the right parameters to a realized service level. Second, we assessed how these models performed under various circumstances. We observed that the Robust Counterpart delivers more stable service levels at lower cost when capacity decreases and demand uncertainty gets higher. In general, the Robust Counterpart is superior of the Stochastic Counterpart.

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Dedicated to the living memory of Richard Geoffrey Jeanne Brounts

Chapter 1

Introduction

1.1 The Capacitated Lot Sizing Problem

In this thesis we will study the Capacitated Lot Sizing Problem (CLSP) under demand uncertainty. More specifically, we will take a Robust Optimization (RO) approach based on the work of Ben-Tal and Nemirovski (1998) to deal with demand uncertainty. We will start with a thorough introduction into the field of RO based on a basic production planning problem. It should provide solid ground to explain our new approach to the CLSP. In a simulation study this model will be compared to two other models: the nominal model and its Stochastic Counterpart. The former of these assumes deterministic demand whereas the latter assumes demand to be randomly distributed. All models will be compared in an simulated environment in a rolling horizon setting where demand is randomly generated from a known probability distribution.

Karimi et al. (2003) define production planning to be "the activity that considers the best use of production resources in order to satisfy production goals (satisfying production requirements and anticipating sales opportunities) over a certain period named the planning horizon". One of the problems in production planning is the lot sizing problem, which according to Karimi et al. (2003) revolves around deciding on "when and how much of a product to produce such that set-up, production and holding costs are minimised". We can account the first lot sizing model to Wagner and Whitin (1958). They considered time-varying but deterministic demand without any capacity restrictions. When a capacity restriction is introduced we get the CLSP. When demand is stationary and randomly distributed we talk about the Stochastic Economic Lot Sizing Problem, while in case demand is non-stationary, but independent, we talk about the Stochastic CLSP.

Production planning typically encompasses three levels of decision making: strategic, tactical and operational. Besides that, it encompasses three time ranges for decision making: long-term, medium-term and short-term. Long-term and strategic usually revolves around deciding on the product mix to offer or where to locate a new facility. In general, one may regard this level as one where the prime focus is on anticipating aggregate needs (Karimi et al. (2003)). Medium-term planning revolves around Material Requirements Planning and determining production plans. This is the level at which the lot sizing decision resides. The production plans are then disaggregated into day-to-day schedules and this is the level of short-term operational planning.

The lot sizing decision can be found in many companies. Winands et al. (2011) state that it is a common problem for glass and paper production, injection molding, metal tamping, semi-continuous chemical processes and in bulk production of consumer products. For example, Fransoo et al. (1995) describe a situation at a glass-containers manufacturing company. There exists a lot sizing decision in this situation, because if a different colour product needs to be produced, then the temperature of the oven in which the glass is heated has to change, which takes fours days. These four days can be regarded as the setup time and this leads to a lot sizing decision. Hence, production has to be planned in a optimal (or near optimal) way while costs are minimized. Since there is a trade-off in inventory holding costs and setup costs, we do not want to produce too long as inventory builds up, nor do we want to switch too often between producing different products as it costs money and consumes valuable capacity.

Clearly, from a practical point of view, the CLSP is worth studying for successful operations planning and control. Besides a practical motivation, there is a scientific one as well. Therefore, we will continue in the next section with providing more background on the problem as well as giving a more thorough motivation for taking a RO approach to the CLSP under demand uncertainty.

1.2 Motivation and Background

In the previous section we gave a brief introduction to the CLSP, its relation to other decision problems in Operations Management and the relevance of studying the problem for practice. In this section we will dive deeper in the background of the problem and we will motivate this research. This motivation is predominantly based on existing research in which future directions are recommended. As we will discuss next, the first reason for this research is that the developments in the field of RO might provide better ways to deal with uncertainty and the second reason stems from the fact that most, if not all, of the research done on the CLSP does not perform a simulation study in which the models are run in a rolling horizon setting.

When studying literature there are two aspects of research on the CLSP that stand out: increase the computation speed and deal with uncertainty. Both Allahverdi et al. (1999) and Jans and Degraeve (2008) state that one of the major limitations of the lot sizing problems they discuss is the assumption of deterministic demand and their inability to deal with demand uncertainty. From the work of Belvaux and Wolsey (2001), Wolsey (2002) and Pochet and Wolsey (2006) we know that we can formulate the CLSP as a Mixed Integer Linear Optimization (LO) problem. Now given the desire to deal with uncertainty, the questions rises if we can deal with these type of problems.

The work of Dantzig (1955) can be regarded as one of the first accounts to incorporate uncertainty into LO problems. This was done by letting the parameters of the LO problem belong to a uncertain but known distribution of demand. A few years later, Wagner and Whitin (1958) proposed their famous model for determining ordering quantities under time-varying deterministic demand. Around that time Manne (1958) came up with a LO problem for economic lot sizing models. Dzielinksi et al. (1963) simulated the models of Manne (1958) in an uncertainty environment and observe that models that don't take uncertainty explicitly into account perform reasonably well in a rolling horizon. Thereafter, attention for the subject was lost for quite a while until it was revived by the work of Soyster (1973). However, the model proposed by Soyster (1973) and others have some undesirable properties: they are either too conservative or non-linear (Soyster (1973); Bertsimas and Sim (2004)).

However, the work of Soyster (1973) has been significantly improved later on by Ben-Tal and Nemirovski (1998, 1999 and 2000). The later authors developed methodologies to deal with uncertainty in LO problems in which conservativeness can be controlled and that are computationally tractable in most cases. By introducing uncertainty into these problems, they become Semi-Infinite Optimization problems. Ben-Tal and Nemirovski show that under certain circumstances these can be rewritten into their so called Robust Counterpart, that are again LO problems in most cases. After the work of the aforementioned authors the field of RO was born and a lot of research followed.

For example, Bertsimas and Sim (2004) propose an approach to solve uncertain LO by means of a robust formulation that is linear and has a parameter to regulate per period the conservativeness. The work of Bertsimas and Thiele (2006) show the usefulness of this technique by applying it to problems in inventory theory. Furthermore, this work can be regarded as the spark that led to the widespread attention for the use of RO. The work of Ben-Tal et al. (2004) introduces multi-stage RO. In terms of production planning this would mean that instead of considering the production quantities in the next T periods as here and now decisions, we would let the production quantities in period t depend in an affine fashion on the realizations of demand in previous periods. Some interesting results based on this technique have been achieved by Ben-Tal et al. (2005) regarding retailer-supplier flexible commitment contracts, and Ben-Tal et al. (2009b) in relation to multi-echelon inventory theory.

Furthermore, it is worth noticing that the work of Ben-Tal et al. (2009a, pp. 4-7) illustrates the impact of minor deviations in the nominal values of the parameters used in LO problems. In their work they show that even the slightest deviation from the chosen parameter might render an optimal solution infeasible and thereby meaningless. In such situation the RO techniques show their capability in coming up with a more robust solution which can deal with uncertainty. All in all, it can be said that RO became a proven technique to deal with uncertainty in Mathematical Optimization problems.

With the ascent of RO we can now distinct two different approaches to deal with uncertainty: one in which uncertainty is defined by means of probability distribution and one where there is a more geometric interpretation of uncertainty. In the former case a probability distribution is underlying demand, while in the latter we extend the polyhedron of the solution space. There are examples of applications of the former to the CLSP like the work of Helber et al. (2013), Rossi et al. (2015) and Tempelmeier and Hilger (2015) shows. They take the nominal CLSP as starting point and deal with it from a stochastic point of view. This means that demand is assumed to be identically and independently normally distributed.

However, in case of taking a stochastic approach two additional complexities are introduced as Winands et al. (2011) and Ben-Tal et al. (2005) point out. The first is that in practice information regarding the demand distribution is unavailable or hard (costly) to obtain. Second, the curse of dimensionality might render it (computationally) impossible to consider multiple products or a realistic number of time periods. When taking a RO approach we do not suffer from these complexities. So, with the possibility to formulate the CLSP as a Mixed Integer LO problem, a RO approach lends itself very well to deal with uncertainty, while avoiding the aforementioned complexities.

The second shortcoming of previous research that we mentioned and that we want to overcome is the fact that little is known about the performance of CLSP models in a rolling horizon setting. Dzielinksi et al. (1963) were most likely one of the first to note the importance of studying production planning problems in a rolling horizon and more specifically, Bookbinder and Tan (1988) in relation to lot sizing. Drexl and Kimms (1997) as well as Helber et al. (2013) mention the importance of future research on simulating lot sizing problems in a rolling horizon setting. This becomes even more clear if we look at the work of Tempelmeier and Hilger (2015). The model proposed herein is run and they only look at the objective value and this tells us nothing on how such model would perform in practice. If we want to assess how this model would perform in practice we should conduct a simulation study in a rolling horizon setting. We are very eager to get to know how this model would perform and for that reason we will investigated their approach.

Above we discussed two shortcomings in present day research on the CLSP. These shortcomings motivate our research and we want to contribute to the field by coming up with solutions for these shortcomings. Hence, our contributions to the field are as follows:

- We will take a novel approach based on RO to deal with demand uncertainty in the nominal CLSP. This has not been done before at the scale we will do it in this research. This step will lead to the so called Robust Counterpart of the nominal CLSP under the assumption that demand is known to range in an interval.

- We will derive our own Stochastic Counterpart of the nominal CLSP and compare it to the one of Tempelmeier and Hilger (2015). Contrary to before, we will now assume demand to randomly distributed.

- We will conduct an extensive simulation study in which the various models will be compared in a rolling horizon setting. Literature is scarce, if not non-existing, on the effects of running CLSP models under demand uncertainty in a rolling horizon, despite the fact that in practice models run in such a setting. This way we hope to shed light on the effects a rolling horizon.

In the next section we translate these contributions in a research proposition. We define the problem we want to study and come up with related research questions.

1.3 Problem Statement and Research Questions

In this section we start by giving a more formal introduction to the CLSP. We start by expressing the problem as a Mixed Integer LO Problem. Thereafter, we discuss relevant aspects to the problem. In combination with the motivation of the research this will lead to the research questions.

From the description of the CLSP in the previous section it became clear that we have a trade-off between holding inventory or setting up production for a specific product more frequently. This trade-off should be represented in the objective function of a Mixed Integer LO formulation. Furthermore, there are some aspects to take into account. First, if demand is deterministic then the resulting lot size will not account for any other realization of demand. Hence, we might get some back-orders if it turns out to be higher than expected in case of demand uncertainty. This means that we should allow for negative inventory contrary to a lot of other models. Second, if we allow for negative inventory there is no incentive to produce anymore. For that reason we will introduce back-order costs. We are fully aware about the controversies that exist in literature, like Tempelmeier and Hilger (2015) arguing that backlog costs are "difficult if not impossible to quantify". However, including back-order costs will turn out to be a great advantage in the models we propose and seems realistic too from the large body of supply chain optimization articles that do assume that back-order costs can be quantified, e.g. de Kok and Fransoo (2003). For that reason we do take back-orders into account. Third, as we will see later on, we will introduce a service level constraint. The introduction of this constraint might have as consequence that we want to produce more than the capacity allows. In order to come up with feasible production plans during the simulation study we allow overtime to relax the problem.

Based on the problem description that we gave in the previous section, taking into account the points mentioned above and based on the work of Helber et al. (2013), we can describe the CLSP as the Mixed Integer LO problem found in Problem 1.1 below.

Problem 1.1 (Nominal CLSP).

$$\textit{min} \sum_{t=1}^{T}\sum_{k\in\mathcal{K}}(s_k^c\gamma_{kt}+y_{kt})+\sum_{t=1}^{T}o^c o_t \tag{1.1}$$

$$\textit{s.t.}\ h_k^c\Big(I_{k0}+\sum_{\tau=1}^{t}(q_{k\tau}-d_{k\tau})\Big)\leq y_{kt} \qquad k\in\mathcal{K},t\in\mathcal{T} \tag{1.2}$$

$$-b_k^c\Big(I_{k0}+\sum_{\tau=1}^{t}(q_{k\tau}-d_{k\tau})\Big)\leq y_{kt} \qquad k\in\mathcal{K},t\in\mathcal{T} \tag{1.3}$$

$$q_{kt}\leq M\gamma_{kt} \qquad k\in\mathcal{K},t\in\mathcal{T} \tag{1.4}$$

$$\sum_{k\in\mathcal{K}}(t_k^p q_{kt}+t_k^s\gamma_{kt})\leq C_t+o_t \qquad t\in\mathcal{T} \tag{1.5}$$

$$\gamma_{kt}\in\{0,1\} \qquad k\in\mathcal{K},t\in\mathcal{T} \tag{1.6}$$

$$q_{kt},o_t\geq 0 \qquad k\in\mathcal{K},t\in\mathcal{T} \tag{1.7}$$

In Problem 1.1 we have s_k^c, h_k^c, b_k^c and o^c representing, respectively, the setup cost, inventory holding cost, back-order cost and overtime cost. Furthermore, we have decision variables y_{kt}, q_{kt}, o_t and γ_{kt} representing, respectively, the upper bound on the maximum cost, the production quantity for product k in period t, the overtime in period t and the setup decision for product k in period t. Besides that we have input parameter d_{kt} representing the per period demand for product k and I_{k0} representing the initial inventory level for product k.

The objective function in 1.1 wants to minimize these costs. Constraint 1.2 and 1.3 are a closed form representation of the inventory balance equation. 1.4 ensures that the binary decision variable γ_{kt} becomes one if there is some production for product k in period t. In Constraint 1.5 we have the processing time t_k^p and setup time t_k^s. These times are multiplied with, respectively, the amount produced and consequently its setup variable to account for capacity consumption. This value can be greater than the available capacity, but then we its extended with overtime.

The objective of the research presented in this thesis is to make a contribution to the development of more effective models for solving the CLSP aided by recent advances in the field of RO. The nominal problem above should be a good starting point. Though, first of all, we should have a thorough understanding about RO. Building on the understanding of the methodologies we will be able to represent the Robust Counterpart to the nominal model introduced above. Furthermore, we are very much interested in a comparison to existing work. For that reason we will compare the nominal model to the Robust Counterpart and to a Stochastic Counterpart that is heavily inspired by the work of Tempelmeier and Hilger (2015). The comparison is conducted in a simulation study in a rolling horizon setting.

As a direct consequence of the objective of this research we arrive at the following main research question:

- What is the relative performance advantage of taking a RO approach to the CLSP over using the nominal model and its Stochastic Counterpart, which is based on the work of Tempelmeier and Hilger (2015), in a rolling horizon setting where demand is identically and independently normally distributed?

In order to answer this question we will formulate the following sub questions,

1. How can we apply the methodologies of RO to production planning problems?
2. How can we derive a Stochastic Counterpart in case we take a stochastic perspective on demand uncertainty?
3. Can we formulate a Robust Counterpart for the nominal CLSP if demand is known to range in a specified interval?
4. How should we setup an experimental framework and measure the relative performance of the models?
5. We know that we can influence the realized β-service level in the Stochastic Counterpart by means of a parameter. However, is the required service level equal to the realized one? If not, how can we fit the right input to the desired output?
6. How can we influence the realized β-service level in the Robust Counterpart?
7. How much does either of the two models, the Stochastic and Robust Counterpart, perform better than the other under different circumstances?

These sub questions conclude this section. We have seen our motivation, research problem, objectives and research questions. In the next sections we will discuss the experimental framework that we use and describe the research design.

1.4 Experimental Framework

In order to be able to make a proper comparison between the various models that we will study in our research project, we need to define a cost structure and certain performance critera. Of course, this depends on the costs we take into consideration and the objective function that we want to minimize. We define the objective function C, and with that the cost structure of the optimization problems introduced in this research, as follows:

$$C = \sum_{t=1}^{T} \sum_{k \in \mathcal{K}} (s_k^c \gamma_{kt} + \max\{h_k^c I_{kt}, -b_k I_{kt}\}) + \sum_{t=1}^{T} o^c o_t \tag{1.8}$$

where $q_{i,t}$ is the amount of units of prodcut k produced in time period t, h_k^c is the inventory holding cost for one unit of prodcut k, b_k^c is the back-order cost for one unit of prodcut k, s_k^k is the setup cost for associated with product k, I_{kt} is the inventory level for product k at time period t, and γ_{kt} is a binary decision variable indicating whether or not prodcut k is produced in time period t.

This cost structure can relate to many production problems in which we take into consideration the inventory holding cost, back-order cost, setup cost as well as overtime cost. Moreover, the structure relates to a minimize the sum of maximum costs per period, hence, the optimal production quantity minimizes the cost function C. Note that in the definition of the nominal CLSP we already eliminated this maximum with the introduction of the piece-wise linear upper bound on it by means of a new decision variable y_{kt}.

In the simulation study we run the various models in a setting where demand is drawn from the Normal distribution. The reason for this is that this distribution is the one most encountered in work on stochastic models on lot sizing, e.g. Helber et al. (2013), Tempelmeier and Hilger (2015) and Rossi et al. (2015). Besides that, we want to give all advantage to the Stochastic Counterpart as we assume demand to be normally distributed in our derivation for the Stochastic Counterpart. Since we use exactly the same distribution in our simulation study, we give all the advantage to one model and this should give better grounds for a good comparison between the models.

We will compare the models cost-wise, that is, the average costs of running the model while rolling forward for a fixed number of time periods. Furthermore, to make a sensible comparison we report the confidence intervals of various statistics like the realized service level, utilization rate, cycle lengths etc.

It is worth saying something about the choice for h_k^c and b_k^c and its relation to service level constraints. Note, that we do not consider lost sales, but deal with insufficient inventory by means of

Table 1.1: The relation between the probability of no stock out and the backorder cost.

$p_{x<}(x_0)$	h_k^c	b_k^c
0.8	1	3
0.9	1	6
0.95	1	9
0.99	1	99
0.995	1	199

back-orders. When we compare the decision on how much to produce in a period as a unconstrained, single-item news vendor problem, we can consider the related decision rule,

$$p_{x<}(Q^*) = \frac{b_k^c}{b_k^c + h_k^c} \tag{1.9}$$

where $p_{x<}(x_0)$ is the probability that the total demand x is less than value x_0 (Silver et al., 1998, pg. 385) or put differently, the probability of no stock out. The latter can be related directly to a service level constraint, because we can state things like the fraction of demand we want to satisfy from inventory. When the inventory holding cost is normalized to $h = 1$ and when taking various values for $p_{x<}$, we can obtain the related values for b_k^c from Equation 1.9. We give an overview for some combinations in Table 1.1.

By now we have explained our experimental framework. The cost function is explained as well as various performance criterion that are of interest. We will continue with describing our the design of our research.

1.5 Research Design and Outline of the Thesis

The aim of the research presented in this thesis is to make a contribution to the development of more effective models for solving the CLSP under demand uncertainty. We are aided in this effort by recent advances in the field of RO. For that reason, we should first have thorough understanding how a RO approach can be taken to the single item production planning problem. Building on the insights obtained from doing so, we will continue with the CLSP presented in a previous section and formulate its Stochastic and Robust Counterpart. These models will then be extensively investigated during an simulation study in a rolling horizon setting.

First of all, we will give a thorough introduction to RO. The topics relevant for this thesis will be discussed and the reader should get enough knowledge about RO to follow along with explanations of the models that use it. Having a clear picture of what RO is about will lead to a better understanding of how to use it in an production planning context. We will give and thoroughly study some basic production planning models in which a RO approach is taken.

We rely on the knowledge obtained when we continue with studying the CLSP. We will start from the nominal CLSP described before and introduce uncertainty into this from a stochastic perspective. Furthermore, a Robust Counterpart will be derived. All models will be thoroughly studied in a simulation study by comparing the nominal model against its counterparts for various settings of the parameters, e.g. the number of periods to commit production in advance, the level of demand uncertainty or the available capacity. We will gain insight in the behaviour of the various models by conducting the thorough simulation studies.

In Chapter 2 we will start with introducing RO. The knowledge obtained herein will be applied in Chapter 3 where we will introduce the CLSP. The models introduced in Chapter 3 will be subject to thorough testing in our simulation study which is discussed in Chapter 4. The final chapter is Chapter 5 in which we will present our research findings, conclusions and suggest future research. Furthermore, the notation and abbreviations used in this thesis can be found in Appendix A and we present some preliminary mathematical knowledge in Appendix B.

Chapter 2

Models for Robust Production Planning

> *"If a man will begin with certainties, he shall end in doubts; but if he will be content to begin with doubts, he shall end in certainties."*
>
> Francis Bacon, *The Advancement of Learning*

2.1 Introduction

In this chapter we give an introduction into RO and explain its concepts using a basic production planning problem. The nominal production planning model that we present in this chapter is one for planning production for a single item, at a single location, under a per period capacity restriction. Demand needs to be satisfied either by the end of the period in which it occurs or at a later stage when put on back-order. This basic problem deals with the trade-off between holding inventory and accepting back-orders. Costs are incurred for holding inventory and putting demand on back-order. Hence, the objective of the problem is to minimize the maximum of inventory holding costs and back-order costs subject to the given constraints.

We start this chapter with discussing multi-stage decision problems in general. This should provide a solid foundation for understanding the nominal production planning problem that we will introduce. This problem serves us in explaining the concepts and methodologies behind RO. Although we could have only referred to the numerous of introductory papers on RO, we will give a thorough introduction ourself by using the nominal production planning model as starting point. Nevertheless, we could certainly recommend the interested reader the work of Ben-Tal et al. (2009a), Bertsimas et al. (2011) and Gorissen et al. (2015) for a more general introduction to the field.

After introducing our nominal problem, we will introduce uncertainty into it. The problem then becomes a semi-infinite optimization problem, which is computationally untractable. However, for specific types of uncertainty sets we will show how to come up with tractable representations, i.e. their Robust Counterparts. The Robust Counterparts range from the most conservative to the least conservative, one where conservatives can be controlled by means of a parameter and the Affinely Adjustable Robust Counterpart (AARC). These Robust Counterparts can be classified under the heading of static or multi-stage RO. The former only considers "here and now" decisions, while the latter lets the current period production quantity depend on the demand in pervious periods in an affine fashion. Using this dependency in the form of Linear Decision Rules results in the so called AARC. We conclude with a future outlook of RO and we especially discuss Distributionally RO and Multi-Stage Adjustable Robust Mixed-Integer Optimization.

2.2 Multi-Stage Decision Problems

Multi-stage optimization problems under uncertainty can be found in numerous fields of study and a wide variety of solutions methods exists to solve them, e.g. exact and approximate dynamic programming, stochastic programming, sampling-based methods and robust (multi-stage) optimization. We start in this chapter from the broad idea of a multi-stage decision problem.

Problem 2.1 (Multi-Stage Decision Problem)**.** *Consider the following one-dimensional, discrete-time, linear dynamical system,*

$$I_{t+1} = \alpha_k I_t + \beta_t q_t + \gamma_t d_t \tag{2.1}$$

where I_t represents the state of system at period t, and given the initial state of the system $I_1 \in \mathbb{R}$. Furthermore, $\alpha_t, \beta_t, \gamma_t \neq 0$ are known scalars. The system is affected by random disturbances d_t which are unknown, but range with certainty in a specified interval centered around $\bar{d}_t$ with half-width $\hat{d}_t$,

$$d_t \in \mathcal{D}_t \stackrel{def}{=} [\bar{d}_t - \hat{d}_t, \bar{d}_t + \hat{d}_t] \tag{2.2}$$

We are interested in finding a sequence of controllers $q_1, q_2, \ldots, q_T$, that are constraint by fixed and known upper and lower bounds for each period,

$$q_t \in [L_t, U_t] \tag{2.3}$$

with $L_t, U_t \in \mathbb{R}$ and that minimizes the following cost function over a finite horizon $1, 2, \ldots, T$,

$$J = c_1 q_1 + \max_{d_1}[f_1(I_2) + c_2 q_2 + \max_{d_2}[f_2(I_3) + \ldots + \max_{d_{T-1}}[c_T q_T + \max_{d_T} f_T(I_{T+1})] \ldots]] \tag{2.4}$$

where the functions $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ are extended real and convex, and scalars $c_t \geq 0$ are fixed and known.

The production planning problems that we study in this chapter are examples of multi-stage decision problems. For instance, to obtain our nominal production planning problem, the case were demand is known, the first step would be to set the half-width of the demand to zero, $\hat{d}_t = 0$. We then let I_t represent the state of the inventory level at time t and we would take $f_t(I_{t+1}) = \max\{hI_{t+1}, -bI_{t+1}\}$ to represent the trade-off between holding inventory and accepting back-orders. We set $\alpha, \beta = 1$, $\gamma = -1$, such that the behavior of the linear dynamical system represented by Equation 2.1 mimics the inventory balance equation. Furthermore, $L_t = 0$, $U_t = C_t$, with C_t being sufficiently large in case we do not want capacity to influence production.

In general, we could take a Dynamic Programming (DP) approach to solve Problem 2.1 based on the work of Bertsekas (2001). Interestingly, the resulting policy would then exactly correspond to the base-stock ordering policies of Clark and Scarf (1960). Though, as we will see in the next section, when demand is certain, a linear optimization formulation would suffice. In the sections thereafter we explore the situation in which demand is made uncertain in the nominal model. We will investigate various Robust Counterparts like the least conservative, the most conservative and the cardinality constraint robust counterparts. Until then, the problems studied only revolve around "here and know" decisions or what we will call static RO. However, a interesting class of policies comes into being when the controllers q_t are affine parameterizations in the observed disturbances, e.g. the past demand:

$$q_t(\mathbf{d}) \in [L_t, U_t], \qquad \forall \mathbf{d} \in \mathcal{D}_1 \times \mathcal{D}_2 \times \ldots \times \mathcal{D}_{t-1} \tag{2.5}$$

As we will see in a later section, the optimal affine policies $q_t(\mathbf{d})$ of these so called disturbance affine feedback control policies, essentially multi-stage RO, can be obtained by deriving the AARC of Problem 2.1. Though, before diving in these complexities, we will start with formulating the nominal production planning problem in the next section.

2.3 Production Planning as Linear Optimization Problem

In our nominal problem we consider planning production for single item under deterministic demand. Costs are incurred for holding inventory and back-orders. The objective of our nominal production planning then becomes,

$$\min \sum_{t=1}^{T} \max\{hI_t, -bI_t\} \tag{2.6}$$

where q_t is the per period production quantity, h is the per unit inventory holding cost, b is the per unit back-order cost, I_t is the inventory level at the end of period t and T indicates the finite time horizon on which we consider the production planning problem. This results in a min-max type of optimization problem where we need to make a decision on the per period production quantities q_t while taking into consideration related costs and subject to the capacity constraint.

Next, we formulate the problem as a linear optimization problem by introducing the related constraints. One of the constraints represents the inventory balance equation, $I_t = I_{t-1} + q_t - d_t$ and the other represents the capacity as upper bound on the per period production quantities.

$$\begin{aligned}
\text{min} \quad & \sum_{t=1}^{T} \max\{hI_t, -bI_t\} && && (2.7)\\
\text{s.t.} \quad & I_t = I_{t-1} + q_t - d_t && t = 1 \ldots T && (2.8)\\
& q_t \leq C_t && t = 1 \ldots T && (2.9)\\
& q_t \geq 0 && t = 1 \ldots T && (2.10)
\end{aligned}$$

Note, the initial inventory level, I_0 is an input parameter. We can eliminate the maximum in the objective function by introducing a new decision variable and two new constraints. The new decision variable y_t is introduced to represent the upper bounds on the piece-wise linear components in the objective function.

$$\begin{aligned}
\text{min} \quad & \sum_{t=1}^{T} y_t && && (2.11)\\
\text{s.t.} \quad & y_t \geq -bI_t && t = 1 \ldots T && (2.12)\\
& y_t \geq hI_t && t = 1 \ldots T && (2.13)\\
& I_t = I_{t-1} + q_t - d_t && t = 1 \ldots T && (2.14)\\
& q_t \leq C_t && t = 1 \ldots T && (2.15)\\
& q_t, y_t \geq 0 && t = 1 \ldots T && (2.16)
\end{aligned}$$

The inventory level at time t can be formulated as the sum of production quantities up to t minus the demand up to t, plus the initial inventory level I_0. Therefore, we can go from a recursive definition of the inventory level at time t to a closed form expression, $I_t = I_0 + \sum_{t=1}^{T} q_t - d_t$. Substituting this for I_t in the respective constraints results in the following linear optimization formulation for our nominal production planning problem.

Problem 2.2 (Nominal Production Planning Problem)**.**

$$\begin{aligned}
\textit{min} \quad & \sum_{t=1}^{T} y_t && && (2.17)\\
\textit{s.t.} \quad & h\big(I_0 + \sum_{s=1}^{t}(q_s - d_s)\big) \leq y_t && t = 1 \ldots T && (2.18)\\
& -b\big(I_0 + \sum_{s=1}^{t}(q_s - d_s)\big) \leq y_t && t = 1 \ldots T && (2.19)\\
& q_t \leq C_t && t = 1 \ldots T && (2.20)\\
& q_t, y_t \geq 0 && t = 1 \ldots T && (2.21)
\end{aligned}$$

The nominal production planning problem defined in Problem 2.2 will serve as the basis for the Robust Counterparts we discuss in the upcoming sections.

2.4 Static Robust Optimization Models

2.4.1 Introducing Uncertainty

We introduce uncertainty into the nominal production planning problem formulated in Problem 2.2 by stating that demand is an element of uncertainty set $\mathcal{U}$ and we define this set as follows,

$$\mathcal{U} = \left\{ \mathbf{d} = d^0 + \sum_{t=1}^{T} \zeta_t d^t : \zeta \in \mathcal{Z} \subset \mathbb{R}^T \right\} \tag{2.22}$$

with,

$$d^0 = \begin{pmatrix} \bar{d}_1 \\ \bar{d}_1 \\ \vdots \\ \bar{d}_T \end{pmatrix}, \qquad d^t = \hat{d}_t \mathbf{e}_t \tag{2.23}$$

Introducing this uncertainty set in Problem 2.2 leads to the semi-infinite optimization problem of Problem 2.3, because we have a finite number of constraints and a infinite amount of possible demand realizations. However, as we will see in the next subsection, depending on how we define the set of perturbation vectors $\zeta \in \mathcal{Z} \subset \mathbb{R}^T$ we are able to derive a tractable representation for Problem 2.3, otherwise known as its Robust Counterpart.

Problem 2.3 (Uncertain Production Planning Problem)**.**

$$\textit{min} \sum_{t=1}^{T} y_t \tag{2.24}$$

$$\textit{s.t.}\ h\big(I_0 + \sum_{s=1}^{t}(q_s - d_s)\big) \le y_t \qquad \mathbf{d} \in \mathcal{U}, t = 1 \dots T \tag{2.25}$$

$$-b\big(I_0 + \sum_{s=1}^{t}(q_s - d_s)\big) \le y_t \qquad \mathbf{d} \in \mathcal{U}, t = 1 \dots T \tag{2.26}$$

$$q_t \le C_t \qquad t = 1 \dots T \tag{2.27}$$

$$q_t, y_t \ge 0 \qquad t = 1 \dots T \tag{2.28}$$

2.4.2 A Conservative Robust Counterpart: l_∞-norm

We now consider the case a l_∞-norm is defined on the set of perturbation vectors. This type of uncertainty is also known as box or interval uncertainty, i.e. we assume the per period realization of demand to range in the interval $d_t = [\bar{d}_t - \hat{d}_t, \bar{d}_t + \hat{d}_t]$. We continue from the uncertain production planning problem found in Problem 2.3 and define the set of perturbation vectors $\mathcal{Z}$ as,

$$\mathcal{Z} = \{\zeta \in \mathbb{R}^T : \|\zeta\|_\infty \le 1\} \tag{2.29}$$

The demand vector is parameterized in an affine fashion by the perturbation vectors,

$$\mathbf{d}(\zeta) = d^0 + \sum_{t=1}^{T} d^t \zeta_t \qquad \|\zeta\|_\infty \le 1 \tag{2.30}$$

For this reason the individual per period demand can be described as follows,

$$d_t(\zeta) = \bar{d}_t + \hat{d}_t \zeta_t \qquad |\zeta_t| \le 1 \tag{2.31}$$

Clearly, when we enforce the l_∞-norm on the perturbation vectors and bound it from above by 1, we obtain interval uncertainty. When we combine this all, we get uncertainty set $\mathcal{U}_\infty$,

$$\mathcal{U}_\infty = \left\{ \mathbf{d} = d^0 + \sum_{t=1}^{T} \zeta_t d^t : \zeta \in \mathcal{Z} = \{\zeta \in \mathbb{R}^T : \|\zeta\|_\infty \le 1\} \subset \mathbb{R}^T \right\} \tag{2.32}$$

We rewrite some of the constraints in the uncertain production planning problem for the sake of convenience and this leads to the following,

$$\min \sum_{t=1}^{T} y_t \tag{2.33}$$

$$\text{s.t.} \quad -y_t + h\big(\sum_{s=1}^{t} q_s - \sum_{s=1}^{t} d_s z_s\big) \leq -hI_0 \qquad \forall \mathbf{d} \in \mathcal{U}_\infty, t = 1 \dots T \tag{2.34}$$

$$-y_t - b\big(\sum_{s=1}^{t} q_s - \sum_{s=1}^{t} d_s z_s\big) \leq bI_0 \qquad \forall \mathbf{d} \in \mathcal{U}_\infty, t = 1 \dots T \tag{2.35}$$

$$q_t \leq C_t \qquad t = 1 \dots T \tag{2.36}$$

$$z_t = 1 \qquad t = 1 \dots T \tag{2.37}$$

$$q_t, y_t \geq 0 \qquad t = 1 \dots T \tag{2.38}$$

We can introduce the parameterized versions of the per period demand, $d_t(\zeta) = \bar{d}_t + \hat{d}_t\zeta_t$, into the uncertain production planning problem and use the definition of uncertainty set $\mathcal{U}_\infty$ to arrive at the following,

$$\min \sum_{t=1}^{T} y_t \tag{2.39}$$

$$\text{s.t.} \quad -y_t + h\big(\sum_{s=1}^{t} q_s - \sum_{s=1}^{t} (\bar{d}_s + \hat{d}_s\zeta_s) z_s\big) \leq -hI_0 \qquad \|\zeta\|_\infty \leq 1, t = 1 \dots T \tag{2.40}$$

$$-y_t - b\big(\sum_{s=1}^{t} q_s - \sum_{s=1}^{t} (\bar{d}_s + \hat{d}_s\zeta_s) z_s\big) \leq bI_0 \qquad \|\zeta\|_\infty \leq 1, t = 1 \dots T \tag{2.41}$$

$$q_t \leq C_t \qquad t = 1 \dots T \tag{2.42}$$

$$z_t = 1 \qquad t = 1 \dots T \tag{2.43}$$

$$q_t, y_t \geq 0 \qquad t = 1 \dots T \tag{2.44}$$

which can be rewritten into,

$$\min \sum_{t=1}^{T} y_t \tag{2.45}$$

$$\text{s.t.} \quad -y_t + h\big(\sum_{s=1}^{t} q_s - \sum_{s=1}^{t} \bar{d}_s z_s - \sum_{s=1}^{t} \hat{d}_s\zeta_s z_s\big) \leq -hI_0 \qquad |\zeta_t| \leq 1, t = 1 \dots T \tag{2.46}$$

$$-y_t - b\big(\sum_{s=1}^{t} q_s - \sum_{s=1}^{t} \bar{d}_s z_s - \sum_{s=1}^{t} \hat{d}_s\zeta_s z_s\big) \leq bI_0 \qquad |\zeta_t| \leq 1, t = 1 \dots T \tag{2.47}$$

$$q_t \leq C_t \qquad t = 1 \dots T \tag{2.48}$$

$$z_t = 1 \qquad t = 1 \dots T \tag{2.49}$$

$$q_t, y_t \geq 0 \qquad t = 1 \dots T \tag{2.50}$$

and from this, we obtain,

$$\min \sum_{t=1}^{T} y_t \tag{2.51}$$

$$\text{s.t.} \quad -y_t + h\Big(\sum_{s=1}^{t} q_s - \sum_{s=1}^{t} \bar{d}_s z_s + \max_{\zeta:\|\zeta\|_\infty \leq 1} \sum_{s=1}^{t} \hat{d}_s \zeta_s z_s\Big) \leq -hI_0 \qquad t = 1 \dots T \tag{2.52}$$

$$-y_t - b\Big(\sum_{s=1}^{t} bq_s - \sum_{s=1}^{t} \bar{d}_s z_s - \max_{\zeta:\|\zeta\|_\infty \leq 1} \sum_{s=1}^{t} \hat{d}_s \zeta_s z_s \leq bI_0 \qquad t = 1 \dots T \tag{2.53}$$

$$q_t \leq C_t \qquad t = 1 \dots T \tag{2.54}$$

$$z_t = 1 \qquad t = 1 \dots T \tag{2.55}$$

$$q_t, y_t \geq 0 \qquad t = 1 \dots T \tag{2.56}$$

In general, and as a consequence of Hölder's inequality (see Appendix B), when $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the norms $\|.\|_p$ and $\|.\|_q$ are conjugates of each other,

$$\|x\|_p = \max_{y:\|y\|_q \leq 1} |\langle x, y \rangle| \tag{2.57}$$

We can use this to find the maximum in our constraints,

$$\max_{\zeta:\|\zeta\|_\infty \leq 1} \sum_{s=1}^{t} \hat{d}_s \zeta_s z_s = \max_{\zeta:\|\zeta\|_\infty \leq 1} |\langle \zeta, \hat{\mathbf{d}}\mathbf{z}^\top \rangle| = \| \begin{pmatrix} \hat{d}_1 z_1 \\ \hat{d}_2 z_2 \\ \vdots \\ \hat{d}_s z_T \end{pmatrix} \|_1 = \sum_{s=1}^{t} |\hat{d}_s z_s| \tag{2.58}$$

Therefore, using $\mathcal{U}_\infty$ as our uncertainty set, we arrive at the following Robust Counterpart of Problem 2.3,

Problem 2.4 (Robust Counterpart l_∞-norm)**.**

$$\textit{min} \sum_{t=1}^{T} y_t \tag{2.59}$$

$$\textit{s.t.} \quad h\Big(I_0 + \sum_{s=1}^{t} q_s - \sum_{s=1}^{t} \bar{d}_s z_s + \sum_{s=1}^{t} \omega_s\Big) \leq y_t \qquad t = 1 \dots T \tag{2.60}$$

$$-b\Big(I_0 + \sum_{s=1}^{t} q_s - \sum_{s=1}^{t} \bar{d}_s z_s - \sum_{s=1}^{t} \omega_s\Big) \leq y_t \qquad t = 1 \dots T \tag{2.61}$$

$$-\omega_t \leq \hat{d}_t z_t \leq \omega_t \qquad t = 1 \dots T \tag{2.62}$$

$$q_t \leq C_t \qquad t = 1 \dots T \tag{2.63}$$

$$z_t = 1 \qquad t = 1 \dots T \tag{2.64}$$

$$q_t, y_t \geq 0 \qquad t = 1 \dots T \tag{2.65}$$

What is of interest is the fact that when introducing uncertainty into our nominal production planning problem, formulated as linear optimization problem, we got a semi-infinite optimization problem which in itself is intractable. However, when using interval uncertainty, we showed how to rewrite this problem to obtain a tractable formulation known as its Robust Counterpart. The beauty in this derivation lies in the fact that its Robust Counterpart happens to be a linear optimization problem again.

Furthermore, we see that the Robust Counterpart is indeed the most conservative approach to deal with demand uncertainty, because for each period it penalizes the constraints in the model by $\sum_{s=1}^{t} |\hat{d}_s z_s|$, being the worst-case deviation. The interested reader might find more about the conservativeness in RO in the work of Gorissen and den Hertog (2013). Though, one should keep in mind, even though the model is conservative, as far as the objective value concerned, nothing is said by that on how the model would perform in a simulation study in a rolling horizon setting. Moreover, as we will see later when discussing optimality, the parameters for the inventory holding cost and back-order cost in Problem 2.4 lend themselves very well to control the conservativeness of the model.

2.4.3 A Less Conservative Robust Counterpart: l_1-norm

In the this subsection we investigate the least conservative Robust Counterpart. We will start again from the uncertain production planning problem of Problem 2.3. Though, now define the set of perturbation vectors as $\mathcal{Z} = \{\zeta \in \mathbb{R}^T : \|\zeta\|_1 \leq \gamma\}$. Therefore, we obtain the following uncertainty set,

$$\mathcal{U}_1 = \left\{ \mathbf{d} = d^0 + \sum_{t=1}^{T} \zeta_t d^t : \zeta \in \mathcal{Z} = \{\zeta \in \mathbb{R}^T : \|\zeta\|_1 \leq \gamma\} \subset \mathbb{R}^T \right\} \tag{2.66}$$

Analogous to what we have done in case of interval uncertainty, we can introduce the parameterized versions of demand, $d_t(\zeta) = \bar{d}_t + \hat{d}_t\zeta_t$, into the problem and arrive at the following,

$$\begin{aligned}
\min \ & \sum_{t=1}^{T} y_t && && (2.67)\\
\text{s.t.} \ & -y_t + \sum_{s=1}^{t} hq_s - h\sum_{s=1}^{t}(\bar{d}_s + \hat{d}_s\zeta_s)z_s \leq -hI_0 && t = 1\ldots T && (2.68)\\
& -y_t - \sum_{s=1}^{t} bq_s + b\sum_{s=1}^{t}(\bar{d}_s + \hat{d}_s\zeta_t)z_s \leq bI_0 && t = 1\ldots T && (2.69)\\
& q_t \leq C_t && t = 1\ldots T && (2.70)\\
& z_t = 1 && t = 1\ldots T && (2.71)\\
& q_t, y_t \geq 0 && t = 1\ldots T && (2.72)
\end{aligned}$$

which can be rewritten in the same fashion as before into,

$$\begin{aligned}
\min \ & \sum_{t=1}^{T} y_t && && (2.73)\\
\text{s.t.} \ & -y_t + \sum_{s=1}^{t} hq_s - h\sum_{s=1}^{t}\bar{d}_s z_s - h\max_{\zeta:\|\zeta\|_1\leq\gamma}\sum_{s=1}^{t}\hat{d}_s\zeta_s z_s \leq -hI_0 && t = 1\ldots T && (2.74)\\
& -y_t - \sum_{s=1}^{t} bq_s + b\sum_{s=1}^{t}\bar{d}_s z_s + b\max_{\zeta:\|\zeta\|_1\leq\gamma}\sum_{s=1}^{t}\hat{d}_s\zeta_s z_s \leq bI_0 && t = 1\ldots T && (2.75)\\
& q_t \leq C_t && t = 1\ldots T && (2.76)\\
& z_t = 1 && t = 1\ldots T && (2.77)\\
& q_t, y_t \geq 0 && t = 1\ldots T && (2.78)
\end{aligned}$$

We again use the fact that the l_1-norm and l_∞-norm are conjugates of each other to obtain the maximum in our constraints,

$$\max_{\zeta:\|\zeta\|_1\leq\gamma}\sum_{s=1}^{t}\hat{d}_s\zeta_s z_s = \max_{\zeta:\|\zeta\|_1\leq\gamma}|\langle\zeta, \hat{\mathbf{d}}\mathbf{z}^\top\rangle| = \| \begin{pmatrix} \hat{d}_1 z_1 \\ \hat{d}_2 z_2 \\ \vdots \\ \hat{d}_s z_T \end{pmatrix} \|_\infty = \gamma \max_s |\hat{d}_s z_s| \tag{2.79}$$

Therefore, when using $\mathcal{U}_1$ as our uncertainty set, we arrive at the following Robust Counterpart of Problem 2.3,

Problem 2.5 (Robust Counterpart l_1-norm).

$$\text{min} \sum_{t=1}^{T} y_t \tag{2.80}$$

$$\text{s.t.} \quad -y_t + \sum_{s=1}^{t} hq_s - h\sum_{s=1}^{t} \bar{d}_s z_s - \gamma h \max_{s:s<t} |\hat{d}_s z_s| \le -hI_0 \qquad t = 1 \dots T \tag{2.81}$$

$$-y_t - \sum_{s=1}^{t} bq_s + b\sum_{s=1}^{t} \bar{d}_s z_s + \gamma b \max_{s:s<t} |\hat{d}_s z_s| \le bI_0 \qquad t = 1 \dots T \tag{2.82}$$

$$q_t \le C_t \qquad t = 1 \dots T \tag{2.83}$$

$$z_t = 1 \qquad t = 1 \dots T \tag{2.84}$$

$$q_t, y_t \ge 0 \qquad t = 1 \dots T \tag{2.85}$$

The Robust Counterpart found in Problem 2.5 gets a penalty for robustness of $\gamma \max_s |\hat{d}_s z_s|$ or the maximum deviation over all possible deviations for the planning horizon $t = 1 \dots T$ times γ, a parameter set to be set by the decision maker. We explore this more in the one of the next subsection when we compare it to the conservative Robust Counterpart.

2.4.4 Cardinality Constrained Robust Counterpart

In this section we assume cardinality constrained uncertainty and derive a robust counterpart for the nominal problem found in Problem 2.2. This type of uncertainty was first introduced by Bertsimas and Sim (2004) and gained widespread attention after showing its usefulness in a supply chain setting by Bertsimas and Thiele (2006). Other examples include the work of Alem and Morabito (2012) in furniture production and the work of Aouam and Brahimi (2013) on integrated production planning and order acceptance.

When using this type of uncertainty the per period demand d_t is still assumed to range in the interval, $[\bar{d}_t - \hat{d}_t, \bar{d}_t + \hat{d}_t]$, though, each period the maximum allowed deviation of the center is different and constraint from above. The set of perturbation is defined in this case as, $\zeta \in \mathcal{Z} = \left\{\zeta \in \mathbb{R}^T : \|\zeta\|_\infty \le 1 \wedge \sum_{s=1}^{t} |\zeta_s| \le \Gamma_t\right\}$ and hence, we obtain the following uncertainty set,

$$\mathcal{U}_c = \left\{ \mathbf{d} = d^0 + \sum_{t=1}^{T} \zeta_t d^t : \zeta \in \mathcal{Z} = \left\{ \zeta \in \mathbb{R}^T : \|\zeta\|_\infty \le 1 \wedge \sum_{s=1}^{t} |\zeta_s| \le \Gamma_t \right\} \subset \mathbb{R}^T \right\} \tag{2.86}$$

Again, demand is parameterized by the perturbation vectors, hence, $d_t(\zeta) = \bar{d}_t + \hat{d}_t \zeta_t$. We can substitute this in the nominal problem and obtain the following,

$$\min \sum_{t=1}^{T} (cq_t + y_t) \tag{2.87}$$

$$\text{s.t.} \quad y_t \ge h\left(I_0 + \sum_{s=1}^{t} (q_s - (\bar{d}_t + \hat{d}_t \zeta_t))\right) \qquad \forall \mathbf{d} \in \mathcal{U}_c, t = 1 \dots T \tag{2.88}$$

$$y_t \ge -b\left(I_0 + \sum_{s=1}^{t} (q_s - (\bar{d}_t + \hat{d}_t \zeta_t))\right) \qquad \forall \mathbf{d} \in \mathcal{U}_c, t = 1 \dots T \tag{2.89}$$

$$q_t \le C_t \qquad t = 1 \dots T \tag{2.90}$$

$$q_t, y_t \ge 0 \qquad t = 1 \dots T \tag{2.91}$$

Clearly, we have to solve the following auxiliary linear programming problem for all periods t,

$$\max \sum_{s=1}^{t} \hat{d}_t \zeta_t \tag{2.92}$$

$$\text{s.t.} \sum_{s=1}^{t} \zeta_t \leq \Gamma_t \tag{2.93}$$

$$0 \leq \|\zeta_t\| \leq 1 \tag{2.94}$$

Following Theorem C.1 we obtain the dual which we can substitute back in the aforementioned problem. It readily follows that the robust counterpart is as follows,

Problem 2.6 (Robust Counterpart Cardinality Constrained Uncertainty)**.**

$$\min \sum_{t=1}^{T} (cq_t + y_t) \tag{2.95}$$

$$\text{s.t. } y_t \geq h(I_0 + \sum_{s=1}^{t}(q_s - \bar{d}_t) + v_t\Gamma_k + \sum_{i=1}^{t} r_{it}) \qquad t = 1 \dots T \tag{2.96}$$

$$y_t \geq b(-I_0 - \sum_{s=1}^{t}(q_s - \bar{d}_t) + v_t\Gamma_k + \sum_{i=1}^{t} r_{it}) \qquad t = 1 \dots T \tag{2.97}$$

$$v_t + r_{it} \geq \hat{d}_i \qquad i < t, t = 1 \dots T \tag{2.98}$$

$$v_t \geq 0 \qquad i < t, t = 1 \dots T \tag{2.99}$$

$$r_{it} \geq 0 \qquad i < t, t = 1 \dots T \tag{2.100}$$

$$q_t \leq C_t \qquad t = 1 \dots T \tag{2.101}$$

2.5 On Optimality of Static Robust Optimization

2.5.1 Notes on Interval Uncertainty

Bertsimas and Thiele (2006) derive some interesting properties for the Robust Counterpart under cardinality constraint uncertainty. In this subsection we extend them to the case of interval uncertainty. Though, we only consider the case where there are no fixed ordering costs, because we didn't take those into account thus far. In order to comprehend these results we have to start with the following definition,

Definition 2.1 (**(S,S) and (s,S) Policies, cf. Bertsimas and Thiele (2006)**)**.** *The optimal policy of an inventory optimization problem is (s, S), or base-stock, if there exists a threshold sequence (s_t, S_t) such that at each period it is optimal to let the linear dynamical system of Problem 2.1 be corrected by $q_t = S_t - d_t$ if $I_t < s_t$ and zero otherwise. If there is no fixed ordering cost, $s_t = S_t$.*

This definition is essential in the theorem and corollary that we will see shortly. Below we repeat a theorem from Bertsimas and Thiele (2006) on the optimal robust policy in case of cardinality constraint uncertainty.

Theorem 2.1 (**Optimal Robust Policy for $\mathcal{U}_c$, cf. Bertsimas and Thiele (2006)**)**.** *1. In case of cardinality constraint uncertainty, the optimal policy for the Robust Counterpart found in Problem 2.6, evaluated at time 1 for the rest of the horizon, is the optimal policy for the nominal problem with the modified demand,*

$$d'_t = \bar{d}_t + \frac{b-h}{b+h}(A_t - A_{t-1}) \tag{2.102}$$

where $A_t = v_t^\star \Gamma_t + \sum_{i=1}^{t} r_{it}^*$ is the deviation of the cumulative demand from its mean at time t, $v_t^\star$ and $r_{it}^\star$ being the optimal variables in Problem 2.6.

2. *If there is no fixed cost, the optimal robust policy is (S, S) with $S_t = d'_t$ for all t.*

The interested reader is referred to Bertsimas and Thiele (2006) for a formal proof of Theorem 2.1. It can readily been seen that this theorem is more generic and interval uncertainty is a specific case as the following corollary shows.

Corollary 2.1 (Optimal Robust Policy for $\mathcal{U}_\infty$). *1. In case of interval uncertainty, the optimal policy for the Robust Counterpart found in Problem 2.4, evaluated at time 1 for the rest of the horizon, it the optimal policy for the nominal problem with the modified demand,*

$$d'_t = \bar{d}_t + \frac{b-h}{b+h}\hat{d}_t \tag{2.103}$$

2. *The optimal robust policy is (S, S) with $S_t = d'_t$ for all t.*

Proof. The proof of this corollary is based on the fact that uncertainty set U_∞ can be expressed as a instance of $\mathcal{U}_c$. This is the case if we take Γ_t to be greater than or equal to the sum of maximum values ζ_s, $s \leq t$, can take. We know that $0 \leq |\zeta_t| \leq 1$, so taking $\Gamma_t \geq t$ should suffice and hence, in this case $\mathcal{U}_c = \mathcal{U}_\infty$. Therefore, without any loss of optimality we can let $A_t - A_{t-1}$ be $\bar{d}_t$ in Theorem 2.1 as it accounts for the deviation of the demand from its mean at time t. □

Corollary 2.1 leads to an interesting result, being that the production quantities in the Robust Counterpart aren't covering the worst-case realization of demand: $\bar{d}_t + \hat{d}_t$. Instead, they are based on the modified demand pattern as expressed by Equation 2.103. In other words, they are based on what seems like a critical fractal that is frequently encountered in stochastic (multi-echelon) inventory optimization (see Clark and Scarf (1960), van Houtum (2006)). Clearly, RO applied to the nominal production planning problem does not lead to a Robust Counterpart which produces for the worst-case, but to a Robust Counterpart where we can influence resulting production plan by means of our input parameters, i.e. the inventory holding cost and the back-order cost.

2.5.2 The Effect of Using the l_∞-norm or l_1-norm

Albeit the l_∞-norm and l_1-norm being the complete opposite of each other, they show some resemblance for certain input parameters. This is especially true when simulating the models in a rolling horizon fashion where only the next period's production quantity has to be committed after planning.

In Proposition 2.1 we show that for certain input parameters the first period production quantities, q_1 are equal in case a l_∞-norm or l_1-norms is used.

Proposition 2.1. *Consider the semi-infinite optimization problem found in Problem 2.3. No matter which of the two uncertainty sets, $\mathcal{U}_\infty$ or $\mathcal{U}_1$, we are using, if $\gamma = 1$ and the demand half-length for period t is equal in both Robust Counterparts, then the resulting first period production quantities from both models are equal.*

Proof. For an arbitrary uncertainty set $\mathcal{U}_p$, $p = 1 \vee p = \infty$, and time period t, consider the constraint related to holding inventory,

$$-y_t + \sum_{s=1}^{t} hq_s - h\sum_{s=1}^{t} d_s z_s \leq -hI_0 \qquad \forall \mathbf{d} \in \mathcal{U}_p \tag{2.104}$$

Remark that the demand vector is parameterized in an affine fashion by the perturbation vectors,

$$\mathbf{d}(\zeta) = d^0 + \sum_{t=1}^{T} d^t \zeta_t \qquad \|\zeta\|_p \leq \kappa \tag{2.105}$$

and that the per period demand can then be described as follows,

$$d_t(\zeta) = \bar{d}_t + \hat{d}_t\zeta_t \qquad |\zeta_t| \leq \kappa \tag{2.106}$$

We can substitute this in Equation 2.104 and rewrite everything in a similar fashion as before,

$$-y_t + \sum_{s=1}^{t} hq_s - h\sum_{s=1}^{t} \bar{d}_s z_s - h \max_{\zeta:\|\zeta\|_p \leq \kappa} \sum_{s=1}^{t} \hat{d}_s \zeta_s z_s \leq -hI_0 \tag{2.107}$$

Again, we use the fact that if $p, q \in [1, \infty]$ and $\frac{1}{q} + \frac{1}{p} = 1$, then $\|.\|_p$ and $\|.\|_p$ are conjugates of each other and,

$$\max_{y:\|y\|_q \leq \kappa} |\langle x, y\rangle| = \|x\|_p \tag{2.108}$$

or specific to this situation,

$$\max_{\zeta:\|\zeta\|_p \leq \kappa} \sum_{s=1}^{t} \hat{d}_s \zeta_s z_s = \kappa \| \begin{pmatrix} \hat{d}_1 z_1 \\ \hat{d}_2 z_2 \\ \vdots \\ \hat{d}_s z_T \end{pmatrix} \|_q \tag{2.109}$$

Then, in case $p = \infty$, $\kappa = 1$, and $t = 1$ we obtain the following,

$$\max_{\zeta:\|\zeta\|_\infty \leq 1} \sum_{s=1}^{1} \hat{d}_s \zeta_s z_s = \| \left(\hat{d}_1 z_1\right) \|_1 = \sum_{s=1}^{1} |\hat{d}_s z_s| = |\hat{d}_1 z_1| \tag{2.110}$$

and in case $p = 1$, $\kappa = \gamma = 1$, and $t = 1$ we obtain the following,

$$\max_{\zeta:\|\zeta\|_1 \leq 1} \sum_{s=1}^{1} \hat{d}_s \zeta_s z_s = \| \left(\hat{d}_1 z_1\right) \|_\infty = \max_{s=1} |\hat{d}_s z_s| = |\hat{d}_1 z_1| \tag{2.111}$$

Equations 2.110 and 2.111 are equal. In an analogous way this can be shown to hold for the constraint regarding the back-order cost. Consequently, this will result in the same penalty for both models with regards to the first period production quantity q_1 and because of that, the first period production quantity will be equal in both cases. This concludes the proof. □

A very important insight follows from Proposition 2.1. If we use the Robust Counterpart resulting from taking the l_∞-norm or the l_1-norm in our uncertain production planning problem and set $\gamma = 1$, then we get equal first period production quantities. As a consequence, if we compare both approaches in a rolling horizon setting during a simulation study and only have to commit the first period after planning, then we would obtain equal first period production quantities and because of that, both models will give the same result.

2.6 Multi-Stage Robust Optimization

2.6.1 Affinely Adjustable Robust Counterpart

A careful reader might have noticed that in static RO the decisions concern "here and now" decisions and because of that, the production quantity for a certain period is not dependent on the realization of demand in the previous periods. However, as one can might imagine, there is a desire to let the current periods decision depend on the realization of the pervious periods disturbances. Of course, this could be done in a rolling horizon setting. Though, even in a rolling horizon setting we want to be able to let a periods production quantity depend on pervious realizations of demand if we had to commit for more than 1 period in advance.

The methodologies developed in Ben-Tal et al. (2004), Ben-Tal et al. (2005), Ben-Tal et al. (2009b) and Bertsimas et al. (2010) enable us to let the current periods production quantity depend on pervious realization in demand. The Robust Counterpart that is able to deal with these previous realization is called the Adjustable Robust Counterpart (ARC). However, in order for this to be computationally tractable, we restrict ourselves in this case to interval uncertainty, i.e. the uncertainty set $\mathcal{U}_\infty$. Furthermore, we define Linear Decision Rules for all t, where the current periods production quantity is depends in an affine fashion on the pervious periods realizations of demand,

$$q_t = q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau \tag{2.112}$$

By letting the production quantities be define by the Linear Decision Rule above, we have to do the same for the upper bounds on the inventory holding costs and back-order costs, Ben-Tal et al. (2009b),

$$y_t = y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \tag{2.113}$$

A detailed explanation why we loss nothing when restricting y_t to be an affine function of all d_t until t instead of T can be found in Ben-Tal et al. (2009b, pp. 426-428). Furthermore, when using Linear Decision Rules, our Robust Counterpart becomes known as the AARC.

We use the Linear Decision Rules we defined above and start from the nominal production planning problem we have seen before in Problem 2.2 to derive the AARC. For the sake of completeness we repeat the nominal problem below,

Problem 2.7 (Nominal Production Planning Problem)**.**

$$\min \ \kappa \tag{2.114}$$

$$\textit{s.t.} \ \kappa \geq \sum_{t=1}^{T} (cq_t + y_t) \tag{2.115}$$

$$y_t \geq h \left(I_0 + \sum_{s=1}^{t} (q_s - d_s) \right) \qquad t = 1 \dots T \tag{2.116}$$

$$y_t \geq -b \left(I_0 + \sum_{s=1}^{t} (q_s - d_s) \right) \qquad t = 1 \dots T \tag{2.117}$$

$$q_t \leq C_t \qquad t = 1 \dots T \tag{2.118}$$

$$q_t \geq 0 \qquad t = 1 \dots T \tag{2.119}$$

Demand is uncertain and is assumed to range within a specific interval, $d_t \in [\bar{d}_t - \hat{d}_t, \bar{d}_t + \hat{d}_t]$, i.e. we have uncertainty set $\mathcal{U}_\infty$. This brings us at the following semi-infinite optimization problem,

Problem 2.8 (Semi-Infinite Optimization Problem)**.**

$$\min \ \kappa \tag{2.120}$$

$$\textit{s.t.} \ \kappa \geq \sum_{t=1}^{T} \left(c(q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau) + (y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau) \right) \qquad \forall \mathbf{d} \in \mathcal{U}_\infty \tag{2.121}$$

$$y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq h \left(I_0 + \sum_{s=1}^{t} \left((q_s^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau) - d_s \right) \right) \qquad t = 1 \dots T, \forall \mathbf{d} \in \mathcal{U}_\infty \tag{2.122}$$

$$y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq -b \left(I_0 + \sum_{s=1}^{t} \left((q_s^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau) - d_s \right) \right) \qquad t = 1 \dots T, \forall \mathbf{d} \in \mathcal{U}_\infty \tag{2.123}$$

$$q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau \leq C_t \qquad t = 1 \dots T, \forall \mathbf{d} \in \mathcal{U}_\infty \tag{2.124}$$

$$q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau \geq 0 \qquad t = 1 \dots T, \forall \mathbf{d} \in \mathcal{U}_\infty \tag{2.125}$$

We rewrite this semi-infinite optimization problem into a linear optimization problem by separately rewriting each of its constraints. We start by rewriting the first constraint which is related to the objective function,

$$\kappa \geq \sum_{t=1}^{T}(cq_t + y_t) \tag{2.126}$$

$$\Leftrightarrow \quad \kappa \geq \sum_{t=1}^{T}\Big(c(q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau) + (y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau)\Big) \tag{2.127}$$

$$\Leftrightarrow \quad \kappa \geq \sum_{t=1}^{T}(cq_t^0 + y_t^0) + \sum_{t=1}^{T}\sum_{\tau=1}^{t-1}(cq_t^\tau d_\tau) + \sum_{t=1}^{T}\sum_{\tau=1}^{t}(y_t^\tau d_\tau) \tag{2.128}$$

We use the fact that,

$$\sum_{t=1}^{T}\sum_{\tau=1}^{t-1} cq_t^\tau d_\tau = \sum_{\tau=1}^{T-1}\sum_{t=\tau+1}^{T} cq_t^\tau d_\tau = \sum_{\tau=1}^{T}\sum_{t=\tau+1}^{T} cq_t^\tau d_\tau, \qquad \sum_{t=1}^{T}\sum_{\tau=1}^{t} y_t^\tau d_\tau = \sum_{\tau=1}^{T}\sum_{t=\tau}^{T} y_t^\tau d_\tau \tag{2.129}$$

Remark that, we can safely let the first summation in $\sum_{\tau=1}^{T-1}\sum_{t=\tau+1}^{T} cq_t^\tau d_\tau$ run from $\tau = 1$ to T instead, because if $\tau = T$ then $\sum_{t=\tau+1}^{T} cq_t^\tau d_\tau$ evaluates to the zero sum. For the sake of convenience, we define an additional variable α_t,

$$\alpha_\tau \equiv \sum_{t=\tau+1}^{T}(cq_t^\tau) + \sum_{t=\tau}^{T}(y_t^\tau) \tag{2.130}$$

Furthermore, in the next derivation and in those that will follow, we will make extensive use of the following equivalences to arrive at the computational tractable representation of the semi-infinite optimization problem.

$$\sum_{t=1}^{T} d_t x_t \leq y, \forall d_t \in [\bar{d}_t - \hat{d}_t, \bar{d}_t + \hat{d}_t] \tag{2.131}$$

$$\Leftrightarrow \quad \sum_{t:x_t<0}(\bar{d}_t x_t - \hat{d}_t x_t) + \sum_{t:x_t>0}(\bar{d}_t x_t + \hat{d}_t x_t) \leq y \tag{2.132}$$

$$\Leftrightarrow \quad \sum_{t=1}^{T}\bar{d}_t x_t + \sum_{t=1}^{T}\hat{d}_t |x_t| \leq y \tag{2.133}$$

$$\Leftrightarrow \quad \sum_{t=1}^{T}\bar{d}_t x_t + \sum_{t=1}^{T}\hat{d}_t \lambda_t \leq y, \text{with } -\lambda_t \leq x_t \leq \lambda_t \tag{2.134}$$

We continue with the objective function using the equivalences mentioned above and our newly defined variable α_τ.

$$\kappa \geq \sum_{t=1}^{T}(cq_t^0 + y_t^0) + \sum_{t=1}^{T}\sum_{\tau=1}^{t-1} cq_t^\tau d_\tau + \sum_{t=1}^{T}\sum_{\tau=1}^{t} y_t^\tau d_\tau \tag{2.135}$$

$$\Leftrightarrow \quad \kappa \geq \sum_{t=1}^{T}(cq_t^0 + y_t^0) + \sum_{\tau=1}^{T}\sum_{t=\tau+1}^{T} cq_t^\tau d_\tau + \sum_{\tau=1}^{T}\sum_{t=\tau}^{T} y_t^\tau d_\tau \tag{2.136}$$

$$\Leftrightarrow \quad \kappa \geq \sum_{t=1}^{T}(cq_t^0 + y_t^0) + \sum_{\tau=1}^{T}\Big(\sum_{t=\tau+1}^{T} cq_t^\tau + \sum_{t=\tau}^{T} y_t^\tau\Big)d_\tau \tag{2.137}$$

(2.138)

$$\Leftrightarrow \quad \kappa \geq \sum_{t=1}^{T}(cq_t^0 + y_t^0) + \sum_{\tau=1}^{T}\alpha_\tau d_\tau \tag{2.139}$$

$$\Leftrightarrow \quad \kappa \geq \sum_{t=1}^{T}(cq_t^0 + y_t^0) + \sum_{\tau=1}^{T}\alpha_\tau \bar{d}_\tau + \sum_{\tau=1}^{T}|\alpha_\tau|\hat{d}_\tau \tag{2.140}$$

$$\Leftrightarrow \quad \kappa \geq \sum_{t=1}^{T}(cq_t^0 + y_t^0) + \sum_{\tau=1}^{T}\alpha_\tau \bar{d}_\tau + \sum_{\tau=1}^{T}\lambda_\tau\hat{d}_\tau \tag{2.141}$$

with $-\lambda_\tau \leq \alpha_\tau \leq \lambda_\tau$. In a similar fashion we continue by rewriting the constraint related to the inventory balance equation, starting with the part related to the inventory holding costs.

$$y_t \geq h\left(I_0 + \sum_{s=1}^{t}(q_s - d_s)\right) \tag{2.142}$$

$$\Leftrightarrow \quad y_t \geq hI_0 + h\sum_{s=1}^{t} q_s - h\sum_{s=1}^{t} d_s \tag{2.143}$$

$$\Leftrightarrow \quad y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq hI_0 + h\sum_{s=1}^{t}(q_s^0 + \sum_{\tau=1}^{s-1} q_s^\tau d_\tau) - h\sum_{s=1}^{t} d_s \tag{2.144}$$

$$\Leftrightarrow \quad y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq hI_0 + h\sum_{s=1}^{t} q_s^0 + h\sum_{s=1}^{t}\sum_{\tau=1}^{s-1} q_s^\tau d_\tau - h\sum_{s=1}^{t} d_s \tag{2.145}$$

$$\Leftrightarrow \quad y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq hI_0 + h\sum_{s=1}^{t} q_s^0 + h\sum_{\tau=1}^{t-1}\sum_{s=\tau+1}^{t} q_s^\tau d_\tau - h\sum_{\tau=1}^{t} d_\tau \tag{2.146}$$

$$\Leftrightarrow \quad -hI_0 \geq h\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t} y_t^\tau d_\tau + h\sum_{\tau=1}^{t-1}\sum_{s=\tau+1}^{t} q_s^\tau d_\tau - h\sum_{\tau=1}^{t} d_\tau \tag{2.147}$$

Again, remark that in case $\tau = t$, then the summation $\sum_{s=\tau+1}^{t} q_s^\tau$ evaluates to the zero sum. Therefore, we can extend the range of the summation in a similar way as before. Furthermore, we define additional variable γ_t^τ,

$$\gamma_t^\tau \equiv y_t^\tau - h\sum_{s=\tau+1}^{t} q_s^\tau + h \tag{2.148}$$

and use this to obtain,

$$\forall t, \quad -hI_0 \geq h\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t}\left((y_t^\tau - h\sum_{s=\tau+1}^{t} q_s^\tau + h)d_\tau\right) \tag{2.149}$$

$$\Leftrightarrow \quad -hI_0 \geq h\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t}\gamma_t^\tau d_\tau \tag{2.150}$$

$$\Leftrightarrow \quad -hI_0 \geq h\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t}\gamma_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t}|\gamma_t^\tau|\hat{d}_\tau \tag{2.151}$$

$$\Leftrightarrow \quad -hI_0 \geq h\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t}\gamma_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t}\pi_t^\tau\hat{d}_\tau \tag{2.152}$$

with $-\pi_t^\tau \leq \gamma_t^\tau \leq \pi_t^\tau$. Analogous to the constraint related to the inventory holding costs we rewrite the part related to the back-order costs.

$$\forall t, \quad y_t \geq -b\left(I_0 + \sum_{s=1}^{t}(q_s - d_s)\right) \tag{2.153}$$

$$\Leftrightarrow \quad y_t \geq -bI_0 - b\sum_{s=1}^{t} q_s + b\sum_{s=1}^{t} d_s \tag{2.154}$$

$$\Leftrightarrow \quad y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq -bI_0 - b\sum_{s=1}^{t}(q_s^0 + \sum_{\tau=1}^{s-1} q_s^\tau d_\tau) + b\sum_{s=1}^{t} d_s \tag{2.155}$$

$$\Leftrightarrow \quad y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq -bI_0 - b\sum_{s=1}^{t} q_s^0 - b\sum_{s=1}^{t}\sum_{\tau=1}^{s-1} q_s^\tau d_\tau + b\sum_{s=1}^{t} d_s \tag{2.156}$$

$$\Leftrightarrow \quad y_t^0 + \sum_{\tau=1}^{t} y_t^\tau d_\tau \geq -bI_0 - b\sum_{s=1}^{t} q_s^0 - b\sum_{\tau=1}^{t-1}\sum_{s=\tau+1}^{t} q_s^\tau d_\tau + b\sum_{\tau=1}^{t} d_\tau \tag{2.157}$$

$$\Leftrightarrow \quad bI_0 \geq -b\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t} y_t^\tau d_\tau - b\sum_{\tau=1}^{t-1}\sum_{s=\tau+1}^{t} q_s^\tau d_\tau + b\sum_{\tau=1}^{t} d_\tau \tag{2.158}$$

We define additional variable ω_t^τ,

$$\omega_t^\tau \equiv y_t^\tau + b\sum_{s=\tau+1}^{t} q_s^\tau - b \tag{2.159}$$

and use this to obtain,

$$\forall t, \quad bI_0 \geq -b\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t}\left((y_t^\tau + b\sum_{s=\tau+1}^{t} q_s^\tau - b)d_\tau\right) \tag{2.160}$$

$$\Leftrightarrow \quad bI_0 \geq -b\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t} \omega_t^\tau d_\tau \tag{2.161}$$

$$\Leftrightarrow \quad bI_0 \geq -b\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t} \omega_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t} |\omega_t^\tau| \hat{d}_\tau \tag{2.162}$$

$$\Leftrightarrow \quad bI_0 \geq -b\sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t} \omega_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t} \phi_t^\tau \hat{d}_\tau \tag{2.163}$$

with $-\phi_t^\tau \leq \omega_t^\tau \leq \phi_t^\tau$. Next is the capacity related constraint, it readily follows from using the equivalences that,

$$q_t \leq C_t \tag{2.164}$$

$$\Leftrightarrow \quad q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau \leq C_t \tag{2.165}$$

$$\Leftrightarrow \quad q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t-1} |q_t^\tau| \hat{d}_\tau \leq C_t \tag{2.166}$$

$$\Leftrightarrow \quad q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t-1} \psi_t^\tau \hat{d}_\tau \leq C_t \tag{2.167}$$

with, $-\psi_t^\tau \leq q_t^\tau \leq \psi_t^\tau$. Likewise for the lower bound on q_t,

$$\forall t, \quad q_t \geq 0 \tag{2.168}$$

$$\Leftrightarrow \quad q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau d_\tau \geq 0 \tag{2.169}$$

$$\Leftrightarrow \quad q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t-1} |q_t^\tau| \hat{d}_\tau \geq 0 \tag{2.170}$$

$$\Leftrightarrow \quad q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t-1} \psi_t^\tau \hat{d}_\tau \geq 0 \tag{2.171}$$

with, $-\psi_t^\tau \leq q_t^\tau \leq \psi_t^\tau$. Finally, we are able to combine each of the constraints that we have rewritten, to obtain the the AARC of the Semi-Infinite Optimization that we started with.

Problem 2.9 (Affinely Adjustable Robust Counterpart for $\mathcal{U}_\infty$).

$$\min \ \kappa \tag{2.172}$$

$$\text{s.t. } \kappa \geq \sum_{t=1}^{T} (cq_t^0 + y_t^0) + \sum_{\tau=1}^{T} \alpha_\tau \bar{d}_\tau + \sum_{\tau=1}^{T} \lambda_\tau \hat{d}_\tau \tag{2.173}$$

$$\alpha_\tau = \sum_{t=\tau+1}^{T} (cq_t^\tau) + \sum_{t=\tau}^{T} (y_t^\tau) \qquad \tau = 1 \dots T \tag{2.174}$$

$$-\lambda_\tau \leq \alpha_\tau \leq \lambda_\tau \qquad \tau = 1 \dots T \tag{2.175}$$

$$-hI_0 \geq h \sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t} \gamma_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t} \pi_t^\tau \hat{d}_\tau \qquad t = 1 \dots T \tag{2.176}$$

$$bI_0 \geq -b \sum_{s=1}^{t} q_s^0 - y_t^0 - \sum_{\tau=1}^{t} \omega_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t} \phi_t^\tau \hat{d}_\tau \qquad t = 1 \dots T \tag{2.177}$$

$$\gamma_t^\tau = y_t^\tau - h \sum_{s=\tau+1}^{t} q_s^\tau + h \qquad \tau = 1 \dots T, t = 1 \dots T \tag{2.178}$$

$$\omega_t^\tau = y_t^\tau + b \sum_{s=\tau+1}^{t} q_s^\tau - b \qquad \tau = 1 \dots T, t = 1 \dots T \tag{2.179}$$

$$-\pi_t^\tau \leq \gamma_t^\tau \leq \pi_t^\tau \qquad \tau = 1 \dots T, t = 1 \dots T \tag{2.180}$$

$$-\phi_t^\tau \leq \omega_t^\tau \leq \phi_t^\tau \qquad \tau = 1 \dots T, t = 1 \dots T \tag{2.181}$$

$$q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t-1} \psi_t^\tau \hat{d}_\tau \leq C_t \qquad t = 1 \dots T \tag{2.182}$$

$$q_t^0 + \sum_{\tau=1}^{t-1} q_t^\tau \bar{d}_\tau + \sum_{\tau=1}^{t-1} \psi_t^\tau \hat{d}_\tau \geq 0 \qquad t = 1 \dots T \tag{2.183}$$

$$-\psi_t^\tau \leq q_t^\tau \leq \psi_t^\tau \qquad t = 1 \dots T, \tau < t \tag{2.184}$$

where $\kappa, \{q_t^\tau\}, \{y_t^\tau\}, \{\alpha_\tau\}, \{\lambda_\tau\}, \{\gamma_t^\tau\}, \{\pi_t^\tau\}, \{\omega_t^\tau\}, \{\phi_t^\tau\}$ and $\{\psi_t^\tau\}$ are the decision variables that together make up the decision vector.

2.6.2 Adjustable Robust Mixed-Integer Optimization

So far we have discussed the techniques and methodologies of RO based on a basic production planning problem. These techniques and methodologies form the state of the art. Though, as in most fields, things will continue to evolve and this will lead to new opportunities. This is also the case for Adjustable RO in case of integer (or binary) decision variables. It must be mentioned that the AARC will not work in case of integer (or binary) decision variables, because this would mean that the integer (or binary) decision variable has be expressed as a Linear Decision Rule and this is clearly not possible. This makes it impossible to use this technique in the next chapter, because we will work with a binary decision variable there to decide on production setups.

However, it is worth mentioning that work has been done to deal with integer (or binary) decision variables in Adjustable RO, but according to Bertsimas and Georghiou (2015) the results are far from optimal. Though, at the time of writing two articles came into being that seem to be promising in solving this issue, but they emerged to late to incorporate in this work. Nevertheless, the interested reader is recommended to look at the work of Bertsimas and Georghiou (2015) or Postek and den Hertog (2015)).

2.7 Distributionally Robust Optimization

We concluded the last section with some kind of a future outlook with regards to Adjustable RO. Besides that development, there is another one worth mentioning, namely Distributionally RO (DRO).

We started in this chapter with a generic representation of multi-stage decision problems. This led to our nominal production planning problem and eventually, we introduced uncertainty in it. We did so in a more or less geometrical way, so to say, because we extended the normal polyhedron to which a optimal solution should belong, e.g. by using interval uncertainty. However, there is a significant part of research that deals with uncertainty by means of a probability distribution, i.e. the techniques around Stochastic Programming. So traditionally, there are two ways to deal with uncertainty, i.e. in a geometrical way as RO does or in a stochastic way as in Stochastic Programming.

However, quite recently, DRO gained widespread attention and can be regarded as a third method. This method appears to be closing the gap between the two fields, because it borrows the probabilistic notion of uncertainty from the stochastic world and combines it with the ability to come up with tractable results from RO. The latter is something Stochastic Programming suffers from for large instances and this issue is known as the curse of dimensionality.

We have discussed before that in case of RO we want our constraints to hold for each possible realization of the parameters z belonging to uncertainty set $\mathcal{U}$,

$$\sup_{z\in\mathcal{U}} f(x,z) \leq 0 \tag{2.185}$$

Though, in case of Stochastic Programming, parameter z would be a random variable belonging to a known probability distribution and we start solving the problem from there. However, it is not strictly the case that this probability distribution is known, or known with certainty. DRO takes a different approach in this. It still assumes parameter z to be a random variable, but the probability distribution is unknown. The only information known are the first few moments, e.g. the mean, variance and perhaps skewness. The random parameter z has a distribution $\mathbb{P}_z$ and belongs to the so called ambiguity set $\mathcal{P}$. In this setting there are two constraints one can distinct: the worst-case expected feasibility constraint,

$$\sup_{\mathbb{P}_z\in\mathcal{P}} E_{\mathbb{P}_z}[f(x,z)] \leq 0 \tag{2.186}$$

and chance constraints,

$$\sup_{\mathbb{P}_z\in\mathcal{P}} P[f(x,z)] \leq \epsilon \tag{2.187}$$

The literature of this type of problem started with the work of Scarf et al. (1958), but meanwhile it gained more and more attention and quite recently a paper by Postek et al. (2015) puts it back on the map. With most work still being of aimed at developing methodologies, the techniques of DRO are in its infancy, but they are worth to keep an eye on in the future.

2.8 Conclusion

In this chapter we started with a generic multi-stage decision problem and showed that a simple production planning problem can be represented as such. This problem took the form of a Linear Optimization problem. We have introduced demand uncertainty herein in a geometrical way. For various types of uncertainty sets we have shown how to come up with its Robust Counterpart. We have even seen one, the one with cardinality constraint uncertainty, where the uncertainty can be controlled per period by means of a parameter. Thereafter, we made the extension to the Affinely Adjustable Robust Counterpart. This enabled us to take into consideration the pervious realizations of demand if we are planning for the next period. Furthermore, we have discussed some interesting properties with regards of the optimality and we concluded this chapter with a future outlook on RO. The outlook is mend to be more of a reminder of the potential of RO. With the knowledge gained in this chapter, we will be able to apply its techniques and methodologies to a more complicated production planning problem which forms the core of this thesis: the CLSP. For sure, we can answer confirmative to the first of our research questions: yes, we can apply the methodologies of RO to production planning problems.

Chapter 3

The Capacitated Lot-Sizing Problem under Demand Uncertainty

3.1 Introduction

In this chapter we study the CLSP in case of deterministic demand and in case of uncertain demand. We start in the next section with formulating the problem as a Mixed Integer LO problem in case of dynamic and time-varying demand. Despite the computational challenges that come with such a formulation, we do stick with it as it provides nice grounds for introducing a stochastic as well as a robust counterpart. In the next chapter we discuss the simulation study in which these models will be subject to thorough testing and analysis.

3.2 The Deterministic Capacitated Lot Sizing Problems

3.2.1 The Problem

The CLSP discussed in this chapter is a finite-horizon, periodic review, multi-item, single location production planning problem. Basically, we extend the single item production planning situation of the previous chapter, into one where we have to plan production for multiple items on one machine with limited capacity, while incurring setup and holding costs. The presence of inventory holding costs and setup costs leads to the trade-off between holding more inventory or switching between producing a certain item more often. Of course, all while taking into account the per period capacity restrictions. The capacity restriction in case of deterministic demand makes the CLSP an extension of the well-known dynamic lot sizing problem by Wagner and Whitin (1958).

In our case it is allowed to produce different types of products in one period. Hence, the CLSP is of a "big bucket" type of problem (Helber et al. (2013)). Contrary to "small-bucket" problems where the periods are set smaller and we have to decide how much periods to produce a certain product before switching to producing another product in a later period.

Capacity is expressed as the amount of time units available for production and setting up production. Capacity is then only consumed by the time required to setup the machine before production can commence and capacity is consumed by the time needed to produce the items of that time period.

As has been mentioned before, we first assume demand to be deterministic and that the available capacity is sufficiently large to satisfy demand. However, when capacity becomes a scares resource and when demand becomes stochastic, we might run into trouble. The reason for this is that we might need more capacity than available due to an extreme realization of demand to satisfy our service level constraint. Therefore, we relax the model in order to overcome this problem and to ensure that we still end up with a feasible production plan. We do so by introducing overtime, which evidently comes at a certain cost. The price of overtime is set in such a way that it is undesirable to use it, because this would result in the most fair way to study the trade of between holding inventory and setting up production. Note, that the aim is to come up with a feasible production plan, not really to have extra capacity.

3.2.2 A Mixed Integer Programming Formulation

Based on the problem description that we gave in the previous subsection and based on the work of Helber et al. (2013), we can describe the CLSP as the Mixed Integer LO problem found in Problem 3.1 below. Note, this is the same nominal model as introduced in Chapter 1, but its repeated for the sake of completeness.

Problem 3.1 (Nominal Capacitated Lot Sizing Problem).

$$\textit{min} \sum_{t=1}^{T} \sum_{k \in \mathcal{K}} (s_k^c \gamma_{kt} + y_{kt}) + \sum_{t=1}^{T} o^c o_t \tag{3.1}$$

$$\textit{s.t.}\ h_k^c \Big(I_{k0} + \sum_{\tau=1}^{t} (q_{k\tau} - d_{k\tau}) \Big) \leq y_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.2}$$

$$-\, b_k^c \Big(I_{k0} + \sum_{\tau=1}^{t} (q_{k\tau} - d_{k\tau}) \Big) \leq y_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.3}$$

$$q_{kt} \leq M \gamma_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.4}$$

$$\sum_{k \in \mathcal{K}} (t_k^p q_{kt} + t_k^s \gamma_{kt}) \leq C_t + o_t \qquad t \in \mathcal{T} \tag{3.5}$$

$$\gamma_{kt} \in \{0, 1\} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.6}$$

$$q_{kt}, o_t \geq 0 \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.7}$$

In Problem 3.1 we have for each product k setup cost s_k^c, inventory holding cost h_k^c, back-order cost b_k^c and related per unit processing time t_k^p and setup time t_k^s. For each time period t we have limited capacity C_t and possibly some overcapacity o_t at cost o^c. We have decision variables q_{kt}, I_{kt}, γ_{kt} and o_t, respectively, representing the production quantity, the inventory level, the binary setup decision for item k in period t and the amount of overtime required. We have set a lower bound on the inventory level for each product k to ensure production over the finite planning horizon, because otherwise the model would put everything on back-order, for which no costs are incurred in this model, to avoid setups and holding inventory.

Constraint 3.6 represents the binary decision variable which can only becomes one if we produce something. This is ensured by Constraint 3.4, in which $M \gg C_t + o_t$, because γ_{kt} will only be one if $q_{kt} > 0$. Constraints 3.2 and 3.3 represent the inventory balance equation as they relate the previous periods inventory level, the amount produced in time period t and the demand in the same time period with the inventory level at the beginning of the next time period. Note, the initial inventory levels for each product are parameters to the model. Constraint 3.5 limits the time spend on setups and production to the available capacity and this includes the overtime.

It is noteworthy to mention that this formulation of the CLSP allows for dynamic and time-varying demand contrary to a large body of formulations that assume stationary demand. In the latter case, demands for the various products arrive according to mutually independent and stationary stochastic processes. This assumption is studied widely and known as the Stochastic Economic Lot Sizing Problem. The work of Sox et al. (1999) and Winands et al. (2011) provide an extensive review of the existing literature. The models discussed herein are out of scope, because they only focus on stationary demand.

3.2.3 Heuristics for the Deterministic Capacitated Lot-Sizing Problem

Solving the CLSP by means of a Mixed Integer LO has its drawbacks, because the problem is known to be non-deterministic polynomial-time complete (NP-complete) (Maes et al. (1991)), which means that a candidate solution to the CLSP can be verified in polynomial time and that every problem in NP can be reduced to the CLSP. Consequently, it is unlikely that we will ever find an algorithm to solve the problem in polynomial time. A vast body of literature exists for improving the computational time of the CLSP with most solutions relying on heuristics.

Next we will discuss some of the most interesting heuristics and the first of them is the ABC-heuristic proposed by Maes and van Wassenhove (1986). Their heuristic takes three steps common to lot sizing problems: the lot sizing step, a feasibility routine and an improvement step. In their lot sizing step they determine whether or not to include a certain time period its demand in the current lot based on certain criterion. One of them is the well known Silver-Meal criterion. Using a lookahead mechanism they build up inventory in earlier periods to avoid shortages in case of insufficient capacity later on. The improvement step looks if there are pairs of lots that can be combined while not avoiding the capacity constraint nor increasing costs. The ABC-heuristic proves to be a great improvement over existing methods at that time, like Manne (1958) or Dzielinksi et al. (1963), because it is very fast and very simple.

Tempelmeier and Herpers (2010) alter the ABC-heuristic in such a way that it is able to deal with stochastic demand. This new heuristic is known as the ABC$_\beta$-heuristic. They introduce a β-service level to account for demand uncertainty. The lot sizes are then set such that the service level is satisfied. The cost criteria are altered as well, because we now have to deal with expected costs. However, the underlying idea behind these criteria still remains. Another interesting aspect of this work is the fact that it uses the "static uncertainty" strategy of Bookbinder and Tan (1988). This means that production is planned for T time periods and that this planning is implemented for each period $t \leq T$, which is in contrast to the "static-dynamic uncertainty" strategy or "dynamic uncertainty" strategy. The static-dynamic uncertainty strategy is one where the periods of production are fixed, but their sizes depend on previous demand realization. The dynamic uncertainty strategy shows a close resemblance to the Affinely Adjustable Robust Counterpart that we have seen before as both tend to "wait-and-see" the actual realizations of demand $d_\tau, \tau < t$ before a decision has to be made for period t, meaning if there will be a setup and if so, the lot size.

Helber and Sahling (2010) propose an optimization-based solution approach that solves a series of Mixed Integer LO problems using a "fix-and-optimize" algorithm. The general structure of the fix-and-optimize algorithm can be found in Algorithm 1 below.

Algorithm 1: Fix-and-Optimize-Heuristic($\mathcal{K}, \mathcal{T}, SCLSP, L$)

Input: set of product indices $\mathcal{K}$, set of time indices $\mathcal{T}$, lot sizing algorithm $SCLSP$, maximum of iterations L
Output: Lot sizes per period

begin
    for $(k,t) \in \mathcal{K} \times \mathcal{T}$ **do**
        FIX($\gamma_{kt} = 1$)
    SOLVE(*SCLSP(*$\mathcal{KT}_0^{fix}$*)*) and obtain $Z^\star$ and γ_{kt}^0 $\forall (k,t)$
    $\gamma_{kt} \leftarrow \gamma_{kt}^0$ $\forall (k,t) \in \mathcal{K} \times \mathcal{T}$
    repeat
        $l \longleftarrow l + 1$
        foreach $s \in S$ **do**
            Define $\mathcal{KT}_s^{fix}$
            for $(k,t) \in \mathcal{KT}_s^{fix}$ **do**
                FIX($\gamma_{kt} = \gamma_{kt}$)
            for $(k,t) \in \mathcal{KT}_s^{opt}$ **do**
                UNFIX(γ_{kt})
            SOLVE(*SCLSP(*$\mathcal{KT}_s^{fix}$*)*) and obtain Z_s and γ_{kt}^s $\forall (k,t)$
            if $Z_s \leq Z^\star$ **then**
                $Z^\star \longleftarrow Z_s$
                $\gamma_{kt} \longleftarrow \gamma_{kt}^s$ $\forall (k,t) \in \mathcal{KT}$
    until *$l = L$ or no improvement has been achieved for the last S subproblems*
end

The algorithm decomposes the problem into subproblems of more practical dimensions, for example by using product-oriented decomposition. In this case, each subproblem s corresponds to a product k. In each of these subproblems the whole planning horizon is considered and lot sizes are determined for it. Computation time is reduced in each subproblem, because most of the binary decision variables are fixed. We can fix each binary variable, because we can exploit the fact that overtime is allowed.

For the remainder of this chapter as well as for the remainder of this thesis, we will leave the heuristics mentioned above out of consideration. We will only focus on comparing methods purely based on mathematical optimization, because these are the ones that lend themselves very well for a Robust Optimization approach. Nevertheless, it should be mentioned that in future research, some of these heuristics, like the fix-and-optimize algorithm, might be used to improve the computation time.

3.3 The Stochastic Capacitated Lot Sizing Problem

3.3.1 The Introduction of Demand Uncertainty

In the previous section we looked at methods to solve the CLSP when the per period demand was known at the beginning of the planning horizon. In this section we only consider the case where demand is uncertain. More specifically, first we let the per period demands be random variables of a known probability distribution, the Normal distribution. From the work of Helber et al. (2013), Rossi et al. (2015) and Tempelmeier and Hilger (2015) it becomes clear that literature is dominated by the use of the Normal distribution. That is why we will use this distribution in our model too. However, it must be noted that the methods and techniques found in this section can be made specific to other probability distributions as well. After considering stochastic models that deal with demand uncertainty, we continue with a Robust Optimization approach to the problem.

With the introduction of uncertainty by means of random variables, we arrive at the stochastic counterpart of the CLSP. For each product k the per period demands are independently and identically distributed with $d_{kt} \sim Normal(\mu_{d_k}, \sigma_{d_k})$. For each product k in each time period t we have, probability density function $f_{d_{kt}}$, cumulative density function $F_{d_{kt}}$, forecasted expected value $E[d_{kt}] = \mu_{d_k}$ and variance $Var[d_{kt}] = \sigma^2_{d_{kt}}$. From now on we no longer talk about demand, back-orders or inventory level, but we have to work with their expected values. Therefore, we have to rewrite the nominal CLSP found in Problem 3.1 to account for the stochastic nature of the per period demand.

First, we take a look at the inventory level I_{kt} variable that occurs in the objective function and the inventory balance equation. The expected inventory level for product k is determined based on the cumulative demand D_{kt} and cumulative production quantity Q_{kt} up to period t. The cumulative demand D_{kt} equals the sum of t independent and identically normally distributed random variables. Hence, cumulative demand is Normal, $D_{kt} \sim Normal(\mu_{D_k} = \sum_{\tau=1}^{t} \mu_{d_k}, \sigma_{D_k} = \sqrt{\sum_{\tau=1}^{t} \sigma^2_{d_k}})$ with $f_{D_{kt}}$ and $F_{D_{kt}}$, respectively, being the probability density function and cumulative density function (Ross, 2010, pp. 67-68). Using these cumulative measures we can express the expected inventory level as follows,

$$E[I_{kt}] = E[\max\{0, Q_{kt} - D_{kt}\}] = \int_0^{Q_{kt}} (Q_{kt} - x) f_{D_{kt}}(x) dx \qquad k \in \mathcal{K}, t \in \mathcal{T} \qquad (3.8)$$

Working with the cumulative measures combined with the fact that the inventory balance equation can be written in a closed form, results in the following expression for the inventory level,

$$I_{kt} = I_{k,t-1} + q_{kt} - d_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \qquad (3.9)$$

$$= I_{k0} + \sum_{\tau=1}^{t} q_{k\tau} - d_{k\tau} \qquad k \in \mathcal{K}, t \in \mathcal{T} \qquad (3.10)$$

$$= I_{k0} + Q_{kt} - D_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \qquad (3.11)$$

This expression can be substituted in the objective function of the nominal CLSP such that we can eliminate the inventory balance equation. For now, we assume that the initial inventory level I_{k0} equals zero. Later on, we will extend the model to work with an arbitrary initial inventory level unequal to zero.

Second, in the Stochastic CLSP we do allow back-orders and we have control over them by means of the β-service level constraint. This means that the back-order cost become obsolete. The use of a service level constraint is rather important in general, because it enables us to control the production quantity with respect to demand. We have an infinitely large tail of probable demand realizations, because demand is normally distributed. This means that if we want to be able to satisfy all possible realizations of demand, we would need to produce an infinite amount of items for each product. Evidently, this is impossible due to physical as well as financial reasons. Hence, the need for a service level becomes clear, because it allows us to specify the fraction of demand we want to satisfy.

Analogous to the expected inventory level, we can express the expected backlog, also known as the first order loss function, as follows,

$$\mathcal{L}^1_{D_{kt}}(Q_{kt}) = E[B^l_{kt}] \qquad k \in \mathcal{T}, t \in \mathcal{T} \tag{3.12}$$

$$= E[\max\{0, D_{kt} - Q_{kt}\}] \qquad k \in \mathcal{T}, t \in \mathcal{T} \tag{3.13}$$

$$= \int_{Q_{kt}}^{\infty} (x - Q_{kt}) f_{D_{kt}}(x)dx \qquad k \in \mathcal{T}, t \in \mathcal{T} \tag{3.14}$$

Note the difference between backlog and back-orders. Back-orders are determined periodically, whereas backlog is a measure for the the cumulative amount of back-orders. So, period t its expected back-orders equals,

$$E[B_{kt}] = \mathcal{L}^1_{D_{kt}}(Q_{kt}) - \mathcal{L}^1_{D_{k,t-1}}(Q_{kt}) \tag{3.15}$$

The most common service level constraints are the α- and β-service level. They are, respectively, expressed as the probability of no stock-out or the percentage of demand directly satisfied from stock. For this reason the latter is also known as the fill-rate constraints. Besides that, both service levels are also known in literature as the P_1 service level and the P_2 service level (see Silver et al. (1998)). The interested reader might consult the work of Helber and Sahling (2010) to get to known more about other service levels.

When combining all of the above and using an arbitrary service level constraint, we can formulate the Stochastic CLSP found in Problem 3.2.

Problem 3.2 (Stochastic Capacitated Lot Sizing Problem)**.**

$$\textit{min} \sum_{t=1}^{T} \sum_{k \in \mathcal{K}} (s^c_k \gamma_{kt} + h^c_k E[I_{kt}]) + \sum_{t=1}^{T} o^c o_t \tag{3.16}$$

$$\textit{s.t.}\ q_{kt} \leq M\gamma_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.17}$$

$$\sum_{k \in \mathcal{K}} t^p_k q_{kt} + t^s_k \gamma_{kt} \leq C_t + o_t \qquad t \in \mathcal{T} \tag{3.18}$$

$$\textit{service level constraint, to be specified} \tag{3.19}$$

$$\gamma_{kt} \in \{0, 1\} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.20}$$

$$q_{kt}, o_t \geq 0 \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.21}$$

In the remainder of this work we will stick to the β-service level for two reasons. First, according to Tempelmeier and Hilger (2015) this is the one most used in practice and second, it is the one most widely used in mathematical programming formulations for the SCLSP (see Tempelmeier (2007), Tempelmeier and Herpers (2010), Tempelmeier (2011) and Tempelmeier and Hilger (2015)). We mentioned before that the β-service level is the percentage of demand that should be directly satisfied from inventory. An exact definition is as follows,

$$1 - \frac{\sum_{t=1}^{T} E[B_{kt}]}{\sum_{t=1}^{T} d_{kt}} \geq \beta \qquad k \in \mathcal{K} \tag{3.22}$$

If we want to work with the β-service level in the SCLSP found in Problem 3.2, then we replace Constraint 3.19 with Equation 3.22.

Tempelmeier and Hilger (2015) however, define another service level constraint as well, i.e. the β_c-service level. This service level ensures that the service level is achieved per cycle, because it relates back-orders and demand during the cycle. They define it as follows,

$$\frac{\sum_{i=1}^{t} E[B_{ki}]}{\sum_{i=1}^{t} E[d_{ki}]} \geq 1 - \beta_c - (1 - \gamma_{k,t+1}) \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.23}$$

$$\frac{\sum_{i=1}^{t} E[B_{ki}]}{\sum_{i=1}^{t} E[d_{ki}]} \leq 1 - \beta_c + (1 - \gamma_{k,t+1}) \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.24}$$

$$\gamma_{k,T+1} = 1 \qquad k \in \mathcal{K} \tag{3.25}$$

A careful reader might already have noticed that the use of the expected inventory level and expected backlog make the problem no longer a Linear Optimization problem, because their respective functions are non-linear in the cumulative production quantity. We will continue in the next section with an explicit derivation for both the expected inventory level and the expected backlog. These explicit derivations will be used to linearize both functions. Using the linearized versions of these function we will be able to formulate the problem again as a Linear Optimization problem.

3.3.2 An Explicit Derivation for the Expected Backlog and Inventory Level

We start out with an explicit derivation for the expected backlog function. The backlog for product k depends on the cumulative demand D_{kt} and cumulative quantity produced Q_{kt} up to period t to fulfil demand. The backlog for product k in period k can be defined as follows,

$$B^l_{kt}(Q_{kt}) = \begin{cases} 0 & \text{if } D_{kt} \leq Q_{kt} \\ D_{kt} - Q_{kt} & \text{if } D_{kt} > Q_{kt} \end{cases} \tag{3.26}$$

Using the definition above we can start our derivation, for any product $k \in \mathcal{K}$ and period $t \in \mathcal{T}$,

$$\mathcal{L}^1_{D_{kt}}(Q_{kt}) = E[B^l_{kt}(Q_{kt})] \tag{3.27}$$

$$= E[\max\{0, D_{kt} - Q_{kt}\}] \tag{3.28}$$

$$= \int_{-\infty}^{\infty} \max\{0, x - Q_{kt}\} f_{D_{kt}}(x) dx \tag{3.29}$$

$$= \int_{Q_{kt}}^{\infty} (x - Q_{kt}) f_{D_{kt}}(x) dx \tag{3.30}$$

The next step in the derivation relies on the fact that the PDF $f(.)$ of the Normal distribution can be written in terms of the standard Normal PDF, $f(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})$. Furthermore, we define $z = \frac{Q_{kt} - \mu_{D_k}}{\sigma_{D_k}}$.

$$\mathcal{L}^1_{D_{kt}}(Q_{kt}) = \int_{Q_{kt}}^{\infty} (D_{kt} - Q_{kt}) \frac{1}{\sigma_{D_k}} \phi\Big(\frac{x - \mu_{D_k}}{\sigma_x}\Big) dx \tag{3.31}$$

$$= \sigma_{D_k} \int_{Q_{kt}}^{\infty} \Big(\Big(\frac{x - \mu_{D_k}}{\sigma_{D_k}}\Big) - z\Big) \frac{1}{\sigma_{D_k}} \phi\Big(\frac{x - \mu_{D_k}}{\sigma_{D_k}}\Big) dx \tag{3.32}$$

In the next steps we, respectively, use the substitution rule for integration as expressed in Equation B.6 and integration by parts which is expressed in Equation B.7. Both rules for integration can be found in Appendix B.

{Using Equation B.6, substitution rule for integration }

$$\mathcal{L}^1_{D_{kt}}(Q_{kt}) = \sigma_{D_k} \int_{z}^{\infty} (t - z)\phi(t) dt \tag{3.33}$$

$$= \sigma_{D_k} \Big(\int_{z}^{\infty} t\phi(t) dt - \int_{z}^{\infty} z\phi(t) dt \Big) \tag{3.34}$$

{Using Equation B.7, integration by parts }

$$= \sigma_{D_k} \Big(\Big[t\Phi(t) - \int_{z}^{\infty} \Phi(t) dt \Big] \Big|_{t=z}^{\infty} - z(1 - \Phi(z)) \Big) \tag{3.35}$$

The best way to understand the step made in going from Equation 3.32 to Equation 3.33, is to look at it the other way around. Then think of $(t-z)\phi(t)$ as being the function $f(x)$ in the integration by substitution rule of Equation B.6. Besides that, remark that the first derivative of z is as follows,

$$\frac{d}{dQ_{kt}}z = \frac{d}{dQ_{kt}}\Big(\frac{Q_{kt}-\mu_{D_k}}{\sigma_{D_k}}\Big) = \frac{1}{\sigma_{D_k}} \tag{3.36}$$

To rewrite the integrand in the last equation we use the fact that the antiderivative of the standard Normal CDF equals, $\int \Phi(t)dt = t\Phi(t) + \phi(t)$.

$$\mathcal{L}^1_{kt}(Q_{kt}) = \sigma_{D_k}\Big(\Big[t\Phi(t) - \big[t\Phi(t) + \phi(t)\big]\Big]\Big|_{t=z}^{\infty} - z(1-\Phi(z))\Big) \tag{3.37}$$

$$= \sigma_{D_k}\Big(\Big[-\phi(t)\Big]\Big|_{t=z}^{\infty} - z(1-\Phi(z))\Big) \tag{3.38}$$

$$= \sigma_{D_k}\Big((-\phi(\infty) + \phi(z)\Big) - z(1-\Phi(z))\Big) \tag{3.39}$$

$$= \sigma_{D_k}\Big(\phi(z) - z(1-\Phi(z))\Big) \tag{3.40}$$

This concludes our explicit derivation for the expected backlog. In an analogous fashion we can derive the expected inventory level for any $t \in \mathcal{T}$ and $k \in \mathcal{K}$. This results in,

$$E[I_{kt}(Q_{kt})] = Q_{kt} - D_{kt} + \mathcal{L}^1_{D_{kt}}(Q_{kt}) \tag{3.41}$$

$$= \sigma_{D_k}\Big(\phi(z) + z(1-\Phi(-z))\Big) \qquad z = \frac{Q_{kt}-\mu_{D_k}}{\sigma_{D_k}} \tag{3.42}$$

Clearly, the functions for the expected backlog and the expected inventory level are non-linear and we have to linearize both to work with them in the Problem 3.2. Therefore, we will continue in the next section with a piecewise linear approximation of both functions.

3.3.3 Linearizing the Expected Backlog and Expected Inventory Level Functions

In this section we describe how to linearize the functions for the expected inventory level and the expected backlog. We can approximate both functions over the range $[0, u^L_{kt}]$ with an arbitrary precision using a sufficient number of L line segments (see Helber et al. (2013), Rossi et al. (2014), Tempelmeier and Hilger (2015) and Rossi et al. (2015)). An illustration of how such linearized functions would look like for $L = 12$, $u^L_{kt} = 200$, $E[D_{kt}] = 100$ and $Var[D_{kt}] = 30$ is shown in Figure 3.1. The figure shows that the expected backlog drops as we produce more, while the expected inventory increases. Since, we only allow a certain number of expected back-orders, we have to produce to ensure the expected backlog is in line with our service level constraint. This directly relates to an increase in the expected inventory level. This increase is not only influenced by the service level constraint, but also by the trade-off between holding inventory and setting up more frequently. Therefore, choosing the optimal cumulative production quantity Q_{kt} depends on achieving the required service level at minimum cost while respecting the capacity constraint.

We can approximate the expected backlog and inventory level as follows. First, we have to define the relevant region and divide this range into L closed intervals with endpoints u^l_{kt}. Endpoint u^0_{kt} equals zero, because is is not necessarily the case that we have to produce and because of that, we should allow for zero production. Of course, this is still under the assumption of zero initial inventory. Furthermore, endpoint u^L_{kt} should be sufficiently large to account for the extra production required by the service level constraint and the trade-off between holding inventory and setting up production. In general it should account for the following: the cumulative production in the past $Q_{k,t-1} = \sum_{\tau=1}^{t-1} d_{k\tau}$, the current periods demand d_{kt} and the minimum of the available capacity and the optimal amount to produce in advance.

We should include future demand $d_{k\tau}$, $\tau > t$, into the current production lot q_{kt}, if some cost criterion tells us that the costs are lower if we include it, then if we were not to include it. For example, we could take the cost criterion of Wagner and Whitin (1958) and define the following cost function,

$$K_{ww}(t,x) = (s_k + h_k\sum_{\tau=t}^{x}(\tau - t)d_{k\tau})/(x-t) \tag{3.43}$$

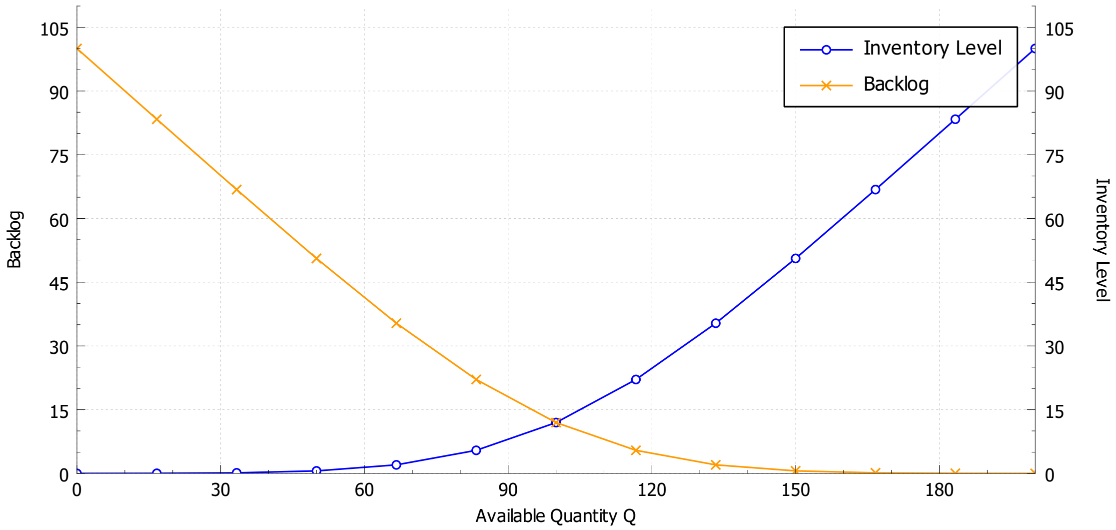

Figure 3.1: Approximation of the expected backlog and inventory level, t=1.

We then would include $d_{k\tau}$, $\tau > t$, in the current period lot q_{kt} if and only if $K_{ww}(t,\tau) \leq K_{ww}(t,\tau-1)$. This criterion can be used in a Boolean function $f_t : [t+1, T] \mapsto \mathbb{B}$, with $\mathbb{B} = \{0,1\}$,

$$f_t(x) = \begin{cases} 1 & \text{if } K_{ww}(t,x) \leq K_{ww}(t,x-1) \\ 0 & \text{if } K_{ww}(t,x) > K_{ww}(t,x-1) \end{cases} \tag{3.44}$$

We can now define the last endpoint u_{kt}^L as follows,

$$u_{kt}^L = Q_{k,t-1} + d_{kt} + \min\left\{C_t, \sum_{\tau=t+1}^{t^\star} d_{k\tau}\right\} \qquad t^\star = \arg\max_x f_t(x) \tag{3.45}$$

However, this upper bound doesn't take stochastic nature of the demand into account. Therefore, we have to make sure that it is sufficiently large such that it satisfies the service level constraint and so, we multiply the upper bound with a factor $g \in \mathbb{R}$.

Accordingly, we can now define the slopes on each interval $[u_{kt}^{l-1}, u_{kt}^l], 0 < l \leq L$. We know from the previous section that the expected inventory level with cumulative production Q_{kt}, is as follows,

$$E[I_{kt}] = E[\max\{0, D_{kt} - Q_{kt}\}] \tag{3.46}$$
$$= Q_{kt} - E[D_{kt}] + \mathcal{L}^1_{D_{kt}}(Q_{kt}) \tag{3.47}$$

We can use this to determine the slope $\Delta^l_{I_{kt}}$ for line segment l of the linearized expected inventory level function,

$$\Delta^l_{I_{kt}} = \Big(\big(u^l_{kt} - E[D_{kt}] + \mathcal{L}^1_{D_{kt}}(u^l_{kt})\big) - \big(u^l_{kt} - E[D_{kt}] + \mathcal{L}^1_{D_{kt}}(u^{l-1}_{kt})\big)\Big)\frac{1}{u^l_{kt} - u^{l-1}_{kt}} \tag{3.48}$$

$$= \Big(\big(u^l_{kt} + \mathcal{L}^1_{D_{kt}}(u^l_{kt})\big) - \big(u^l_{kt} + \mathcal{L}^1_{D_{kt}}(u^l_{kt})\big)\Big)\frac{1}{u^l_{kt} - u^{l-1}_{kt}} \qquad \begin{cases} k \in \mathcal{K}; \\ t \in \mathcal{T}; \\ l = 1, 2, \ldots L \end{cases} \tag{3.49}$$

We continue with the slopes for the expected back-orders function. We know from the previous section that the expected backlog is as follows,

$$\mathcal{L}^1_{kt}(Q_{kt}) = E[B^l_{kt}(Q_{kt})] \tag{3.50}$$
$$= E[\max\{0, D_{kt} - Q_{kt}\}] \tag{3.51}$$
$$= \sigma_{D_k}\Big(\phi(z) - z(1 - \Phi(z))\Big) \qquad z = \frac{Q_{kt} - \mu_{D_k}}{\sigma_{D_k}} \tag{3.52}$$

Furthermore, we know that the expected back-orders for period t equal $\mathcal{L}^1_{D_{kt}}(Q_{kt}) - \mathcal{L}^1_{D_{k,t-1}}(Q_{kt})$. Hence, the expected back-orders function can be approximated on the same region whereby the slopes are,

$$\Delta^l_{B_{kt}} = \frac{\left(\left(\mathcal{L}^1_{D_{kt}}(u^l_{kt}) - \mathcal{L}^1_{D_{k,t-1}}(u^l_{kt})\right) - \left(\mathcal{L}^1_{D_{kt}}(u^{l-1}_{kt}) - \mathcal{L}^1_{D_{k,t-1}}(u^{l-1}_{kt})\right)\right)}{u^l_{kt} - u^{l-1}_{kt}} \quad \begin{cases} k \in \mathcal{K} \\ t \in \mathcal{T} \\ l = 1, 2, \ldots L \end{cases} \tag{3.53}$$

In the next section we will use these linearized versions of the expected backlog and expected inventory level functions in Linear Optimization formulation for the Approximated Stochastic CLSP.

3.3.4 The Approximated Stochastic Capacitated Lot Sizing Problem

In the previous section we linearized both the expected backlog as well as the expected inventory level function. In this section we continue to integrate this knowledge into the Stochastic CLSP found in Problem 3.2 to arrive at an approximation or what we will call its Stochastic Counterpart.

We introduce a new decision variable w^l_{kt} representing the production quantity for product k in period t associated with interval l. We want w^l_{kt} to be set to their maximum capacity for $l = 1, \ldots, l^\star$ and set to zero for $l = l^\star + 1, \ldots, L$. This can be represented with the following equations,

$$w^l_{kt} = u^l_{kt} - u^{l-1}_{kt} \qquad l = 1, 2, \ldots, l^\star - 1 \tag{3.54}$$

$$w^l_{kt} = \sum_{\tau=1}^{t} q_{k\tau} - u^{l-1}_{kt} \qquad l = l^\star \tag{3.55}$$

$$w^l_{kt} = 0 \qquad l = l^\star + 1, l^\star + 2 \ldots, L \tag{3.56}$$

For all t and k, these equations translate into the following constraints,

$$w^l_{kt} \leq W \lambda^l_{kt} \qquad l = 2, 2, \ldots, L \tag{3.57}$$

$$w^{l-1}_{kt} = (u^l_{kt} - u^{l-1}_{kt})\lambda^l_{kt} \qquad l = 2, 3, \ldots, L \tag{3.58}$$

$$\lambda^l_{kt} \in \{0, 1\} \qquad l = 2, 3, \ldots, L \tag{3.59}$$

If a w^l_{kt} gets larger than zero, it activates the binary decision variable λ^l_{kt}. This decision variable then ensures that w^{l-1}_{kt} is filled to the maximum, which is $u^l_{kt} - u^{l-1}_{kt}$.

We can now use the new decision variables w^l_{kt} in combination with the slopes found in the previous section to obtain the piecewise linear approximation to the expected back-order and inventory level functions. This results in the following approximation for the expected inventory level function,

$$E[I_{kt}] = \Delta^0_{I_{kt}} + \sum_{l=1}^{L} \Delta^l_{I_{kt}} w^l_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.60}$$

and in the following approximation for the expected backlog function,

$$E[B_{kt}] = \Delta^0_{B_{kt}} + \sum_{l=1}^{L} \Delta^l_{B_{kt}} w^l_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.61}$$

where $\Delta^0_{I_{kt}}$ and $\Delta^0_{B_{kt}}$ represent the functions their value at u^0_{kt}. Until now we have assume that we have not initial inventory. However, from a practical point of view, this is rarely the case. Introducing an initial inventory I_{k0} would correspond to a translation for both the expected backlog and inventory level function. The direction would depend upon the sign of I_{k0}. That is, if $I_{k0} < 0$ then we have to translate both functions to the right and if $I_{k0} > 0$ we need to do the opposite. Consequently, if there is a negative initial inventory position Q_{kt} would increase, and it would decrease otherwise.

Moreover, we can now express the cumulative production quantity as the sum of these new decision variables, $Q_{kt} = \sum_{l=1}^{L} w^l_{kt}$, and this can be used to express the per period production quantity q_{kt} as follows,

$$\sum_{l=1}^{L} w_{kt}^{l} - \sum_{l=1}^{L} w_{k,t-1}^{l} = q_{kt} \tag{3.62}$$

Besides that, we can use the summation to introduce the inequality stating that the cumulative production quantity of the previous period cannot exceed that of the current one,

$$\sum_{l=1}^{L} w_{k,t-1}^{l} \leq \sum_{l=1}^{L} w_{kt}^{l} \tag{3.63}$$

We can combine all of these equations, inequalities and approximations and substitute them into the Stochastic CLSP found in Problem 3.2 to arrive at the approximated version that can be found below.

Problem 3.3 (Approximated Stochastic Capacitated Lot Sizing Problem).

$$\textit{min} \sum_{t=1}^{T} \sum_{k \in \mathcal{K}} (s_k^c \gamma_{kt} + h_k^c [\Delta_{I_{kt}}^0 + \sum_{l=1}^{L} \Delta_{I_{kt}}^l w_{kt}^l]) + \sum_{t=1}^{T} o^c o_t \tag{3.64}$$

$$\textit{s.t.} \; w_{kt}^l \leq W \lambda_{kt}^l \qquad \begin{cases} k \in \mathcal{K} \\ t \in \mathcal{T} \\ l = 1, 2, \ldots, L \end{cases} \tag{3.65}$$

$$w_{kt}^{l-1} = (u_{kt}^l - u_{kt}^{l-1}) \lambda_{kt}^l \qquad \begin{cases} k \in \mathcal{K} \\ t \in \mathcal{T} \\ l = 2, 3, \ldots, L \end{cases} \tag{3.66}$$

$$\sum_{l=1}^{L} w_{k,t-1}^{l} \leq \sum_{l=1}^{L} w_{kt}^{l} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.67}$$

$$\sum_{l=1}^{L} w_{k,t}^{l} - \sum_{l=1}^{L} w_{k,t-1}^{l} = q_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.68}$$

$$q_{kt} \leq M \gamma_{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.69}$$

$$\sum_{k \in \mathcal{K}} t_k^p q_{kt} + t_k^s \gamma_{kt} \leq C_t + o_t \qquad t \in \mathcal{T} \tag{3.70}$$

$$\frac{\sum_{i=1}^{t} E[B_{ki}]}{\sum_{i=1}^{t} E[d_{ki}]} \geq 1 - \beta_c - (1 - \gamma_{k,t+1}) \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.71}$$

$$\frac{\sum_{i=1}^{t} E[B_{ki}]}{\sum_{i=1}^{t} E[d_{ki}]} \leq 1 - \beta_c + (1 - \gamma_{k,t+1}) \qquad k \in \mathcal{K}, t \in \mathcal{T} \tag{3.72}$$

$$\gamma_{k,T+1} = 1 \qquad k \in \mathcal{K} \tag{3.73}$$

$$\gamma_{kt}, \lambda_{kt} \in \{0, 1\} \qquad \begin{cases} k \in \mathcal{K} \\ t \in \mathcal{T} \\ l = 1, 2, \ldots, L \end{cases} \tag{3.74}$$

$$\lambda_{kt} \in \{0, 1\} \qquad \begin{cases} k \in \mathcal{K} \\ t \in \mathcal{T} \\ l = 2, 3, \ldots, L \end{cases} \tag{3.75}$$

$$w_{kt} \geq 0 \qquad \begin{cases} k \in \mathcal{K} \\ t \in \mathcal{T} \\ l = 1, 2, \ldots, L \end{cases} \tag{3.76}$$

This brings us at the end of our journey to obtain a Linear Optimization formulation where the expected backlog and inventory level function are piecewise linearly approximated. Remarkably, there is a slight and very important difference from work of Tempelmeier and Hilger (2015) even though they had the same goal. We will touch upon this subject in the next subsection.

3.3.5 Corrigendum to the Work of Tempelmeier and Hilger (2015)

We have mentioned before that we are after a comparison between the nominal CLSP and its related counterparts that take uncertainty into account. We noted that there are two ways to deal with uncertainty, in a stochastic way and a more geometrical way. We took the work of Tempelmeier and Hilger (2015) as a starting point, because they did some work on the Stochastic Counterpart. Though, in order not to blindly copy their model and to ensure its correctness, we came up with the derivation ourselves as has been meticulously explained in the previous subsections. However, surprisingly there is a striking difference between the models.

Contrary to Tempelmeier and Hilger (2015), in our derivation we explicitly enforce Equations 3.57, 3.58 and 3.59 to hold by using some constraints. These constraints ensure the correct filling of the decision variables w_{kt}^k. By introducing binary decision variable λ_{kt}^l, we make sure that the values w_{kt}^l are filled to maximum capacity before filling any w_{kt}^j, $j > l$. Moreover, if w_{kt}^{l+1} has to be filled, but $w_{kt}^{l+1} < u_{kt}^l - u_{kt}^l$ (its not to be filled to its maximum), then $w_{kt}^j = 0$, for all $j > l+1$. When this is not strictly enforced, like in the work of Tempelmeier and Hilger (2015), we obtain the wrong results. We will explain the consequences of this to the best of our abilities and as means to an end, we will make use of Figure 3.2 below.

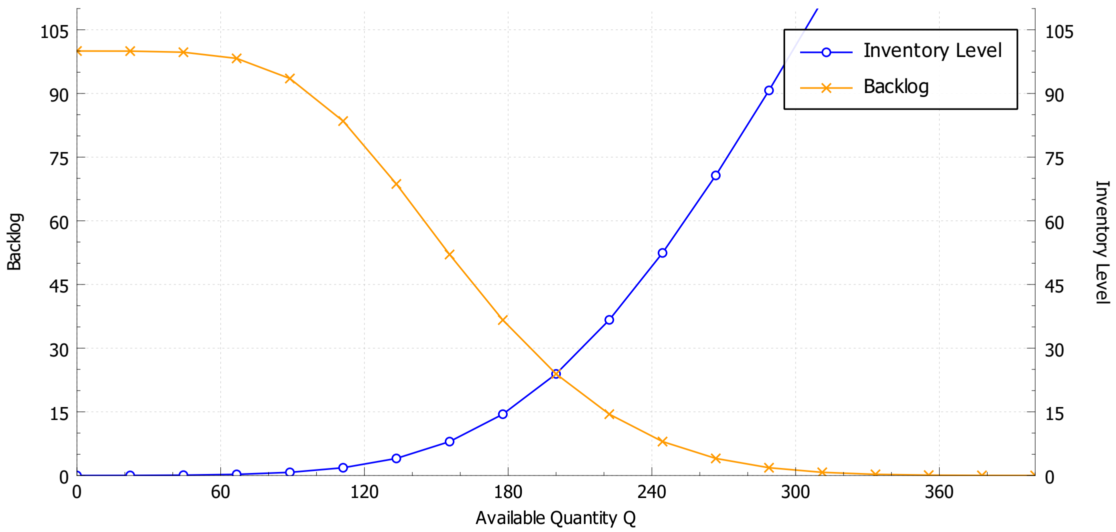

Figure 3.2: Approximation of the expected backlog and inventory level, t=2.

Figure 3.2 and Figure 3.1 look different, but both figures show a close resemblance. The difference is in the time period they consider, which is time period $t = 1$ and $t = 2$ for, respectively, Figure 3.1 and Figure 3.2. Only when we drew Figure 3.2 we obtained a better understanding of the consequence of omitting the the aforementioned constraints. Let us explain why.

In the objective function we have the linear approximation for the expected inventory level and in the β_c-service level constraints we have the linear approximation for the expected backlog. In these approximations we have slopes $\Delta_{I_{kt}}^l$ and $\Delta_{B_{kt}}^l$. Maximum effect can be achieved when setting $w_{kt}^l > 0$ for those slopes l where $\Delta_{I_{kt}}^l$ (or $\Delta_{B_{kt}}^l$) is greater (or less) than others. Then we would not have to set any w_{kt}^l greater to zero where only little can be achieved. This can be illustrated by using Figure 3.2.

For example, consider the case where we want to reduce the expected backlog from 100 to 90. One way this can be achieved is by filling w_{kt}^l, $l \leq 4$, to their maximum value and fill $w_{kt}^5 > 0$ to account for the remainder. However, when we consider slope Δ_{kt}^7 we note that the slope is large enough to cover the desired 10 units decrease in the expected backlog right away. Hence, if we only fill w^7_{kt} we would still achieve the 10 units decrease, but produce significantly less. This is an interesting insight, because apparently we can reduce the expected backlog by "producing" less and this is of interest, because we incur setup costs if we want to produce and inventory holding costs for excessive amounts of products produced. Please, note that this way of hacking the model does not violate any of the constraints.

Now we know that we do not necessarily have to "produce" a lot to significantly reduce the expected backlog function, we just have to choose the intervals that correspond to steep slopes. From the definition of the β_c-service level constraint we know that a reduction in the expected backlog function corresponds to a better service level. This is a desirable effect and the insight obtained above would imply that we can do so by producing less. Of course, is should be duly noted that this is abuse of terms in some sense, because setting w^l_{kt} larger than zero does not necessarily corresponds to producing products. The result is interesting, to say the least, but its consequence became fully clear during an extensive investigation of this problem by means of a simulation study.

The simulation study confirmed the idea that the Tempelmeier and Hilger (2015) selectively fills w^l_{kt}, i.e. they are not filled to their maximum for increasing numbers in l. Moreover, it does so while satisfying each of the constraints, also the service level constraint. First of all, it fills those w^l_{kt} that reduces the expected backlog the most, but second it manages to let the constraint $\sum_{l=1}^{L} w^l_{k,t} - \sum_{l=1}^{L} w^l_{k,t-1} = q_{kt}$, result in zero production for $2 \leq t$. This is because for $t = 1$ the constraint would read $\sum_{l=1}^{L} w^l_{k,t} = q_{kt}$ and we have some products k produced. However, since we are able to freely chose w^l_{kt}, it is possible to let $\sum_{l=1}^{L} w^l_{k,t} - \sum_{l=1}^{L} w^l_{k,t-1} = q_{kt} = 0$. Consequently, we do not have any setup costs in periods $2 \leq t$, because $q_{kt} = 0$ in those periods. We can conclude that the model of Tempelmeier and Hilger (2015) has its flaws, because it does not strictly enforce the w^l_{kt} to be filled in the correct way. Clearly, the constraints mentioned above are of utmost importance and should be incorporated in a stochastic approximation to the CLSP for things to work out correctly.

3.4 A Robust Counterpart of the Capacitated Lot Sizing Problem under Interval Uncertainty

Deriving the robust counterpart of the nominal CLSP introduced in Problem 3.1 is rather straight forward based on the knowledge we obtained in the previous chapter. In a similar fashion as has been done in case of the single item production planning problem, can derive the Robust Counterpart.

We start from the nominal CLSP found in Problem 3.1. In typical formulations of the CLSP the incentive to produce comes from the fact that negative inventory is not allowed. However, as we have explained before, negative inventory, or back-orders, are unavoidable in the case of demand uncertainty. If demand for each product is identically and independently normally distributed and we want to have zero expected back-orders each period, then we should have an infinite amount of inventory. In case of stochastic uncertainty we have introduced the β_c-service level as an incentive to produce. In case of the robust approach we will rely on the back-order cost parameter.

In the nominal CLSP problem we can introduce uncertainty by means of uncertainty sets. In our case we will only restrict ourselves to the most conservative Robust Counterpart, i.e. taking the l_∞-norm on the demand which is otherwise known as interval uncertainty. This means that the per period demand ranges with a certain interval, $d_t \in [\bar{d}_t - \hat{d}_t, \bar{d}_t - \hat{d}_t]$. More formally, like before, the demand vector is parameterized in an affine fashion by the perturbation vectors,

$$\mathbf{d}(\zeta) = d^0 + \sum_{t=1}^{T} d^t \zeta_t \qquad \|\zeta\|_\infty \leq 1 \tag{3.77}$$

In a similar way as before, we have the following uncertainty set $\mathcal{U}_\infty$,

$$\mathcal{U}_\infty = \left\{ \mathbf{d} = d^0 + \sum_{t=1}^{T} \zeta_t d^t : \zeta \in \mathcal{Z} = \{\zeta \in \mathbb{R}^T : \|\zeta\|_\infty \leq 1\} \subset \mathbb{R}^T \right\} \tag{3.78}$$

We can substitute this uncertainty set in Problem 3.1 to arrive at,

Problem 3.4 (Uncertain Capacitated Lot Sizing Problem).

$$\text{min} \sum_{t=1}^{T}\sum_{k\in\mathcal{K}}(s_k^c\gamma_{kt} + y_{kt}) + \sum_{t=1}^{T} o^c o_t \tag{3.79}$$

$$\text{s.t. } h_k^c\Big(I_{k0} + \sum_{s=1}^{t} q_{ks} - \sum_{s=1}^{t} d_{ks}\Big) \le y_{kt} \qquad \forall \mathbf{d}\in\mathcal{U}_\infty, k\in\mathcal{K}, t\in\mathcal{T} \tag{3.80}$$

$$-b_k^c\Big(I_{k0} + \sum_{s=1}^{t} q_{ks} - \sum_{s=1}^{t} d_{ks}\Big) \le y_{kt} \qquad \forall \mathbf{d}\in\mathcal{U}_\infty, k\in\mathcal{K}, t\in\mathcal{T} \tag{3.81}$$

$$q_{kt} \le M\gamma_{kt} \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.82}$$

$$\sum_{k\in\mathcal{K}}(t_k^p q_{kt} + t_k^s\gamma_{kt}) \le C_t + o_t \qquad t\in\mathcal{T} \tag{3.83}$$

$$\gamma_{kt}\in\{0,1\} \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.84}$$

$$q_{kt}, o_t \ge 0 \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.85}$$

We have uncertainty in exactly the same constraints as we had in the previous chapter. The only difference is that we have k of such constraints, for each product one. Since the other constraints are not affected by the uncertainty, we can leave them out of consideration with regards to determining the Robust Counterpart of the problem above. This means that we can borrow heavily from the derivation in the previous chapter to come up with the Robust Counterpart found in Problem 3.5 below,

Problem 3.5 (Robust Counterpart Capacitated Lot Sizing Problem with an l_∞-norm).

$$\text{min} \sum_{t=1}^{T}\sum_{k\in\mathcal{K}}(s_k^c\gamma_{kt} + y_{kt}) + \sum_{t=1}^{T} o^c o_t \tag{3.86}$$

$$\text{s.t. } h_k^c\Big(I_{k0} + \sum_{s=1}^{t} q_{ks} - \sum_{s=1}^{t} \bar{d}_{ks}z_{ks} + \sum_{s=1}^{t}\omega_{ks}\Big) \le y_{kt} \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.87}$$

$$-b_k^c\Big(I_{k0} + \sum_{s=1}^{t} q_{ks} - \sum_{s=1}^{t} \bar{d}_{ks}z_{ks} - \sum_{s=1}^{t}\omega_{ks}\Big) \le y_{kt} \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.88}$$

$$-\omega_{kt} \le \hat{d}_{kt}z_{kt} \le \omega_{kt} \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.89}$$

$$q_{kt} \le M\gamma_{kt} \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.90}$$

$$\sum_{k\in\mathcal{K}}(t_k^p q_{kt} + t_k^s\gamma_{kt}) \le C_t + o_t \qquad t\in\mathcal{T} \tag{3.91}$$

$$\gamma_{kt}\in\{0,1\} \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.92}$$

$$z_t = 1 \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.93}$$

$$q_{kt}, o_t \ge 0 \qquad k\in\mathcal{K}, t\in\mathcal{T} \tag{3.94}$$

3.5 Conclusion

In this chapter we discussed models for the CLSP in case of deterministic demand and uncertain demand. We started by discussing the case where demand is known in advance and this has led to the nominal problem. We have seen that for computational reasons various heuristics have been developed. Nevertheless, we stayed with mathematical programming formulations and basically, we extended the production planning problem of the pervious chapter by considering multiple items, setup costs and possibly, setup times.

We introduced demand uncertainty in the nominal problem in two ways. First, we took a stochastic approach and second, we took a Robust Optimization approach. The introduction of demand uncertainty gave rise to a challenge, because if demand would realize in such a way that we can't satisfy it, we would end up in having back-orders. However, only a certain amount of back-orders is allowed and this may mean that the β_c-service level could be violated if there is not enough capacity. Therefore, we relaxed the problem by introducing overtime. We meticulously described the derivation of the stochastic counterpart as well as flaws that were discovered in existing approaches.

At last, we took a Robust Optimization approach to the problem and introduced interval demand uncertainty. The derivation of the robust Counterpart was merely a straight forward execution of the same steps taken in the previous chapter, because most of the complexities were already discussed that chapter.

So, we have been able to formulate the a Stochastic and Robust Counterpart of the nominal CLSP. This means that we can confirm two more of our research questions. In the next chapter we will compare the models introduced in this chapter to assess their performance in case of demand uncertainty during a simulation study in a rolling horizon setting.

Chapter 4

Simulation Study

4.1 Introduction

This chapter deals with a comparison between the models that we have introduced in the previous chapter for solving the Capacitated Lot Sizing Problem. Our primary goal is to achieve a certain β-service level for a product when demand is uncertain and secondary to that we will try to minimize the costs involved. Hence, we are interested in the model that achieves the required service level at minimum cost. Since we investigate how these models perform under demand uncertainty, we expect the Stochastic and Robust Counterpart of the nominal problem to perform better under demand uncertainty than the nominal problem, because both have mechanisms to deal with uncertainty.

In order to establish statistically significant results, we conducted a simulation study. In this study we run the various production planning models in a rolling horizon setting. We start this chapter with discussing the methodology behind this simulation study. We will explain the type of simulation study conducted, convince you as reader about the validity of the implementation of the models, explain the simulation framework we have build and the statistics we will measure. Thereafter, we continue with experimental design and the results we have obtained.

4.2 Methodology

4.2.1 A Discrete-Event Simulation in a Rolling Horizon Setting

By definition of Law (2015, pp. 5-6), we conduct a dynamic, stochastic, discrete-event simulation which in our case concerns the modeling of a production system as it evolves over time. State variables such as the inventory level change instantaneously and at only a countable number of points in time, i.e. our review period. The stochastic component stems from the fact that demand is uncertain and randomly distributed. We will assume demand to be identically and independently normally distributed. However, the distribution is truncated such that no demand will be realized lower than zero, i.e. negative demand is not allowed.

We will employ our lot sizing models in a rolling horizon setting. The reason for this is becuase previous research by Drexl and Kimms (1997) and Helber et al. (2013) suggest that research should be done on how lot sizing models perform in a rolling horizon setting. Furthermore, lot sizing models are inherently employed in practice in a rolling horizon setting. Therefore, in order to asses the whether or not our models are suitable for practice we have to simulate them in a rolling horizon setting.

In essence, a rolling horizon settings means the following. At $t = 1$, we have a finite planning horizon $[t, t+T-1]$ for which we want to determine the lot sizes. We employ our models and receive a production plan. Of the resulting production plan we have to commit the first T_c periods. With commit we mean that we have to implement production for these periods and even as time increases and new information comes available, we are not allowed to alter the production quantities set for those specific periods. Hence, after we have planned production, we observe the next T_c periods before planning production again. This means that we have rolled T_c periods forward. We continue to roll forward as long as $t+T \leq H$. Note, it goes without saying that all models will be subject to the same realizations of demand for the sake of comparison.

For the sake of clarity, we will define some relevant terms that we will use throughout this chapter.

Replication In a replication we plan production for a finite horizon H in a rolling horizon setting.

Run A run consists of multiple replications. In some cases we need to have multiple replications to state something about statistics that are measured over the full horizon considered in one replication, e.g. the service level. Different runs are conducted in which one specific property is varied. For example, in different runs we look at various values for the coefficient of variation for the demand.

Simulation A simulation is the all encompassing entity holding various runs and the replications related to each run.

Law (2015) makes some excellent suggestions on how to compare the behaviour of different models in the same system. One of the most important aspects is whether or not the goal of a simulation study the statistical analysis for steady state behavior. This behavior only becomes clear if the simulation runs for rather large number of time periods and under various initial conditions (Law, 2015, pp. 491-492). However, the question then rises, does it make sense to run our models for an large number of periods? The answer to this lies in characteristics of our system. We do think that for various reasons the characteristics of our system will change over time, which is in line with statements of Law (2015): "in a manufacturing system the production-scheduling rules and the facility layout may change from time to time". Therefore, we will conduct a terminating simulation in which the natural event to terminate is a specific future time period H. This time period H should be chosen to mimic real life production planning. For example, assume that we have found some state of the art production planning problem, that time periods are measured in weeks and that we have to commit production on a monthly basis ($T_c = 4$). In this case it would be a reasonable assumption that after four years, $H = 208$, our system changes, for example, because of a change in the product-mix or the availability of a new production planning model.

4.2.2 Simulation Framework

Structure

For this simulation study we have written our own simulation framework. This way we were able avoid the overhead that comes with more general simulation software and obtain a dedicated program. Because of this we were also able to design the structure of the code in the way we wanted it. Figure 4.1 shows the full structure of the code.

At the core of the framework resides the the `Simulaton` class. This class is responsible for loading the parameters under which the system will be simulated. Furthermore, it runs the actual simulation and when finished it writes the data and statistics to a file and a SQLite 3 database. It relies heavily on mathematical support functions that are found in the package Mathematical Support. For example, this package contains the `Statistics` class that can be used for various calculations such as calculating the mean of a list of numbers, or determining confidence intervals.

The simulation is started by invoking the `run`-function on the `Simulaton` class. This happens from the Graphical User Interface (GUI) we have made. Because we have chosen for C++ as our programming language, we were able to work with the Qt 5.4 library[1]. This is a sophisticated library that allowed us to develop our GUI in a quick and easy way.

A production planning model will be run based on the settings that have been loaded into the `Simulaton` class. The models are children of the abstract `AbstractProductionPlanningModel` class. This class dictates the structure of the `AbstractProductionPlanninModel.plan`-function to its children. The children have to implement this function according to its specifications. This is really useful as the code in the `Simulaton` class only have to invoke the abstract production planning class, but at run-time, this class will be instantiated to one of the models specified in the input parameters. The big advantage is that nothing to the actual code related to the simulation have to change if we want to add another model. For example, if we come up at a later stage with another model, then we only have to subclass the `AbstractProductionPlanningModel` class and implement the `AbstractProductionPlanningModel.plan`-function. Then we specify in our input parameters this

[1]Website of Qt: http://www.qt.io/.

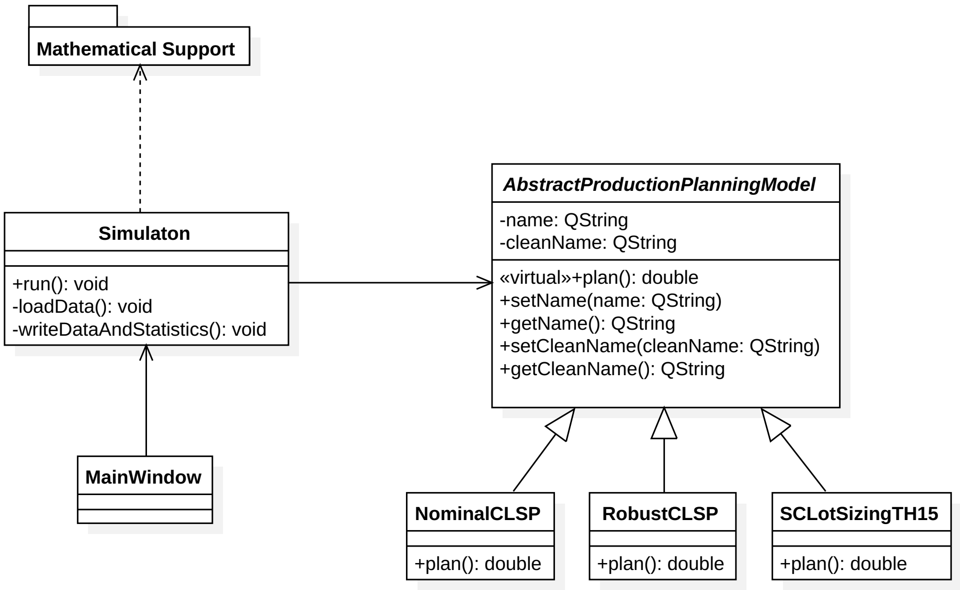

Figure 4.1: Program structure of our simulation framework as a UML diagram.

new model and get it run at run-time. The only limit there is, are the parameters passed on to the `AbstractProductionPlanninModel.plan`-function. However, for now the is a pretty extensive list, being: holding cost h_k^c, back-order cost b_k^c, production cost p_k^c, setup cost s_k^c, overtime cost o^c, capacity C_t, initial inventory I_{k0}, processing times t_k^p, setup times t_k^s, demand means μ_{d_k}, demand standard deviations σ_{d_k}, demand half widths $\hat{d}_k$, and required service level β. As a result, the function returns the per period the amount to produce q_{kt}, i.e the lot sizes.

Optimization Engine

Our simulation framework uses Gurobi Optimizer 6.5[2]. We chose this optimizer, because of its ease of use in our C++ programming environment. The C++ interface allowed us to integrate the Gurobi libraries very easily into our project. Gurobi allows for the definition of a mathematical optimization problem in terms of C++ variables and statements. Therefore, we were able to obtain results significantly faster than when using more general software packages, e.g. AIMMS or Matlab. By programming ourself we were able to reduce the overhead these packages typically come with.

Input File

A simulation is run based on the inputs in the simulation initialization file. This file contains different types of parameters. It contains the static parameters that describe how much times replications to do, which models to run as well as the different parameters. These parameters will not change during various runs. On the other hand, we have so called variates. These are the parameters that take a different value in each run. Table 4.1 shows the different variates we have.

Based on the specified variates different permutations are determined and each permutation is associated with a run. For example we could have run $(4, 0.1-0.1, 0.1-0.1, 0.80, 20-20, 4000) \in R_n$ as a permutation and this would mean that $T_c = 4$, we have two products that each have a forecasted and realized coefficient of variation 0.1, we want to achieve a service level of 80% and this corresponds to a demand half width of 20.

[2]Website of Gurobi: http://www.gurobi.com/.

Table 4.1: Variates in the simulation initialization file.

Variable	Description	Type	Input	Example
commit	Number of periods to commit from the production plan	Integer	Comma seperated	1, 4, 12
cvarf	Forecasted coefficient of variation	Real	Bar and comma seperated	0.1-0.1, 0.5-0.5
cvard	Actual coefficient of variation	Real	Bar and comma seperated	0.1-0.1, 0.5-0.5
beta	Desired β_c-service level in Stochastic CLSP	Real	Comma seperated	0.80, 0.98
halfie	Demand half-width to be used in Robust CLSP	Integer	Bar and comma seperated	20-20, 40-40
capacity	Available capacity for production and setups	Integer	Single value	4000

Model Credibility and Validity

We will continue with describing what we have done to increase the credibility of the models we implemented and discuss their validity. For sure, the code was written in a neat and tidy way to avoid "spaghetti"-code and since we used C++ as our programming language we were able to use the benefits of an Object Oriented Programming language. However, employing sound testing methods is of essence as mistakes are made pretty easily. Mistakes made in the syntaxis of the programming used are quickly noticed at compile time and various run time errors such as an index out of bounds is discovered pretty easily as well. The real challenge lies in the conceptual errors made during programming. An example could be the calculation of the normalized random variable z, if we were to type, $z = (Q_{kt} - \sigma_{D_{kt}})/\sigma_{D_{kt}}$ nothing would go wrong. However, this is conceptually wrong and to avoid those errors we employed various techniques.

To ensure the correctness of our code we used boundary value testing in combination with special value testing. This means that we checked the results of various function for a number of input values. For example, the first-order loss function was tested for various production quantities. Especially, for values less than zero, zero and greater than zero. The results were checked against a manual calculation of the function. By using boundary value testing we were able to validate our functions.

Moreover, with some functions relying on multiple other function, we had to do some kind of integration testing. For example, the piece-wise linear approximation function relies on normalizing the random variables, calculating the expected inventory level and backlog, etc. To verify the correctness we again checked its output for various input values. Moreover, we plotted the results for various time periods. We have seen the usefulness of these plots in the previous chapter where the came in handy to explain the structure of the functions.

Finally, we have ran the models for various parameters and analyzed their outcome. This way we were able to check their correctness, or not. If we would not have done such an elaborate testing, most likely, we would not have found the mistakes made in the model of Tempelmeier and Hilger (2015) as well as other insights that we will discuss below.

Statistics

During the simulation study we have to store data which is needed for the derivation of the relevant statistics. There are various statistics of interest at different levels. For each replication, we store the relevant information related to each product and the information related to the system as whole. For example, at product level we store its associated setup costs, while at system level we store the aggregated setup costs over all products. Then based on the information gathered during a replication, we are able to derive more general statistics. For example, we calculate the mean, standard deviation and confidence intervals of the β-service level for each run based on the service level obtained in the associated replications. This way we will be able to obtain statistically justified results as long as the number of replications and periods to plan for are set correctly.

4.3 Experimental Design

In the first two experiments we will only consider one item and we assume sufficient capacity. This way we are able to get some useful insights before diving into the more challenging problem of planning production for multiple items. Later on we will look at more items and decrease capacity. This way we are able to see how the system behaves for the different models under high utilization. Note, that as long as we have sufficient capacity for producing a certain number of items, each product behaves independently of each other. Only as capacity decreases the production of the different items starts to interact.

The first question we will answer is, what would happen if we employ the nominal model for various settings? Our aim is to achieve a certain service level and to see which model delivers this at minimum cost. Therefore, we will continue experimenting to find our under which settings the Stochastic and Robust counterpart achieve a this service level. Thereafter, we can make a comparison on which model achieves a this service level in the best way, cost-wise and planning-wise.

Unless otherwise stated, we will plan production over a 4 year interval and the relevant time period is one week. We will replicate 30 times. The inventory holding cost is normalized and hence, $h_k^c = 1$. Other costs are set, $b_k^c = 20, s_k^c = 5000, o^c = 10000$, and relevant times are $t_k^p = 1$ and $t_k^s = 0$. Demand is assumed to be normally distributed with a mean of $\mu_{d_k} = 1000$ and we don't have any seasonality.

4.4 Experimental Results

4.4.1 Experiment 1: Setting the Stage

Our first experiment is aimed at setting the stage. We are interested in what would happen if we employ the nominal model in case demand is uncertain and if we vary different parameters. Since there is no explicit parameter to control the required service level, we expect the model to perform worse when demand uncertainty is high and for increasing values of the number of periods to commit.

Table 4.2 shows the parameters that we vary in this experiment. We will look at how the nominal model performs under low demand uncertainty ($c_{var} = 0.25$) and high demand uncertainty ($c_{var} = 1$). Besides that, we will vary the number of periods to commit between one week, one month and three months.

Table 4.2: Parameters for experiment 1.

Parameter	**Symbol**	**Experimental values**
Coefficient of variation of demand	c_{var}	$\{0.25, 1\}$
Number of products	N	$\{1\}$
Periods to commit	T_c	$\{1, 4, 12\}$
Capacity	C	$\{6000\}$

The results for this first experiment can be found in Table 4.3. We have 6 different runs in this simulation for which we reported the input parameters and the 95 percent confidence interval for the mean of the realized β-service level (calculated according to Law (2015, pp. 234-235))

Table 4.3: Results experiment 1.

	Input				**Output (95% CI)**
Run	c_{var}	**N**	T_c	**C**	β**-service level**
1	0.25	1	1	6000	[98.63, 98.93]
2	1	1	1	6000	[90.01, 91.11]
3	0.25	1	4	6000	[96.14, 96.84]
4	1	1	4	6000	[72.48, 75.22]
5	0.25	1	12	6000	[90.13, 91.93]
6	1	1	12	6000	[39.23, 44.75]

The results confirm the expectation that the service level drops when demand uncertainty increases or when we have to commit more periods. The reason for this to happen has to do with the trade-off between setup costs and inventory holding costs by the lot sizing decision. Typically, setup costs lead to producing in advance for future demand. However, one should note that the same is achieved within production planning models that do not take setup costs into account, but that deal with fixed ordering costs. This brings us back to the definition of the (s, S)-policy we gave in Chapter 2. Whenever there is a fixed ordering cost, we have a (s, S)-policy. Hence, our lot size effectively behaves as a safety stock. Thus, with the lot size taking on the role of safety stock, we can deal with demand uncertainty up to a certain level. We see that for low demand uncertainty and various T_c, the service level stays surprisingly high and only when the demand uncertainty is high the service level starts to drop dramatically for increasing values of T_c. This is a very interesting result and we will refer to it as the "base-stock" effect.

4.4.2 Experiment 2: Influencing the Service Level

Lot Sizing using the Stochastic Counterpart

In the previous experiment we obtained some very interesting results regarding the nominal problem. We came to understand how the problem behaves for various parameters and that the resulting service level is merely depending on the situation. However, if we want to obtain a specific service level, the model proves to be inadequate because it doesn't have a parameter to control it. In the previous chapter though, we introduced the Stochastic Counterpart based on the work of Tempelmeier and Hilger (2015) that does have a control parameter. We can specify the β_c-service level that we require. In the following we will investigate if a certain required service level corresponds to a realized service level. We more or less took the same input parameters for the variates as we did in the previous experiment. Though, in addition, we specified the β_c-service level we require: $\beta_c = 0.90$.

When we use a required 90% β_c-service level into the Stochastic Counterpart and run the simulation, then we obtain the results found in Table 4.4.

Table 4.4: Results for a required 90% β-service level.

	Input					Output (95% CI)
Run	c_{var}	**N**	T_c	**C**	β_{req}	**β-service level**
1	0.25	1	1	6000	0.90	[100.00, 100.00]
2	1	1	1	6000	0.90	[100.00, 100.00]
3	0.25	1	4	6000	0.90	[93.48, 94.46]
4	1	1	4	6000	0.90	[89.21, 90.99]
5	0.25	1	12	6000	0.90	[91.99, 93.26]
6	1	1	12	6000	0.90	[79.65, 82.10]

From these results it becomes clear that we have better control over the realized β-service level than in case we use the nominal model. However, these results aren't completely satisfying. The results for $T_c = 1$ do not come as a surprise, because we get the "base-stock"-effect as we have found in the previous experiment. In this case, we get even better results, because this model produces a bit more just to be on the safe side regarding satisfying the service level. But, if we look at the case were $T_c = 12$ and demand uncertainty is high, than we have a significant drop in the realized service level. Detailed examination of the associated production quantities give rise to think that the extra amount produced with regards to satisfying the service level are nothing in comparison to the demand uncertainty. Which makes sense, because we can't protect ourself against all possible realizations of demand, especially the extreme ones.

Keep in mind the primary goal of our system, we want to achieve a certain service level. Since, the required service levels do not correspond directly to the realized ones, we have investigated for which input values we obtain the desired realization. After extensive testing were we iteratively looked at various levels we came up with the ones found in Table 4.5.

Two important things were found during the experiment in this subsection. First, we can confirm that it is possible to influence the realized service level by means of our input parameter and second,

Table 4.5: Results for fitting a realized 90% β-service level (Stochastic Counterpart).

	Input					Output (95% CI)
Run	c_{var}	**N**	T_c	**C**	β_{req}	**β-service level**
1	0.25	1	4	6000	0.865	[89.38, 90.52]
2	1	1	4	6000	0.90	[89.21, 90.99]
3	0.25	1	12	6000	0.875	[89.03, 90.77]
4	1	1	12	6000	0.945	[88.70, 91.064]

we found out for which input parameters a required service level realizes. It goes without saying that this can be repeated in case we have other required service levels.

Lot Sizing using the Robust Counterpart

In this subsection we will look at the Robust Counterpart. This model does not have such an intuitive parameter to set the required β-service level as the Stochastic Counterpart does. Instead, we have to set the demand half-width to influence the realized β-service level. Hence, we iteratively look at various demand half-widths in order to find the one that corresponds to the service level we interested in. As before, we are want to obtain a 90% β-service level. The results of this process are listed in Table 4.6

Table 4.6: Results for fitting a realized 90% β-service level (Robust Counterpart).

	Input					Output (95% CI)
Run	c_{var}	**N**	T_c	**C**	$\hat{d}_{kt}$	**β-service level**
1	0.25	1	1	6000	0	[98.65, 98.91]
2	1	1	1	6000	0	[90.88, 91.86]
3	0.25	1	4	6000	0	[96.92, 97.52]
x	1	1	4	6000	485	[88.71, 90.85]
5	0.25	1	12	6000	0	[90.40, 92.70]
6	1	1	12	6000	555	[88.33, 91.49]

4.4.3 Experiment 3: Relative Performance under Low and High Ratios of Demand to Capacity

We have mentioned before that we can consider each product individually if there is enough capacity. The products are independent of each other in this case, because they do not influence each other, i.e. they have access to sufficient capacity to produce whenever and whatever they decide to produce. However, things change if capacity becomes tighter. Then the ratio between demand and capacity goes up and consequently, the utilization rate of the system increases. In this experiment we deliberately tighten the capacity such that we can investigate how the models react under various settings for the system. In this experiment we will consider the different combinations of the variates listed in Table 4.7.

Table 4.7: Parameters for experiment 3.

Parameter	Symbol	Experimental values
Coefficient of variation of demand	c_{var}	$\{0.25, 1\}$
Number of products	N	$\{3\}$
Periods to commit	T_c	$\{4, 12\}$
Capacity	C	$\{4000, 5000, 10000\}$

In this experiment we strive to achieve a realized β-service level of 90 percent. Therefore, we will use the fitted parameters for the Stochastic and Robust Counterpart that were found in the previous experiment. We repeat them for the sake of completeness in Table 4.8.

Table 4.8: Fitted parameters for a 90 percent β service level.

Required β-service level	Setting		Fitter parameters	
	T_c	c_{var}	β	$\hat{d}_{kt}$
0.90	4	0.25	0	0.865
0.90	4	1	485	0.90
0.90	12	0.25	0	0.875
0.90	12	1	555	0.945

This experiment was set up in exactly the same way as before, e.g the number of replications stays the same as well as the horizon. The only difference is in the combination of variates in each run and we have increased the number of products to three. To give a first glance of the results, we provide the obtained service levels before going into detail. The results for $T_c = 12$ are given in Table 4.9 and for $T_c = 4$ they are given in 4.10. Note, for traceability reasons these tables also show the actual identifier for each run.

Table 4.9: Resulting service levels experiment 3 ($T_c = 12$)

Input								Output (means)		
Run	c_{var}	**N**	T_c	**C**	β	$\hat{d}_{kt}$	Model	β**-service level**	$\rho_{d/c}$	ρ_u
803	0.25	3	12	10000	0.875	0-0-0	RC	92, 91, 92	30	30
803	0.25	3	12	10000	0.875	0-0-0	SC	91, 90, 91	30	30
804	0.25	3	12	5000	0.875	0-0-0	RC	91, 91, 90	59	60
804	0.25	3	12	5000	0.875	0-0-0	SC	91, 91, 89	59	60
805	0.25	3	12	4000	0.875	0-0-0	RC	89, 90, 89	74	75
805	0.25	3	12	4000	0.875	0-0-0	SC	91, 92, 87	74	75
800	1	3	12	10000	0.945	555-555-555	RC	92, 92, 92	39	39
800	1	3	12	10000	0.945	555-555-555	SC	88, 87, 79	39	39
801	1	3	12	5000	0.945	555-555-555	RC	92, 92, 93	77	75
801	1	3	12	5000	0.945	555-555-555	SC	87, 71, 34	77	78
802	1	3	12	4000	0.945	555-555-555	RC	80, 78, 78	96	97
802	1	3	12	4000	0.945	555-555-555	SC	86, 42, 17	96	92

From the table above it becomes clear that if the demand to capacity ratio $\rho_{d/c}$ increases (capacity decreases) and the demand uncertainty increases, then the Stochastic Counterpart will yield lower β-service levels for the products. Important in this case is to notice that the Robust Counterpart yields more stable service levels when this happens. Nevertheless, under certain circumstances, both yield sufficiently large service levels or even larger service levels than required. When demand uncertainty is low and $T_c = 12$, both models yield more or less the same service level and satisfies the required 90 percent β-service level. But when $T_c = 4$ the Robust Counterpart will yield high service levels. The reason for this is, because the Robust Counterpart then behaves like the nominal model whereas the Stochastic Counterpart purposely goes for back-orders to approach the 90 percent service level.

We are very much interested in the reason why the Robust Counterpart performs better with regards to the service levels reported above and therefore, we analysed all kinds of statistics. We used paired-t confidence intervals to make a valid comparison between the two models employed. This technique is based on the work of (Law, 2015, pp. 560-569) and they explain that it is a better method than using hypothesis testing. This is because in hypothesis testing there is no more information than a hypothesis being accepted or rejected, while using paired-t confidence intervals we get information about how much better or worse one model performs over the other if there is a significant difference.

We will consider run 808 as an example. We start by looking at the paired-t confidence interval for the inventory holding cost h^k and we explain how to come up with these. First we create a new statistic based on subtracting the mean inventory holding cost for the Robust Counterpart from the one for the Stochastic Counterpart. This is done for each replication. We then determine the 95 percent interval for this new statistic. If this interval contains zero, then we know that there is no significant difference

Table 4.10: Resulting service levels for experiment 3 ($T_c = 4$)

Input								Output (means)		
Run	c_{var}	**N**	T_c	**C**	β	$\hat{d}_{kt}$	Model	β**-service level**	$\rho_{d/c}$	ρ_u
809	0.25	3	4	10000	0.865	0-0-0	RC	97, 96, 96	30	31
809	0.25	3	4	10000	0.865	0-0-0	SC	91, 90, 91	30	31
810	0.25	3	4	5000	0.865	0-0-0	RC	96, 96, 95	60	62
810	0.25	3	4	5000	0.865	0-0-0	SC	92, 92, 92	60	62
811	0.25	3	4	4000	0.865	0-0-0	RC	95, 95, 96	74	76
811	0.25	3	4	4000	0.865	0-0-0	SC	92, 92, 92	74	76
806	1	3	4	10000	0.9	485-485-485	RC	92, 92, 92	39	41
806	1	3	4	10000	0.9	485-485-485	SC	91, 90, 91	39	41
807	1	3	4	5000	0.9	485-485-485	RC	90, 91, 91	74	77
807	1	3	4	5000	0.9	485-485-485	SC	89, 88, 98	74	78
808	1	3	4	4000	0.9	485-485-485	RC	75, 76, 74	99	98
808	1	3	4	4000	0.9	485-485-485	SC	89, 69, 11	99	97

for the two means, but if there is, then we know which of the two models performs better. In this case the interval turns out to be: $[141924, 318800]$ and this means that the inventory holding costs for the Robust Counterpart are between 141924 and 318800 units higher than for the Stochastic Counterpart.

The paired-t confidence interval is determined for the setup cost as well. In this case there is no significant difference, because the interval is $[-59491, 47158]$. We have repeated this for the back-order costs and the total cost. It is not surprising that the back-order cost are about 46.2 and 65 million higher for the Stochastic Counterpart seen the realized service level. This results higher total cost for the Stochastic Counterpart.

We looked at the various paired-t confidence intervals for each individual run. We reported which of the two models had the highest cost based on the paired-t confidence interval in Table 4.11 and 4.11. To no surprise, the Stochastic Counterpart incurs more back-orders costs and thereby more total cost in general. If the service level drops as the number back-orders grows, then more cost are incurred. Furthermore, we looked at the average inventory level over all products in these runs. Clearly we can see a relation between the holding cost and the average inventory level (I_k). In general we may say that the Robust Counterpart has more inventory when needed, i.e. if the demand uncertainty is high and capacity scares, then we have higher inventory holding cost, because we have need more inventory on average to keep the service level high.

Table 4.11: Resulting cost and average inventory level for experiment 3 ($T_c = 12$)

Input					Highest cost				Mean I_k	
Run	c_{var}	**N**	T_c	**C**	h^c	s^c	b^c	t^c	**RC**	**SC**
800	1	3	12	10000	RC	RC	SC	SC	2830	2198
801	1	3	12	5000	RC	SC	SC	SC	2791	-786
802	1	3	12	4000	RC	RC	SC	SC	1512	-4525
803	0.25	3	12	10000	SC	RC	SC	SC	1032	1372
804	0.25	3	12	5000	SC	RC	SC	SC	904	1201
805	0.25	3	12	4000	SC	RC	0	0	808	994

So, our primary aim was to achieve a certain service level and secondary to that to keep costs low. We may conclude that the Robust Counterpart yields more steady results, because its keeps the service level among the products reasonably stable and it does not drop as dramatically as the Stochastic Counterpart when capacity tightens and demand uncertainty increases. This is purely an observation, but now we are interested in the reason behind it. Recall that the Stochastic Counterpart works with a service level in one of its constraints. In order to make things work we had to relax the problem, otherwise we would not obtain a feasible production plan. However, as the system gets under pressure, it has a strong desire in at least satisfying the service level of one product. All capacity is

Table 4.12: Resulting cost and average inventory level for experiment 3 ($T_c = 4$)

Input					**Highest cost**				**Mean I_k**	
Run	c_{var}	**N**	T_c	**C**	h^c	s^c	b^c	t^c	**RC**	**SC**
806	1	3	12	10000	SC	SC	SC	SC	1978	2156
807	1	3	12	5000	RC	SC	SC	SC	1826	1490
808	1	3	12	4000	RC	0	SC	SC	843	-4387
809	0.25	3	12	10000	SC	RC	SC	SC	1186	1270
810	0.25	3	12	5000	SC	SC	SC	SC	1012	1190
811	0.25	3	12	4000	RC	SC	SC	SC	1019	933

allocated to this product. Hence, more back-orders are incurred for the other products. Clearly, this method does not incorporate the knowledge of back-orders and their associated costs into the model as the Robust Counterpart does. Because it specifically incorporate these costs, it tries to minimize the back-orders, while ensuring a sufficiently large average inventory level to deal with demand uncertainty. Hence, besides having a trade-off between inventory holding costs and setup costs, there is a triangular trade-off with the back-orders costs.

4.5 Conclusion

In this chapter we conducted a thorough simulation study on our models. First, we look at the situation where the nominal model was employed in a stochastic setting. We discovered that things go wrong as demand uncertainty or the periods to commit increase. We then looked at the Stochastic and Robust Counterpart and tried to determine for which input parameters we would obtain a certain required service level. Using this information we were able to shift up gears and look at the multi-item situation. From this last experiment it became clear that the Robust Counterpart results in better service levels in general at less cost.

Chapter 5

Conclusions and Future Research

5.1 Introduction

In this chapter we will go over the main research findings and make suggestions for future research. First, we will go over the main research question and the related sub questions that we posed in the first chapter. In the last three chapters we have been able to answer them. Closely related to these answers are the research findings, or implications, that we will discuss in this chapter. We conclude this chapter by discussing possible future research.

For the sake of completeness, we will repeat the main research questions and the related sub questions below,

- What is the relative performance advantage of taking a RO approach to the CLSP over using the nominal model and its Stochastic Counterpart, which is based on the work of Tempelmeier and Hilger (2015), in a rolling horizon setting where demand is identically and independently normally distributed?

and the sub questions,

1. How can we apply the methodologies of RO to production planning problems?
2. How can we derive a Stochastic Counterpart in case we take a stochastic perspective on demand uncertainty?
3. Can we formulate a Robust Counterpart for the nominal CLSP if demand is known to range in a specified interval?
4. How should we setup an experimental framework and measure the relative performance of the models?
5. We know that we can influence the realized β-service level in the Stochastic Counterpart by means of a parameter. However, is the required service level equal to the realized one? If not, how can we fit the right input to the desired output?
6. How can we influence the realized β-service level in the Robust Counterpart?
7. How much does either of the two models, the Stochastic and Robust Counterpart, perform better than the other under different circumstances?

5.2 Main Research Findings

5.2.1 Using Robust Optimization in Production Planning Models

We started our research by explaining the methodologies behind Robust Optimization in 2. Based on a basic production planning problem we have been able to discuss how this approach deals with uncertainty. Moreover, we have been able to discuss some of the implications of using this method.

Based on results from the work of Bertsimas and Thiele (2006) we have been able to show that the optimal robust policy for this basic problem in case of interval uncertainty is a base-stock policy. More on the implications of this result later. From this chapter it became clear that we could answer the first of our sub research questions, Question 1, because we can apply the methodologies of Robust Optimization to production planning models.

In Chapter 3 we again took a Robust Optimization approach, but this time for the Capacitated Lot Sizing Problem. We started from the nominal problem and showed that it is possible to come up with a Robust Counterpart. So, again we were able to apply the methodologies of Robust Optimization. This strengthens the answer to the Question 1 and answers Question 3.

5.2.2 Deriving a Stochastic Counterpart

So, we derived a Robust and Stochastic Counterpart. The differences between the two is in the way the look at demand uncertainty. The Robust Counterpart looks at uncertainty in a more or less geometrical way, because it extends the polyhedron in which it looks for solutions. Contrary, the Stochastic Counterpart deals with uncertainty in a stochastic way. This means that it assumes demand to be randomly distributed. In this case, it assumes demand to be identically and independently normally distributed. However, we saw that it is not straightforward to incorporate this notion of uncertainty in the nominal CLSP. We had to approximate the functions for the expected inventory level and back-orders.

We started the approximation of the functions for the expected inventory level and back-orders by defining them in terms of cumulative production quantities. Then we derived an explicit formulation for both and these formulations allowed us to linearize both. Finally, we were able to come up with the Approximated Stochastic Capacitated Lot Sizing Problem in which these linearizations were incorporated. Hence, we ended up with the Stochastic Counterpart to the nominal problem. This answers Question 2, yes we can derive a Stochastic Counterpart for the nominal Capacitated Lot Sizing Problem.

An interesting result followed from preliminary research on this model in our simulation framework. Our model is based predominantly on the work of Tempelmeier and Hilger (2015). But when we implemented this model we encountered some counter-intuitive behavior. To give a small recapitulation on what goes wrong, the initial model does not properly fill the intervals that are associated with the parts of the linearized functions. This allows the model to select only those that yield more favorable values for the expected inventory level and back-order function. It still satisfies the service level constraint, but it no longer results in a sensible production plan. Hence, we showed that the model of Tempelmeier and Hilger (2015) has its flaws and explained how this could be improved. Of course, our model incorporated these improvements.

5.2.3 Influencing the Realized β-Service Level

Two of our sub research questions, Question 6 and 5 we aimed at investigating if we could influence the realized β-service level. Before we started researching this question in the simulation study, we already knew that the Stochastic Counterpart had a parameter for the required β-service level. We filled in some values to check whether the required service levels correspond to the realized service levels. We found a relation between input and output. This enabled us to fit the desired output with the required input. We repeated this for the Robust Counterpart as well and we discovered a relation between the outputs and the required demand half-widths as inputs. So, yes, we can influence the realized β-service level the way we want it to be.

5.2.4 Relative Performance under Low and High Ratios of Demand to Capacity

We looked at how the models perform if we have three products in our system and if we were to tighten capacity. From the last experiment in the previous chapter it became clear that the Robust Counterpart is superior to the Stochastic Counterpart. It makes sense that the service levels drop as capacity tightens, but clearly, they dropped more in case of the Stochastic Counterpart. As reason for this phenomenon we gave the strong desire of the Stochastic Counterpart to satisfy the service level constraint. This model doesn't have the trade-off with back-order costs as the Robust Counterpart has.

This gives reason to think that the Robust Counterpart aims at minimizing back-orders costs, inventory holding cost and setup costs, while the Stochastic Counterpart wants to minimize inventory holding costs and setup costs, while satisfying the service levels. However, when the models end up in the position of significant back-orders the Robust Counterpart acknowledges the importance of reducing the back-orders for all products, while the Stochastic Counterpart only focusses on achieving the service levels (most of the times at cost of the other products).

5.2.5 General Conclusion

In general we may conclude that a Robust Optimization lends itself for the use in production planning models, e.g. our single-item problem and the CLSP. Furthermore, we showed how to derive a Stochastic Counterpart. By doing so we found out that the model of Tempelmeier and Hilger (2015) has it flaw. Hence, we suggested how to improve this. Furthermore, it is safe to say that the Robust Optimization approach leads to a better β-service level at less cost than the Stochastic Counterpart.

We conclude this section by repeating the contributions we made to the field. We showed how to take a Robust Optimization approach to the Capacitated Lot Sizing Problem, we corrected an existing stochastic model and came up with our own Stochastic Counterpart, we simulated our models in a rolling horizon setting and gave insight in how to take a Robust Optimization approach.

5.3 Future Research

In this final section of this final chapter we will make recommendations for future research regarding production planning and more specific to the Capacitated Lot Sizing Problem. First, we want to say something about further use of Robust Optimization. Then we state something about the use of a rolling horizon setting in a simulation study. Then we recommend where to go next with regards to studying a Robust Optimization approach to lot sizing problems.

We touched slightly upon the new developments around Distributionally Robust Optimization in Chapter 2 and argued that this might become a third paradigm besides looking at uncertainty from a Stochastic or Robust Optimization approach. Though, the word paradigm might be an overstatement as all three methods are closely related. It seems that DRO combines the best of both worlds. It borrows from the notion of probability distributions from the stochastic side, i.e. uncertainty is specified by moments and other deviation measures. Based on this information a formulation is made using the Robust Optimization methodologies. Although these developments are in its infancy, they are worth keeping an eye on, because in practice it is hard (and costly) to obtain information about the demand distribution. Though, the first moments are generally easy to obtain. Hence, DRO sounds promising with this respect.

Then with regards to the rolling horizon setting that we took in our simulation study. Previous research already suggested to do so, but it is seldom, if not non-existing, encountered in works on the Capacitated Lot Sizing Problem. This is quite remarkable seen the fact that most models employed in practice run in a rolling horizon. Therefore, in our opinion, performing a simulation study in a rolling horizon is of utmost importance and intrinsically bound together with proper research. If one doesn't know how a model performs in a rolling horizon it is to be expected that nothing can be said about its relevance for practice. Hence, we highly recommend future research to run simulations in a rolling horizon setting.

In our simulation study we have looked at low and high ratios of demand to capacity. While most works do not consider high ratios, we explicitly did, because it is important for practice to do so. The work of Fransoo et al. (1995) pointed out the importance of stable cycle lengths at high ratios of demand to capacity. To illustrate this take a look at Figure 5.1. In this figure we have plotted the inventory levels for three products. Clearly, this system is under high pressure, because each setup is followed by the respective production of the product. And as production ends, the setup of the next product immediately commences.

In this system it is interesting to see if demand becomes uncertain and we have a higher realization of demand for a certain product. This situation is illustrated in Figure 5.2. We see an increase in the demand rate which depletes the inventory for this product much faster. Hence, a setup is required to produce new products. However, to do so, we have to make a shorter production run for the third product. Which depletes faster in this way as well and hence, requires a setup earlier than expected. If this effect cascades through the system, we will end up with shorter cycle lengths for the products, less time is spend on production and more on setups. Thus, the utilization rate of the system drops. To restore this longer cycles have to be made to build up inventory again. Though, this problem could have been avoided in the first place by not setting up production right a way, but to maintain a "stable cycle length". With the importance of stable cycle lengths being clear, we could imagine that it is of interest to study our models again under high ratios of demand to capacity to determine if the Robust Optimization approach naturally guarantees stable cycle lengths.

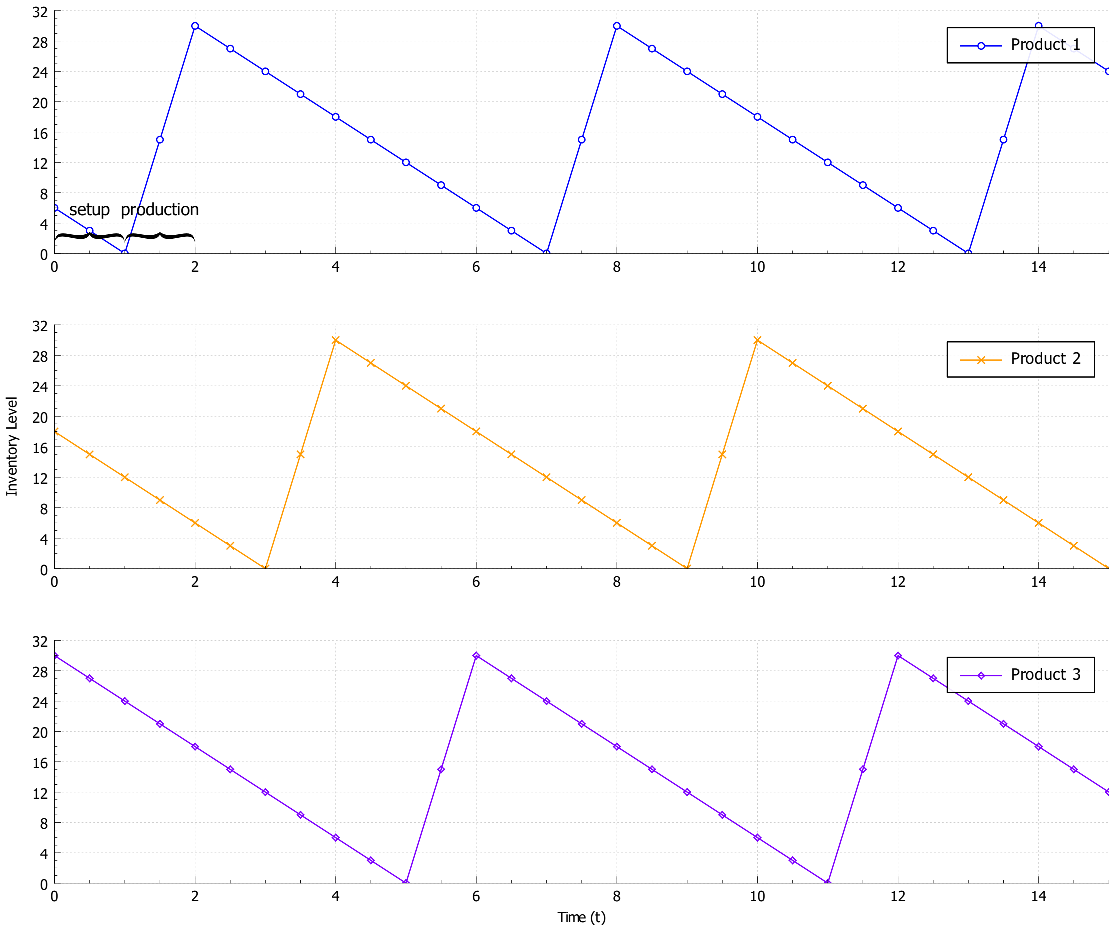

Figure 5.1: Stable cycle length, constant demand.

A final recommendation for future research is about extending the problem. In this we could go in two directions. First, we could make the models more of a sequencing type by dealing with sequence dependent setups. Or we could go more towards multi-level models. In the latter multiple locations are considered. Examples of Multi-Level Capacitated Lot Sizing can be found in Stadtler (1996) as well as Stadtler (2003). Spitter (2005) noted that these models represent the Supply Chain Operations Planning problem studied by de Kok and Fransoo (2003). de Kok and Fransoo (2003) define the objective of Supply Chain Operations Planning as "to coordinate the release of materials and resources in the supply network under consideration such that customer service constraints are met at minimal cost". In the models of de Kok and Fransoo (2003) the order release function is explicitly modeled. Hence, if a multi-level multi-item Capacitated Lot Sizing formulation is used for the Supply Chain Operations Planning problem, the it is assumed that material release decisions are not decoupled from resource consumption. Nevertheless, a link can be discovered from the CLSP to SCOP and as Spitter (2005) notes, mathematical programming formulations can be made to work well for the Supply Chain Operations Problem. This gives reason to think that a Robust Optimization approach lends itself well in this case too.

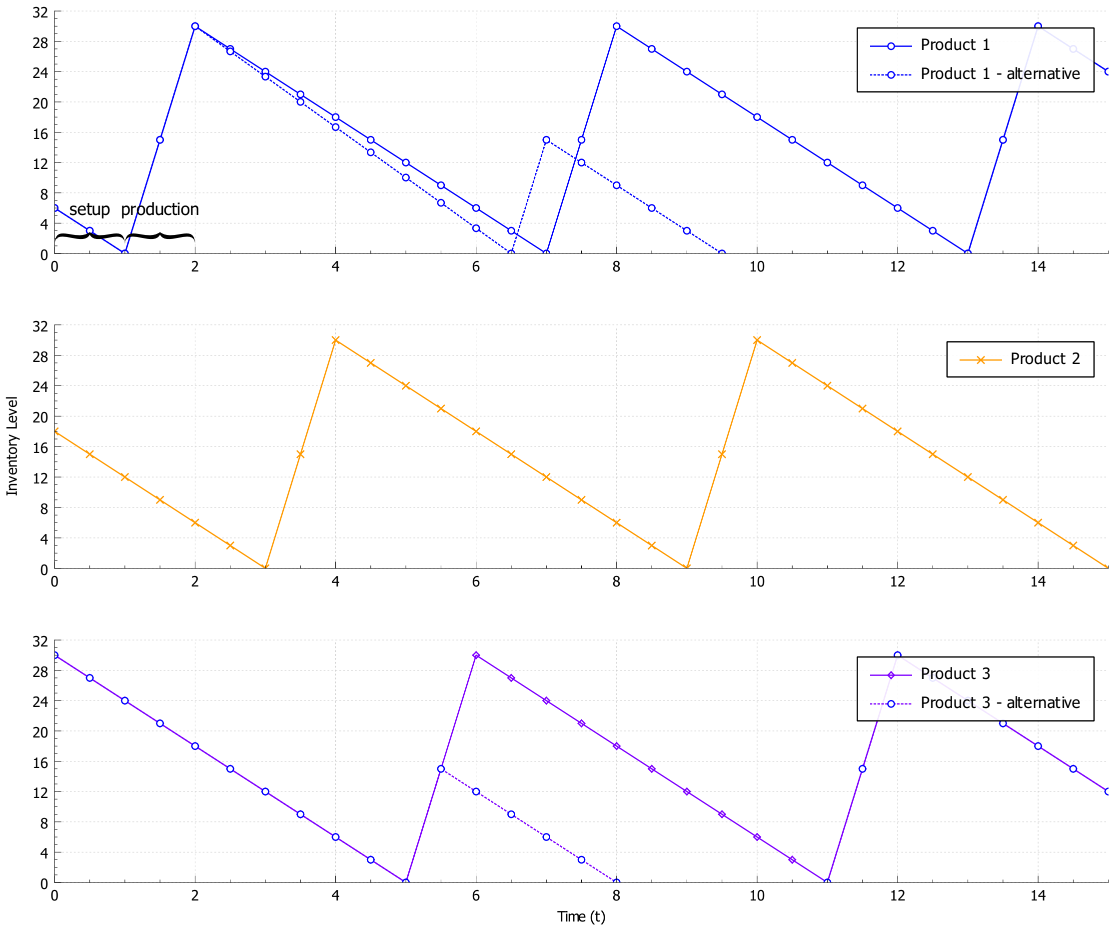

Figure 5.2: Effect of increasing demand rate.

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Summary

In this thesis the Capacitated Lot Sizing Problem under demand uncertainty is considered. This problem deals with determining a production plan for one machine over a finite horizon for a fixed number of products while being constraint by per period capacity restrictions. Costs are incurred for setting up production, holding inventory and incurring back-orders. First a general introduction is given to the thesis. In this introduction we see that the the problem can very well be formulated as a Mixed Integer Linear Optimization problem. Besides that we gave a motivation for this research. The most important motivations for this research are the fact that a thorough study on how lot sizing models behave in a rolling horizon is not found in literature and not many models that uncertainty into account. Hence, various researchers have suggested to conduct such a study (see Karimi et al. (2003), Allahverdi et al. (1999) and Jans and Degraeve (2008)). Both are of importance for practice as well, because most demand in practice is inherently uncertain and models are most likely to be employed in a rolling horizon setting.

Two different methods have been taken to deal with uncertainty, i.e. a Robust Optimization approach and a Stochastic approach. The methodologies regarding the former originates from the work of Ben-Tal and Nemirovski (1998) and recently gained widespread attention. The reason for this is that a Robust Optimization doesn't suffer from the curse of dimensionality as Stochastic Programming does for example, because it deals with uncertainty in a geometrical way. What this means is that the polyhedron with possible solutions to a mathematical problem is extended. A solution is said to be robust if it resides in this new polyhedron. Another big advantage of this method is that a problem can be formulated as (Mixed Integer) Linear Optimization problem and stays as such when the Robust Counterpart is derived in most cases.

When dealing with the Stochastic approach we have to take into account that we can no longer talk about inventory level, demand or back-orders. We have to express those in their expected values. However, we can't deal with these straight away in a Mixed Integer Linear Optimization problem. Therefore, we have determined the Piece-Wise Linear Approximations of the functions for these expected values. These approximations have then be substituted back into the problem to obtain again a linear problem. Interestingly, while deriving our own model we compared it to the one of Tempelmeier and Hilger (2015). We found that their model does not correctly fills the intervals corresponding to the different slopes in the approximation. Their model can be hacked in such a way that only those intervals are chosen that have the largest slope. This way we can still achieve the requirements of the service level constraint in this model, while incurring less cost. We have tackled this problem and came up with the so called Stochastic Counterpart.

All models have been compared in an simulated environment in a rolling horizon setting where demand is randomly generated from the Normal distribution. First, the nominal problem is investigated and some interesting property came to light. The results confirm the expectation that the service level drops when demand uncertainty increases or when we have to commit more periods. The reason for this to happen has to do with the trade-off between setup costs and inventory holding costs by the lot sizing decision. As a consequence of this trade-off the lot size will be larger than the required demand for one period. Hence, our lot size effectively behaves as a safety stock. So, under various circumstances we have seen reasonable service levels. Though, a decrease in the realized service level is observed as demand uncertainty increases or we have to commit more periods.

The realized service level when using the Stochastic Counterpart can be influenced by means of a β-service level constraint whereas for the Robust Counterpart we can set the back-order cost and the size of the demand interval. Therefore, we continued with fitting the right parameters to a realized service level. This enabled us to asses how these models perform under various circumstances. We

observed that the Robust Counterpart delivers more stable service levels at lower cost when capacity decreases and demand uncertainty gets higher. By careful investigation of the results in this experiment we found out the reason for this, which we will explain next.

Our primary aim was to achieve a certain service level and secondary to that to keep costs low. The Robust Counterpart yields more steady results, because its keeps the service level among the products reasonably stable and it does not drop as dramatically as the Stochastic Counterpart when capacity tightens and demand uncertainty increases. To understand the reason behind it, recall that the Stochastic Counterpart works with a service level in one of its constraints. Then as the system gets under pressure, it has a strong desire in at least satisfying the service level of one product. All capacity is allocated to this product. Hence, more back-orders are incurred for the other products. Since it doesn't take back-order cost into account it has no incentive to reduce the back-orders. Contrary, the Robust Counterpart does take the knowledge of the back-orders and their associated cost into account. Because it specifically incorporates this cost, it tries to minimize the back-orders, while it ensures a sufficiently large average inventory level to deal with demand uncertainty. Therefore, besides having a trade-off between inventory holding costs and setup costs, it has a triangular trade-off with the back-orders costs.

In short, we can summarize our contributions as follows. We have taken two approaches to deal with demand uncertainty and this has led to the Robust and Stochastic Counterpart. The former is based on the methodologies from Robust Optimization, and such an approach have not yet been taken, while the latter is inspired by the work of, Tempelmeier and Hilger (2015). We have shown that the model introduced in Tempelmeier and Hilger (2015) does not work correctly. Hence, we have improved this model. Thus we contribute two new models to deal with demand uncertainty for the Capacitated Lot Sizing Problem. Furthermore, we have studied how the nominal model, the Robust and Stochastic Counterpart perform in a rolling horizon setting under demand uncertainty. Surprisingly, the nominal model performs reasonably well in certain circumstances as its lot size effectively behaves like safety stock. In another experiment we have shown that the realized β-service level can be influenced for both counterparts. This made them suitable for a comparison under various circumstances. It was found that the Robust Counterpart is more stable and results in less cost. Therefore, we have shown that a Robust Optimization approach is not only possible, but leads to better results in general.

Appendices

Appendix A

Notation and Abbreviations

Notation

The notation used in this thesis is summarized below:

$\mathcal{T}$	Index set for the time periods ($t \in \{1, 2, \ldots, T\}$)
$\mathcal{K}$	Index set for the products ($k \in \{1, 2, \ldots, K\}$)
C_t	machine capacity in period t.
v_k	processing time for item k.
h_k	inventory holding cost for item k.
b_k	back-order cost for item k.
s_k	setup costs for item k.
d_{kt}	demand for item k in period t.
D_{kt}	cumulative demand for item k up to period t.
I_{kt}	inventory level for item k at the end of period t.
q_{kt}	decision variable for the production quantity of item k in period t.
Q_{kt}	cumulative production quantity of item k up to period t.
γ_{kt}	binary decision variable to produce item k in period t.

Remark, in the remainder of this thesis index i might be omitted in the case of a single item production planning problem.

Abbreviations

The abbreviations used in this thesis are summarized below:

ARC	Adjustable Robust Counterpart
AARC	Affinely Adjustable Robust Counterpart
CLSP	Capacitated Lot Sizing Problem
DRO	Distributionally Robust Optimization
LO	Linear Optimization
RO	Robust Optimization
RC	Robust Counterpart
SC	Stochastic Counterpart
SCLSP	Stochastic Capacitated Lot Sizing Problem
SP	Stochastic Programming

Appendix B

Mathematical Preliminaries

B.1 From Mathematical Optimization to Linear Optimization

In this section we start with a general form for mathematical optimization problems and gradually work towards the subclass of linear optimization problems. Conform Boyd and Vandenberghe (2004), a mathematical optimization (MO) problem has the following form,

$$MO : \min_x \{f_0(x) : f_i(x) \leq b_i, \forall i = 1, \ldots, m\} \tag{B.1}$$

where x is the vector of optimization variables, $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective function, $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are the constraint functions and the constants b_i are the upper bounds on these constraint functions.

We can distinguish classes of optimization problems if we look at the particular forms of the functions $f_0, \ldots f_m$. An interesting class of optimization problems are the so called convex optimization (CO) problems, i.e. the functions adhere to the definition of convexity.

Definition B.1 (Convex and Concave Functions). *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called convex if for every $x, y \in \mathbb{R}^n$, and every $\alpha \in [0, 1]$, we have,*

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).$$

Contrary, a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called concave if for every $x, y \in \mathbb{R}^n$, and every $\alpha \in [0, 1]$, we have,

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y).$$

It is worth noticing that functions of the form $f(x) = a_0 + \sum_{i=1}^{n} a_i x_i$, where a is a vector of scalars, are called affine functions and that these affine functions are both convex and concave.

In this paper we are particulary interested in a subclass of these convex optimization problems, namely, linear optimization (LO) problems. We call mathematical optimization problems linear optimization problems if the functions $f_0, \ldots, f_m$ adhere to the definition of linearity.

Definition B.2 (Linear Functions). *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is linear if for every $x, y \in \mathbb{R}^n$, and every $\alpha, \beta \in \mathbb{R}$, we have,*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y).$$

We can write these linear optimization problems in their canonical form as follows,

$$LO : \min_x \{c^\top x + d : Ax \leq b\} \tag{B.2}$$

where $x \in \mathbb{R}^n$ is the vector of decisions variables, $c, d \in \mathbb{R}^n$ are parameters in the cost function, $A \in \mathbb{R}^{m \times n}$ is the constraint matrix and $b \in \mathbb{R}^m$ is the *right hand side*. In matrix A, a_{ij}, $i \in [1, n]$ and $j \in [1, m]$, indicates an element of the matrix at row i and column j. Together, inequalities $a_i^\top x \leq b_i$ constrain the possible values for x and hence, they influence the value of the objective function.

B.2 Norms

Definition B.3 (Norms). *For $p \in [1, \infty]$, we define the p-norm $\|.\|_p$ on $\mathbb{R}^n$ by the relation,*

$$\|x\|_p = \begin{cases} (\sum_i |x_i|^p)^{1/p}, & 1 \leq p < \infty \\ \lim_{p\to\infty} \|x\|_p = \max_i\{x_i\}, & p = \infty. \end{cases} \tag{B.3}$$

When $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the norms $\|.\|_p$ and $\|.\|_q$ are conjugates of each other,

$$\|x\|_p = \max_{y:\|y\|_q \leq 1} |\langle x, y\rangle| \tag{B.4}$$

In particular, Hölder inequality is true, which means that,

$$|\langle x, y\rangle| \leq \|x\|_p \|y\|_q \tag{B.5}$$

B.3 Integration

Integration by substitution

$$\int_{\psi(a)}^{\psi(b)} f(x)dx = \int_a^b f(\psi(d))\psi'(d)dd \tag{B.6}$$

Integration by parts

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \tag{B.7}$$

Appendix C

Proofs of Theorems

This chapter of the appendix contains proofs for various theorems.

Theorem C.1 (Robust Counterpart Cardinally Constraint Uncertainty, cf. Bertsimas and Sim (2004))**.** *The uncertain linear optimization problem,*

$$\left\{\min_x\{c^\top x : Ax \leq b\}\right\}_{A\in\mathbb{A}} \tag{C.1}$$

has the following robust linear counterpart,

$$\begin{aligned}
&\textit{min } c^\top x && && \text{(C.2)}\\
&\text{s.t. } \sum_j \bar{a}_{ij}x_j + q_i\Gamma + \sum_{j:(i,j)\in J} r_{ij} \leq b_i && \forall i && \text{(C.3)}\\
&q_i + r_{ij} \geq \hat{a}_{ij}y_j && \forall (i,j)\in J && \text{(C.4)}\\
&-y \leq x \leq y && && \text{(C.5)}\\
&l \leq x \leq u && && \text{(C.6)}\\
&q_i \geq 0, r_{ij} \geq 0, y \geq 0 && \forall i,j && \text{(C.7)}
\end{aligned}$$

Proof. We can write the i^{th} constraint of the uncertain linear problem as,

$$max_{A\in\mathbb{A}} \sum_j a_{ij}x_j \leq b_i \tag{C.8}$$

Using the fact that a_{ij} can be written as $\bar{a}+z_{ij}\hat{a}_{ij}$ with $z_{ij} \in [-1,1]$, we come up with the following auxiliary linear optimization problem that need to be solved for every i^{th} constraint,

$$\begin{aligned}
&\text{max } \sum_j(\bar{a} + z_{ij}\hat{a}_{ij})x_j && && \text{(C.9)}\\
&\text{s.t. } \sum_{(i,j)\in J} |z_{ij}| \leq \Gamma && && \text{(C.10)}\\
&|z_{ij}| \leq 1 && \forall (i,j)\in J && \text{(C.11)}\\
&|z_{ij}| \geq 0 && \forall (i,j)\in J && \text{(C.12)}
\end{aligned}$$

We rewrite the auxiliary linear optimization problem, use vector notation and remove the absolute values using $|z_{ij}| = z_{ij}^+ + z_{ij}^-$ (Bertsimas and Tsitsiklis, 1997, pg. 18),

$$\max \begin{pmatrix} \hat{a}_{i1}x_1 & -\hat{a}_{i1}x_1 & \cdots & \hat{a}_{ij}x_j & -\hat{a}_{ij}x_j \end{pmatrix} \begin{pmatrix} z_{i1}^+ \\ z_{i1}^- \\ \vdots \\ z_{ij}^+ \\ z_{ij}^- \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} z_{i1}^+ \\ z_{i1}^- \\ \vdots \\ z_{ij}^+ \\ z_{ij}^- \end{pmatrix} \leq \begin{pmatrix} \Gamma \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$z_{ij}^+, z_{ij}^- \geq 0$$

The feasible set is non-empty and bounded ($z = \mathbf{0}$ is a solution), and hence, we can apply strong duality to arrive at the following,

$$\min \begin{pmatrix} \Gamma & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} q_i \\ r_{i1} \\ r_{i2} \\ \vdots \\ r_{ij} \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} q_i \\ r_{i1} \\ r_{i2} \\ \vdots \\ r_{ij} \end{pmatrix} \geq \begin{pmatrix} \hat{a}_{i1}x_1 \\ -\hat{a}_{i1}x_1 \\ \vdots \\ \hat{a}_{ij}x_j \\ -\hat{a}_{ij}x_j \end{pmatrix}$$

$$q_i \geq 0, r_{ij} \geq 0$$

We can introduce the absolute values again, because $q_i + r_{ij} \geq \hat{a}_{i1}x_1$ and $q_i + r_{ij} \geq -\hat{a}_{i1}x_1$ (Bertsimas and Tsitsiklis, 1997, pg. 18) and this results in,

$$\min \begin{pmatrix} \Gamma & 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} q_i \\ r_{i1} \\ r_{i2} \\ \vdots \\ r_{ij} \end{pmatrix}$$

$$\text{s.t.} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} q_i \\ r_{i1} \\ r_{i2} \\ \vdots \\ r_{ij} \end{pmatrix} \geq \begin{pmatrix} \hat{a}_{i1}|x_1| \\ \hat{a}_{i2}|x_1| \\ \vdots \\ \hat{a}_{ij}|x_j| \end{pmatrix}$$

$$q_i \geq 0, r_{ij} \geq 0$$

We can rewrite the this into the following problem without vector notation,

$$\begin{aligned} \min\ & q_i\Gamma + \sum_{j:(i,j)\in J} r_{ij} \\ \text{s.t.}\ & q_i + r_{ij} \geq \hat{a}_{ij}|x_j| \qquad \forall j : (i,j) \in J \\ & q_i \geq 0, r_{ij} \geq 0 \qquad \forall j : (i,j) \in J \end{aligned}$$

This is dual linear optimization program to the primal one we started with and we can substitute this in the original uncertain linear optimization problem to arrive at its robust counterpart,

$$\min c^\top x \tag{C.13}$$

$$\text{s.t.} \sum_j \bar{a}_{ij} x_j + q_i \Gamma + \sum_{j:(i,j)\in J} r_{ij} \le b_i \qquad \forall i \tag{C.14}$$

$$q_i + r_{ij} \ge \hat{a}_{ij} y_j \qquad \forall (i,j) \in J \tag{C.15}$$

$$-y \le x \le y \tag{C.16}$$

$$l \le x \le u \tag{C.17}$$

$$q_i \ge 0, r_{ij} \ge 0, y \ge 0 \qquad \forall i,j \tag{C.18}$$

□