MASTER

The dynamical behavior of a dipolar vortex near sharp-edged boundaries

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The Dynamical Behavior of a Dipolar Vortex near Sharp-Edged Boundaries

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Abstract

In this study the evolution of a dipolar vortex colliding with different solid objects was investigated. In the absence of any obstructions a dipolar vortex moves along a straight path. In general, the presence of a solid object implies the production of secondary vortices that can greatly affect the trajectories of the primary vortices. In this study, the focus is on the collision of a dipolar vortex with sharp-edged walls. The resulting dynamics were observed and studied in rotating tank experiments, where the presence of background rotation resulted in a quasi-two-dimensional flow field. The dynamics observed in the laboratory were found to be described accurately by numerical simulations of the relevant equations for two-dimensional fluid motion. To complement the experimental results the numerical study was extended for geometries and flow initializations that are not feasible a rotating tank experiment. Furthermore, a model based on potential flow theory was implemented in order to describe the kinematics of the primary and secondary vortices. This so-called point vortex model was optimized to reproduce the results from the numerical simulations. For certain cases, this highly simplified model could accurately describe characterizing aspects of the flow field evolutions that were observed in the experiments and numerical simulations.

Key words: Rotating tank experiment, direct numerical simulation, potential flow, point vortex model
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1. Introduction

1.1. Vortices in (quasi) two-dimensional fluid flow

In two-dimensional (2D) turbulence, vortex structures are commonly observed. Large-scale atmospheric flows are a fascinating example of this 2D turbulence. The presence of a density stratification and background rotation combined with the shallowness of the atmosphere causes large scale atmospheric flows to behave quasi-2D. Vortex structures play an important role in the dynamics of these flows. In weather reports, the presence of high and low pressure cells is often mentioned. These areas are associated with swirling fluid motion. These vortices have a significant influence on the weather patterns observed in the atmosphere. For improving the accuracy of weather forecasts, it is crucial to predict the trajectories and formation of these vortices. The motion of these vortices is influenced by other vortices. A pair of two vortices with oppositely signed circulation is known as a vortex dipole. These structures advect themselves through a fluid whilst entraining a certain portion of this fluid. Therefore, a dipolar vortex can be an effective transport mechanism for heat, contamination or other properties that a fluid might carry [1]. Figure 1.1 shows such a vortex structure above the Atlantic Ocean transporting sand from the Sahara desert.

![Figure 1.1 A dipolar vortex above the Atlantic Ocean visualized by entrained Saharan sand [2]](image)

As a dipolar vortex moves through a fluid it could encounter objects in the flow domain. These objects can greatly affect the dynamics of the original dipole. In earlier research on this subject, Barker and Crow [3] did experiments in which a dipolar vortex collided with a solid wall. A numerical study on the dipole-wall collision was performed by Orlandi [4]. For the example of the dipolar vortex in the atmosphere, this wall could be a model for a mountain ridge. It turned out that so-called secondary vorticity can be produced in a boundary layer near a wall and when this boundary layer detaches from the wall it can enter the flow domain forming new (secondary) circulation. This can in turn greatly alter the dynamics of the primary (i.e., the original) dipole vortices. Figure 1.2 shows another example of a dipolar vortex. Here this fluid structure is formed at the takeoff of an airplane, such that these vortices may interact with the runway (i.e., a solid boundary). The evolution of this vortex-wall collision is of great importance for subsequent airplanes that land in this turbulent flow field.
The interactions of vortices with their surroundings in a quasi-2D flow are not only encountered in geophysical or other atmospheric flows. In many industrial applications vortex-object collisions are observed. Vortices trailing from a fan blade can interact with other fan blades. These dynamics can be observed in a wide range of industrial applications, from helicopter rotors to turbocharger machinery applied in automotive engineering.

1.2. Research objective

In order to understand the complex dynamics of a dipolar vortex colliding with arbitrary solid obstacles, simplifications have to be made. First the fluid flow is assumed to be 2D and secondly the dynamics of the flow are characterized by the dynamics of the vortices. It is well known that new vortices can emerge in a dipole induced flow by detachment of the boundary layers at obstacle walls. This is especially likely to occur at a sharp edge of an object [5] [6]. Therefore, the first objective of this project is to better understand secondary vorticity production at the sharp edge of a finite wall by a dipole induced flow. Furthermore, interesting dynamics are found in [7], where a dipolar vortex approached an opening between two finite walls. As this appears to be an extension of the single finite wall the next step is to study the evolution of a dipolar vortex approaching the opening between two walls.

1.3. Methods and report outline

In this thesis vortex dynamics in quasi 2D flows are studied in a rotating tank experiment. The governing equations for fluid motion are known as the Navier-Stokes equations and the numerical simulation of this set of equations provides an alternative method for studying fluid flows. The relevant theory for these approaches is presented in chapter 2. Furthermore, section 2.5 of this chapter describes a so-called point vortex model for the motion of vortices and production of secondary vortices at sharp edges. In the following three chapters the methods regarding the experiments, numerical simulations and point vortex modeling are discussed. In the subsequent chapter 6, a comparison between the experimentally and numerically obtained results is presented. Results of more in-depth numerical analysis are presented in chapter 7. Finally the results of the point vortex model are presented in chapter 8. In the final chapter 9 the conclusions of this thesis will be discussed.
2. Theory

2.1. Governing equations
The governing equations of fluid motion are the law of conservation of momentum and the law of conservation of (incompressible) mass, respectively:

\[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v} + \mathbf{f}, \tag{2.1} \]

\[ \nabla \cdot \mathbf{v} = 0, \tag{2.2} \]

where \( \mathbf{v} = (u, v, 0) \) in a 2D flow. Now we define two useful scalar fields; the vorticity \( \omega \) and the stream function \( \psi \):

\[ \omega = \mathbf{k} \cdot (\nabla \times \mathbf{v}), \tag{2.3} \]

\[ \mathbf{v} = -\mathbf{k} \times \nabla \psi, \tag{2.4} \]

where \( \mathbf{k} \) is the unit vector normal to the flow field \( (u, v) \). In terms of these scalar fields, equations (2.1) and (2.2) can be rewritten in the so-called vorticity-stream function formulation,

\[ \frac{\partial \omega}{\partial t} + (\nabla \times \nabla \psi) \cdot \mathbf{k} = \nu \nabla^2 \omega, \tag{2.5} \]

\[ \omega = -\nabla^2 \psi. \tag{2.6} \]

For a barotropic fluid and in the absence of a non-conservative force field, the combination of equations (2.5) & (2.6) forms an alternative formulation for the combination of equations (2.1) & (2.2). A useful quantity to characterize vortex structures in a fluid flow is the circulation \( \Gamma \) associated with a surface \( \Phi \) and its anticlockwise orientated boundary \( \partial \Phi \),

\[ \Gamma = \oint_{\partial \Phi} \mathbf{v} \cdot d\mathbf{l} = \iint_{\Phi} \omega \, dA. \tag{2.7} \]

The circulation \( \Gamma \) of a vortex patch is a measure for the strength of a vortex.

2.2. The Lamb-Chaplygin dipole
In order to study a dipolar vortex structure colliding with a wall it is important to have a mathematical description of a physical dipole [8]. This is a vortex structure that consists of two vortices that can move through a fluid without deformation and as one coherent structure. This means that in the co-moving frame the fluid flow is steady. A well-known mathematical model for a physical dipolar vortex is the so-called Lamb-Chaplygin dipole. This model describes a steadily moving dipolar vortex that satisfies the equation (2.6) and the inviscid \( (\nu = 0) \) and steady version of equation (2.5). Furthermore it assumes a linear relation between \( \omega_i \) and \( \psi_i \) inside the dipole atmosphere and an irrotational flow outside the atmosphere, which is assumed to be circular with radius \( R \),
\( r \leq R \) \hspace{1cm} \omega_i = k^2 \psi_i, \hspace{1cm} (2.8) \\
\( r > R \) \hspace{1cm} \omega_e = 0, \hspace{1cm} (2.9) 

such that the inviscid and steady version of equation (2.5) is automatically satisfied. Combining equations (2.6) and (2.8) results in a Poisson equation for the stream function inside the dipolar atmosphere \( (\psi_i) \):

\( r \leq R \) \hspace{1cm} -\nabla^2 \psi_i = k^2 \psi_i. \hspace{1cm} (2.10) 

By demanding that the flow at \( r = R \) matches the irrotational flow outside the atmosphere with translation velocity \( U \), the result for the interior and exterior stream function in the co-moving frame is [8]:

\( r \geq R \):
\[
\psi_e(r, \theta) = U \left( \frac{R^2}{r} - r \right) \sin(\theta), \hspace{1cm} (2.11)
\]

\( r < R \):
\[
\psi_i(r, \theta) = \left( \frac{-2U}{k J_0(kR)} \right) J_1(kr) \sin(\theta). \hspace{1cm} (2.12)
\]

This describes a circular dipole with radius \( R \) and translation velocity \( U \), \( J_0 \) and \( J_1 \) are the Bessel functions of the first kind of zeroth and first order, respectively, and:

\[
kR = 3.83 ..., \hspace{1cm} (2.13)
\]

is the first non-trivial zero of \( J_1 \). In order to illustrate the Lamb-Chaplygin dipole structure a graphical representation of this dipolar vortex model is shown in Figure 2.1, here the stream function contours in the co-moving frame are plotted, along with the vorticity distribution.

![Graphical representation of the Lamb-Chaplygin dipolar vortex model showing the stream function contours and vorticity distribution (color) for a dipole atmosphere with \( R = 1 \)](image-url)
The Lamb-Chaplygin dipolar vortex propagates through an inviscid fluid without deformation, or dissipation. In a (realistic) fluid with viscosity the viscous forces affect the time evolution of the dipolar structure and energy in the flow field is getting dissipated. The effect of the viscous forces can be assessed by a comparison with the internal forces. The so-called Reynolds number represents the ratio between the characteristic magnitudes of these forces in a flow field. If a Lamb-Chaplygin dipolar vortex in initialized in a fluid with kinematic viscosity ($\nu$), the Reynolds number associated with the dipole radius ($R$) and translation speed ($U$) is defined as:

$$\text{Re} = \frac{UR}{\nu} \quad (2.14)$$

For high Reynolds numbers ($\text{Re} \gg 1$) the effects of the viscous forces are small, and therefore will only influence the dipolar vortex structure at a slow rate.

### 2.3. Effects of background rotation

#### 2.3.1. Taylor-Proudman theorem

The experiments described in this thesis are conducted in a rotating tank. This implies the presence of a Coriolis and centrifugal force in the co-rotating frame. The latter can be written as a gradient and is therefore included in the pressure gradient force term. The Navier-Stokes equations (2.1) in this co-rotating frame of reference are:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} - 2\Omega \times \mathbf{v}, \quad (2.15)$$

where $\Omega$ is the angular velocity associated with the background rotation. The equation can be made non-dimensional with the introduction of the following non-dimensional quantities and operator:

$$\tilde{\mathbf{v}} = \frac{\mathbf{v}}{U}, \quad \tilde{p} = \frac{p}{\rho \Omega UL}, \quad \tilde{t} = t\Omega, \quad \tilde{r} = \frac{r}{L}, \quad \tilde{\nabla} = L\nabla \quad (2.16)$$

here $U$ is a typical velocity in the fluid and $L$ a typical length scale in the system. By substituting these non-dimensional variables into equation (2.15) one obtains:

$$\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + \frac{U}{\Omega L} (\tilde{\mathbf{v}} \cdot \tilde{\nabla}) \tilde{\mathbf{v}} = -\tilde{\nabla} \tilde{p} + \frac{\nu}{\Omega L^2} \tilde{\nabla}^2 \tilde{\mathbf{v}} - 2k \times \tilde{\mathbf{v}}. \quad (2.17)$$

Here $k$ is the unit vector in the direction of $\Omega$, i.e. $k = \frac{\Omega}{\Omega}$. The equation contains two non-dimensional numbers:

$$\text{Ro} = \frac{U}{\Omega L} = \frac{\omega}{f} ; \quad \text{E} = \frac{\nu}{\Omega L^2} \quad (2.18)$$

The Rossby number $\text{Ro}$ represents the ratio between inertial forces and the Coriolis force. The representation involving $\omega$ and $f$ is the Rossby number for the fluid in a vortex structure, where $f = 2\Omega$ is the so-called Coriolis parameter. The Ekman number $\text{E}$ represents the ratio of viscous forces and the Coriolis force. In the case of a quasi-stationary flow (this is discussed in the Appendix A, $\left(\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} = 0\right)$ and
the Rossby number and Ekman number have very small values, i.e. Ro \ll 1 and E \ll 1. Equation (2.15) then becomes:

\[-\nabla \tilde{\rho} = 2\mathbf{k} \times \tilde{\mathbf{v}}.\]  \hspace{1cm} (2.19)

Taking the curl of this equation, we derive:

\[
\nabla \times (\nabla \times \tilde{\rho}) = \nabla \times (\nabla \times 2\mathbf{k} \times \tilde{\mathbf{v}}) = 2\{k(\nabla \cdot \tilde{\mathbf{v}}) - \tilde{\mathbf{v}}(\nabla \cdot k) + (\nabla \cdot \mathbf{v})k - (\mathbf{k} \cdot \nabla)\tilde{\mathbf{v}}\}
\]

\[
= 2\left\{0 - 0 + 0 - \frac{\partial \tilde{\mathbf{v}}}{\partial z}\right\}.
\]  \hspace{1cm} (2.20)

resulting in the so-called Taylor-Proudman theorem:

\[
\frac{\partial \tilde{\mathbf{v}}}{\partial z} = 0.
\]  \hspace{1cm} (2.21)

This implies that if the flow satisfies the conditions mentioned earlier the flow profile is independent of the z-coordinate (i.e. 2D). This result was first derived by Proudman in 1916 [9] and has been experimentally verified by Taylor in 1923 [10] and is therefore known as the Taylor-Proudman theorem.

2D flows with high Reynolds numbers behave significantly different from three-dimensional (3D) flows. An important feature of a 2D turbulent flow is that it is characterized by the inverse energy cascade [11]. This means that the energy containing eddies grow in size by coalescing. Therefore an eddy rich flow organizes in large structures, as opposed to the energy cascade observed in 3D turbulent flows. Typically here vortex structures tend to break up into smaller structures until a characteristic length scale is reached where the dissipative nature of the viscous forces dominates [12].

### 2.3.2. Two-dimensionality of a flow with a dominant vorticity direction

An extensive experimental study on the stability of vortices in a rotating fluid, characterized by a wide range of Rossby numbers, is presented in [13]. Here it was concluded from a 2D instability analysis that the sign of the vorticity in a vortex as observed in a co-rotating frame is of great relevance for its stability. Furthermore it was found that cyclonic vortices characterized by a high Rossby number (Ro > 1) usually exhibit a stable and 2D character. In contrast, the Taylor-Proudman theorem is limited to flows that are assumed to be characterized by a small Rossby number. Interestingly, stable, 2D vortices characterized by a high Rossby number are commonly observed (e.g. [5], [13], [14], [15] and are encountered in this thesis as well). This “unexpected” two-dimensionality can still be explained by the Taylor-Proudman theorem. Rather than evaluating the equations of motion in a frame co-rotating with the background, a frame of reference can be chosen such that it co-rotates with the angular velocity of the fluid in a vortex. If this vortex is swirling in the same direction as the background (i.e. cyclonic motion), the fluid in this vortex is again characterized by a low Rossby number (Ro = \(\frac{\omega}{f}\)). Therefore the flow inside this vortex is 2D according to the Taylor-Proudman theorem. This is of course not necessarily the case for a vortex that swirls in the opposite direction as the background rotation (i.e. anti-cyclonic motion), as the Coriolis parameter (\(f\)) might vanish in this frame. Here we present yet another analysis of the 2D character of a flow in a vortex (i.e. a flow in a rotating table as seen from the lab-frame). Rather than evaluating the applicability of the Taylor-Proudman theorem for different sections of a flow
field in different rotating frames, we consider the flow as observed in an inertial frame of reference (i.e. the laboratory frame).

Consider a quasi-steady flow where viscosity plays no role of importance (see Appendix A). In the laboratory frame, equation (2.1) reduces to:

\[(\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho}, \tag{2.22}\]

describing a balance between inertial and pressure gradient forces. It proves insightful to take the curl of this equation. Assuming a barotropic fluid results in:

\[
\nabla \times (\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \times \left( -\frac{\nabla p}{\rho} \right) \rightarrow
\]

\[
\nabla \times \left( \frac{1}{2} \nabla (\mathbf{v} \cdot \mathbf{v}) + (\nabla \times \mathbf{v}) \times \mathbf{v} \right) = 0 \rightarrow
\]

\[0 + \nabla \times (\omega \times \mathbf{v}) = 0. \tag{2.23}\]

Using a vector cross product identity, this can be rewritten into:

\[
\omega (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \omega) + (\nabla \times \mathbf{v}) \omega - (\omega \cdot \nabla) \mathbf{v} = 0. \tag{2.24}\]

The first term of this equation is equal to zero due to the incompressibility of the fluid (eq. (2.2)) and since the divergence of the curl of any vector field is always zero, the second term can be truncated as well. Resulting in:

\[(\mathbf{v} \cdot \nabla) \omega - (\omega \cdot \nabla) \mathbf{v} = 0. \tag{2.25}\]

We arrive at the so-called, barotropic, steady and inviscid version of the vorticity equation. Now we consider a flow with a vorticity field having one dominant component, say the $z$-component \((\omega_z \gg \omega_x, \omega_y)\), such that

\[
\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}, \tag{2.26}\]

and furthermore, assuming not trivially,

\[
(\mathbf{v} \cdot \nabla) \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \approx (\mathbf{v} \cdot \nabla) \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}. \tag{2.27}\]

With this we can evaluate the $x$ and $y$ component of equation (2.25), whilst defining $\mathbf{v} = (u, v, w)$,

\[x: \quad -\omega_z \frac{\partial u}{\partial z} = 0 \rightarrow \frac{\partial u}{\partial z} = 0, \tag{2.28}\]

\[y: \quad -\omega_z \frac{\partial v}{\partial z} = 0 \rightarrow \frac{\partial v}{\partial z} = 0. \tag{2.29}\]
Apparently the velocity field components perpendicular to the dominant vorticity direction \((z)\) are independent of the \(z\)-coordinate. In order to analyze the vertical velocity field component \((w)\) dependence on the spatial coordinates \((x,y,z)\), we first look into the vorticity in the \(x\) and \(y\) direction,

\[
\begin{align*}
0 &= \omega_x = \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \Rightarrow \frac{\partial w}{\partial y} = 0, \\
0 &= \omega_y = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \Rightarrow \frac{\partial w}{\partial x} = 0,
\end{align*}
\]

concluding that \(w\) can only be a function of the \(z\)-coordinate,

\[
w = w(z).
\]

The \(z\)-dependence can be further analyzed by taking the \(z\)-derivative of the incompressibility equation (2.2).

\[
\frac{\partial}{\partial z}(\nabla \cdot \mathbf{v}) = \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2} = 0 \Rightarrow \frac{\partial^2 w}{\partial z^2} = 0.
\]

Leading to the conclusion that \(w\) can only be a linear function of \(z\),

\[
w = az + b,
\]

where \(a\) and \(b\) are constants. The derivation above shows that a flow characterized by a dominant vorticity component in an inertial frame is quasi-2D, without the necessity of limiting the range of vorticity in the flow field. It also shows that in general an anti-cyclonic vortex (or a part of it) as observed in a frame co-rotating with the background, cannot have a larger absolute vorticity than the background vorticity \((\Omega)\) to maintain a 2D character.

2.3.3. Force balance for vortices

Consider –in a rotating frame of reference- a fluid parcel in a steady circular vortex. This fluid parcel experiences three forces that are in balance:

1. Centrifugal force
2. Coriolis force
3. Pressure force

For a fluid parcel in circular motion with azimuthal velocity \(V\), radius \(R\), density \(\rho\), inward pressure gradient \(\frac{\partial p}{\partial n}\) and Coriolis parameter \(f\) the force balance is:

\[
\frac{V^2}{R} + fv + \frac{1}{\rho} \frac{\partial p}{\partial n} = 0.
\]
Possible configurations of force balances for circular motion are shown in Figure 2.2.

The (c) and (d) configuration are anomalous as they describe strong anticyclonic motion, and therefore are usually not observed in rotating tank experiments [13].

The requirement that the root in equation (2.36) must be real-valued implies a condition for the pressure gradient:

\[
\frac{\partial p}{\partial n} < \frac{1}{4} R \rho f^2 \tag{2.37}
\]

Apparently for the high pressure anti-cyclonic vortex the pressure gradient is limited \(\frac{\partial p}{\partial n} > 0\), in and around the center the horizontal pressure gradient is close to zero. This results in moderate motions in the vortex center as compared to the low pressure cell. For the low pressure cyclonic vortex \(\frac{\partial p}{\partial n} < 0\) the condition in equation (2.37) does not imply a maximum absolute pressure gradient.
2.3.4. The Effect of the boundary layer at the bottom

In the rotating tank experiments the flow domain is necessarily 3D. For example, the bottom imposes a no-slip condition, so that a boundary layer is formed. The assumptions made in section 2.3.1 and 2.3.2 do not apply in this boundary layer. The characteristics of this boundary layer, perpendicular to the rotation axis are different from those of a boundary layer parallel to the rotation axis (e.g. those found at objects in the 2D flow domain). The boundary layer at the bottom of a rotating tank is referred to as the Ekman boundary layer.

Vortices “living” in a rotating tank are affected by this boundary layer [17]. Not only is there additional viscous dissipation (i.e. bottom drag) compared to the truly 2D case, it also appears that fluid from the boundary layer and the interior region is exchanged if the bulk flow has vorticity. This effect is called Ekman blowing or suction. Depending on the sign of vorticity a vortex can be associated with a high or low pressure field according to section 2.3.3. The boundary layer under a low-pressure cyclonic vortex blows fluid into the vortex, whereas the boundary layer underneath a high-pressure anticyclonic vortex sucks in fluid from the interior flow. This effect results in a faster decay of cyclonic vortices compared to the anticyclonic vortices. This non-linear effect can be understood by virtue of conservation of angular momentum. The inward motion present in an anti-cyclonic vortex (to compensate the outward flux at the bottom) causes fluid parcels to gain vorticity as they move towards the center in order to maintain angular momentum, and vice versa for cyclonic vorticity.

The influence of these 3D effects can be modeled for a 2D flow by adding two Ekman related terms to the vorticity stream function formulation (2.5). For details see [18]. The result is:

\[
\frac{\partial \omega}{\partial t} + (\nabla \omega \times \nabla \psi) \cdot \mathbf{k} - \frac{1}{2} \sqrt{E} \nabla \psi \cdot \nabla \omega = \nu \nabla^2 \omega - \frac{1}{2} \sqrt{E} \omega (\omega + f). \tag{2.38}
\]

This equation describes the evolution of a 2D fluid flow taking into account 3D bottom Ekman effects that are present in a rotating laboratory fluid flow. The last term on the right-hand side of the equation describes the non-linear effect of the Ekman boundary layer.

This in combination with the effects described in section 2.3.3 has major implications for the evolution of the dipolar vortex. Apparently the experimental setup contains a built in asymmetry between the cyclonic and anticyclonic vortex patches. Therefore, it is easy to see that a dipolar vortex according to the symmetric Lamb-Chaplygin dipolar vortex model cannot exist in a rotating tank experiment. To avoid the asymmetric effects mentioned in section 2.3.2, 2.3.3 and 2.3.4 the ratio of vorticity and background rotation \((\omega/f)\) should be as small as possible.

2.4. Birth of new vortex structures

2.4.1. Kelvin’s circulation theorem for 2D flows

Consider the circulation along a material curve \(C\):

\[
\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{l}. \tag{2.39}
\]
The rate of change of the circulation can be expressed as follows:

\[
\frac{D\Gamma}{Dt} = \oint_c \frac{Dv}{Dt} \cdot dl + \oint_c v \cdot \frac{Ddl}{Dt}.
\]  

(2.40)

The first term in the right hand-side can be evaluated using equation (2.15):

\[
\oint_c \frac{Dv}{Dt} \cdot dl = \oint_c -\frac{\nabla p}{\rho} + v\nabla^2 v - 2\Omega \times v \cdot dl = \oint_c -\frac{\nabla p}{\rho} \cdot dl + \oint_c v\nabla^2 v \cdot dl - \oint_c 2\Omega \times v \cdot dl.
\]  

(2.41)

Under the assumption that viscous effects are small (i.e. in the interior of a high Reynolds number flow) we can neglect the second term on the right hand side. By applying the Stokes theorem to the first and last term on the right hand side the following relation can be derived:

\[
\oint_c \frac{Dv}{Dt} \cdot dl = \iint_c \nabla \times -\frac{\nabla p}{\rho} \cdot dA - \iint_c \nabla \times 2\Omega \times v \cdot dA = 0 - 2\Omega \iint_c \frac{\partial u}{\partial x} e_x + \frac{\partial v}{\partial y} e_y - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) e_x \cdot dA = 2\Omega \iint_c \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) e_x \cdot dA = 0,
\]  

(2.42)

the last two equalities are under the assumption that we consider an incompressible flow and material curve C in a plane (i.e. x−y plane).

The second term on the right hand side of equation (2.39) can be written as:

\[
\oint_c v \cdot \frac{Ddl}{Dt} = \oint_c v \cdot [(dl \cdot \nabla) v] = \oint_c v |\nabla^2| \cdot dl = 0.
\]  

(2.43)

Again the last equality is obtained by applying the Stokes theorem. Combining the results above leads to the famous Kelvin’s circulation theorem for an inviscid fluid flow:

\[
\frac{D\Gamma}{Dt} = 0.
\]  

(2.44)

This means that fluid parcels in the interior of a high Reynolds number flow maintain their vorticity (see equation (2.7)) and that therefore iso-vorticity lines can act as a fluid tracer. This also leads to the conclusion that secondary vorticity structures that may arise in the flow domain originate from flow regions where viscous effects cannot be neglected (e.g. boundary layers).

2.4.2. Secondary vorticity

The path of coherent dipolar vortex structures can be affected by the presence of other vortices. In order to describe the flow evolution of a dipolar vortex the “birth” of new vortex structures during the flow evolution is very important. These so-called secondary vorticity structures concerned in this thesis owe their existence to viscous vorticity/circulation production in a flow region where viscous effects are important. Two of such regions are boundary layers near a no-slip wall and the shear layer originating
from a boundary layer at a sharp edge due to flow separation. The boundary layer is the region where a non-zero flow profile adjusts to the no-slip condition at the wall. This region is often characterized by high spatial derivatives in the velocity profile and therefore high vorticity values are observed here.

As a dipolar vortex approaches a solid no-slip wall the resulting flow induces a boundary layer. It has been found in experiments and numerical investigations that this vorticity rich region can detach from the wall [4][19]. This subsequent roll-up of the detached boundary layer vorticity results in a new vortex structure. Interestingly these new vortex structures can greatly affect primary vortices in the dipole. More information on these dynamics can be found in [20].

An example of the collision of a dipole with a no-slip wall was studied experimentally by van Heijst & Flór (1989) [19] and numerically by Orlandi (1990) [4]. Figure 2.3 shows the flow evolution observed in both the dye visualized experiment and a numerical simulation.

![Figure 2.3](image)

Figure 2.3 Results of the experiment (upper row) by van Heijst and Flór [19] and vorticity contours obtained from numerical simulations by Orlandi [4] (bottom row)

The detachment of boundary layers from a wall and the resulting subsequent secondary vortices are also observed for flows near sharp edges: here so-called flow separation is observed, which causes vorticity from the boundary layer to be advected into the interior flow domain, also resulting in vortices. This is due to the fact that fluid passing the ridge of a sharp edge cannot follow the contours of the sharp edge. This effect can be understood by a simple analysis of the forces on fluid parcels that are necessary to create a non-separating fluid flow around a sharp edge. For this purpose, let us consider a potential flow around a sharp edge of a perfect fluid (i.e. \( \nu = 0 \)). The velocity components of such a flow can be obtained by conformal mapping an uniform parallel flow along an infinite wall into a domain with a sharp-edged wall [21]. The resulting cylindrical velocity components \((v_\theta, v_r)\) are presented below and the corresponding streamline pattern is shown in Figure 2.4.
Figure 2.4 Stream function contours around a sharp edge (in red) according to potential flow theory

It is clear that near the sharp edge the azimuthal velocity becomes infinite and thereby a fluid parcel experiences a centrifugal force that also becomes infinitely large. This centrifugal force is compensated by a pressure gradient force (accordingly to the Bernoulli laws). However, the singularity at \( r = 0 \) remains. For real fluids (i.e. \( \nu \neq 0 \)) viscous forces generally oppress large spatial gradients in the flow field. Furthermore, the no-slip condition states the the fluid at the fluid-wall interface is at rest (i.e. \( \nu = 0 \)). This means that the pressure gradient force near the tip is limited and thereby the centrifugal force is limited. Therefore, if a fluid flows around a sharp edge, the fluid parcels cannot follow the streamline contours as shown in Figure 2.4 and flow separation is likely to take place. This separated stream advects vorticity from the boundary layer into the interior flow, thus creating a shear layer. The vorticity from such a shear layer can also form into a new vortex structure. Again, these secondary vortices can affect the evolution of flow field.
2.5. Potential flow

In order to model the collision of a dipole colliding with a wall, 2D potential flow theory can be used, this theory describes the motion of an inviscid fluid. This section aims to clarify how this theory is used to construct a model that is able to describe the motion and production of vortices in an analytical manner.

2.5.1. Complex velocity potential and point vortex modeling

Potential flow theory is elegantly described by velocities in the complex plane. The complex 2D velocity field is conveniently described by a complex scalar field, called the complex velocity potential: \( w \). Let \( z \) be the complex coordinate and \( u \) and \( v \) be the velocity in the direction of the real and imaginary axis, respectively, then:

\[
\frac{dw}{dz} = u - iv,
\]

with ‘\( i \)’ representing the imaginary unit (i.e. \( i = \sqrt{-1} \)). It can be easily shown that the flow is irrotational and incompressible if \( w \) is an analytical function of \( z \). Therefore finite sized vortex patches (e.g. the Lamb-Chaplygin dipole) cannot be described with such a complex velocity potential. To model the flow field due to a vortex the vorticity is considered concentrated entirely in a single point, thereby introducing circulation in the flow field. The complex velocity potential for a so-called point vortex located at \( z = z_0 \) is given by:

\[
w(z) = -iy \log(z - z_0),
\]

where \( \gamma \) is the strength of the point vortex. It is clear that the flow field described by this potential contains a pole in \( z = z_0 \) with residue \( -iy \). Therefore, according to Cauchy-Riemann integration rules in the complex plane the flow is now rotational with circulation \( \Gamma' = 2\pi\gamma \).

In the case of \( N \) vortices on an infinite plane the motion of each vortex is determined by the sum of the velocities induced by all other vortices, as described by the equation:

\[
\frac{dz_j}{dt} = u - iv \big|_{z=z_j} = \sum_{l=1, l \neq j}^{N} \frac{dw_l}{dz} \big|_{z=z_j} = \sum_{l=1, l \neq j}^{N} \frac{-i\gamma_l}{z_j - z_i}
\]

Here the subscripts indicate properties of a specific vortex (labeled 1 to \( N \)), while the * operator denotes the complex conjugate.

A simple dipole model can be constructed by the superposition of two oppositely-signed point vortices. These point vortices induce a velocity in the other such that the vortices form a moving pair with constant intermediate distance.

2.5.2. Wall modeling

The modeling of a no-slip boundary (i.e. a wall) is a major problem. Boundary layers at the wall due to the no-slip condition owe their existence to viscosity and are absent in any potential flow description. Therefore the no-slip condition cannot be adopted correctly in this model. However, the no-penetration
condition at the walls can be included by the process of adding mirror images of the flow inducing entities. By artificially adding for each point vortex an oppositely signed vortex mirror image, the no-penetration condition at the wall is satisfied. This concept requires an infinite, straight wall. Since the present study concerns geometries with finite walls, the method of conformal mapping has to be used.

2.5.3. Conformal mapping
In potential flow theory conformal mapping can be used to map a complex physical geometry to a more simple geometry. In this report conformal mapping is used to map geometries with semi-infinite walls to a geometry with infinite walls such that the method of mirror images can be readily applied. The mapping consists of a complex analytic function that maps the physical $z$-plane to a corresponding $\zeta$-plane (for details see chapter 5):

$$\zeta = f(z).$$

A mapping is considered conformal when it preserves local angles and the inverse function exists (i.e. the function is one-to-one). The mapping function $f(z)$ is specific for the geometry that is mapped. The strategy for solving a problem with conformal mapping is to initialize the problem in the $z$-plane, map it conformally to the $\zeta$-plane and solve the equations of motions in this plane (see section 2.5.4). Afterwards the positions of the point vortices over time can be mapped back to the $z$-plane via the inverse mapping function.

2.5.4. Equations of motion in the $\zeta$-plane
If $w_z$ and $w_\zeta$ are the complex potentials in the $z$-plane and the mapped $\zeta$-plane describing the motions of point vortices, respectively, then in general [22]:

$$w_z(z) \neq w_\zeta(\zeta).$$

Therefore the velocities are not conformal maps of each other under the mapping function $f$. Evidently the equations of motion in the $\zeta$-plane differ from those in the $z$-plane. This rather mathematical exercise was first solved for a single vortex by Kirchhoff and Routh in 1881 [23] and in 1941 it was generalized for $N$ vortices by Lin [24] and [25]. The result is known in literature as the Routh-Kirchhoff path function or Routh rule, a comprising and legible derivation of this result can be found in [26]. Accordingly the equations of motion for the $j$-th vortex with a time independent strength in a geometry with in total $N$ vortices and $M$ mirror images in a mapped frame is given by:

$$\frac{d\zeta_j}{dt} = \left[ \sum_{l=1, l \neq j}^N \frac{-iy_l}{\zeta_j - \bar{\zeta}_l} + \sum_{k=1}^M \frac{-iy_k^m}{\zeta_j - \zeta_k^m} + \frac{i\gamma_j}{2} \frac{d}{d\zeta} \left( \log \left( \frac{dz}{d\zeta} \right) \right) \right]_{\zeta = \zeta_j} \left( \frac{1}{\frac{dz}{d\zeta}} \right)_{\zeta = \zeta_j}^{\left. \frac{dz}{d\zeta} \right|_{\zeta = \zeta_j}},$$

where the superscript $m$ indicates properties of the mirror image vortices. Apparently this path-function contains the derivative $\frac{dz}{d\zeta}$ that obviously originates from the conformal mapping function $f$ (as in equation (2.49)).
2.5.5. The Kutta condition

Potential flow theory as presented so far is not suitable for the study of the collision of a dipolar vortex with sharp-edged objects [5]. It is known that at a sharp edge in a flow domain new vortex structures can easily emerge. This effect is not included in the potential flow approach. The unphysical nature of the potential flow model can be illustrated by the fact that the potential flow around a sharp edge contains a singularity at the location of the sharp edge (see section 2.4.2). In order to construct a more physically accurate model an alteration has to be made. The singularity can be removed by demanding the flow to satisfy the so-called “Kutta condition”. The Kutta condition states that fluid in a flow will create a circulation with such strength that the tip of the sharp is a stagnation point [27]. A stagnation point in a fluid flow is a point where streamlines coincide. This stagnation point facilitates the modeling of a separating fluid flow from the wall. Here the streamline along a wall boundary coincides with a streamline that reaches into the flow domain. Mathematically the Kutta condition states [28]:

\[
\left. \frac{dw}{dz} \right|_{z = z_{tip}} = 0. \tag{2.52}
\]

This condition can be fulfilled by adding a so-called Kutta vortex with specific strength \( (\gamma_K) \) and position \( (z_K) \). This means that in a flow domain with N other vortices and in total M mirror vortices (including the Kutta vortex mirror(s) itself) the following relation applies.

\[
\frac{i \gamma_{Kutta}}{\zeta_{tip} - z_{Kutta}} = \sum_{l=1}^{N} \frac{-i \gamma_l}{\zeta_{tip} - \zeta_l} + \sum_{j=1}^{M} \frac{-i \gamma_j}{\zeta_{tip} - \zeta_j}. \tag{2.53}
\]

Now the Kutta vortex can be used to model the secondary vorticity that is shed from the sharp-edged wall.

2.5.6. Modeling secondary vorticity

From experiments and numerical simulations it is known [5] that when a dipole approaches a sharp-edged wall secondary vorticity that is generated at by detachment of the boundary layer vorticity from the straight part of the wall and from the sharp edge. The vorticity from the straight part of the wall is not modeled, but the circulation at the tip can be modeled by the Kutta vortex. The observed secondary vortex gains strength whilst it is moving from the wall. This secondary vorticity can be modeled by releasing the Kutta vortex and let it gain strength whilst it is moving by applying the Kutta condition. The vorticity center position (modeled by \( z_K \) in the point vortex model) is not the same as the location of the originated new vorticity (\( \zeta_{tip} \)). This new circulation is modeled as if it is instantaneously transported to the vortex center. Therefore according to the so-called zero-force model the equation of motion for a moving point vortex with time dependent strength contains an extra term [29]. For convenience we use: \( j_{Kutta} = K \) for the number of the Kutta vortex with time dependent strength.
\[
\frac{d\zeta^*_K}{dt} = \sum_{i=1, l \neq K}^N \frac{-i\Gamma^*_l}{\zeta_l - \zeta^*_K} + \sum_{k=1}^M \frac{-i\gamma_k m}{\zeta_k - \zeta^*_K} + \frac{i\gamma_K}{2} \frac{d}{d\zeta} \left( \log \left( \frac{dz}{d\zeta} \right) \right)_{\zeta_K} \left[ \frac{1}{|dz/d\zeta|^2} \right]_{\zeta_K}^1 \left( \zeta^*(\zeta_K) - \zeta^*_{tip} \right) \frac{1}{\gamma_K} \frac{dz^{*}(\zeta_K)^*}{dt} \frac{1}{(dz^{*}(\zeta_K)^*)_{\zeta_K}}.
\]  

(2.54)

The use of the zero force model is suggested in [28]. In this model the shear layer is considered to be a sheet with zero vorticity connected to the Kutta vortex where the circulation is entirely concentrated. The extra term (second row) in equation (2.54) under the assumption that the global force exerted on the feeding shear layer and the Kutta vortex is null [30]. Note that \( \frac{dy_K}{dt} \) depends on the complex velocity of the Kutta vortex itself. Therefore, an implicit relation must be solved to obtain the complex velocity of the Kutta-vortex.

The zero-force model is not the only option to describe the generation of vorticity with the use of the Kutta condition. Another method is to release many vortices subsequently such that the Kutta condition is fulfilled [31]. The secondary vortex is then modeled by a combination of many Kutta vortices. A method with a large number of vortices is not preferred as computational costs rise to maintain acceptable numerical accuracy [32].

### 2.5.7. Free parameters

Equations (2.53) and (2.54) describe the motion and strength of the Kutta vortex as a function (among other variables) of its location. When solving the equations numerically an initial location \( \zeta_K \) has to be chosen. This is a so-called free parameter. The term “free” refers to the fact that the value of this model parameter has no distinct relation to the physical flow field properties. Therefore, it is free to be determined. It is obvious that the vorticity generated at the sharp edge is located at the sharp edge, so the initial position should be chosen close to the edge. For symmetry reasons it is chosen not to consider \( \zeta_K \) with a perpendicular offset to the sharp-edged wall. To avoid the production of many weak Kutta vortices the secondary vortex position is fixed until a threshold strength \( (\gamma_c) \) is reached. The critical strength before release \( (\gamma_c) \) is also a free parameter. Finally, in [28] it is suggested that the Kutta vortex gains strength as long as it continuously gains in absolute strength. This scheme would make it possible for a Kutta vortex to gain strength at any distance from the sharp edge. As the Kutta vortex can be advected away from the sharp edge, it is unphysical to model the strength of this vortex by the conditions at the sharp edge. So at some critical distance \( (\delta) \) the strength of the Kutta vortex is fixed and a new Kutta condition satisfying vortex is placed near the edge. This gives the modeling of the secondary vorticity 3 free parameters, these are listed below.

1. Initial position of the Kutta vortex \( (\zeta_K) \)
2. Critical vortex strength for release \( (\gamma_c) \)
3. Maximum strengthening distance from sharp edge \( (\delta) \)

A method for determining these parameter values is presented in chapter 8.
3. Experimental method
This chapter presents the experimental method that is used to study the collision of a dipolar vortex against solid obstacles different shapes. An important aspect of the flow studied flow here is its 2D character. Any experiment carried out in a laboratory is obviously 3D. However, background rotation promotes a 2D flow field, as expressed by to the Taylor-Proudman theorem in section 2.3.1. Therefore a quasi-2D flow can be achieved by performing the experiment in a rotating tank.

3.1. Experimental set-up
A schematic overview of the experimental setup is shown in Figure 3.1.

A rectangular cuboid water tank with dimensions 100 x 150 x 40 cm$^3$ is placed on a rotating table and filled with tap water. Before the experiment can start the fluid is brought into solid body rotation at an angular velocity of $\Omega = 0.70$ rad s$^{-1}$. This rotation speed value is chosen high enough to ensure the general validity of the assumptions made for the Taylor-Proudman theorem, yet slow enough for the experiment to be practically feasible and safe.

Prior to the experiment the fluid is allows to reach solid-body rotation during at least 60 min. Once the fluid is solid body rotation the centrifugal force will result in a parabolic free surface. The effects of a position dependent water height [17] should be corrected for. In order to have a fluid of uniform depth,
a parabolic bottom is used such that the depth of the fluid layer is the same everywhere. A dipolar vortex is conveniently created by dragging an open, thin-walled cylinder horizontally along a straight line through the fluid while simultaneously lifting it slowly out of the fluid. During this process the flow is unsteady and the Taylor-Proudman theorem does not apply. As the typical velocity of the cylinder is 10 cm/s and the diameter of the cylinder is 6 cm the Reynolds number associated with the cylinder lifting process is \( \text{Re} \approx 10^4 \). This implies that the wake behind the cylinder is turbulent. After the cylinder is lifted out the inverse energy cascade [11] causes the small vortex structures to form larger vortex structures and the result is a quasi-2D dipolar vortex structure. This dipole is characterized by a typical length scale associated with its radius \( R = 10 - 15 \text{ cm} \) and a typical propagation speed \( U = 1 - 2 \text{ cm/s} \).

In the experiments the collision of a dipole with two different geometries is studied. For the experiments with a sharp edge a flat aluminum plate with a thickness of 2 mm is used. Since this thickness is much smaller than the typical length scale associated with the dipole the edge of this plate is considered to be “sharp”. For the collision with an opening in a wall the fact that the ends of the wall are not sharp is important [7]. Therefore cylinders with a diameter of 3 cm are placed at the edge of the walls. A photo of the used “opening in a wall” geometry is shown below.

![Figure 3.2 Plan-view photograph of the gap between two walls (white) surrounded by water that is partially dyed (yellow). Cylinders are attached to the walls such that the edged have a larger radius of curvature](image)

During the experiment the fluid flow is recorded by a digital camera. The camera co-rotates with the water tank. A way to visualize the fluid flow is to add dye that acts as a visual tracer in the fluid by virtue of contrast between fluid with different colors or transparency. By adding dye near the geometry and in the cylinder before it is lifted out the tank the flow can be recorded by a color camera. It turns out that a large portion of the dye in the cylinder is entrained by the dipole and the evolution of the dipole can thus be monitored. The dye visualizations do not give any quantitative data for comparison with the numerical simulations, neither does it provide flow information for areas where there is no visible contrast. For a quantitative and entire flow field measurement Particle Image Velocimetry (PIV) measurements are done. These measurements are carried out by tracking suspended particles that are
passively advected by the flow using a 1600 x 1200 pixel$^2$ gray-scale camera. The camera records at a frame rate of 30 frames per second. Subsequent frames show details of the flow field by deformation of the particle distribution between these frames. A length calibration is necessary to correct for optical defects and gain information of the magnification factor for the images (i.e. mm/pixel). To do this a sheet with a grid is placed in the tank, which is filled with water at the depth of the light sheet that illuminates the particles. The camera takes a snapshot of a frame and this image is then analyzed by PIVMAP 1.2 software. Together with the known grid spacing a reshape mapping and a magnification factor are calculated. After carrying out this process once, all frames can be reshaped. The individual frames from this process are fed to the software program PIVview 3.5, together with magnification factor and the frame rate. The software is able to calculate the flow field by correlating small sections of an image with the corresponding section of the next image. More information on the PIV technique can be found in [33]. In practice the obtained flow field data is noisy. Therefore, outliers in the vorticity field are averaged out by applying a Gaussian filter. Due to the presence of relative weak convection cells in the flow field [15] a non-zero vorticity field is observed outside the primary and secondary vortices. The associated circulation is filtered out by applying thresholding. This post-processing influences the flow field locally, but the extracted data (vortex locations and strengths) concern more global flow field properties that appeared to be independent on the chosen filter parameters. Therefore, the quantitative data from PIV appears well determined despite of the errors in the raw flow field data. An actual error analysis of these quantities is not presented in this thesis.

Due to the dimensions of the rotating tank, the used fluid and the tank’s angular velocity the experiment is limited to only a finite range of experimental parameters. This section gives typical values of important quantities that have an effect on what can be investigated experimentally.

The experimental setup and the dipolar vortex produced in the experiment are both characterized by some typical values that give an estimate of dimensionless numbers discussed in the theory section. The typical values are:

$$L = 10 - 15 \text{ cm}, \quad U = 1 - 2 \text{ cm s}^{-1}, \quad \Omega = 0.7 \text{ rad s}^{-1}, \quad v = 10^{-6} \text{ m}^2 \text{ s}^{-1}$$  \hspace{1cm} (3.1)

From these we obtain the following typical values for the dimensionless numbers:

$$\text{Re} \approx 1000 - 3000,$$
$$\text{Ro} \approx 0.1 - 0.5,$$
$$E \approx 5 \cdot 10^{-5} - 10^{-4}.$$  \hspace{1cm} (3.2)

In the theory both $E$ and $\text{Ro}$ were assumed to be small enough for the Taylor-Proudman theorem to be applicable (i.e. $E << 1$ and $\text{Ro} \ll 1$). Although $E << 1$ in the experiments, since $\text{Ro} = O(1)$ it cannot be concluded that the Taylor-Proudman theorem will apply in the rotating tank experiment and therefore further investigation is needed. The Reynolds number $\text{Re} \gg 1$. This indicates the relative small effect of the viscous forces on the flow field evolution in the interior region of the flow (i.e at some distance from any solid boundaries).
3.2. Visual conformation of the Taylor-Proudman theorem

In the previous section it became apparent that based on a typical order of magnitude flow analysis the Taylor-Proudman theorem does not necessarily apply in the used experimental regime. As this is an important prerequisite for the experimental setup a specific experiment was run in order to verify the applicability of the theorem in the experiments. In this experiment the results of the well-known collision of a dipolar vortex with a straight wall collision from [19] (shown in Figure 2.3) are reproduced. This time however, in a rotating tank experiment with a wall that has only half the height associated with the free surface water level. If the flow is truly independent of the vertical coordinate the no-slip condition at the wall should also be applied in the column above the wall. In order to check if this is indeed the case the dipole vortex evolution will be traced with dye. A schematic overview of the experiment is shown in Figure 3.3.

Figure 3.3 Artist impression of the experimental set-up in which a dipolar vortex is made to collide with a submerged wall

Snapshots of the flow evolution are shown in Figure 3.4. The camera was placed under an angle in order to visualize the column-like structure of the flow.

Figure 3.4 Snapshots of a dye visualization experiment showing the evolution for a dipolar vortex that collides with a submerged wall.
In Figure 3.4 the wall is obscured by green dye and the fact that it is submerged does not promote its visibility either. However the presence of the wall is revealed by the fact that the orange dyed dipolar vortex rebounds in a similar fashion as presented in [4] and [19]. Evidently the flow did indeed organize in so-called Taylor-columns, and that boundary layer vorticity is produced at locations above the submerged wall. This experimental evidence leads to the conclusion that even though \( \text{Ro} = O(1) \) a quasi-2D flow is achieved by the used experimental set-up.

### 3.3. Formation of the dipolar vortex

The formation of the dipole was briefly discussed in section 3.1. Figure 3.5 shows a sequence of snapshots of a dye experiment that visualizes the formation of a dipole. 50 seconds after the start of the cylinder lifting a 2D laminar dipolar vortex has formed. The initial turbulent eddies that are not aligned with the background rotation tilt due to gyroscopic precession such that they align with the background rotation direction. This results in a quasi 2D flow field. It can be observed that the dipolar vortex entrains dye that was initially located inside the hollow cylinder. As the table rotates clockwise the upper and lower vortices are anticyclonic and cyclonic, respectively. The effects of the Ekman boundary layer are visible. The cyclonic vortex is subjected to Ekman blowing from the bottom and therefore fluid entering the vortex causes dyed fluid to leave the vortex, which is visible from the green tail. The anticyclonic upper vortex exhausts fluid via the Ekman suction mechanism at the bottom, which causes fluid entrainment of un-dyed fluid from around the vortex. This is visible by the black finger that seems to enter the dipole. It is also visible that the cyclonic vortex has formed earlier than its anticyclonic counterpart. As the vorticity characterizing the turbulent eddies that will form into the anticyclonic vortex is smaller than the theoretical limit discussed in section 2.3.2 and [13], the merged eddies are unstable. This is not the case for the merging eddies forming the cyclonic part, here the mentioned threshold does not apply. A stable high-pressure anticyclonic vortex is formed after the minimum vorticity of the merged eddies is above the mentioned threshold (regular flow, see section 2.3.3).

![Figure 3.5 Snapshots of a dye visualization experiment showing the dipolar vortex process, with clockwise background rotation direction.](image)

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3.4. Experimental challenges
The experimentally obtained flow evolution deviates from the evolution that would be observed in a purely 2D case. In sections 2.3.3 and 2.3.4 it is explained that this is related to the background rotation. Furthermore technical problems are also encountered, the most notable being the non-constant rotation speed of the table. The electronic control system that controls the rotation speed of the water tank can only do so with a certain tolerance. This means that the control electronics are constantly adjusting the rotation speed ever so slightly to maintain the desired rotation speed. Due to the inertia of the water in the tank the flow does not instantly adjust to the new rotation speed. This results in an alternating swirling flow in the tank. Due to the presence of a wall with a sharp edge the swirling fluid also flows around the sharp edge and this can lead to the production of secondary vortices. These vortices are not the result of a dipole induced flow and can therefore obscure the results in a non-transparent manner.

Counter measures have been taken to reduce these effects, the addition of in total 100 kg extra weight, above the camera and symmetrically distributed 2 meters from the rotation axis, effectively suppressed the production of vortices at the sharp edge due to the alternating flow. However, this method was only partially successful in suppressing the alternating flow effects as will be discussed in the results chapters.

In practice the process of obtaining good results is a challenge, for the dye-visualization experiments the results can be analyzed by eye in real time, leading to a direct decision on whether the experiment should be repeated or not. For the PIV experiment this poses a bigger practical problem, as only after the analysis of the results it can be concluded whether the experiment was a success or should be repeated.
4. Numerical method

In this Chapter the method that is used to obtain the numerical results is presented.

4.1. Navier-Stokes based numerical simulations

Experimental results are confronted with direct numerical simulations (DNS) of the relevant set of equations: (2.1) and (2.2). More specifically, we will use the commercial software package COMSOL Multiphysics® Modeling Software version 4.4 [34] that applies the finite element method to solve the incompressible Navier-Stokes equations.

The software calculates the flow evolution in a flow domain, resulting from an initialized flow field and according to the applied boundary conditions. The solution is calculated with a certain spatial resolution that is determined by the size of the finite elements. All numerical simulations are performed with a Reynolds number of \( \text{Re} = 2500 \) (as defined by equation (2.14)) in order to match the experimental Reynolds number. In the numerical simulations two basic geometries are investigated: the sharp-edged wall (shown in Figure 4.1 (a)) and the opening between two walls (shown in Figure 4.1 (b)).

![Figure 4.1 Numerical flow domain (colored) and geometries: sharp-edged wall (a) and a wall with an opening (b). The offset parameter \( d \) and gap opening parameter \( g \) are introduced.](image)

The dipolar flow field is initialized with the velocity field described by the stream function from equations (2.11) and (2.12), and evaluated in a non-co-moving frame, in such a way that the dipolar vortex is located and orientated at the desired position and orientation. For the fluid-geometry interface (black lines) the no-slip boundary condition is applied. In order to minimize the influence of the edges of the domain a stress-free boundary condition is applied at these walls such that there is no formation of boundary layers here. Furthermore, a numerical grid is defined. The software allows an irregular grid that enables the user to define a finer mesh in certain areas where high spatial derivatives are observed. For example, the grid at the no-slip walls (the viscous boundary layer is characterized by high spatial derivatives) is more refined than the mesh near the edge of the domain (where the fluid is quiescent).
order to find out when the flow domain is resolved with a high enough resolution (i.e. a fine enough mesh) a convergence study was performed. The convergence study aims to show what spatial resolution is needed in order to correctly describe the flow field evolution. The numerical solution should be independent of the chosen grid. So by iteratively increasing the number of mesh elements (i.e. decreasing the mesh element size) and comparing the results, the maximum mesh element size can be determined. The convergence study concerns a collision of a dipole with a sharp-edged wall with an offset value of $d = 0$. Figure 4.2 shows the evolution of the total kinetic energy of the flow for different numbers of mesh elements.

![Energy convergence study](image)

**Figure 4.2** Time evolution of the kinetic energy in the flow field for simulations with different grid refinements. The legend gives the used number of mesh elements.

Figure 4.2 shows that the total kinetic energy converges to a single value for decreasing mesh element sizes and that the numerically obtained flow field becomes mesh independent when the total number of mesh elements is for the order of 1329777 or higher. Therefore, in this report all the numerical results are based on simulations with the mentioned number of mesh elements.

### 4.2. 2D Navier-Stokes and the vorticity-stream function formulation

Rather than solving the Navier-Stokes based set of equations (2.1) and (2.2), the 2D flow field can also be obtained from the set of equations (2.5) and (2.6). The so-called vorticity-stream function formulation makes it possible to model the effects imposed by the Ekman layer at the bottom according to equation (2.38). Therefore it seems beneficial to use this formulation in order to simulate the flow more realistically for the conditions in a rotating tank experiment. However, there are two main reasons for choosing the classical Navier-Stokes solver method. First the Ekman theory by itself does not include all the important 3D effects mentioned in section 2.3.2, 2.3.3 and 3.3. Second, the Navier-Stokes simulation software is optimized for solving fluid flow problems. Solving the vorticity-stream function formulation equations with a general partial differential equation solver may result in unphysical flow fields [7].
5. Point vortex model method

In this chapter the method that is used to obtain results from the point vortex model is presented. The basics of this model are presented in section 2.5.

5.1. Sharp-edged wall

The sharp edge geometry can be implemented with the use of a correct conformal mapping and the accompanying placement of mirror images. The mapping function should map the entire complex plane with the exception of a semi-infinite slit along the positive real axis to the upper half plane. The mapping is described by the following transformation:

\[ z: \mathbb{C} \setminus (0, \infty) \rightarrow \zeta: \{ \text{im } \zeta > 0 \}. \]  

(5.1)

The used mapping function and inverse mapping function that result in such a transformation are:

\[
\begin{align*}
    z &= \zeta^2, \\
    \zeta &= i(-z)^{1/2}.
\end{align*}
\]  

(5.2)

Here the fractional power denotes the principal value square root operator i.e. with a branch cut along the negative real axis. A visual representation of the conformal mapping is given in Figure 5.1.

![Figure 5.1 Graphical representation of conformal mapping described by (5.1) and (5.2), the colored markers indicate corresponding positions in the z and \( \zeta \)-plane.](image-url)

The location of the infinite wall in the \( \zeta \)-plane corresponds with the location of the semi-infinite slit in the \( z \)-plane. In the \( \zeta \)-plane the method of placing mirror images can be applied such that the flow does not penetrate the semi-infinite wall in the \( z \)-plane. The location of the j-th vortex in the \( \zeta \)-plane can be split up in a real and imaginary part according to:

\[
\zeta_j = \xi_j + i\eta_j,
\]  

(5.3)

where \( \eta_j > 0 \). The j-th vortex in the \( \zeta \)-plane has one mirror image located according to:
the strength of the j-th mirror image $\gamma_j^m$ is given by:

$$\gamma_j^m = -\gamma_j.$$  \hfill (5.5)

With all vortices in place the equations of motion as given in section 2.5 can be evaluated. Equation (2.51) gives the complex velocity of a vortex labeled 'j' that has a time independent strength, $\gamma_j$. In total N vortices are present in the flow domain (thus not counting mirrors). Substituting equation (5.2), (5.4) and (5.5) gives for the complex velocity of the j-th vortex:

$$\frac{d\zeta_j}{dt} = \left[ \sum_{l=1,l \neq j}^{N} \frac{-iy_l}{\zeta_j - \zeta_l} + \sum_{l=1}^{N} \frac{iy_l}{\zeta_j - \zeta_l} + \frac{i\gamma_j}{2\zeta_j} \right] \frac{1}{|2\zeta_j|^2}. \hfill (5.6)$$

According to equation (2.53) the vortex with time dependent strength (i.e. labeled with number N) has a strength $\gamma_N$ given by:

$$\gamma_N = \frac{\zeta_N \zeta_N^*}{\eta_N} \sum_{l=1}^{N-1} \gamma_l \eta_l.$$ \hfill (5.7)

Once this vortex is released, its complex velocity is not given by equation (5.6), but it is obtained from solving a linear system of equations (see section 2.5.6). Substitution of equations (5.1) – (5.7) in equation (2.54) results in:

$$\left[ 1 + \text{Re}\left\{ \frac{2\xi_N}{\xi_N^* + \eta_N} \frac{1}{(2\zeta_N)^2} \right\} - \text{Re}\left\{ \frac{(\xi_N^2 - \eta_N^2)}{\eta_N(\xi_N^2 + \eta_N^2)} \frac{1}{(2\zeta_N)^2} \right\} \right] \frac{d\xi_N}{dt} = \frac{d\eta_N}{dt}.$$  \hfill (5.8)

In this expression, $\text{Re}\{z\}$ and $\text{Im}\{z\}$ denote the real and imaginary part of the complex quantity $z$, respectively.
5.1.1. Numerical method
Now that the used conformal mapping and the equations of motions are known, a numerical method can be formulated that describes the kinematics of the point vortices. This method is explained in the flow chart in Figure 5.2. The actual MATLAB code that is used for obtaining the results is given in Appendix B.
Initialization of input parameters:
Initialize dipole vortices \( (z_j=1, z_j=2, \gamma_1, \gamma_2) \)
set start location of Kutta vortex \( z_K \)
set threshold strengthening distance \( \delta \).
set threshold release strength \( \gamma_c \)

Map initial positions to \( \zeta \)-plane

Determine Kutta vortex strength \( \gamma_N(n=1) \),
with equation (5.7)

Calculate complex velocities \( \frac{d\zeta_j^n}{dt} \) of
vortices \( j=1,2,\ldots,N-1 \). Equation (5.6)

Is the Kutta vortex released? Or should it
be released\( |\gamma_N(n)| > \gamma_c \)?

No

\( \frac{d\zeta_N}{dt}(n)=0 \)

Solve equation (5.8) for complex
velocity of vortex with time
dependent strength \( \begin{bmatrix} \frac{d\xi_N}{dt} \\ \frac{d\eta_N}{dt} \end{bmatrix} \) (n)

Use time integration method for
calculating vortex positions in new time
step \( \zeta_j(n+1) = \zeta_j(n) + f\left( \frac{d\zeta_j}{dt}(n) \right) \)

Determine Kutta vortex strength
(\( \gamma_N(n+1) \)) with updated positions.

Map back locations \( \zeta_j \rightarrow z_j \)
-end-

Is \( n-1 \) indexing the
final time step?

No

Raise time step index

\( n \rightarrow n + 1 \)

Store vortex positions and vortex
strengths for analysis

\( \zeta_j(n) \) & \( \gamma_j(n) \)

Raise \( N \) and define new
Kutta vortex:

\( N \rightarrow N + 1 \)
\( \zeta_N(n+1) = \zeta_{K,\text{start}} \)
\( \gamma_N(n+1) = 0 \)

Is \( |\gamma_N(n+1)| < |\gamma_N(n)| \) and is the Kutta
vortex released? Or \( |z_N(n+1)| > \delta ? \)

Yes

No

Figure 5.2 Flowchart showing the steps taken by the solver to evaluate the point-vortex kinematics
5.2. Gap in a wall

The ‘gap in a wall’ geometry can be interpreted as a geometry with two semi-infinite walls. Therefore a similar approach can be used as with the single semi-infinite wall (i.e. a sharp-edged wall).

The mapping function that is used to study a flow in a geometry with two semi-infinite walls maps two semi-infinite walls to a geometry with two infinite walls. A possible transformation is described by;

$$z: \mathbb{C}\setminus\left((\infty, -\frac{g}{2}] \cup [\frac{g}{2}, \infty)\right) \to \zeta: \{0 < \text{Re } \zeta < D\}. \quad (5.9)$$

where \(g\) is the opening space between the walls in the \(z\)-plane, and \(D\) the distance between the two infinite walls in the \(\zeta\)-plane (the kinematics are not dependent on \(D\)). The following conformal mapping function and inverse mapping function is used:

$$\zeta = \frac{D}{\pi} \arcsin\left(\frac{2z}{g}\right) + \frac{D}{2},$$

$$z = \frac{g}{2} \sin\left(\frac{\pi \zeta}{D} - \frac{\pi}{2}\right). \quad (5.10)$$

A graphical representation of this mapping is given in Figure 5.2.

![Graphical representation of conformal mapping described by (5.9) and (5.10).](image)

Note that the gap spacing \((g)\) defines the opening geometry in the \(z\)-plane, whereas the distance of the infinite walls in the \(\zeta\)-plane \((D)\) can be chosen freely and does not influence the resulting dynamics. For visualization reasons Figure 5.2 shows an example mapping with gap opening \(g = 4\), in order to match the opening geometry used in the DNS (presented in Figure 4.1 (b)), an gap opening \(g = 1.8\) is used in the point vortex modeling. In order to satisfy the no-penetration condition on both walls, mirror vortices have to be added. Each vortex in the \(\zeta\)-plane has two mirror vortices: one for each wall on either side. These mirror vortices cancel out the perpendicular velocity component induced by the original vortex everywhere on one of the walls. However the mirror vortices will induce a perpendicular velocity component in the other wall, which is not corrected for unless a mirror vortex is placed as if the first
mirror is an original. This effect (due to opposing mirrors) is called the opposing mirror effect and is illustrated in Figure 5.3.

**Figure 5.3** French philosopher Gilles Deleuze demonstrating the opposing mirror effect

Figure 5.3 shows a man standing in a room with two opposing parallel mirrors; it appears that two opposing mirrors create an infinite number of mirror images. Now consider a point vortex in the $\zeta$-plane at location $\zeta_j$. As can be seen in Figure 5.3, the images can be grouped in pairs. The location of the vortices in the n-th pair is $\zeta_{j,n}$.

$$\zeta_{j,n} = (\pm \xi_j) + (-1)^{n+1} 2D(n - 1) + i\eta_j$$  \hspace{1cm} (5.11)

Note that the original vortex is part of the pair ($n = 1$).

The equations of motion for vortices in the $\zeta$-plane with two parallel walls are different from those in a domain with a single wall. This is mainly due to the presence of an infinite number of mirror vortices. However, still the equations from section 2.5 apply. In order to calculate the induced complex velocity field of a point vortex and all its mirrors an infinite summation must be evaluated. In [35] it is shown that this series converges and this result in a complex velocity field in the mapped $\zeta$-plane $(u + iv)\zeta$ induced by a point vortex with strength $\gamma_j$, located at $\zeta_j$ that is given by:

$$(u + iv)\zeta = -\frac{i\gamma_j \pi}{2D} \left\{ \cot \left( \frac{\pi(\zeta - \zeta_j + 2\xi_j)}{2D} \right) - \cot \left( \frac{\pi(\zeta - \zeta_j)}{2D} \right) \right\}. \hspace{1cm} (5.12)$$

This complex velocity field possesses –as expected- a singularity in $\zeta = \zeta_j$, the velocity induced in the j-th vortex by itself (i.e. its mirrors) $\left[ \frac{d\zeta_j}{dt} \right]_j$ is:

$$\left[ \frac{d\zeta_j}{dt} \right]_j^* = -\frac{i\gamma_j \pi}{2D} \cot \left( \frac{\pi\xi_j}{D} \right). \hspace{1cm} (5.13)$$
which is purely imaginary valued (i.e. the motion is parallel to the imaginary axis in the \( \zeta \)-plane). With this result equation (2.51) can be rewritten to obtain an expression for the velocities of the vortices with time-independent strength.

\[
\frac{d\zeta_j}{dt} = \sum_{l=0}^{\max(N,M)} \left( \frac{iy_j \pi}{2D} \left( \cot \left( \frac{\pi \left( \zeta_j - \zeta_l + 2\zeta \right)}{2D} \right) - \cot \left( \frac{\pi \left( \zeta_j - \zeta_l \right)}{2D} \right) \right) \right) + \frac{iy_j \pi}{2D} \cot \left( \frac{\pi \zeta_j}{D} \right) + \frac{iy_j \pi}{2} \tan \left( \frac{\pi \left( \zeta_j - \frac{D}{2} \right)}{D} \right) \left| \frac{\pi g}{2D \csc \left( \frac{\pi \zeta_j}{D} \right)} \right|^2.
\tag{5.14}
\]

Here \( N \) and \( M \) refer to the index of the Kutta vortex at the left and right side of the flow domain, respectively. For readability of this report the equations for the strengths and velocities of these Kutta vortices can be found in the Appendix C. Apparently, the equations contain more terms compared to the case of the single sharp edge. This is due to the fact that now two time-dependent vortices are present and these also interact with each other. The increased size of the linear system indicates a challenge for other investigations on this topic when more vortices with time-dependent strength are added. The size of the linear system that has to be solved in order to obtain the velocities and strengths of the Kutta vortices scales with the square of the number of Kutta vortices. This problem might be surpassed by approximating the rate of change of the Kutta vortex strength, rather than determining it exactly.

### 5.2.1. Numerical method

The numerical method used for the geometry with a gap in a wall is similar to the method used for the sharp edge. The main difference is due to the presence of two Kutta-vortices. It gives rise to two indices for vortices with time dependent strength, \( N \) and \( M \), rather than one. Correctly programming in all expressions in Appendix C can be a tedious exercise and might be surpassed by approximating the rate of change of the Kutta vortex strength by setting it equal to the rate of change during the last time step (first order backward approximation). The errors due to this approximation affect the resulting kinematics. However, since the present study only concerns the general characteristics of a dipole approaching an opening in a wall the method with the approximation generates acceptable results.

Gourjii [32] suggests defining a method using time-integrating an expression for the rate of change of the Kutta vortex strength to obtain its time derivative using a higher order Runge-Kutta method time integration method. This method seems preferable with respect to numerical accuracy. However, when constructing a more elaborate time integrator it seems best to use a so-called symplectic time integration method, such that the Hamiltonian structure of the equations is inherited by the numerical solution [36].

The time integration method used to obtain the results presented in this thesis is the rather crude and unsophisticated forward-Euler method. However, a typical, well converged solution of a dipole collision is obtained within 2 seconds of computing time. Therefore the effort associated with the implementation of a higher order method (e.g. 4-th order Runge-Kutta) is not compensated by the advantage of its numerical efficiency. This is actually a major advantage of the zero-force model, which describes the generation of secondary vorticity with a single Kutta vortex. In this way good numerical accuracy can be achieved by a simple time integrating scheme.
6. Experimental and numerical results

In order to make a comparison between the experimental and numerical results this chapter contains results from both studies. This offers the possibility to compare the evolution of a purely 2D flow described by the Navier-Stokes equations (2.1) and (2.2), with the physical flow observed in the laboratory. First a comparison between the initialized dipolar vortex from the DNS and in the experiment is made. Then the results from the collision of a dipolar vortex with a sharp-edged wall are presented. Finally, the case where a dipole approaches an opening in a wall is studied.

6.1. Comparison of the initialized dipolar vortex

In the numerical simulations the 2D flow field is initialized with a Lamb-Chaplygin dipole according to equations (2.11) and (2.12). The dipolar vortex used in the experiment originates from a 3D turbulent flow behind a cylinder that is lifted out of the water. Therefore it is interesting to compare the characteristics of both dipolar vortices.

Figure 6.1 visualizes the structure of the dipoles with color plots for the vorticity ($\omega$) and iso lines for the stream function ($\psi$) for a Lamb-Chaplygin (a) and an experimentally obtained dipole (b).

![Color plot of the vorticity field ($\omega$) and constant stream function ($\psi$) contours for the Lamb-Chaplygin dipole model (a) and the a experimentally initialized dipole (b), the bold black curves denote the dipole atmospheres](image)

In Figure 6.1 the iso-$\psi$ lines are plotted in the co-moving frame. A bold black curve denotes the separatrix. It was found that the experimental dipole exchanges fluid with the exterior flow field (see section 3.3). Therefore, the atmosphere drawn in Figure 6.1 (b) is approximated and not exactly determined. It is clear that the experimental dipole is asymmetric whereas the Lamb-Chaplygin dipole is perfectly symmetric. The oval shape of the experimentally realized dipole is a commonly observed characteristic of physical dipolar vortices [14] [15] [37]. A more quantitative analysis can be made by a so-called $\omega - \psi$ scatter plot.
Figure 6.2 $\omega - \psi$ scatter plot of the experimental data shown in Figure 6.1 (b) inside the atmosphere. A bold black line indicates a linear least squares fit for these data points.

The markers in Figure 6.2 indicate the $\omega$ and $\psi$ value associated with each PIV-interrogation area that is located inside the atmosphere. It appears that there is a clear correlation between the $\omega$ and $\psi$ values inside the experimental dipolar vortex. This indicates that the vortex dipole is a coherent structure (see section 2.2). It can also be observed that the correlation appears to be stronger for the fluid in the cyclonic vortex ($\omega > 0$) when compared to the fluid in the anticyclonic half ($\omega < 0$). Furthermore, the maximum absolute vorticity value is larger for the cyclonic half of the dipole than for the anticyclonic part. The deviation from the plotted linear trend line is different for both cyclonic and anticyclonic half: It appears that slope of the cyclonic $\omega - \psi$ tangent is larger than for the anticyclonic case. According to the Lamb-Chaplygin model this coefficient can be linked to a typical size. A higher slope indicates a smaller vortex structure (eq. (2.13)). This suggests that the cyclonic vortex is more compact. It can be verified by the plot in Figure 6.1 (b) that this is indeed the case. According to the Lamb-Chaplygin dipole model the slope of the $\omega - \psi$ tangent ($k^2$, according to equation (2.8)) is related to the radius of the dipole according to equation (2.13). This radius can in turn be linked to the distance of the weighed centers of vorticity of both negative and positive vortex ($\varepsilon$).

$$
\varepsilon_{\text{Lamb}} = 0.92 \ldots \cdot R = \frac{0.92 \ldots \cdot 3.83 \ldots}{\sqrt{k^2}}
$$

(6.1)

For the experimental dipole the slope can be determined from the linear fit and the $\varepsilon$ value can be determined from the data plotted in Figure 6.1 (b). To compare the Lamb-Chaplygin model dipole and the experimental dipole the dimensionless product $\varepsilon \cdot \sqrt{k^2}$ can be analyzed.
Lamb – Chaplygin experiment

\[ \varepsilon_{\text{Lamb}} \cdot \sqrt{k^2_{\text{Lamb}}} = 3.52, \quad (6.2) \]
\[ \varepsilon_{\text{Exp}} \cdot \sqrt{k^2_{\text{Exp}}} = 3.43. \quad (6.3) \]

Apparently the noted values are within a 5% deviation. Even though the experimental dipole is neither circular nor symmetric, the linearity of the \( \omega - \psi \) scatter ensures that the experimental dipole inherits some of the Lamb-Chaplygin model characteristics concerning the vorticity distribution.

Another important parameter related to the dynamics of a dipolar vortex is the ratio of circulation of both dipole halves. When this ratio is not equal to minus one (-1) the vortex will move along a curved path as it bends towards the side of the stronger vortex [5]. According to equation (2.38) the cyclonic vorticity will show a faster decay than the anticyclonic vorticity due to Ekman effects. In the numerical simulations these Ekman effects are neglected and therefore the ratio of the positive and negative circulation of an initialized Lamb-Chaplygin dipole will remain -1. Note that the actual circulations of the vortices composing the dipole do decrease due to viscous dissipation. The evolution of the ratio of circulation for the positive and negative patch for the experimental and numerical dipole is shown in Figure 6.3.

![Circulation ratio evolution](image)

**Figure 6.3** Evolution of the circulation ratio of the vortices composing the experimental and numerical dipoles

Figure 6.3 shows the time evolution of the ratio of positive circulation and negative circulation. The increasing negative value for the experimental dipole indicates that the anticyclonic vortex gains in relative strength. This is in compliance with the Ekman theory presented in section 2.3.4. Another effect that imposes an asymmetric result is that the cyclonic vortex is more compact than the anti-cyclonic vortex. The accompanying strain fields are therefore not identical for both vortices, resulting in different viscous dissipation rates. In conclusion, the dipole initialized in the experiments is not identical to the
initialized dipole in the numerical simulations. Furthermore, the dynamics are not identical. These differences should be kept in mind when analyzing the results.

6.2. Collision with sharp-edged wall
In this section a comparison is made between the numerical and experimental results for the case where a dipolar vortex collides with a sharp-edged wall with a certain offset value \( d \) (see Figure 4.1 (a)).

6.2.1. Dye visualization and numerical results
Experimentally obtained results are compared with the numerically obtained results. Comparing the flow field visible via the entrainment of dye and the numerically derived vorticity field gives insight in the similarities and the differences between the experimental and numerical approach in a qualitative way. This is done by comparing figures showing typical stages of the flow evolution from both the experimental and numerical results. The distance of the primary vortices to the sharp edge is taken as reference in order to find corresponding frames in both approaches. The actual time that passes in between the frames is linked to this distance via the vortex velocities. The vortices in the rotating tank are subjected to drag originating from the bottom, which results in a slow-down of the experimental vortices. In the numerical simulation these effects are neglected, implying that the time evolution of comparable vortex positions is not linearly linked between both approaches. However, for a qualitative description of the flow field the exact position of the vortex structure and corresponding time are not essential.

6.2.1.1. The effect of background rotation direction on secondary vorticity
In chapters 2 and 3 it was explained that an anti-cyclonic vortex with vorticity values lower than the background vorticity \( \omega < -2\Omega \) is not necessarily 2D and therefore might be unstable, resulting in 3D turbulence (i.e. dissipative). As the vorticity originating from the boundary-layer is characterized by very high values for high Reynolds number flows a vortex arising from this boundary layer might be anomalous (see Figure 2.2). An experiment was conducted in order to show the differences in the production of cyclonic and anti-cyclonic secondary vorticity. Figure 6.4 shows the flow evolution of a dye-experiment by virtue of sequential snapshots taken during the experiment.
Figure 6.4 Dye visualizations showing a flow field evolution where a dipole collides with a sharp-edged wall for different background rotation directions. The resulting secondary vorticity is cyclonic (upper row) and anti-cyclonic (bottom row).

Figure 6.4 shows in general similar vortex dynamics in both experiments. However, details of the flow evolutions are crucially different. It is clearly visible that the paths of the positive primary vortices deviate, especially after the collision. This can be explained by a possible pairing with different strength secondary vortex from the sharp edge. In the experiment with negative background rotation (upper row) the secondary vorticity originating from the tip is cyclonic, and therefore not limited, resulting in a regular vortex. In the experiment with positive background rotation (bottom row) the secondary vorticity is anti-cyclonic, and the vorticity characterizing the shear layer is possibly below the mentioned threshold that would result in an anomalous vortex characterized by 3D turbulence. A zoom-in on these secondary vortices right after the formation reveals that this appears to be indeed the case.

Figure 6.5 Zoom-in on dye visualization snapshots showing the formation of a secondary vortex at the sharp edge, for both cyclonic (left) and anti-cyclonic (right) secondary vorticity.
Figure 6.5 shows that the dye distribution inside the anti-cyclonic secondary vortex is more irregular in comparison to the cyclonic secondary vorticity. This also explains the stronger curved path in the experiment with positive background rotation. In this experiment the secondary vorticity originates from 3D (i.e. dissipative) turbulence.

From this experiment it becomes clear that for optimal results the secondary vortex originating from the sharp edge should always be cyclonic in order to stay clear from the 3D regime. Furthermore, it can be concluded that in the experiment the primary (reference) vorticity should be as small as possible compared to the background vorticity to minimize these 3D effects. This resulted in an alteration of the dipole forming mechanism, such that it would produce dipole halves with lower characteristic vorticity. This also improved the symmetric appearance of the dipole.

### 6.2.1.2. Offset \( d = -0.2 \)

Figure 6.6 shows the evolution of a dipole-sharp edge collision for a offset value of \( d = -0.2 \)

![Flow field evolutions for a collision of a dipolar vortex with a sharp-edged wall with the offset parameter \( d = -0.2 \). The sub-figures show snapshots of a dye visualization from an experiment (top) and the vorticity field from a numerical simulation (bottom)](image)

It appears that the dynamics of the dipole and the creation of secondary vorticity in the rotating tank experiment show similar characteristics as found in the numerical simulation. The dye experiments show that the fluid inside the original dipole is still entrained by the new vortex structures. The numerically obtained vorticity plots reveal in great detail how the secondary vortex structure from the sharp edge is formed. Vorticity from the boundary layer detaches at the sharp edge and enters the interior flow domain. As it does it rolls up into a vortex whilst vorticity from to boundary layer is still entering the interior of the flow domain and is advected toward the new born vortex (see Figure 6.7). This causes the secondary vortex to gain strength. After a while the vorticity in the shear layer is not “captured” by the new vortex anymore and the shear layer breaks up. At this stage the original dipole half and the secondary vorticity have paired into a new dipolar vortex. Its trajectory is curved towards the side of the primary vorticity, indicating that the secondary circulation is weaker than the primary circulation. A remarkable difference is that in the final stage two dipolar vortices are visible, and both of the new experimental dipoles seem to move along a more curved path than those in the numerical results. The
curvature of the path of a dipolar vortex is a function of the ratio of circulation of the individual vortex patches in the dipole. From this stronger curvature in the dye experiment it appears that the secondary circulation in both new dipoles is weaker compared to the primary circulation in the original dipole halves when compared to the numerical results. Quantitative results from the PIV measurements concerning this topic are presented later in this chapter (section 6.2.2).

Figure 6.7 Zoom-in on the secondary vortex formation process illustrated by red arrows that indicate the instantaneous flow field direction: The shear layer is ‘feeding’ vorticity from the boundary layer that detaches at the sharp edge towards the secondary vortex.

6.2.1.3. Offset $d = 0.4$

Figure 6.8 shows numerical and dye experimental results for the case of an offset value of $d = 0.4$.

Figure 6.8 Snapshots of the flow field evolution observed in an experiment (upper row) and DNS (bottom row) where a dipolar vortex collides with a sharp-edged wall with offset parameter value $d = 0.4$. The experimentally obtained flow field is visualized by dye and the DNS results are visualized by vorticity distributions. The A and B markers indicate the location of secondary and tertiary vorticity, respectively.
For this case, the evolution of experimentally obtained flow field shows a large deviation from the results observed in the numerical simulation. It appears that the secondary vorticity originating from the sharp edge does not behave identically. In the numerical simulations the dipole formed at the sharp edge follows a less curved path compared to the experimental secondary dipole. As a reference the path of the right-hand side dipole can be compared with the path of the left-hand side dipole. This can be explained by the fact that during the collision in the experiment two vortex structures have emerged rather than one (as in the numerical simulation). The ‘A’ and ‘B’ markers indicate the position of the first and second vortex that originated from the sharp edge, respectively. It appears that the shear layer in the experiment has broken up in two vortices, the first vortex (A) is advected along the primary vortex. The second vortex (B) seems to follow a similar path. In the final frame it appears that a tri-polar vortex structure has formed.

6.2.1.4. Offset $d = -3$

Figure 6.9 shows the results for a dipole passing a sharp-edged wall, with the offset parameter $d = -3$.

![Figure 6.9 Snapshots of the flow field evolution observed in an experiment (upper row) and DNS (bottom row) where a dipolar vortex passes the sharp-edged wall with offset parameter value $d = -3$. The experimentally obtained flow field is visualized by dye and the DNS results are visualized by vorticity distributions.](image)

At this offset the original dipole does not collide with the sharp-edged wall. It appears that the dipole path is unaffected by the presence of the wall. However, the dipole-induced flow creates a vortex at the sharp edge. This secondary vortex does not directly interact with one of the original dipole halves. The middle frame reveals an important characteristic of the creation of secondary vorticity. Figure 6.10 shows a zoom of the secondary vortex during its forming process. The shear layer in the experiment is not a straight feeding layer as it appears to break up into several smaller vortices (Kelvin-Helmholtz-like instability). These vortices then consolidate into a larger vortex.
A red line is drawn to accentuate the wiggles visible in the dye-experiment shear layer. These wiggles are not visible in the numerical simulations. Two possible explanations for this discrepancy are noted below:

1. The wiggles are the result of a Kelvin-Helmholtz-like instability that breaks up the shear layer. This instability might not be observed in the numerical simulations due to the absence of small (physical) perturbations that trigger the instability. Furthermore, the temporal and spatial resolution required to resolve the instability mechanism might not be present in the numerical simulation.

2. The wiggles are the result of the non-constant rotation speed of the table (as discussed in section 3.4.). As the table alternatively accelerates and decelerates, the resulting alternating flow around the sharp edge causes the shear layer itself to detach and break up in several smaller vortices. This effect is obviously not present in the numerical simulations.

The second explanation is supported by the observation that the shear layer in the \( d = -3 \) experiment breaks up into several smaller vortices, whereas the feeding shear layer in the \( d = 0.4 \) experiment results in only two vortices. It appears that each secondary vortex is formed over a similar period of time, but since the shear layer in the \( d = -3 \) experiment exists much longer it endures more periods associated with the rotation speed oscillations. The break up of the shear layer into several smaller vortices explains the occurrence of vortices A and B from the previous section.

### 6.2.2. PIV experimental and numerical results

In the previous section several qualitative aspects of the experimentally and numerically obtained flow field evolution are highlighted. The PIV experimental results enable the comparison of quantitative data from the flow field obtained in the experimental and numerical studies. The comparison is split up into a few different offset values. For each offset value the general flow field evolution is given by several frames that illustrate characteristic stages of the collision dynamics. A more in-depth analysis of the
secondary vorticity is provided by an evolution plot of its circulation. Furthermore, vortex trajectories can be evaluated from the PIV results.

6.2.2.1. Offset $d = 0.8$

Figure 6.11 shows the flow evolution by means of vorticity field plots for offset parameter $d = 0.8$.

![Vorticity field plots](image)

Figure 6.11 Snapshots of the vorticity fields during a collision of a dipolar vortex with a sharp-edged wall, with offset parameter value $d = 0.8$. The results are obtained from a PIV experiment (top) and a numerical simulation (bottom).

The vorticity field evolution shows a phenomenological correlation between the PIV experiment and the numerical results. In the third and fourth frame of the PIV measurements it is visible that two secondary vortices originate from the sharp edge. A more quantitative analysis of the production of this secondary vorticity is presented in Figure 6.12. Here a plot shows the strength of the secondary vortex that pairs with the positive half of the original dipole normalized by the strength of this primary dipole half (this is a time-dependent normalization). The time is non-dimensionalized with the advective timescale $R/U$, with $U$ and $R$ the primary dipole speed and radius, respectively. An offset it added in the case of the experimental dipole to make sure the collision is temporally aligned with the numerical simulation. The experimental dipole is not yet in the camera’s field of view until Time = 1. A wider field of view would have resulted in a decreased spatial resolution. However, it appears that until Time = 5 the secondary vortex gains in relative strength in a similar fashion in both numerical and experimental results. However, in the final stage a strong departure is observed. This can be explained by the fact that in this experiment two secondary vortices have emerged and that until Time = 5 both are advected along with the primary vorticity (e.g. in Figure 6.8). After that time the second secondary vortex is not advected along anymore and is left behind.
Figure 6.12 Plot of the relative secondary vortex circulation evolution during a collision of a dipolar vortex with a sharp edged wall with offset value $d = 0.8$. The green and blue line indicate the results obtained from the PIV experiment and numerical simulation, respectively.

In order to study the vortex dynamics of this collision, the trajectories of the positive primary and negative secondary vortex can be analyzed. Figure 6.13 shows a comparison between the paths obtained from the experiment and from the numerical simulation. The path is obtained by tracking the center of vorticity of the particular vortex.
The vortex paths show similar characteristics. As with the circulation evolution plot in Figure 6.12, a notable difference is observed in the later stage of the collisions. It seems that the dipole in the numerical simulations has travelled further from the sharp edge, even though the experimental dipole is plotted for a longer time after the collision. Also the dipole path in the final stage of the experiment shows a stronger curvature than the numerically obtained dipole path. This can be explained by the differences between relative circulations between the vortex patches making up the dipoles (see Figure 6.12). Furthermore, the fluid flow in the experiment is slowed down by the effect of bottom friction, which is absent in the numerical simulations. Therefore experimental vortices move slower in the later stage of the collision evolution.

6.2.2.2. Offset $d = 0.4$

Experimental results were also obtained with the PIV method for slightly different offset. Figure 6.14 shows the flow field evolution with the vorticity fields obtained from both PIV experiment and numerical simulation.
Figure 6.14 Snapshots of the vorticity fields during a collision of a dipolar vortex with a sharp-edged wall, with offset parameter value \( d = 0.4 \). The results are obtained from a PIV experiment (top) and a numerical simulation (bottom).

The comparison of the vorticity fields shows a resemblance between the experimentally and numerically obtained flow evolutions. However, it again appears that the shear layer has broken up into several smaller vortices that are not advected along with the primary vorticity. Therefore the new experimental dipole follows a more curved path than the dipole in the numerical simulations. In Figure 6.15 the time evolution of the relative vorticity is shown for both the experimental and numerical simulation.

![Secondary circulation evolution, \( d=0.8 \)](image)

Figure 6.15 Plot of the relative secondary vortex circulation evolution during a collision of a dipolar vortex with a sharp edged wall with offset value \( d = 0.4 \). The green and blue line indicate the results obtained from the PIV experiment and numerical simulation, respectively.

It appears that the evolution of the circulation in both the numerical and experimental study is similar until Time = 3. At later stages a large deviation is visible, which originates from the fact that the vortices...
forming from the shear layer are not all advected along with the primary vortex. Figure 6.16 presents a graphical comparison of the vortex paths.

Figure 6.16 Plot of positive primary (red) and negative secondary (blue) vortex paths during a collision of a dipolar vortex with a sharp-edged wall with offset parameter value \( d = 0.4 \). The results are obtained from a numerical simulation and from a PIV experiment.

It appears that the trajectories show similar characteristics. However, in the final stage the path of the experimental dipole is more curved than the numerically obtained path. This is to be expected according to the ratio of circulation of the experimental dipole, as presented in Figure 6.15.

**6.3. Collision with opening in a wall**

In [7] an intriguing result was found with the use of numerical simulations: a dipolar vortex approaching an opening in a wall resulted in a rebound of the primary vorticity patches, that later recombined and were able to pass through the opening. This flow evolution was observed for a specific opening geometry. This section aims to verify this numerical result in an experiment.
6.3.1. Collision with opening between peninsulas

The geometry used for this experiment is shown in Figure 3.2 and Figure 4.1 (b). The flow domain shown in these figures is simply-connected. Therefore these walls are a model of peninsulas. A comparison between numerical and dye-experiment results is shown in Figure 6.17.

![Figure 6.17 Snapshots of the flow field evolution for a dipole approaching an opening in a wall. The images show the dye distribution observed during an experiment (top) and the vorticity field from a numerical simulation (bottom).](image)

In the dye-experiment it is clear that no green dye (representing the fluid in the original dipole) passes through the opening between the peninsulas, as was predicted by the numerical results. In the dye-visualization it is visible that the rebound and recombination of the original dipole halves is indeed observed as in the numerical simulation. However, the path of the new dipole that has formed from the rebound dipoles is not identical. The numerically obtained flow field evolution appears to be perfectly symmetric, and the new dipole approached the middle of the opening. The experimental evolution is not perfectly symmetric and the dipole that originates from the recombination follows a path that leads to the edge of the left peninsula. The asymmetry originates from the asymmetric initialization (i.e. at a slight offset), the asymmetric initial dipole and the asymmetric production and decay of cyclonic and anti-cyclonic vorticity. In fact, the dye-experimental results presented above are chosen out of many attempts to produce a symmetric collision. The fact that the dipole cannot pass through the opening can be understood by virtue of mass conservation: as the dipole approaches the opening between the peninsulas the mass inside the dipole atmosphere is transported through the opening. This mass transport is compensated by the flow around the dipole in opposite direction. When the dipole arrives at the opening between the peninsulas the compensating fluid flow must “squeeze” itself between the wall and the dipole. This so-called black-flow results in two secondary vorticity patches. It turns out that this backward flux is not sufficient to facilitate the pass through the opening of the primary dipole. The presence of the secondary vortex structures results in the formation of two dipoles that advect the primary vorticity away from the wall (i.e a rebound of primary vorticity).
6.3.2. Collision with opening between islands

By creating an alternate path for the compensating fluid flow different dynamics are observed. During experimental investigations it came to the attention that a dipole approaching an opening between two islands has different dynamics than the approach of a similar opening between peninsulas. Therefore, a numerical simulation was performed to verify this result. The geometry of these “two islands” is shown in Figure 6.18.

![Figure 6.18 Graphical representation of the used geometry in the numerical simulation where the dipolar vortex approaches an opening between two islands (in white). Length scales are noted as a multiple of the initial dipole radius (R).](image)

Details of the opening geometry are given in Figure 4.1. To better replicate the experimental situation the dipole is targeted with an offset towards the opening between the islands. Figure 6.19 shows a comparison between the evolution of the flow in the numerical simulation (lower row) and the dye-experiment (upper row).
It is clear that in this experiment green dye has passed from one side of the islands to the other. Also in the numerical simulation vorticity carried by the primary dipole has passed the opening between islands. Apparently the fact that the flow domain is now multiply-connected results in different dynamics compared to the “peninsulas” case (see Figure 6.17). How these different dynamics depend on the geometry is studied in detail in chapter 7.

6.4. **Experimental recommendations**

It is clear that the experimental method used is able to give valuable information about the physics concerning 2D flows. However some experimental difficulties also came to light and discrepancies from the purely 2D numerical simulation are observed. The three most notable differences are listed below:

1. Bottom drag in the experiment causes extra viscous dissipation
2. Due to the background rotation in the experiment positive and negative (e.g. cyclonic and anti-cyclonic) vortices have distinctively different characteristics
3. The effect of oscillations in the background rotation speed leads to different boundary layer separation dynamics

Changes can be made in order to better approximate a purely 2D flow in the experiment.

- The effect of extra drag originating from the bottom can be reduced by increasing the inertia in the dipolar vortex. This can be done by increasing the water depth.
- The asymmetries between cyclonic and anti-cyclonic vortices are related to the Rossby number. Decreasing this number to a value much smaller than 1 will ensure the validity of linear Ekman theory. Also the asymmetry due to the basic force balances explained in section 2.3.3 would vanish for lower Rossby numbers. This also leads to a more 2D behavior of anti-cyclonic vortices. Therefore, increasing the rotation speed would give results closer to the purely 2D flow in the numerical simulations. This strategy was not applied in the experiments presented in this report.
since increasing the rotation speed would require another parabolic bottom plate with another curvature. Such a plate was not available.

- The oscillations in the background rotation speed should be reduced; this could be done by improving the electronic control system that regulates the rotation speed. However, the process of simply increasing the inertia of the rotating set-up also resulted in a significant improvement.

- Another experimental set-up can be used that is able to produce a quasi-2D flow. A possibility is the use of a linear stratified fluid as in [19]. Furthermore, Wong [37] presents a short study of the collision of a dipolar vortex with a sharp-edged wall in a shallow layer experiment. Both alternatives do not rely on background rotation to achieve a quasi-2D flow. However, other technical and fundamental drawbacks must be taken in account when choosing an alternative. This topic is discussed in more detail by Berkvens in [5].

In order to reduce the discrepancies between the numerical and experimental results, the numerical simulations could be made more realistic by incorporating the 3D effects.
7. Further numerical analysis
In the previous chapter the experimental results are compared with the numerical results. It was shown that the numerical simulation could only partially reproduce the physical flow field observed in the rotating tank experiment. Important reasons for deviations have been noted. It became apparent that the numerical simulations give the flow field evolution of a purely 2D flow. This chapter contains a numerical study that complements the experimental study. The numerically obtained results presented in this chapter are not compared with experimental data because of two reasons: Obtaining experimental results requires significantly more time that is not available and the exact geometric set-up and initialization can be easily defined in a numerical simulation, whereas it would lead to many practical difficulties in a physical experiment. In this chapter further numerical analysis of the collision of a dipolar vortex with a sharp edge is presented. Furthermore, the dynamics resulting from a dipolar vortex approaching an opening in a wall is studied in more detail.

7.1. Classification of dipole collisions with a sharp edge
In [5], an extensive study on the dipolar vortex collision with a sharp edge is presented including a DNS approach. The results found here lead to a classification of dipolar vortex collisions with sharp-edged walls for different offset parameter values \( d \). It was concluded that the dynamics regarding the primary vorticity for different lateral offsets can be characterized in 3 regimes. In these regimes similar dynamics were observed regarding the vortices of the initial dipole. In this section a further study on the dynamics of the secondary vortex originating from the sharp edge is presented.

From the numerical results it appears the dynamics of the secondary vortex are characterized by five rather than three regimes, as was the case for the primary vorticity dynamics. This section describes each regime and gives an example to illustrate the associated dynamics. This is done by continuously increasing the offset parameter value starting from a large negative offset parameter value towards a large positive offset value, thereby sweeping the entire offset parameter range.

The first domain is found for offset values \( d < -1.5 \). In this domain a secondary vortex with positive vorticity is created at the tip of the wall due to the dipole induced fluid flow passing the sharp edge. The secondary vortex does not interact with the primary vorticity directly, and appears to stay in the proximity of the sharp edge. An example is shown in section 2.4.1.4.

The second regime is observed for offset parameters in the domain \( d = [-1.5, -0.5] \). In this regime several complex dynamics are observed. A shared characteristic of the collisions in this regime is that the negative primary vortex is typically torn apart as only a part of this vortex interacts with the wall, or with secondary vorticity from the sharp edge. Another shared characteristic is that the positive primary vortex passes the wall and pairs with either primary or secondary negative circulation, or a combination of both. Figure 7.1 illustrates the different dynamics associated with 4 different values of the offset parameter by showing snapshots at \( t = 3 \) and \( t = 5 \).
The third regime can be found for offset parameters \( d = [-0.5, 1.25] \). In this regime the positive primary vortex pairs-up with negative secondary circulation and forms a new dipole that follows a path that is curved towards the side of the (stronger) positive primary vortex half. The negative primary vortex collides with the wall, and a positive secondary vortex is produced by the detachment of the boundary layer from the wall. A detailed analysis of some examples can be found in section 6.2.2.

For \( d = [1.25, 2] \) a fourth regime can be distinguished. The positive primary vortex is accompanied by a boundary layer at the wall with negative vorticity. This boundary layer at the wall detaches, at the same time another negative vortex is formed at the sharp edge. These two negative vortices combine into a larger vortex. The negative primary vortex shows identical dynamics as in the previous regime.

In the final and fifth regime \((d > 2)\) the primary vortex collision is similar to a collision with an infinite wall as presented in [4]. However a weak negative vortex is created at the sharp edge with rapidly decreasing strength for increasing offset values. This secondary vortex from the tip of the sharp edge is not entrained by the negative secondary vorticity resulting from the boundary layer detachment from the wall.
In all regimes the positive primary vortex follows a distinctive path after the collision, such that certain path characteristics can serve to distinguish between the regimes. As the positive primary vortex is always accompanied by a negative vortex, the path can be characterized by a relative radius of curvature. This radius is not necessarily constant after the collision, which means it is a time dependent value. This is illustrated in Figure 7.3, where it is shown that different radii can be fitted to characterize the path of the positive primary vortex after the collision.

![Radii fit vortex path (d = 0)](image)

**Figure 7.3** Graph indicating the different radii evaluated from different locations in time

Figure 7.4 shows the time averaged radius of curvature \((C_+)\) found for different offset parameters. The time averaging is done over the time domain \(t = [4, 6]\). Note that on the y-axis the reciprocal of the non-dimensionalized value is plotted in order to omit large values on the y-axis.
Figure 7.4 Plot showing the averaged curvature of the positive vortex path after the collision with a sharp edge for different offset values

The three different regimes for primary vortex dynamics as found in [5] can be distinguished. It can be clearly seen that the positive primary vortex path curvature only depends on its relative lateral distance to the sharp edge in a finite offset parameter domain, i.e. \([-2 < d < 2]\). In general the dependence of the reciprocal curvature on the offset parameter can be summarized by a limited ramp function (purple line). However the strongest deviation from this trend can be found in the second and fourth regime.
7.2. Dipolar vortex approaching an opening between peninsulas: The effect of an asymmetry

The evolution of a dipolar vortex that approaches an opening in a wall is discussed in [7]. Here the dipolar vortex was aligned perfectly centered with respect to the opening. Therefore, the resulting flow field evolution is symmetric with respect to the axis of symmetry of the geometry. This result is presented in Figure 6.17. It appears that the initial dipolar vortices rebound from the opening, and subsequently recombine into a smaller sized dipole that is able to pass through the opening. If this result is to be replicated in an experiment the evolution should also be near symmetric. In order to find the dependence of the flow evolution on a small asymmetric initialization of the flow DNS is used. In these simulations the dipolar vortex was given a small lateral offset (e) with respect to the axis of symmetry of the geometry, as illustrated in Figure 7.5.

![Figure 7.5 Graphic illustrating the asymmetric offset parameter (e)](image)

Figure 7.6 shows the evolution of the vorticity distribution for different asymmetric offset parameter values (e ∈ {0.001 0.0033 0.01 0.033 0.1}), the values are non-dimensionalized with the dipole radius R. The initial approach phase of the dipole is not plotted, as here the dipole path is trivial and shows no visible asymmetry.
It can be concluded that for increasing $e$-values the flow evolution has an increasingly asymmetric character. A critical offset value that determines whether or not the recombined primary vortices will pass the gap is found to be around $e \approx 0.0033$. However, for this case, the flow field is clearly not symmetric after the pass through the opening of the original vortices. This $e$-value is so small that it
would require a very well controlled environment to reproduce the symmetric result in an experiment. With this in mind the results presented in section 6.3.1 are surprisingly symmetric, especially considering the fact that the dynamics involved in the experiment are fundamentally asymmetric.

In order to give a more quantitative interpretation of the flow evolution shown in Figure 7.6, we consider the temporal evolution of the total enstrophy that is present in the upper side of the flow domain (i.e. above the wall), $\Lambda$. According to:

$$\Lambda = \frac{1}{2} \iint_{\text{top}} \omega^2 \, dA,$$  \hspace{1cm} (7.1)

where the surface integral is evaluated over the upper side of the flow domain (i.e. above the walls). Figure 7.7 shows a plot of the time evolution of $\Lambda$, normalized by the enstrophy of the initialized Lamb-Chaplygin dipolar vortex ($\Sigma$), for the different asymmetric offset parameters ($e$).

![Figure 7.7 Graph showing the normalized enstrophy evolution behind the opening after $t = 10$ for different offset parameter values ($e$)](image)

Interestingly for all offset parameters no enstrophy passes the wall during the initial approach of the dipole. It is only after the rebound and subsequent recombination ($t > 10$) that enstrophy is present behind the gap. For the smallest offset parameters, the moment that the dipole passes the wall is clearly visible by the rapid increase of enstrophy between $t = 14$ and $t = 15$. For these cases, after $t = 15$, the effect of viscous dissipation of enstrophy is visible. The lines corresponding to $e = 0.01$ and $e = 0.033$ reflect the more complex evolution that is presented in the vorticity field evolution in Figure 7.6. Here most of the enstrophy present in the upper side of the flow domain is due to vorticity that is generated at the wall. However, for the largest studied offset value ($e = 0.1$) no enstrophy is present behind the gap during the flow evolution.
7.3. Dipolar vortex approaching an opening between two Islands: The effect of different geometries

In section 6.3.2 it was mentioned that the dynamics of a dipolar vortex that approaches an opening between two islands is crucially different from the collision with the opening between two peninsulas (as studied in the previous section). The fact that the fluid flow domain is now multiply-connected appears to be of great influence on the evolution of the flow. In this section we only focus on the perfect symmetric case. An example were the dipolar vortex is able to pass through the opening on first approach is shown in Figure 7.8. This process is henceforth referred to as “the pass through the opening”.

These different dynamics can be accounted for by the presence of a flow circulating around the islands during the pass through of the dipole, such that the necessity of backflow through the gap is released. Figure 7.9 shows a close-up of the vorticity distribution and the velocity field during the pass through the opening at $t = 3.5$.

It is clear that when the dipole enters the gap a flow circulating around the islands is set in motion. This facilitates a pass through of the dipole, which was not possible in the simply-connected flow domain (i.e. peninsula case). The next step is to study the effect of different island geometries.

As a first study to illustrate the dependence of the collision dynamics on the geometric parameters, it is chosen to start with the assessment of the effect of a single geometric parameter on the observed
dynamics. By increasing the island's length \((L)\) and setting the distance from the islands to the domain boundaries \((A\) and \(B)\) fixed, having the same values as before, the effect of the island length can be studied. The parameters \(A\), \(B\) and \(L\) define the large scale geometry, as shown in Figure 7.10. Note that the opening geometry is always kept as presented in Figure 4.1.

Figure 7.10 Graph showing the geometry used for studying the effect of the island length \(L\)

Figure 7.11 shows the dynamics of the dipolar collision for geometry parameters \(A = 2.6\), \(B = 4.8\) (as in Figure 7.8) and different island lengths \(L \in \{4, 16, 32, 128\}\).
Figure 7.11 Snapshots of the temporal evolution of the vorticity distribution associated with a dipolar vortex approaching an opening between two islands with length $L = 4, 16, 32$ and $128$
Figure 7.11 shows that for a short island length \( L = 4 \) the dipolar vortex can pass through the opening on first approach. However as the island length increases the pass through of the primary vortices is delayed. For large island lengths \( L > 32 \) the dipole seems unable to pass through the opening during the initial approach. Only after a rebound and subsequent recombination the primary vorticity passes through the opening. From the case with very large island length \( L = 128 \) it appears that the vortex dynamics from the simply-connected domain are recovered.

Again we examine the amount enstrophy present in the upper side flow domain as a function of time, for different island lengths. The result are shown in Figure 7.12.

![Figure 7.12 Evolution of enstrophy present in the upper side of the flow domain for different island lengths](image)

The enstrophy present behind the opening appears to be enhanced when compared to the ‘peninsula case’ with symmetric initialization. Here no enstrophy was present in the upper domain until \( t = 12 \). The delay in the presence of enstrophy in the upper flow domain associated with long islands is visualized in Figure 7.12, by the fact that for increasing island lengths the rapid increase of \( \Lambda \) takes places at a later moment in time. Furthermore the \( L = 4 \) case (purple line) shows additional increase of the enstrophy that is present behind the opening. This is due to the secondary vorticity that is also present in the upper side of the flow domain (see Figure 7.11). However, it appears that for increasing island lengths the dynamics of the opening between peninsulas are recovered.

Concluding from Figure 7.11 and Figure 7.12 different regimes associated with the primary dipole evolution can be identified.

1. Apparent free pass through the opening of the dipolar vortex on first approach (e.g. \( L = 4 \))
2. Some delay in the pass through the opening as the initial dipole slows down as it enters the opening (e.g. L = 16)
3. Notable delay due to rebound of initial dipole with secondary vorticity and subsequent recombination (e.g. L = 32)
4. Maximum delay. Retrieval of dynamics as observed by in the case with an simply-connected domain. (e.g. L = 128)

It appears that the effect of the possible presence of a circulating flow field around the islands on the pass through the opening of the dipolar vortex on first approach decreases for longer islands. This might be due to the fact that the surface flux associated with the flow field around the islands ($\phi$) varies for different island geometries, resulting in different effects of the circulating flow on the vortex dynamics. $\phi$ is defined as the rate of surface transport through the opening (i.e. the surface flux around the islands). According to:

$$\phi = \int_{-g/2}^{g/2} v^y dx,$$

(7.2)

where a straight integration path connects one side of the opening to the other and $v^y$ is the velocity component in the $y$-direction (i.e. perpendicular with respect to the integration path). The evolution of $\phi$ can be studied for different island lengths. Therefore, Figure 7.13 shows the time evolution of $\phi$ normalized by a surface flux associated with the Lamb-Chaplygin dipolar vortex ($\Delta$), defined as the ratio of the dipolar atmosphere surface ($\pi R^2$), and the advective time scale ($\frac{R}{U}$),

$$\Delta = \frac{U}{R} \pi R^2.$$

(7.3)

Figure 7.13 Surface flux ($\phi$) evolution profiles during the DNS of the case where a dipole initially approaches an opening between two islands. Profiles are drawn for different island lengths.
The flux profiles plotted in Figure 7.13 show a distinctive trend for increasing island lengths. The total volume flux appears to be gradually spread out over time with increasing island lengths. Furthermore, the moment of maximum surface flux shifts towards a later moment in time for increasing island lengths. For a more quantitative comparison of the circulating flow around the islands the total surface transport during the first 9 non-dimensional time units \( \Xi \) can be evaluated for the different island lengths. \( \Xi \) is obtained by integrating the flux \( \phi \) over time. Figure 7.14 shows the transported surface \( \Xi \) normalized by the surface of the initialized dipole during the first 9 non-dimensional time units for different island lengths.

![Graph showing the total transported surface until \( t = 9 \) for different island lengths](image)

It is apparent that in general the mass flux decreases for increasing island lengths. This can be understood by the fact that there is a limited amount of kinetic energy stored in the flow field. Therefore, only a limited amount of energy can be present in the flow domain where the fluid circulates around the islands (i.e. at some distance from the opening). Intuitively, a circulating flow characterized with a certain flux requires different kinetic energies for different island geometries. More specifically, a flow circulating a longer island requires more energy to achieve the same flux. This can simply be seen by the fact that for longer islands more fluid has to be set in motion at approximately similar velocities.

It is demonstrated that the dynamics of a dipole approaching an opening between two islands depends on the island geometry. The next step is to study the effect of other geometric parameters than the island length. A geometry as defined in Figure 7.10 has three geometrical parameters of interest \( (A, B \) and \( L \) leading to many different possible combinations of these parameter values. Even more so, considering this geometry is defined by much more aspects than only the \( A, B \) and \( L \) parameter values (e.g. rectangular domain, straight islands, constant island width, etc.). In other words, it appears that a geometry is defined by a location in an arbitrary high-dimensional parameter space. So in order to give a general description of the effect of different geometries on the vortex dynamics observed when a dipolar vortex approaches an opening between two islands, a more general approach is needed. Note that the opening geometry is always kept the same (i.e. as defined in Chapter 4 in Figure 4.1 (b)).
In order to find such a general approach, we study the implications of the following Ansatz:

Since the circulating flows are always set in motion by the same dipolar vortex flow field, the kinetic energies associated with the fluid flows around different island geometries are always in the same order of magnitude. Therefore, the transported surface (\(\Sigma\)) is a function of the geometry.

This Ansatz motivates to study a possible correlation between the surface flux and the circulating flow field energy as a function of the geometry. It will prove insightful to work in terms of the mass-flux (\(\rho \phi\)) rather than the surface flux (\(\phi\)). The fluid is assumed isopycnic, so that the mass flux associated with a certain surface flux only differs by a constant factor \(\rho\) (i.e. density). A mass-flux efficiency parameter (\(T\)) can be introduced that represents the ratio of mass flux and kinetic energy of the part of the flow field that circulates the island (i.e. the flow field at some distance from the opening), according to:

\[
T = \frac{\rho \phi}{E} = \frac{\int u^2 \, dl}{\frac{1}{2} \iint_{\text{flow field}} u^2 \, dA}, \tag{7.4}
\]

where \(E\) is the kinetic energy of the circulating flow field. Before we analyze the meaning of this \(T\)-value in more detail, we first consider some aspects of the circulating flow field.

According to Kelvin’s circulation theorem (see section 2.4.1), in a fluid domain where the flow is initially irrotational, the flow remains irrotational unless it is affected by viscous forces. Therefore, we assume that the interior region of the circulating flow (at some distance from the viscous boundary layer) that is set in motion around the islands is irrotational (\(\omega = 0\)). It is well known that this part of the velocity field can be described by the gradient of a velocity potential (\(\Phi\)), according to:

\[
\mathbf{u} = \nabla \Phi. \tag{7.5}
\]

For an incompressible flow this results in a Laplace equation for the velocity potential \(\Phi\),

\[
\nabla^2 \Phi = 0. \tag{7.6}
\]

In general the relevant boundary conditions accompanying this equation are chosen such that the flow field does not penetrate through the domain boundaries. The existence and uniqueness of the solution to this problem is discussed in [38].

To illustrate the interpretation of the \(T\)-value we consider a 2D flow field \(\mathbf{u}_1(x,y)\) according to the velocity potential \(\Phi_1\) that satisfies equation (7.6) with the relevant boundary conditions, such that \(\mathbf{u}_1\) describes a flow around an island geometry. From this flow field the associated \(\phi_1, E_1\) and \(T_1\) values can be calculated. Another flow field \(\mathbf{u}_2\) in the same geometry can be introduced. This flow field is described by a velocity potential \(\Phi_2 = c \Phi_1\) (where \(c\) is a constant) such that it is also a solution to equation (7.6). The surface flux \(\phi_2 = c \phi_1\), and the kinetic energy \(E_2 = c^2 E_1\) differ from the values associated with the flow field \(\mathbf{u}_1\). Furthermore, \(T_2 = \frac{T_1}{c}\) is also different. Apparently the \(T\)-value is not only dependent on geometric parameters. For a value that assesses the geometry and not the actual flow field for circulation efficiency, the \(T\)-value can be evaluated for a flow with a normalized energy. Consider again
the flow field \( u_1(x,y) \), in order to calculate the surface flux around the geometry with a normalized energy in the flow field, \( u_n(x,y) = \frac{u_1(x,y)}{\sqrt{|E_1|}} \) can be defined (the \( \llbracket \ldots \rrbracket \) indicates an operator that truncates the dimensions of its argument). This flow field has an associated surface flux: \( \phi_n = \frac{\phi_1}{\sqrt{|E_1|}} \). Now the \( T \)-value with a normalized energy can be calculated from the flow field \( u_1(x,y) \). To stress the difference with the earlier defined \( T \)-value this non-dimensional value is labeled \( O \) (Greek Omicron).

\[
O = \frac{\phi_1 \sqrt{\rho}}{\sqrt{|E_1|}} = \frac{\int u_1^+ dl}{\frac{1}{2} \sqrt{\iint u_1^2 dA}}. \quad (7.7)
\]

The flow field \( u_2 = cu_1 \) has a \( O \)-value that is independent of \( c \).

To summarize: The non-dimensional omicron value represents the mass flux efficiency for a certain geometry. It was assumed that the flow around the islands is set in motion with an amount of energy this is independent of the islands geometry. This Omicron value might represent the influence of a specific geometry on the resulting transported surface (\( \Theta \)) in one single value.

The next step is to verify the Ansatz and find out how the Omicron value can be used in a practical way to assess a geometry without DNS of the actual flow field evolution. To obtain the \( O \)-value of a flow circulating an island efficiently it is possible to solve the equations of potential flow [21]. To approximate the flow structure circulating the islands (i.e at some distance from the opening) with potential flow, the geometry is slightly altered by adding a slit that closes the opening such that no fluid can stream through the opening. A steady fluid flow profile is obtained by setting an arbitrary velocity potential difference between the upper and bottom side of the slit. This results in a fluid flow around the islands. By demanding that the flow field and accompanying velocity potential satisfy equations (7.5) and (7.6) with the relevant boundary conditions, the found fluid profile flow might be a good approximation of the actual fluid circulating flow profile (neglecting the actual magnitude of the velocities).

Figure 7.15 shows the resemblance of the potential flow field \( (a) \) & \( (c) \) and the actual fluid flow field during the first approach of the dipole towards the opening \( (b) \) & \( (d) \) in a complex geometry.
Figure 7.15 Comparison between the absolute velocity field (a) & (b) and streamline pattern (c) & (d) obtained from potential flow (a) & (c) and the flow field from the dipole induced flow at $t = 3.6$ (b) & (d).

Figure 7.15 (a) and (b) shows the absolute velocity fields. Arrows with normalized lengths are plotted that show the direction of the vector fields. These figures show that indeed the structure of the actual fluid flow around the islands (b) can be approximated by the potential flow solution (a). The red parts in the figure indicate a high absolute velocity in the flow field. These sections indicate the presence of
bottle necks for the fluid flow around the islands. It appears that the location, structure and relative severity of these bottle necks observed during the DNS of the actual flow field evolution are described quite well in the potential flow approach. The largest deviations are found at the location of the vortex structures and the boundary layers in the actual fluid flow field. Figure 7.15 (c) & (d) show the structures of both flow fields by virtue of stream line patterns.

The Omicron values obtained from the potential flow approach are compared to the surface transport that was achieved during a dipole collision with the opening ($\Xi$) in Figure 7.16.

The scatter plot reveals a correlation between the transported surface ($\Xi$) and the $\Omega$-value for different geometries. The diamond, square and triangle markers indicate geometries according to Figure 7.10, for island lengths $L = 8$, $L = 16$ and $L = 32$, respectively, with a variety of different $A$ and $B$ values. The round markers indicate more complex geometries (e.g. in Figure 7.15). Apparently for $\Omega < 0.2$ a correlation and trend is visible. As expected a higher $\Omega$-value results in more mass transport during the dipolar vortex collision. However, this trend is broken for $\Omega > 0.2$. Here a higher Omicron value does not necessarily result in larger mass flux. This can be understood by the fact that for flows with high mass fluxes, vortices are usually found in the flow circulating the island. These vortices contain kinetic energy, whereas they do not transport fluid efficiently. The vortices are off course not found in the irrotational approximation that was used to find the Omicron value. Furthermore, the assumption that with every collision the same amount of energy is transferred into the circulating flow does not hold for the geometries with a large mass transport. It can be argued that as the dipolar vortex passes through the opening with minimal delay (e.g. in Figure 7.8), it is actually the transported surface that is limited, rather than the energy in the circulating flow field.
On the used computer system, simulating the dipolar flow evolution as it approaches an opening costs approximately 2-5 hours of computation time, depending on the exact geometry. Finding the Omicron value from a potential flow typically costs 20 seconds, with the same hardware. Therefore, it can be concluded that the flow around the islands when a dipolar vortex approaches an opening between can be efficiently characterized by the Omicron value.

Note that the applicability of this method is still limited to specific geometries. The studied geometries were always symmetric to ensure a symmetric flow evolution and the bottle necks in the geometries are always much wider than typical boundary layer thicknesses to exclude the viscous effects associated with these layers as much as possible. Furthermore, the edges where always smoothed (i.e. not sharp) such that the singular effects as presented in section 2.4.2 are evaded. However, a Omicron value analysis can be Applied in more general cases. Everywhere where a fluid is set into motion with a certain energy the transport efficiency of the geometry can be assessed by the Omicron value.
8. Point vortex modeling results

This chapter presents the result from the point vortex model that models the production of secondary vorticity originating from a sharp edge and the accompanying kinematics. Since the model is 2D, it is not expected to reproduce any of the 3D effects that were encountered in the experiment. However, the DNS considered a 2D flow field evolution and was able to describe the total production of secondary circulation at the sharp edge as was observed in the experiment. Therefore, it is sensible to compare the point vortex model with DNS rather than with experimentally obtained results.

8.1. Comparison of the initialized vortex dipoles

In this section a comparison between the Lamb-Chaplygin and point vortex dipole is presented. In the DNS the dipolar flow field is initialized according to the Lamb-Chaplygin model, whereas in the point vortex model a dipole is constructed by the superposition of two point vortices. When constructing a symmetric dipole from two point vortices two parameters must be set, namely, the separation distance and the absolute strength of the positive and negative vortices. The separation distance defines a length scale in the problem and together with the strength of the vortices it also determines the velocity at which the dipole propagates through a fluid. It seems an obvious choice to set the strength such that the velocity of the dipole is the same as the Lamb-Chaplygin dipole. Figure 8.1 shows the stream function contours for both the Lamb-Chaplygin and point vortex dipole, with normalized length scale and translation velocity.

![Lamb-Chaplygin dipole Iso-ψ lines](image1)

![Point vortex dipole Iso-ψ lines](image2)

Figure 8.1 Stream function contours for Lamb-Chaplygin (left) and point vortex (right) dipole model in the co-moving frame, blue contours indicate the negative stream function values values: $\psi = (-2, -1.75, \ldots, -0.25)$, whereas red contours indicate the positive stream function values $\psi = (0, 0.25, \ldots, 2)$. $\psi = 0$ is chosen to coincide with the dipole atmosphere (bold black curve).

Red and blue lines indicate positive and negative stream function contour values, respectively. The bold black curves indicate the dipoles’ atmospheres. For the point-vortex dipole the atmosphere is closely approximated by an ellipse with major and minor radius 1.044 and 0.866, respectively. The singular character of the point vortex dipole is visible by the fact that the contour density reaches high values near the location of the point vortices. A quantity that can be compared is the dimensionless circulation...
associated with a dipole half. For the Lamb-Chaplygin dipole with normalized non-dimensional radius and translation velocity, and the point vortex dipole with normalized non-dimensional separation distance and translation velocity this is:

\[
\Gamma_{+,(L-C)} \approx 6.79, \quad (8.1) \\
\Gamma_{+,(PV)} = 2\pi \approx 6.28. \quad (8.2)
\]

The values are within 8% deviation, indicating a shared characteristic.

8.2. Choosing free parameters

In section 2.5.7 and Chapter 5 it was mentioned that the point vortex model relies on so-called free parameters. These free parameters are: Starting location of the Kutta vortex \(z_K\), threshold strength before release \(y_c\) and the maximum strengthening distance from the sharp edge for the Kutta vortex \(\delta\). These parameter values have great influence on the kinematics of the point vortices. Since these parameters do not have a distinct relation with physical properties of a flow field, a method for determining the values of these parameters was devised. This method consists of an optimization process to find a suitable Free Parameter Value Combination (FPVC). In order to assess the results obtained with a certain FPVC the point vortex model results can be compared with a benchmark result from the DNS.

8.2.1. The benchmark result obtained from the DNS

In order to make a realistic decision on the desired quantitative results that the point vortex model should be able to produce, first the short-comings of this simplified model are considered. The most notable aspects that limit the applicability of this model are listed below:

1. In the point vortex model vorticity is concentrated in a point rather than spread out over a finite area. This makes it impossible to model the deformation, splitting and merging of vortices that are observed in both experimental and numerical investigations.
2. Viscous dissipation of the kinetic energy in the flow field is not present in the inviscid approach that is inherited from potential flow theory. Therefore the lagging behind of vortex motion observed in DNS with respect to the predicted point vortex motion will increase over time.
3. Boundary layers and the vorticity that originates from the detachment of the wall owe their existences to viscous effects. The Kutta condition applied in the point vortex model only concerns the circulation that originates from the sharp edge and not the vortex structures that cause the rebound of a dipole during its interaction with the boundary layer as illustrated in section 2.4.2.

Note that it might be possible to include these effects in the model by artificially introducing new specific modeling rules (like the Kutta condition/vortex). However, this would oppress the main advantage of the point vortex model, simplicity. The model is optimized for a dipole collision with a sharp edge rather than also include the collision with an opening in a wall. This is chosen because the single sharp edge setup can be seen as a fundamental building block for several geometries. Furthermore it is chosen to only focus on the secondary circulation produced at the sharp edge during the dipole collision. This choice
seems legitimate as it is known from the results presented in chapters 6 and 7 that it is this secondary circulation that can affect the dynamics of the primary vortices. For comparison sake the circulation ratio between the primary and secondary vorticity at non-dimensional time \( t = 6 \) is chosen as a reference. Where \( t = 0 \) corresponds to the situation where the primary dipole is at a non-dimensional longitudinal distance \( y = 3 \) from the wall. A set of typical collision offset parameter \( (d) \) values is chosen, which is used for determining the vorticity ratio at \( t = 6 \). The offset parameter list is:

\[
d \in \{-3, -2.5, -2, -1.75, -1.5, -1.25, -1, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 1, 1.25, 1.5\}
\]

(8.3)

The list starts from a large negative value, where the original dipole passes the sharp edge. However, in this case secondary vorticity is still produced at the sharp edge by the dipole induced flow. No offset values are chosen between \( d = -1 \) and \( d = -0.4 \), as the collision dynamics for these values are known to be characterized by splitting and merging of vortices. Then the offset is incrementally increased until \( d = 1.5 \). No larger offset values are considered because for \( d > 1.5 \) the detachment of the boundary layer at the wall is of great influence on the dynamics (see section 7.1).

A plot of the circulation ratios at \( t = 6 \) for different offset values obtained from DNS is shown in Figure 8.2. The presented results will serve as a benchmark result to assess the performance of the point vortex model.

![Benchmark result obtained from DNS](image)

**Figure 8.2** Ratio of the secondary and positive primary circulation at \( t = 6 \) obtained from DNS. These results will serve as a benchmark to assess the performance of the point vortex model with different FPVCs

### 8.2.2. Simulated annealing

The process of finding a good performing FPVC is made difficult by the fact the free parameters span up a 3D parameter space and the effect of changing a single parameter is determined by the value of the other parameters as well. In order to minimalize deviation from the benchmark result without checking all possible FPVCs it is chosen to optimize the FPVC by using the optimization method of simulated annealing [39]. The algorithm scans the parameter space spanned by the three free parameters and tries
to check only the performance of combinations that give the best results. The procedure is stopped after $5 \cdot 10^4$ parameter combinations. The error percentage (that is minimized by the procedure) is the average relative deviation from the benchmark result and is given by:

$$\text{error (\%)} = \frac{100}{17} \sum_{i=1}^{17} \left( \frac{\gamma_{C, PV} - \gamma_{C, DNS}}{\gamma_{C, PV} - \gamma_{C, DNS}} \right)^2. \quad (8.4)$$

Here $i$ indicates one of the 17 offset values (see (8.3)), the subscripts $s$ and $p$ refer to the secondary and primary vortex strength ($\gamma$) from the PV-model or circulation from DNS ($\Gamma$), evaluated at non-dimensional time $t = 6$.

Figure 8.3 shows the locations in the parameter space of FPVCs that give the best results (i.e. Error < 5%).

![Figure 8.3 Scatter plots showing projections of locations in parameter space from the best performing free parameter value combinations.](image)

Figure 8.3 shows that in the allowed parameter space (indicated by the axis) the best performing free parameter value combinations are grouped together in a single “blob”. This indicates the existence of a range of sensible FPVCs. From the left and middle plot it appears that the strengthening distance ($\delta$) should be chosen between 0.55 and 0.75. The left plot shows a rather large relative spread in good performing Kutta vortex start position values ($z_K$). Good performance is found for nearly all values where $|z_K| < 0.05$. The same can be concluded for the absolute Kutta vortex threshold release strength ($\gamma_C$). For $\gamma_C < 0.02$ good performance is found. However, the right-hand side plot shows a correlation between suitable threshold strengths and start z-position values. It appears that as the Kutta vortex is placed closer to the tip, the threshold strength should decrease for good performance. This can be explained by the fact that as the Kutta vortex is placed closer to the tip, the Kutta condition is fulfilled with a weaker vortex. So in order to let it detach at a sensible moment in time the threshold strength should be lowered. It was argued in section 2.5 that a too low threshold strength leads to large numerical errors (a low $\gamma_K / z_K$ ratio should be avoided). It can also be seen that even though the positions of low error FPVCs seem to span a large fraction of the allowed parameter space, the intermediate errors do not differ much. They perform all within a 4.7% - 5.0% error range. This makes choosing a specific set of FPVC seem arbitrary. However, $z_K = -0.02$ is suggested in [28]. Accordingly, a threshold strength of $\gamma_C = 0.008$
is chosen, such that the numerical accuracy after release is not crucial. From the first and second plot, a strengthening distance $\delta = 0.62$ appears to be a correct choice.

The used free parameters are then given by

$$ (z_K, \gamma_c, \delta) = (-0.02, 0.008, 0.62). \quad (8.5) $$

This FPVC is located in the center of the “blob”, indicating that in this region in the parameter space the good performance of the point vortex model is independent on the actual FPVC.

### 8.3. Collision with sharp-edged wall

Now that a FPVC is chosen, results can be obtained with the point vortex (PV) model and compared with the DNS results. First the benchmark results are presented in Figure 8.4.

![Figure 8.4 Comparison of the ratios of secondary and positive primary vortex circulation at $t = 6$ obtained from DNS (blue and green dots) and the point vortex model (red and magenta lines).](image)

In Figure 8.4 the secondary vorticity obtained from both DNS and point vortex modeling are compared. In the positive secondary vorticity branch (i.e. for $d < -1$) a remarkable correspondence in found. The secondary vortex circulation is well described by the point vortex model. For the branch with negative secondary circulation the secondary vortex strength obtained from the point vortex model is not able to reproduce the trend that was found in the numerical simulations. However, in general the secondary circulation is of the same order of magnitude here and therefore the predicted kinematics could still be similar.

In order to study the collision of the dipole with a sharp-edged wall in more detail, the trajectories of vortices, the evolution of the secondary vortex circulation and flow field visualizations are considered next.
8.3.1. Vortex trajectories
Vortex paths as found from the point vortex model for 9 different offset parameters are shown in Figure 8.5.

In general it appears that more than one vortex originates from the sharp edge. However, for the case with negative secondary circulation ($d > -.5$), a single secondary point vortex pairs with the primary
positive vortex, whereas the other (tertiary, quaternary, etc...) vortices stay in the proximity of the sharp edge. It appears that the application of the zero-force model (see section 2.5.6) does result in a description where a single point vortex models the secondary vorticity originating from the sharp edge. A low number of Kutta vortices during the collision results in low computational costs (that scale with the square of the number of vortices) and therefore provides an efficient yet reliable description of the collision. However, this is clearly not the case for the offset value $d = -0.8$, where many vortices emerge from the sharp edge. It was already argued that the dynamics observed in the DNS in the offset parameter regime $[-1 < d < -0.5]$ cannot be reproduced by the point vortex model. Therefore it is chosen not to look further into the complex but irrelevant point vortex dynamics observed here. Other (predictable) unphysical behavior is shown by the negative primary vortex for $d > -0.5$. Due to the absence of secondary vorticity at the wall in this model, the negative primary vortex travels along the wall whilst maintaining a constant distance to the wall. This description also applies for the positive primary vortex with offset parameter $d > 1.5$, only as this vortex reaches the sharp edge it pairs with secondary vorticity that originates there. For $d < -1$ the original dipole halves are still paired after the collision. Therefore the dipole will continue to move along a straight (i.e. uncurved) path, although it is possible that the initialized dipole changes direction during the interaction with the secondary vorticity originating from the sharp-edged wall. Furthermore, it is possible that the separation distance between the primary vortices decreases (e.g. for offset parameter value $d = -1$). This results in an enhanced translation velocity of the initial point vortex dipole.

In section 7.1 it was mentioned that the positive primary vortex has a specific path for different offset values ($d$). In order to check in more detail how the point vortex model performs in comparison to the DNS results, the path of the primary positive vortex can be studied. Figure 8.6 shows the paths for the positive primary vortex found from both DNS (up to $t = 9$) and point vortex modeling (up to $t = 7$) for different offset parameters. The trajectories of the positive primary vortex from the DNS are obtained by tracking the location of maximum vorticity in this vortex.
Figure 8.6 Positive primary vortex trajectories for different offset parameters \( (d) \), obtained via point vortex model (green lines) and DNS (magenta lines).

Figure 8.6 reveals that for some cases the vortex trajectories of the positive primary vortex obtained from DNS and the point vortex model show a truly remarkable resemblance. For \( d < -1.5 \) the vortex paths are in both cases near straight lines. However, a retardation due to viscous dissipation in the DNS dipolar flow field can be observed. Because the positions of the vortex obtained via DNS lag behind the positions of the point vortex, the equi-temporal markers are not equi-distant. Again, the result in the regime \(-1.25\)
< \(d < -0.5\) (i.e. for \(d = -0.75\)) shows a discrepancy. This is due to the fact that the split up of the negative primary vorticity is not modeled and the production of secondary vorticity is incorrectly described in the point vortex model. However, in the regime \(-0.5 < d < 1\) the point vortex model is able predict the path of the positive primary vortex with remarkable accuracy. However, the DNS vortex path seems to have decreased curvature in the final stage of the evolution, which is not the case for the point vortex model. This can be understood by the vorticity distribution of the secondary dipole (e.g. in Figure 6.8). The primary positive vorticity appears to be spread-out whereas the negative secondary vorticity has a more compact circular structure. This difference in vorticity distribution is not modeled in the point vortex model. For \(d > 1.5\) the point vortex model is again not able to reproduce the kinematics observed in the DNS. The positive primary vortex in the DNS simulations shows a rebound from the wall, whereas the point vortex only rebounds as it reaches in the proximity of the sharp edge.

To conclude this section, Figure 8.7 shows the point vortex paths alongside snap shots of the vorticity distribution obtained from DNS for \(d = 0\). This is to underline the resemblance of the results that are obtained by DNS and the point vortex model.

![Figure 8.7 Snapshots (colored plots) of the evolution of vorticity distribution obtained via DNS and in the middle corresponding results from the point vortex model \((d = 0)\)](image)

8.3.2. Secondary circulation evolution

Now that the applicability domain of the point vortex method is established and understood, we can focus on the time evolution of the secondary vortex originating from the sharp edge. Figure 8.8 shows the time evolution of the secondary circulation \((\Gamma_s)\) that originates at the sharp edge. The secondary circulation is non-dimensionalized by the circulation of the positive primary vortex \((\Gamma_p)\). The plot shows the circulation evolution for offset parameters within the applicability domain of the point vortex method.
In Figure 8.8 it appears that the trend observed in the DNS simulations is reproduced by the point vortex model, which is a remarkable result as this model was only optimized to reproduce the circulation at time $t = 6$. Apparently the Kutta condition in combination with the zero-force model as discussed in theory sections 2.5.5 and 2.5.6, adequately describe the generation of secondary vorticity. However, for the cases $d > -0.5$, the strengthening distance limit ($\delta$) is necessary to stop the strengthening phase. The strengthening process in the point vortex model ends abruptly when the point vortex is at a certain distance from the sharp edge. The secondary circulation evolution obtained from DNS shows a more gradual transition at the end of the strengthening process. This smooth transition can be understood; as the secondary vortex gains strength and moves away from the wall the shear layer that transports new vorticity towards the vortex is dissipating a part of the circulation (see appendix A). As the shear layer extends, the dissipation takes place over a longer distance and in general for a longer time. Therefore a
smaller fraction of the vorticity originating at the boundary layer is captured in the secondary vortex, leading to a slower strengthening rate in the final stage of the secondary vortex formation. This effect is off course not taken into account by the zero-force model, here the new “vorticity” from the sharp edge is instantaneously captured in the point vortex. Furthermore any viscous dissipation is neglected in the inviscid approach.

Note that for \( d < -1 \), the strengthening stops before the point vortex is out of strengthening range. In this case a new Kutta vortex is placed generally around \( t = 3 \), because the absolute secondary vortex strength has reached a maximum. Therefore the results in this regime do not depend on the strengthening distance free parameter \( (\delta) \). Interestingly, for the cases \( d > 0 \), the strength of the secondary vortex is already above the release threshold strength \( (\gamma_c) \) at \( t = 0 \), leading to the release of the secondary vortex at initialization. The largest deviation in secondary circulation evolution is found for \( d = -0.4 \). As was already shown in Figure 8.4, the secondary vortex in the point vortex model is significantly stronger than the secondary vortex in the DNS. This can be explained by the fact that in the DNS simulations a small fraction of the negative primary vorticity is advected along with the positive primary vortex (see Figure 7.2). This is off course not the case in the point vortex model, leading to a somewhat different flow field evolution. Also in the initial phase \( (t < 3) \), the point vortex model is not able to describe the secondary vortex strength. For this case it might be possible to capture the circulation evolution for this case better by choosing other free parameter values.

8.3.3. Atmosphere tracking

The temporal evolution of a point vortex dipole can also be visualized by adding passive tracers to the flow field. By tracking these tracers in time we obtain a visualization of the flow field from the point vortex model. This can be used for a comparison with the flow fields obtained from DNS and the dye visualization experiments. Figure 8.9 illustrates this flow field visualization method with tracers that are initially located on a curve inside the dipole atmosphere (see Figure 8.1). The tracers are not located exactly on the atmosphere edge, as this would result in a convergence of tracers towards a stagnation point. In this case a dipolar point vortex moves in a straight and unperturbed path.

![Figure 8.9 Plot showing the evolution of 3200 tracers initially placed on a closed curve inside the dipole’s atmosphere that is denoted by the red curve in the left-hand plot](image-url)
Figure 8.9 shows a deformation of the initial curve drawn by the tracers. It appears these tracers visualize the contours of the fluid that is entrained by the vortices.

Figures 8.19, 8.11 and 8.12 show combined results obtained with DNS, the experiment and the point vortex model using passive tracers, respectively, for $d = -0.2, 1, -1.25$. In these figures the focus is on the formation of the secondary vorticity originating from the sharp-edged wall.

Figure 8.10 Snapshots of flow field evolution obtained with DNS (upper row), experimental dye visualization (middle row) and the point vortex model (bottom row), with offset parameter value $d = -0.2$
Figure 8.10 shows that indeed the different approaches result in very similar dynamics of the vortices. This is truly remarkable as the DNS uses a simplified (i.e. 2D) version of the equations of motion to reproduce the experimental dynamics, and in turn, the point vortex model is based on an even more simplified description of the fluid motion. The most notable difference is observed for the positive secondary vortex that originates from the detachment of the boundary layer at the wall. As discussed, this boundary layer vorticity is not modeled in the point vortex simulations. Differences are also observed between the DNS and experimental results. This can be accounted for by the fact that the secondary vorticity originating from the wall in the experiment is anti-cyclonic therefore the 2D detachment (as observed in DNS) might not fully represent the 3D dynamics for the anti-cyclone in the experiment (see section 2.3.2).

Figure 8.11 shows the evolution for the different approaches with an offset parameter value \( d = 1 \).

![Figure 8.11 Snapshots of flow field evolution obtained with DNS (upper row), experimental dye visualization (middle row) and the point vortex model (bottom row), with offset parameter value \( d = 1 \)](image)

The production of secondary vorticity at the sharp edge and the accompanying vortex motion observed in the different approaches shows a clear resemblance.

It is interesting to compare the results for offset parameter \( d = -1.25 \). From DNS it was found (see section 7.1) that in this regime the negative primary vortex is torn apart. The point vortex model is not able to reproduce these type of results regarding the vorticity distributions. On the other hand the point vortex
model is able to predict the strength of the secondary vortex correctly (see Figure 8.4). Unfortunately, no rotating-tank experiments were successfully conducted in this regime. Nevertheless experimental results in this regime are presented in [5], and show good agreement with the DNS results presented here. Therefore Figure 8.12 shows a comparison between the DNS and point vortex model results.

![Figure 8.12 Plots of flow field evolution for two different approaches, DNS vorticity distributions (top) and point vortex tracer plots (bottom). Offset parameter $d = -1.25$](image)

The DNS vorticity distribution plots reveal that the negative primary vortex interacts with the secondary vortex. This interaction results in the detachment of the fluid layer from the edge of this vortex. This fluid possesses vorticity and it therefore visible by the blue tail. This results in a non-symmetric dipole with a stronger positive half. In the point vortex model, the tracers reveal that fluid inside the original dipolar atmosphere also detaches during interaction with the secondary vortex that originates from the wall. However, since this fluid does not possess vorticity the resulting original dipole remains symmetric. It also appears that the secondary vortex in the point vortex model has moved further towards the side of the original dipole as in DNS.

### 8.4. Collision with opening in a wall

In section 5.2 a method of applying the point vortex model to a geometry with an opening in an infinite wall is presented. The relevant DNS results for this geometry are presented in section 7.2. It is interesting to see how the point vortex model performs for this geometry. Note that in the DNS the walls are not infinitesimal thin but have a finite width of 0.4R (as shown in Figure 4.1 (b)). Furthermore, the edges of the walls are rounded. This section presents the results obtained with the point vortex model. In view of time it is chosen not to elaborate on these results as detailed as the previous section. Therefore, only the vortex paths for the symmetric case are discussed in this section.
8.4.1. Vortex paths
The evolution of the flow field where a dipolar vortex approaches an opening between peninsulas is studied experimentally (section 6.3.1) and with DNS (section 7.2). Since the geometry used in the DNS differs from the one used in the point vortex model, it is not expected that the same free parameter value combination as used with the sharp edge will give an accurate description. However as a first guess, the free parameter values as presented in equation (8.5) are used again. The paths of the primary and secondary vortices are plotted in Figure 8.13.

![Figure 8.13 Point vortex dipole collision with an opening according to the FPVC presented in equation (8.5)](image)

It is clear that the point vortex dipole does not pass through the opening on first approach and the production of secondary circulation that is located beneath the wall (i.e. $y < 0$) is predicted. Furthermore the primary vortices move apart as these vortices pair with the secondary vortices leading to a rebound (see Figure 8.13). Nevertheless, the recombination of the primary vortices (e.g. in Figure 6.17) is not observed (i.e. not within the plotted spatial and temporal domain). This appears to deviate from the observed results in the DNS. It is clear that the rebounded dipoles in the point vortex model move along a different path compared to those obtained from the DNS. The point vortex trajectories are less curved than those observed in the DNS. This curvature is determined by the circulations of the point vortices. A pair of point vortices with a large relative difference in strengths results in a strongly curved path of the pair. So in order to model the curvature of the point vortex paths of the rebounded dipoles better according to the DNS results, the secondary circulation should decrease. This can be conveniently achieved by decreasing the strengthening distance ($\delta$). The resulting vortex paths for different $\delta$-values is plotted in Figure 8.14. The other free parameters are not altered.
Figure 8.14 Trajectories of primary and secondary vortices for collision with opening in a wall with different strengthening distances (δ)

Note that in Figure 8.14 only the paths of the primary and secondary vortices are plotted. Tertiary vortices are not plotted to give the plot a clean appearance. As expected, for decreasing strengthening distance (δ), the rebounded dipoles move along a stronger curved path. Remarkably the recombined primary vortices are able to pass through the opening. It appears that the location of recombination can be tuned by altering the strengthening distance (δ). In the DNS it is observed that the dipoles rebounding from the opening recombine at approximately \( y = -3 \). This corresponds with a strengthening distance \( \delta = 0.38 \). This value is much smaller than the strengthening distance that was used for modeling the collision of a dipolar vortex with a sharp-edged wall (i.e. \( \delta = 0.62 \)). This can be attributed to the different wall geometries used in the DNS (see Figure 4.1). The result is in compliance with the result found in [40], where it was concluded that the curvature of the walls’ edges is of great influence on the circulation of the vortex originating by detachment of the boundary layer vorticity at this edge. In general it appears that indeed a smaller radius of curvature results in more intense vortices.

To conclude this section, Figure 8.15 shows the point vortex paths alongside snap shots of the vorticity distribution obtained from DNS.
Figure 8.15 shows that the point vortex model is able to reproduce important aspects of the flow field evolution derived from the DNS. Again only the primary and secondary vortices are plotted.

8.5. Final thoughts on the point vortex model

The results presented in this chapter indicate that the concept of point vortices and mirror images combined with the Kutta condition and the zero-force model (see section 2.5) are able to approximate certain characteristics of the dynamics observed in the experiments and DNS remarkably well. Furthermore, the model provides an efficient description of the vortex kinematics. A DNS typically costs 2 - 5 hours of computing time (depending on simulated time, geometry, computer, etc.), whereas the equations concerning the point vortex model typically take 2 - 5 seconds to evaluate numerically. However, the point vortex model is not a substitute for DNS or experiments. Certain cases are pointed out where the point vortex model is not able to describe the vortex motion and secondary vortex strength as was observed in the DNS. Furthermore, the ambiguity of the free parameter values limits the applicability of this model as a fully self-contained approach for studying vortex kinematics. Other types of flows around a sharp edge (e.g. in [30]) might also induce vortices that originate from the sharp edge. The kinematics found here might very well be described by other free parameters.
9. Conclusion

This thesis reports on a study of a dipolar vortex that collides with different solid objects. The research comprises experiments and numerical simulations. Furthermore, a point vortex model was implemented. The conclusions regarding these three different aspects are separately discussed in this chapter.

9.1. Experimental investigations

In the experiments a quasi-2D flow was achieved by setting up the experiment in a rotating tank. Dipolar vortices where targeted at a sharp-edged wall and in later experiments at an opening in a wall. The flow evolution was studied and analyzed both qualitatively and quantitatively by applying dye-visualization and PIV experiments, respectively.

In the sharp-edged wall case, the generation of a secondary vortex that originated from the sharp edge was observed. For some offset parameter values (\(d\)) the secondary vortex paired with a primary vortex (i.e. a vortex in the initial dipole) and formed a new dipole that moved along a curved trajectory. For large negative offset parameters, the secondary vortex stayed in the proximity of the sharp edge whilst the primary dipole was free to pass the wall. It appeared in most cases that several vortices emerged from the sharp edge. Furthermore, it became clear that the circulation direction of the secondary vortex with respect to the background rotation influenced its strength.

In the case where the dipolar vortex approached an opening in a wall an interesting dynamical difference was found when the dipole approached the opening between two peninsulas or an opening between two islands. For the simply-connected flow domain (i.e. the peninsula case) a rebound and the subsequent recombination of the primary vortices was observed and none of the dye carried by the original dipole passed through the opening. However, for the multiply-connected flow domain (i.e. with two islands) a pass through the opening was observed. It was argued that the fluid from the dipole entering the other side of the flow domain was compensated by a flow circulating around the islands, rather than flowing back through the opening. This is not possible in the simply-connected domain.

9.2. Numerical simulation

In order to check the validity of the numerical simulations, DNS results were compared with experimentally obtained results, see chapter 6. It appeared that the simulation results are able to describe the production of secondary vorticity at the wall(s) and the edge(s) of the wall(s). More specifically, the secondary circulation originating from the sharp edge during a dipole collision is simulated correctly. However, in the numerical simulations all the secondary circulation is concentrated in a single patch of vorticity, whereas in the experiment the shear layer appears to break up periodically. This results in several smaller secondary vortices, leading to different trajectories of the secondary dipoles observed in the experiment than in the numerical simulations. This was explained by the presence of an alternating flow around the sharp edge in the experiment due to a non-constant angular velocity of the rotating tank.

DNS was also used to further investigate several aspects of the dipolar flow in more detail. Three different aspects where studied with an accuracy that could not be matched in a physical experiment. First the dipolar collision with a sharp edge wall was studied. It was found that the collision offset range...
can be classified in five regimes regarding the secondary circulation dynamics. Secondly, the approach of a dipolar vortex towards an opening between two peninsulas with a small asymmetric offset was studied. It appeared that the evolution of the initial dipolar vortices is highly dependent a small offset with respect to the symmetry-axis of the geometry. Finally, the dynamics of a dipolar vortex approaching a similar opening in a multiply-connected domain were studied. It appeared that the geometry of the islands in such a domain is of great influence on the vortex dynamics observed at the opening. It was argued that this is due to the transport of fluid around the islands. More specifically, the moment when the primary vorticity passes through the opening is delayed for increasing island lengths. Furthermore, it was shown that a multiply-connected domain can be classified by a so-called Omicron value ($\Omega$). It was found that the amount of transported fluid around the islands during the dipolar flow field evolution is a function of this value.

9.3. Point vortex model
In the theory section 2.5 a model is discussed that describes the kinematics of vortices in terms of point vortices. This model relies on so-called free parameter values that have no distinct relation to physical flow field properties. The free parameter values were optimized to reproduce the relative secondary vortex strength that was found in DNS after a collision of a dipolar vortex with a sharp-edged wall. The results of the optimization process showed that in order to obtain good results, the free parameter values can be chosen within a certain range of the respective parameter space. In some offset parameter regimes the secondary circulation evolution, the trajectory of the positive primary vortex and the entrained fluid by the vortices appeared to be accurately described by the point vortex model. These regimes were restricted to offset parameter values where splitting and merging of vortices is not observed. Furthermore, the secondary vorticity originating from detachment of the boundary layer was not factored-in in the presented model.
10. References


28. L. Zannetti, V.V. Meleshko, A.A. Gourjii, G.J.F. van Heijst, Behaviour of a two-dimensional vortex pair approaching a solid wedge. *To be published*.


32. A.A. Gourjii, Personal communications via E-mail (Kiev, Ukraine & Eindhoven, The Netherlands, 2014).


11. Appendix

Appendix A) Steady and inviscid flow approximation

In some mathematical derivations in chapter 2 the flow is assumed to be quasi-steady and inviscid such that the magnitude of the force associated with their respective terms in the Navier-Stokes equation can be neglected. In order to check this assumption a grey-scale plot of the magnitude of the four individual force fields from equation (2.1) are plotted in Figure 11.1. These fields are obtained from the numerical simulation of this equation for a flow field that is typical for this thesis. We analyze the case where a dipole collides with a sharp-edged wall with offset parameter \( d = 0 \), at time \( t = 3.5 \) (see Chapter 4).

\[
\begin{align*}
\left| \frac{\partial \mathbf{v}}{\partial t} \right| & \quad \left| (\mathbf{v} \cdot \nabla) \mathbf{v} \right| & \quad \left| - \frac{\nabla p}{\rho} \right| & \quad \left| \nu \nabla^2 \mathbf{v} \right|
\end{align*}
\]

Figure 11.1 Comparison of the magnitude of the different force terms in equation (2.1) as observed in a numerical simulation. The grey scale intensity represents a linear scale with arbitrary units.

It appears that the bulk flow is dominated by the forces that are described by the advection and pressure gradient term. Therefore is seems that indeed the assumption of a steady and inviscid flow are valid in the interior region of the flow. However it is concluded that the inviscid flow approach is not valid in boundary layers and in the shear layers feeding the secondary vortices (i.e. detached boundary layers).
Appendix B) MABLAB code, point-vortex model for sharp edge geometry

clc
clear all;
close all;
tic
%Mapping parameter
n = 2;
% "Free" parameters
threshdist = 0.62;
thresh = 0.008;
startkutta = -0.02;
% Time integrator timestep
dt = 0.0005;
%start position, separation distance and total simulated time
startx = -0.0;
starty = -3;
b = 1;
tijd = 6;
%No more input required
startz = (startx)+(1i*starty);
startksi = 1i*((startz-(b/2)*startz+(b/2)*startkutta)).^(1/2);
eind = tijd/dt;
T = 0:dt:tijd;
loc(:,1) = startksi;
iks = 0:100:100;
R = [1; -1; 0];
[N,~] = size(R);
for j=1:1:eind+1
kpn = loc(:,j).*conj(loc(:,j));
dqdz = n*loc(:,j).^(n-1);
R(N) = -((sum(((R(1:N-1)).*imag(loc(1:N-1,j)))./(kpn(1:N-1))))./(imag(loc(N,j)))/(kpn(N)));
Rk(j) = R(N);
Rkutta(j) = R(3);
%pre allocating for added speed
Rw=zeros((2*N)-1,1);
w=Rw;
for m=1:1:N
a = (loc(m,j)-[loc(:,j); conj(loc(:,j))]);
c = a(a~=0);
Rw(:,m) = ([R(a(1:N-1)==0); -R]);
w(:,m) = c;
end
Vacc = (sum(1i*Rw./w)+transpose((1i*R/2*(n-1)))./loc(:,j)))./transpose(abs(dqdz).^2);
V = Vacc';
loc(1:N-1,j+1) = loc(1:N-1,j)+dt*V(1:N-1);
if abs(R(N)) <= thresh
loc(N,j+1)=loc(N,j);
end
if abs(R(N)) > thresh
\[ k_{\text{mn}} = (\text{real}(\text{loc}(,j)).^2 - (\text{imag}(\text{loc}(,j)).^2); \]
\[ \text{DDRDT1} = k_{\text{pn}}(N)/(R(N) * \text{imag}(\text{loc}(N,j))) * (\sum((R(1:N-1))./(k_{\text{pn}}(1:N-1).^2)).(2*\text{imag}(\text{loc}(1:N-1,j)).\text{real}(\text{loc}(1:N-1,j)).\text{real}(V(1:N-1)) - (k_{\text{mn}}(1:N-1).\text{imag}(V(1:N-1)))) \];

\[ A = \begin{bmatrix} 1 + \text{real}(\text{loc}(N,j)) * (2/n) * \text{real}(\text{loc}(N,j))/(k_{\text{pn}}(N)) - \text{real}(\text{loc}(N,j)) * k_{\text{mn}}(N)/(n * (\text{imag}(\text{loc}(N,j)) * k_{\text{pn}}(N))) & \text{imag}(\text{loc}(N,j)) * \text{real}(\text{loc}(N,j)) * (2/n)/(k_{\text{pn}}(N)) + (-\text{imag}(\text{loc}(N,j)) * k_{\text{mn}}(N)/(n * (\text{imag}(\text{loc}(N,j)) * k_{\text{pn}}(N))) \end{bmatrix}; \]

\[ B = \begin{bmatrix} \text{real}(V(N)) - (\text{real}(\text{loc}(N,j)) * \text{DDRDT1}/n); \text{imag}(V(N)) - (\text{imag}(\text{loc}(N,j)) * \text{DDRDT1}/n) \end{bmatrix}; \]

\[ X = \text{A}\text{B}; \]
\[ \text{loc}(N,j+1) = \text{loc}(N,j) + \text{dt} * (X(1) + 1i * X(2)); \]

If \( j > 0.15/\text{dt} \)
\[ \text{if} \text{abs}(\text{loc}(N,j)) >= (\text{thresh} \text{dist})^(1/n) || \text{abs}(Rk(j)) - \text{abs}(Rk(j-round(0.15/\text{dt})) < 0 \]
\[ N = N+1; \]
\[ R(N) = 0; \]
\[ \text{loc}(N,j+1) = \text{startksi}(3); \]
\[ \text{end} \]
\[ \text{end} \]
\[ \text{end} \]
\[ \text{Rkutta}(j+1) = \text{Rkutta}(j); \]
\[ \text{locz} = \text{loc}.*n; \]

For \( g = 1:1:N \)
\[ \text{plot} (\text{real}(\text{locz}(g,:)), \text{imag}(\text{locz}(g,:)), '-', 'Color', 'green'); \]
\[ \text{hold on} \]
\[ \text{if} \text{R}(g) > 0; \]
\[ \text{scatter} (\text{real}(\text{locz}(g,1:1/\text{dt}:j)), \text{imag}(\text{locz}(g,1:1/\text{dt}:j)), \]
\[ '\text{filled}', 'red') \]
\[ \text{hold on} \]
\[ \text{end} \]
\[ \text{if} \text{R}(g) <= 0; \]
\[ \text{scatter} (\text{real}(\text{locz}(g,1:1/\text{dt}:j)), \text{imag}(\text{locz}(g,1:1/\text{dt}:j)), \]
\[ '\text{filled}', 'blue') \]
\[ \text{hold on} \]
\[ \text{end} \]
\[ \text{end} \]

\[ \text{plot} (\text{iks}, \text{zeros(size(iks))}, 'k'); \]
\[ \text{hold on} \]
\[ \text{axis} ([\text{startx}-5 \text{startx}+5 -5 5]*b) \]
\[ \text{axis square} \]
\[ \text{xlabel}'x' \]
\[ \text{ylabel}'y' \]
\[ \text{title}([''{\text{\textbackslash ltd}} = ' num2str(startx)]); \]
\[ \text{drawnow} \]
\[ \text{hold off} \]
\[ \text{toc} \]
Appendix C) Equations of motion for the point vortex method with an opening in a wall geometry

Complex velocity of \( j \)-th vortex with time independent strength:

\[
\frac{d\zeta_j}{dt} = \sum_{i=0,j\neq j}^{\max(N,M)} \left( \frac{iy_\pi}{2D} \cot \left( \frac{\pi (\zeta_j - \zeta_i + 2ix_0)}{2D} \right) - \cot \frac{\pi (\zeta_j - \zeta_i)}{2D} \right) + \frac{iy_\pi}{2D} \cot \frac{\pi (\zeta_j - \zeta_i)}{2D} + i\frac{R_j}{2} \tan \left( \frac{\pi (\zeta_j - \zeta_i)}{2D} \right) \left| \frac{\pi g}{2D \csc \left( \frac{\pi x_0}{D} \right)} \right|^2.
\]

Linear system for velocities of both M and N th vortex (i.e. with time dependent strength):

\[
\begin{bmatrix}
1 - \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \xi_N} + \frac{\partial y_M}{\partial \xi_N} \right) & \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \eta_N} + \frac{\partial y_M}{\partial \eta_N} \right) & \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \xi_M} + \frac{\partial y_M}{\partial \xi_M} \right) & \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \eta_M} + \frac{\partial y_M}{\partial \eta_M} \right) & \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \xi_j} + \frac{\partial y_M}{\partial \xi_j} \right) & \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \eta_j} + \frac{\partial y_M}{\partial \eta_j} \right) & \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \xi_j} + \frac{\partial y_M}{\partial \eta_j} \right) & \left( \frac{D}{2y_N} \text{Re}[G(\zeta_N)] \frac{\partial y_N}{\partial \eta_j} + \frac{\partial y_M}{\partial \xi_j} \right) \\
-\frac{\partial y_M}{\partial \xi_N} & -\frac{\partial y_M}{\partial \eta_N} & -\frac{\partial y_M}{\partial \xi_M} & -\frac{\partial y_M}{\partial \eta_M} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} \\
-\frac{\partial y_M}{\partial \xi_M} & -\frac{\partial y_M}{\partial \eta_M} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} \\
-\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} \\
-\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} & -\frac{\partial y_M}{\partial \xi_j} & -\frac{\partial y_M}{\partial \eta_j} \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_N}{\partial \xi_N} \\
\frac{\partial y_N}{\partial \eta_N} \\
\frac{\partial y_N}{\partial \xi_M} \\
\frac{\partial y_N}{\partial \eta_M} \\
\frac{\partial y_N}{\partial \xi_j} \\
\frac{\partial y_N}{\partial \eta_j} \\
\frac{\partial y_M}{\partial \xi_M} \\
\frac{\partial y_M}{\partial \eta_M} \\
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial y_N}{\partial \xi_j} \\
\frac{\partial y_N}{\partial \eta_j} \\
\frac{\partial y_M}{\partial \xi_j} \\
\frac{\partial y_M}{\partial \eta_j} \\
\frac{\partial y_M}{\partial \xi_j} \\
\frac{\partial y_M}{\partial \eta_j} \\
\frac{\partial y_M}{\partial \xi_j} \\
\frac{\partial y_M}{\partial \eta_j} \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial y_N}{\partial \xi_N} \\
\frac{\partial y_N}{\partial \eta_N} \\
\frac{\partial y_N}{\partial \xi_M} \\
\frac{\partial y_N}{\partial \eta_M} \\
\frac{\partial y_N}{\partial \xi_j} \\
\frac{\partial y_N}{\partial \eta_j} \\
\frac{\partial y_M}{\partial \xi_M} \\
\frac{\partial y_M}{\partial \eta_M} \\
\end{bmatrix}
\begin{bmatrix}
G(\zeta_N) \\
F(\zeta_N) \\
G(\zeta_M) \\
F(\zeta_M) \\
G(\zeta_j) \\
F(\zeta_j) \\
G(\zeta_j) \\
F(\zeta_j) \\
\end{bmatrix}

where,

\[
G(\zeta) = \tan \left( \frac{\pi (\zeta_N - D)}{2D} \right) + \cot \left( \frac{\pi (\zeta_N - D)}{2D} \right),
\]

\[
F(\zeta) = \tan \left( \frac{\pi (\zeta_N - D)}{2D} \right) - \cot \left( \frac{\pi (\zeta_N - D)}{2D} \right).
\]
Strength of M and N th vortex:

\[
\gamma_{MN} = \frac{1}{ad - bc} \left[ dE - bF \right],
\]

where,

\[
a = \cot \left( \frac{\pi (\xi_M - i\eta_N)}{2D} \right) + \cot \left( \frac{\pi (\xi_M + i\eta_N)}{2D} \right),
\]

\[
b = \cot \left( \frac{\pi (\xi_M + i\eta_N)}{2D} \right) + \cot \left( \frac{\pi (\xi_M - i\eta_N)}{2D} \right),
\]

\[
c = \cot \left( \frac{\pi (D + \xi_N - i\eta_M)}{2D} \right) + \cot \left( \frac{\pi (\xi_N + i\eta_M - D)}{2D} \right),
\]

\[
d = \cot \left( \frac{\pi (\xi_M + i\eta_N)}{2D} \right) + \cot \left( \frac{\pi (\xi_M - i\eta_N)}{2D} \right),
\]

\[
E = \sum_{j=0, j \neq 1,2}^{\max(N,M)-1} \gamma_j \cot \left( \frac{\pi (\xi_j - i\eta_j)}{2D} \right) + \cot \left( \frac{\pi (\xi_j + i\eta_j)}{2D} \right),
\]

\[
F = \sum_{j=0, j \neq 1,2}^{\max(N,M)-1} \gamma_j \cot \left( \frac{\pi (D + \xi_j - i\eta_j)}{2D} \right) + \cot \left( \frac{\pi (\xi_j + i\eta_j - D)}{2D} \right).
\]

Derivatives of vortex strength:

\[
\frac{\partial \gamma_{MN}}{\partial \xi_N} = \frac{\pi}{2D} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N - i\eta_M)}{2D} \right) - 1} + \frac{2}{\cos \left( \frac{\pi (\xi_N + i\eta_M)}{2D} \right) - 1} \right] \frac{d(bF - dE)}{(ad - bc)^2} + \frac{\pi}{2D} \left[ \frac{2}{\cos \left( \frac{\pi (D + \xi_N - i\eta_M)}{2D} \right) - 1} + \frac{2}{\cos \left( \frac{\pi (\xi_N + i\eta_M - D)}{2D} \right) - 1} \right] \frac{b(E d - bF)}{(ad - bc)^2},
\]

\[
\frac{\partial \gamma_{MN}}{\partial \eta_N} = \frac{\pi}{2D} \left[ \frac{2}{1 - \cos \left( \frac{\pi (\xi_N - i\eta_M)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N + i\eta_M)}{2D} \right)} \right] \frac{d(bF - dE)}{(ad - bc)^2} - \frac{\pi}{2D} \left[ \frac{2}{1 - \cos \left( \frac{\pi (D + \xi_N - i\eta_M)}{2D} \right)} + \frac{2}{1 - \cos \left( \frac{\pi (\xi_N + i\eta_M - D)}{2D} \right)} \right] \frac{b(E d - bF)}{(ad - bc)^2}.
\]
Furthermore,

\[
\frac{\partial \gamma_N}{\partial \xi_M} = \frac{\pi}{2D} \left[ \frac{2}{\cos \left( \frac{\pi (D + \xi_M - i \eta_M)}{2D} \right) - 1} + \frac{2}{\cos \left( \frac{\pi (D + \xi_M + \eta_M - D)}{2D} \right) - 1} \right] \frac{E}{2D} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_M - i \eta_M)}{2D} \right) - 1} + \frac{2}{\cos \left( \frac{\pi (\xi_M + i \eta_M)}{2D} \right) - 1} \right] \frac{F}{(ad - bc) + c(Fb - Ed)} \right.

\[
\frac{\partial \gamma_N}{\partial \eta_M} = -i \pi \left[ \frac{2}{1 - \cos \left( \frac{\pi (\xi_M + i \eta_M)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_M - i \eta_M)}{2D} \right)} \right] \frac{F}{ad - bc} \left[ \frac{2}{1 - \cos \left( \frac{\pi (\xi_M - i \eta_M)}{2D} \right)} + \frac{2}{1 - \cos \left( \frac{\pi (\xi_M + i \eta_M)}{2D} \right)} \right] \frac{E}{(ad - bc) + (ad - bc)^2}

\[
\frac{\partial \gamma_M}{\partial \xi_N} = \frac{\pi}{2D} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right) - 1} + \frac{2}{\cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right) - 1} \right] \frac{d(Fa - Ec)}{(ad - bc)^2} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right)} \right] \frac{F}{(ad - bc) + b(Fa - Ec)}

\[
\frac{\partial \gamma_M}{\partial \eta_N} = \frac{i \pi}{2D} \left[ \frac{2}{1 - \cos \left( \frac{\pi (\xi_N + i \eta_N)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} \right] \frac{d(Fa - Ec)}{(ad - bc)^2} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right)} \right] \frac{E}{(ad - bc) + (ad - bc)^2}

\[
\frac{\partial \gamma_M}{\partial \xi_N} = -i \pi \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right) - 1} + \frac{2}{\cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right) - 1} \right] \frac{C(Fa - Ec)}{(ad - bc)^2} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} \right] \frac{a(Fa - Ec)}{(ad - bc)^2}

\[
\frac{\partial \gamma_M}{\partial \eta_N} = -i \pi \left[ \frac{2}{1 - \cos \left( \frac{\pi (\xi_N + i \eta_N)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} \right] \frac{C(Fa - Ec)}{(ad - bc)^2} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} \right] \frac{a(Fa - Ec)}{(ad - bc)^2}

Furthermore, 

\[
\zeta_j \frac{\partial \gamma_N}{\partial \xi_j} + \eta_j \frac{\partial \gamma_N}{\partial \eta_j} = \frac{\pi}{2D} \left[ \frac{2}{\cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right) - 1} + \frac{2}{\cos \left( \frac{\pi (\xi_N + i \eta_N)}{2D} \right) - 1} \right] \frac{c}{bc - ad} + \frac{2}{\cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right) - 1} \frac{a}{ad - bc}

\[
+ \eta_j \left[ \frac{i \pi}{2D} \left[ \frac{2}{1 - \cos \left( \frac{\pi (\xi_N + i \eta_N)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} \right] \frac{c}{bc - ad} + \frac{2}{\cos \left( \frac{\pi (\xi_N + i \eta_N - D)}{2D} \right)} - \frac{2}{1 - \cos \left( \frac{\pi (\xi_N - i \eta_N)}{2D} \right)} \right] \frac{a}{ad - bc}

Typesetting and printing errors reserved, Good luck ;)}