MASTER

An axiomatisation for rooted branching bisimulation with explicit divergence

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Abstract

Branching bisimulation with explicit divergence is an extension of branching bisimulation which preserves divergence. A proces is divergent if from that process an infinite sequence of τ-transitions can be done. We prove that rooted branching bisimulation with explicit divergence is a congruence and we develop a sound and complete axiomatisation for rooted branching bisimulation with explicit divergence.

Contents

Preface

In the paper *Axiomatising Divergence* by M. Lohrey et al. [1] an axiomatisation was developed for the divergence sensitive spectrum of weak bisimulation equivalence. In the paper *Branching bisimilarity with explicit divergence* by R.J. van Glabbeek et al. [11] a finer divergence preserving equivalence was studied which is based on branching bisimilarity. The goal of my project was to find a sound and complete axiomatisation of the coarsest congruence included in branching bisimulation with explicit divergence. The language which is used is the same as in [1]. It contains a constant nil, action prefix, the choice operator and recursion. Also, an operator that explicitly adds divergence is used.

The coarsest congruence included in branching bisimulation with explicit divergence is rooted branching bisimulation with explicit divergence. Before an axiomatisation could be made for this relation we needed to show that the relation is a congruence. A relation is desired to be a congruence since this facilitates equational reasoning. A congruence is a relation which is an equivalence and which is compatible with the defined grammar. In [11] it was shown that branching bisimulation with explicit divergence is an equivalence relation. So in order to show that rooted branching bisimulation with explicit divergence is an equivalence relation only the root condition needed to be considered. To prove compatibility for the recursion operator a standard technique is to use an up-to relation. This technique was also used in [1] and we also used this technique.

After proving that the relation is a congruence a sound and complete axiomatisation was found. An axiomatisation is sound if everything which can be derived using the axioms is true. An axiomatisation is complete if everything which is true can also be derived using the axioms. First we had to determine which axioms in [1] were still valid for rooted branching bisimulation with explicit divergence. In order to obtain completeness some axioms needed to be added. Some of these axioms where taken from the paper *A Complete Axiomatization for Branching Bisimulation Congruence of Finite-State Behaviours* by R.J. van Glabbeek [8]. In this paper an axiomatisation for branching bisimulation congruence was developed.

Lastly, I want to thank my supervisor Bas Luttik for helping me during the project. I also want to thank the other members of my graduation committee Pieter Cuijpers and Tim Willemse for providing me with feedback on this document.

Chapter 1

Introduction

The model-checking approach to verifying the correctness of systems is to verify the correctness of a system by checking a formula in modal logic. The systems we consider are systems which can be viewed as a discrete system consisting of states and transitions between states. Such a system can be modeled mathematically using labelled transition systems. The kind of formula which is checked can for instance be a modal formula of the form: *Eventually a state satisfying a formula ϕ will be reached*. Before checking whether a formula is true for a labelled transition system we reduce the labelled transition system to a smaller labelled transition system which is behaviourally equivalent. Many different behavioural equivalences have been defined on labelled transition systems, each with their own level of abstraction. In [9] van Glabbeek has made a classification of different behavioural equivalences.

The best known behavioural equivalence is bisimulation congruence [2], which takes a notion of observability into account. Two processes are equal if the same behaviour can be observed. However, abstraction is used in labelled transition systems by replacing actions which are considered unimportant by τ-actions, which are then treated as unobservable. This makes it necessary to have a behavioural equivalence with the possibility to abstract from these invisible τ-actions. From the classification in [9] branching bisimulation, which was proposed by van Glabbeek and Weijland [12], is the finest congruence. It preserves the branching structure of processes, in [10] van Glabbeek formulates what it means for a model to respect branching time. If two processes are branching bisimilar they can also be related using coarser equivalences. Such a coarser equivalence which was introduced before branching bisimulation is weak bisimulation. Weak bisimulation, developed by Milner [2] holds a very coarse equivalence, this means that it relates more elements.

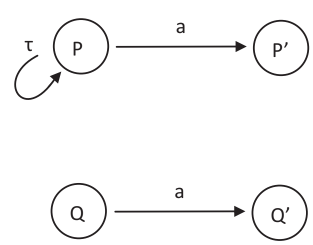

Figure 1.1: States P and Q.

Since branching bisimulation abstracts from divergence (an infinite sequence of τ-steps) it is not compatible with the eventually property. Take for example a state P from which it is possible to do a τ-loop or do an a-transition to 0 ($P = recX.(\tau.X + a.0)$) and a state Q from which it is only possible to do the a-transition to 0 ($Q = a.0$). This is shown in Figure 1.1. Then P and Q are branching bisimilar. However, it is not always true that from P eventually the state 0 will be reached, while from Q this is true. A solution is to assume a notion of fairness where it is assumed that a process will not infinitely often choose an internal action over a visible one. It is often reasonable to assume this notion of fairness. Take for instance the communication between a sender and receiver over a lossy channel where the sender keeps sending the same message until an acknowledgement has been received from the receiver. The behaviour that occurs as long as the acknowledgement has not been received can be modeled by a τ-loop. However it is highly likely to assume that the channel will not always lose or corrupt data, such that the acknowledgement will eventually be sent. So in this case it is realistic to make a fairness assumption. It is however not always desirable to make this assumption of fairness.

Branching bisimulation with explicit divergence was proposed in [12]. It is the finest reasonable equivalence in the classification of behavioural equivalences with silent moves in [9]. Branching bisimulation with explicit divergence is an extension of branching bisimulation which abstracts from internal activity, while preserving divergence. In [11] several conditions that can be added to the notion of branching bismulation in order to make it divergence sensitive are proposed after which it is shown that these conditions lead to the same equivalence. It was also shown that branching bisimulation with explicit divergence is an equivalence relation. Showing that branching bisimulation with explicit divergence is an equivalence was non-trivial, the difficulty here lay in proving transitivity. In order to relate two processes using branching bisimulation with explicit divergence all states in the infinite τ-path should be related to each other, which is a property that is not automatically preserved when composing relations. We have encountered similar difficulties when proving that rooted branching bisimulation with explicit divergence is a congruence.

Writing a labelled transition system as an expression offers a clear and short manner to describe the behaviour of a labelled transition system. It also makes it possible to describe infinite behaviour using recursion. These expressions can also be used in tools. In order to show that two expressions are branching bisimilar with explicit divergence the definition for branching bisimulation with explicit divergence can be used. It is however much faster and clearer to have a set of axioms which can be used to rewrite an expression into an equal expression. These axioms can also give insight in the equivalence. So a sound and complete axiomatisation for branching bisimulation with explicit divergence is needed. Soundness means that everything which can be proven using the axioms is true. Completeness means that everything which is true can be shown using the axioms.

Divergence occurs when from a state a sequence of τ-transitions can be done after which the initial state is reached again. This can be described mathematically using recursion. Recursion is however hard to axiomatise. In [5] Milner has developed a complete axiomatisation for weak congruence using recursion. Weak congruence was adjusted to make it divergence sensitive in [1] where a complete axiomatisation was given. In [8] van Glabbeek has developed a complete axiomatisation for branching congruence also using recursion. We will develop a complete axiomatisation for rooted branching bisimulation with explicit divergence. However, axiomatising divergence in the scope of branching bisimulation is more complex than axiomatising divergence for weak congruence; where divergence holds an infinite τ-path for weak bisimulation, in order to relate two processes using branching bisimulation with explicit divergence all states in the infinite τ-path should be related to each other.

In order to use axioms to show that two expressions are equal we need to show that we can replace equals by equals in a context. This means that we have to show that branching bisimulation with explicit divergence is a congruence. However, just like branching bisimulation, branching bisimulation with explicit divergence is not a congruence. Branching bisimulation was made a congruence by adding a root condition, the same is done for branching bisimulation with explicit divergence. We have shown that rooted branching bisimulation with explicit divergence is a congruence using an up-to relation (see chapter 6 in [7] for an introduction on up-to relations). This relation is based on the up-to relations used in [1] and [8] in order to show that divergence preserving weak congruence and branching congruence are indeed congruences. However an extra condition needs to be added to the up-to relation for divergence preserving branching congruence.

Completeness is shown using the same structure as used by Milner in [5]. Preserving divergence however makes these steps harder. In [1] this was solved by adding an extra operator Δ, we use the same method.

In Section 2 the grammar and operational semantics which we use to describe a labelled transition system are introduced. In Section 3 the definitions for several bisimulation relations are given, these definitions lead to the definition for rooted branching bisimulation with explicit divergence. Section 4 holds the congruence proof for rooted branching bisimulation with explicit divergence. The axioms are introduced in Section 5, followed by the soundness proof for these axioms in Section 6. In Section 7 the completeness proof is given. This report is concluded in Section 8 with the conclusions and options for future work.

Chapter 2

Preliminaries

Writing a labelled transition system as an expression offers a clear and short manner to describe the behaviour of a labelled transition system. It also makes it possible to describe infinite behaviour using recursion. Here we define the grammar and operational semantics which we use to describe labelled transition systems. The grammar and operational semantics are the same as used in [1].

The set $\mathbb{E}$ of expressions for labelled transition systems is generated by the following grammar:

$$\mathcal{E} = 0 \mid a.\mathcal{E} \mid \mathcal{E} + \mathcal{E} \mid recX.\mathcal{E} \mid X \mid \Delta(\mathcal{E})$$

where $a \in A$, with A a set of actions with a special element $\tau \in A$, and $X \in V$, with V a set of variables. The special element τ is an invisible action used to abstract from activity which is considered unobservable. Let $E \in \mathbb{E}$, the expression $\Delta(E)$ has the same behaviour as E, except that a divergence is explicitly added to the root of E in the form of a τ-loop. The Δ-operator is redundant, it is however convenient for the first step of the completeness proof where an expression is rewritten into a guarded expression. Below we explain when an expression is guarded and why we need the Δ-operator.

A variable X occurs free in E if it occurs in E outside the scope of any binding $recX$-operator. The set of all free variables in E is denoted by $\mathbb{V}(E)$. The set $\mathbb{P}$ of closed expressions is defined as $\mathbb{P} = \{E \in \mathbb{E} | \mathbb{V}(E) = \emptyset\}$.

The variable X is guarded in E if every free occurrence of X in E lies within a subexpression of the form $a.F$ with $a \in A \backslash \{\tau\}$, otherwise we call X unguarded. The variable X is weakly guarded in E if every free occurrence of X in E lies within a subexpression of the form $a.F$ with $a \in A$, i.e. allowing $a = \tau$, otherwise we call X totally unguarded. The expression E is guarded if for every subexpression $recX.F$ of E the variable X is guarded in F.

Divergence occurs when from a state a sequence of τ-transitions can be done after which the initial state is reached again. So if a process is divergent this process can be described by an expression using recursion. This expression will then have an unguarded occurrence of X within the scope of a binding $recX$-operator, so this expression will be unguarded. The only way to describe divergent behaviour using a guarded expression is by using the Δ-operator.

The transition relation is defined by the following operational rules. Here $E, E', F \in \mathbb{E}$.

$$\frac{}{a.E \xrightarrow{a} E} \qquad \frac{E \xrightarrow{a} E'}{E + F \xrightarrow{a} E'} \qquad \frac{E \xrightarrow{a} E'}{F + E \xrightarrow{a} E'}$$

$$\frac{E\{recX.E/X\} \stackrel{a}{\to} E'}{recX.E \stackrel{a}{\to} E'} \qquad \frac{E \stackrel{a}{\to} E'}{\Delta(E) \stackrel{a}{\to} E'} \qquad \frac{}{\Delta(E) \stackrel{\tau}{\to} \Delta(E)}$$

Here $E\{F/X\}$ denotes the expression that results from E by simultaneously replacing all free occurrences of X in E by F. If necessary we rename bound variables in E to avoid free variables in F getting bound.

The syntax and operational semantics define a transition system $(\mathbb{E}, \to)$, with as set of states $\mathbb{E}$ the set of expressions according to the syntax above, and as transition relation $\to$ the least relation that satisfies the operational rules.

Let $E, E' \in \mathbb{E}$ and let $a \in A$, then we write $E \stackrel{a}{\to} E'$ for $(E, a, E') \in \to$ and $E \stackrel{(a)}{\to} E'$ for $E \stackrel{a}{\to} E'$ or $(a = \tau$ and $E = E')$. The transitive closure of the binary relation $\stackrel{\tau}{\to}$ is denoted by $\to^+$ and its reflexive-transitive closure by $\twoheadrightarrow$.

Let $n \in \omega$, we inductively define b^n as follows:
$b^0 = 0$
$b^{n+1} = b.b^n$

In Appendix A we state some useful properties about the transition relation of a labelled transition system.

Chapter 3

Bisimulations

Here the definitions for several bisimulation relations are given. First the definitions for strong bisimulation and branching bisimulation are given. These are followed by the definitions for branching bisimulation with explicit divergence and rooted branching bisimulation with explicit divergence, which are taken from [11] and [13] respectively. They are defined for closed expressions. For branching bisimulation with explicit divergence several alternative definitions are given. In [11] branching bisimulation with explicit divergence is studied, where several conditions that can be added to the notion of branching bisimulation in order to make it divergence sensitive are proposed after which it is shown that these conditions lead to the same equivalence. Therefore all definitions can be used when proving that a relation is a branching bisimulation with explicit divergence.

Definition 3.0.1. (Strong bisimulation) A symmetric binary relation $\mathcal{R}$ on $\mathbb{P}$ is a strong bisimulation if it satisfies the following condition for all $P, Q \in \mathbb{P}$ and $a \in A$:

(S) if $P \mathrel{\mathcal{R}} Q$ and $P \xrightarrow{a} P'$ for some state P', then there exists a state Q' such that $Q \xrightarrow{a} Q'$ and $P' \mathrel{\mathcal{R}} Q'$.

To indicate that a strong bisimulation $\mathcal{R}$ exists such that $P \mathrel{\mathcal{R}} Q$ we write $P \underline{\leftrightarrow} Q$.

Definition 3.0.2. (Branching bisimulation) A symmetric binary relation $\mathcal{R}$ on $\mathbb{P}$ is a branching bisimulation if it satisfies the following condition for all $P, Q \in \mathbb{P}$ and $a \in A$:

(T) if $P \mathrel{\mathcal{R}} Q$ and $P \xrightarrow{a} P'$ for some state P', then there exist states Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, $P \mathrel{\mathcal{R}} Q''$ and $P' \mathrel{\mathcal{R}} Q'$.

To indicate that a branching bisimulation $\mathcal{R}$ exists such that $P \mathrel{\mathcal{R}} Q$ we write $P \underline{\leftrightarrow}_b Q$.

Definition 3.0.3. (Branching bisimulation with explicit divergence) A symmetric binary relation $\mathcal{R}$ on $\mathbb{P}$ is a branching bisimulation with explicit divergence if it is a branching bisimulation (i.e., it satisfies condition (T) of Definition 3.0.2) and in addition satisfies the following condition for all $P, Q \in \mathbb{P}$:

(D) if $P \mathrel{\mathcal{R}} Q$ and there is an infinite sequence of states $(P_k)_{k \in \omega}$ such that $P = P_0, P_k \xrightarrow{\tau} P_{k+1}$ and $P_k \mathrel{\mathcal{R}} Q$ for all $k \in \omega$, then there exists an infinite sequence of states $(Q_l)_{l \in \omega}$ such that $Q = Q_0, Q_l \xrightarrow{\tau} Q_{l+1}$ for all $l \in \omega$, and $P_k \mathrel{\mathcal{R}} Q_l$ for all $k, l \in \omega$.

Theorem 3.0.1. (Branching bisimulation with explicit divergence) A symmetric binary relation $\mathcal{R}$ on $\mathbb{P}$ is a branching bisimulation with explicit divergence if it is a branching bisimulation (i.e. it satisfies condition (T) of Definition 3.0.2) and in addition satisfies the following condition for all $P, Q \in \mathbb{P}$:

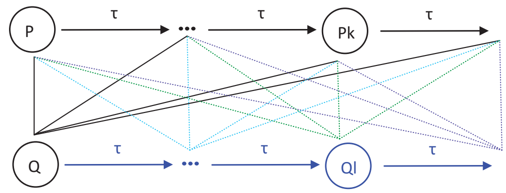

Figure 3.1: Condition (D) from Definition 3.0.3.

(D) if $P \mathcal{R} Q$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists a state Q' such that $Q \xrightarrow{\tau} Q'$ and $P_k \mathcal{R} Q'$ for some $k \in \omega$.

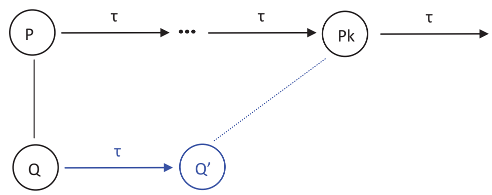

Figure 3.2: Condition (D) from Theorem 3.0.1.

Theorem 3.0.2. (Branching bisimulation with explicit divergence) A symmetric binary relation $\mathcal{R}$ on $\mathbb{P}$ is a branching bisimulation with explicit divergence if it is a branching bisimulation (i.e. it satisfies condition (T) of Definition 3.0.2) and in addition satisfies the following condition for all $P, Q \in \mathbb{P}$:

(D) if $P \mathcal{R} Q$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists an infinite sequence of states $(Q_l)_{l\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $Q = Q_0$, $Q_l \xrightarrow{\tau} Q_{l+1}$ and $P_{\sigma(l)} \mathcal{R} Q_l$ for all $l \in \omega$.

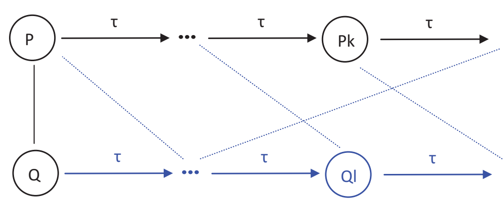

Figure 3.3: Condition (D) from Theorem 3.0.2.

To indicate that a branching bisimulation with explicit divergence $\mathcal{R}$ exists such that $P \mathcal{R} Q$ we write $P \underline{\leftrightarrow}_b^{\Delta} Q$.

Lemma 3.0.1. *Branching bisimulation with explicit divergence is an equivalence*

Proof. This follows from Corollary 4.2. in [11]. □

Just like branching bisimulation, branching bisimulation with explicit divergence is not a congruence for alternative composition. It is also not a congruence for the Δ-operator. For instance, $\tau.0 \underline{\leftrightarrow}_b^{\Delta} 0$, while $\Delta(\tau.0) \not\underline{\leftrightarrow}_b^{\Delta} \Delta(0)$, since $\Delta(\tau.0)$ can do a τ-transition to 0, while $\Delta(0)$ cannot. A root condition is added to branching bisimulation with explicit divergence to create a congruence relation.

Definition 3.0.4. (Rooted branching bisimulation with explicit divergence) A branching bisimulation with explicit divergence $\mathcal{R}$ is *rooted with respect to* P *and* Q if it satisfies the following root condition for all $a \in A$:

(R) if $P \xrightarrow{a} P'$ for some state P', then there exists a state Q' such that $Q \xrightarrow{a} Q'$ and $P' \mathcal{R} Q'$.

To indicate that a branching bisimulation with explicit divergence $\mathcal{R}$ exists such that $P \mathcal{R} Q$ and $\mathcal{R}$ is rooted with respect to P and Q we write $P \underline{\leftrightarrow}_{rb}^{\Delta} Q$.

The relation $\underline{\leftrightarrow}_{rb}^{\Delta}$ is defined for closed expressions. The relation is lifted from $\mathbb{P}$ to $\mathbb{E}$ as described in [1]: Let $\underline{\leftrightarrow}_{rb}^{\Delta} \subseteq \mathbb{P} \times \mathbb{P}$ and $E, F \in \mathbb{E}$. Let $\overrightarrow{X} = (X_1, ..., X_n)$ be a sequence of variables that contains all variables from $\mathbb{V}(E) \cup \mathbb{V}(F)$. Then $E \underline{\leftrightarrow}_{rb}^{\Delta} F$ if for all $\overrightarrow{P} = (P_1, ..., P_n)$ with $P_i \in \mathbb{P}$ we have $E\{\overrightarrow{P}/\overrightarrow{X}\} \underline{\leftrightarrow}_{rb}^{\Delta} F\{\overrightarrow{P}/\overrightarrow{X}\}$.

In Appendix B we prove in Lemma B.0.10 that the relation $\underline{\leftrightarrow}_{b}^{\Delta}$ satisfies Condition (T) from Definition 3.0.2 and Condition (D) from Definition 3.0.3 for expressions $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. In Lemma B.0.11 it is proven that the relation $\underline{\leftrightarrow}_{rb}^{\Delta}$ satisfies Condition (R) from Definition 3.0.4 for expressions $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. From this we can conclude Corollary 3.0.1 and 3.0.2.

Corollary 3.0.1. *The relation* $\underline{\leftrightarrow}_{b}^{\Delta}$ *satisfies Condition* (T) *from Definition 3.0.2 and Condition* (D) *from Definition 3.0.3 for* $E, F \in \mathbb{E}$ *such that* $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$.

Corollary 3.0.2. *The relation* $\underline{\leftrightarrow}_{rb}^{\Delta}$ *satisfies Condition* (R) *from Definition 3.0.4 for* $E, F \in \mathbb{E}$ *such that* $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$.

From Corollary 3.0.1 and 3.0.2 it follows that Condition (T) from Definition 3.0.2, Condition (D) from Definition 3.0.3 and Condition (R) from Definition 3.0.4 also hold for open expressions E and F. This is used in the congruence proof and the completeness proof.

Chapter 4

Congruence

To show that rooted branching bisimulation with explicit divergence is a congruence we need to show that it is an equivalence relation and that it is compatible with the defined grammar. In Section 4.1 we show that rooted branching bisimulation with explicit divergence is an equivalence relation. In Section 4.2 we show that it is compatible with the defined grammar. We finish this chapter by proving that rooted branching bisimulation with explicit divergence is the coarsest congruence included in branching bisimulation with explicit divergence. We prove this in Section 4.3.

4.1 Equivalence

A relation is an equivalence relation if it is reflexive, symmetric and transitive. Branching bisimulation with explicit divergence is an equivalence (see Lemma 3.0.1). In order to show that rooted branching bisimulation with explicit divergence is an equivalence, we need to investigate the effect of condition (R) of Definition 3.0.4 for each of the mentioned properties. In Lemma 4.1.1 we show that rooted branching bisimulation with explicit divergence is an equivalence relation.

Lemma 4.1.1. *Rooted branching bisimulation with explicit divergence is an equivalence*

Proof. This is proven in Lemma C.0.12 in Appendix C. □

4.2 Compatibility

A relation $\mathcal{R}$ is compatible with the defined grammar if for each operator the result of applying that operator on a pair which is in the relation is still in the relation. E.g., for the grammar defined in this report the following should hold for $P_1, P_2, P_3, P_4 \in \mathbb{P}$ and $E_1, E_2 \in \mathbb{E}$

$P_1 \mathrel{\mathcal{R}} P_2$ implies $a.P_1 \mathrel{\mathcal{R}} a.P_2$
$P_1 \mathrel{\mathcal{R}} P_2$ and $P_3 \mathrel{\mathcal{R}} P_4$ imply $P_1 + P_3 \mathrel{\mathcal{R}} P_2 + P_4$
$E_1 \mathrel{\mathcal{R}} E_2$ implies $recX.E_1 \mathrel{\mathcal{R}} recX.E_2$
$P_1 \mathrel{\mathcal{R}} P_2$ implies $\Delta(P_1) \mathrel{\mathcal{R}} \Delta(P_2)$

In Lemma 4.2.5 we show that rooted branching bisimulation with explicit divergence is compatible with the defined grammar. The hardest part here is showing compatibility for the recX-operator. Here we explain how we do that. Lemma 4.2.4 holds the proof that recX is compatible with the defined grammar.

In order to show compatibility for recX we assume for $E, F \in \mathbb{E}$ that $E \underline{\leftrightarrow}_{rb}^{\Delta} F$, we now want to show that $recX.E \underline{\leftrightarrow}_{rb}^{\Delta} recX.F$. Assume $recX.E \xrightarrow{a} E''$, we want to show that $recX.F \xrightarrow{a} F''$, where $E'' \underline{\leftrightarrow}_b^\Delta F''$. Since $recX.E \xrightarrow{a} E''$ we know by Lemma A.0.6 that $E \xrightarrow{a} E'$ for some E' and $E'' = E'\{recX.E/X\}$. Since $E \underline{\leftrightarrow}_{rb}^\Delta F$ we know $F \xrightarrow{a} F'$ where $E' \underline{\leftrightarrow}_b^\Delta F'$. By Lemma A.0.3 we know $F\{recX.F/X\} \xrightarrow{a} F'\{recX.F/X\}$ and by the operational semantics we have $recX.F \xrightarrow{a} F'\{recX.F/X\}$. What remains is showing that $E'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$. Using the way the relation is lifted we can obtain $E'\{recX.F/X\} \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$, however we could not show $E'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$. This is shown in Figure 4.1.

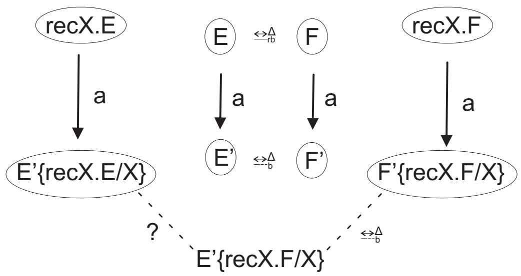

Figure 4.1: Compatibility for recX.

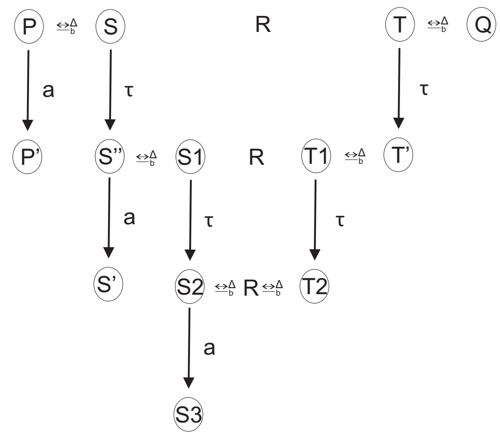

Figure 4.2: Adding Condition $(U2)$

A solution is to define an up-to relation. The "up-to" technique is described in [6] and chapter 6 of [7]. We define rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ in Definition 4.2.1. We then prove that any relation which is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ also is a rooted branching bisimulation with explicit divergence, this is shown in Lemma 4.2.1. Therefore it is sufficient to show that a relation is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ to conclude that it is a rooted branching bisimulation with explicit divergence.

First we defined the up-to relation in Definition 4.2.1 using only Condition $(U1)$, this condition is based on the up-to relation used in [1]. However, we were not able to show that any relation which is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ also is a rooted branching bisimulation with explicit divergence. This is shown in Figure 4.2. We want to prove that $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta \subseteq \underline{\leftrightarrow}_b^\Delta$, so assume $P \underline{\leftrightarrow}_b^\Delta S \mathcal{R} T \underline{\leftrightarrow}_b^\Delta Q$. In order to show that $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta \subseteq \underline{\leftrightarrow}_b^\Delta$ we have to show Condition (T) from Definition 3.0.2. So we have to show that if $P \xrightarrow{a} P'$, then $Q \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, where $P \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q''$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$. Assume $P \xrightarrow{a} P'$, by Condition (T) we know $S \twoheadrightarrow S'' \xrightarrow{(a)} S'$, now assume $S \xrightarrow{\tau} S'' \xrightarrow{a} S'$, as shown in Figure 4.2. Then we know by Condition $(U1)$ from Definition 4.2.1 that $T \xrightarrow{\tau} T'$, where $S'' \underline{\leftrightarrow}_b^\Delta S_1 \mathcal{R} T_1 \underline{\leftrightarrow}_b^\Delta T'$, for some S_1 and T_1. We have $S'' \underline{\leftrightarrow}_b^\Delta S_1 \mathcal{R} T_1 \underline{\leftrightarrow}_b^\Delta T'$ and $S'' \xrightarrow{a} S'$. Again it is possible that $S_1 \xrightarrow{\tau} S_2 \xrightarrow{a} S_3$, as shown in Figure 4.2. This leads to $T_1 \xrightarrow{\tau} T_2$, but it is possible that from T_1, and thus from T the τ-steps are never followed by an a-step. So we cannot show that $Q \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$. In order to solve this problem Condition $(U2)$ was added, this is the same condition as used in [8].

In order to show that $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta \subseteq \underline{\leftrightarrow}_b^\Delta$ we also have to show the branching bisimulation with explicit divergence condition. The easiest condition to use is Condition (D) from Theorem 3.0.1. So we have to show that if $P \underline{\leftrightarrow}_b^\Delta S \mathcal{R} T \underline{\leftrightarrow}_b^\Delta Q$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists a state Q' such that $Q \xrightarrow{\tau} Q'$ and $P_k \mathcal{R} Q'$ for some $k \in \omega$. So assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. Then there also is an infinite sequence of states

$(S_l)_{l\in\omega}$ such that $S = S_0$ and $S_l \xrightarrow{\tau} S_{l+1}$ for all $l \in \omega$. Using either Condition $(U1)$ or $(U2)$ from Definition 4.2.1 it is possible that $T \xrightarrow{\tau} T'$, where $S_1 \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta T'$, as shown in Figure 4.3. Using Condition (T) from Definition 3.0.2 it is possible that $T' \underline{\leftrightarrow}_b^\Delta Q$. So we cannot show that $Q \xrightarrow{\tau} Q'$ for some Q'. In order to solve this problem Condition $(U3)$ was added.

In Definition 4.2.1 we give the definition for rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$. In Lemma 4.2.1 we prove that every relation which is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ also is a rooted branching bisimulation with explicit divergence. Therefore it is sufficient to prove that a relation is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ to conclude that the pairs in the relation are rooted branching bisimilar with explicit divergence.

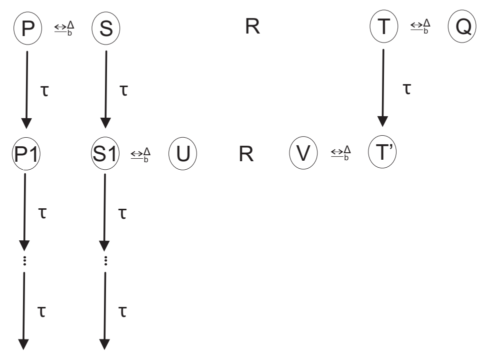

Figure 4.3: Adding Condition $(U3)$

Definition 4.2.1. (Rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$) A symmetric binary relation $\mathcal{R}$ between $P, Q \in \mathbb{P}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ if it satisfies the following conditions for all $a \in A$:

(U1) if $P\mathcal{R}Q$ and $P \xrightarrow{a} P'$ for some state P', then there exists a state Q' such that $Q \xrightarrow{a} Q'$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.

(U2) If $P\mathcal{R}Q$ and $P \twoheadrightarrow P'' \xrightarrow{(a)} P'$ for some states P' and P'', then there exist states Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, $P'' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q''$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.

(U3) if $P\mathcal{R}Q$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists an infinite sequence of states $(Q_l)_{l\in\omega}$ such that $Q = Q_0$ and $Q_l \xrightarrow{\tau} Q_{l+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $P_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_l$ for all $l \in \omega$.

Lemma 4.2.1. *If $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$, then $\mathcal{R} \subseteq \underline{\leftrightarrow}_{rb}^\Delta$*

Proof. If $P \mathcal{R} Q$, then by the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we know $P \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q$. So $\mathcal{R} \subseteq \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta$. If we prove that $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta \subseteq \underline{\leftrightarrow}_b^\Delta$ and that $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta$ is rooted with respect to the pairs from $\mathcal{R}$ we can conclude that $\mathcal{R} \subseteq \underline{\leftrightarrow}_{rb}^\Delta$.

Assume $P \mathcal{R} Q$, then we know by Condition $(U1)$ from Definition 4.2.1 that if $P \xrightarrow{a} P'$, then there exists a Q' such that $Q \xrightarrow{a} Q'$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$. So the root condition from Definition 3.0.4 holds for pairs in $\mathcal{R}$.

We continue with proving that $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta \subseteq \underline{\leftrightarrow}_b^\Delta$. Since both $\underline{\leftrightarrow}_b^\Delta$ and $\mathcal{R}$ are symmetric relations $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta$ is also symmetric. So it suffices to consider a pair (P, Q).
In order to prove that $\underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta \subseteq \underline{\leftrightarrow}_b^\Delta$ we need to show that, as stated in Definition 3.0.2, if $P \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q$ and $P \xrightarrow{a} P'$, then there exist states Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, $P \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q''$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.
We also need to show that, as stated in Theorem 3.0.2 if $P \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists an infinite sequence of states $(Q_l)_{l\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $Q = Q_0$, $Q_l \xrightarrow{\tau} Q_{l+1}$ and $P_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_l$ for all $l \in \omega$.

Assume $P \underline{\leftrightarrow}_b^{\Delta} S \mathcal{R} T \underline{\leftrightarrow}_b^{\Delta} Q$ and $P \xrightarrow{a} P'$. Then by Definition 3.0.2 we know $S \twoheadrightarrow S'' \xrightarrow{(a)} S'$, $P \underline{\leftrightarrow}_b^{\Delta} S''$ and $P' \underline{\leftrightarrow}_b^{\Delta} S'$. Since $S \mathcal{R} T$ we know by Definition 4.2.1 that $T \twoheadrightarrow T'' \xrightarrow{(a)} T'$, $S'' \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} T''$ and $S' \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} T'$. Since $T \underline{\leftrightarrow}_b^{\Delta} Q$ and $T \twoheadrightarrow T'' \xrightarrow{(a)} T'$ we know using Definition 3.0.2 that $Q \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, $T'' \underline{\leftrightarrow}_b^{\Delta} Q''$ and $T' \underline{\leftrightarrow}_b^{\Delta} Q'$. Using the transitivity of $\underline{\leftrightarrow}_b^{\Delta}$ we conclude that $P \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} Q''$ and $P' \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} Q'$.

Assume $P \underline{\leftrightarrow}_b^{\Delta} S \mathcal{R} T \underline{\leftrightarrow}_b^{\Delta} Q$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. By Theorem 3.0.2 we know that there is an infinite sequence of states $(S_l)_{l\in\omega}$ such that $S = S_0$ and $S_l \xrightarrow{\tau} S_{l+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $P_{\sigma(l)} \underline{\leftrightarrow}_b^{\Delta} S_l$ for all $l \in \omega$. Since $S \mathcal{R} T$ we know by Definition 4.2.1 that there exists an infinite sequence of states $(T_m)_{m\in\omega}$ such that $T = T_0$ and $T_m \xrightarrow{\tau} T_{m+1}$ and there exists a mapping $\upsilon : \omega \to \omega$ such that $S_{\upsilon(m)} \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} T_m$ for all $m \in \omega$. Since $T \underline{\leftrightarrow}_b^{\Delta} Q$ we know by Theorem 3.0.2 that there exists an infinite sequence of states $(Q_n)_{n\in\omega}$ and a mapping $\rho : \omega \to \omega$ such that $Q = Q_0$, $Q_n \xrightarrow{\tau} Q_{n+1}$ and $T_{\rho(n)} \underline{\leftrightarrow}_b^{\Delta} Q_n$ for all $n \in \omega$. From this we conclude that by the transitivity of $\underline{\leftrightarrow}_b^{\Delta}$ there is a mapping $\sigma\circ\upsilon\circ\rho : \omega \to \omega$ such that $P_{\sigma(\upsilon(\rho(n)))} \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} Q_n$ for all $n \in \omega$.

So $\underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} \subseteq \underline{\leftrightarrow}_b^{\Delta}$ and thus $\mathcal{R} \subseteq \underline{\leftrightarrow}_{rb}^{\Delta}$. □

In Lemma 4.2.4 we prove that if $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \underline{\leftrightarrow}_{rb}^{\Delta} F$, then the relation $\mathcal{R} = \{(G\{recX.E/X\}, G\{recX.F/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^{\Delta}$. If we take $G = X$ we can conclude that $recX.E \underline{\leftrightarrow}_{rb}^{\Delta} recX.F$. The hardest part of the proof is showing Condition ($U3$) from Definition 4.2.1, here we explain how we prove this.

In order to show Condition ($U3$) we have to show that if there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{recX.E/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists an infinite sequence of states $(Q_l)_{l\in\omega}$ such that $G\{recX.F/X\} = Q_0$ and $Q_l \xrightarrow{\tau} Q_{l+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $P_{\sigma(l)} \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} Q_l$ for all $l \in \omega$. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{recX.E/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$.

Then we have that either there is an infinite sequence starting from G, so there is an infinite sequence of states $(G_k)_{k\in\omega}$ such that $G = G_0$ and $G_k \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$. By Lemma A.0.3 we have that $G_k\{recX.F/X\} \xrightarrow{\tau} G_{k+1}\{recX.F/X\}$. This is shown in Figure 4.4. We have $G_k\{recX.E/X\} \mathcal{R} G_k\{recX.F/X\}$ by how $\mathcal{R}$ is defined and by the reflexivity of $\underline{\leftrightarrow}_b^{\Delta}$ we have $G_k\{recX.E/X\} \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} G_k\{recX.F/X\}$ for all $k \in \omega$.

It is also possible that $G \twoheadrightarrow G'$ such that X is totally unguarded in G'. Then the infinite sequence can continue from $recX.E$, since $G'\{recX.E/X\} \xrightarrow{\tau} E'\{recX.E/X\}$ if $recX.E \xrightarrow{\tau} E'\{recX.E/X\}$. If $recX.E \xrightarrow{\tau} E'\{recX.E/X\}$ then $recX.F \xrightarrow{\tau} F'\{recX.F/X\}$ since $E \underline{\leftrightarrow}_{rb}^{\Delta} F$ as shown in Figure 4.1. Since $G \twoheadrightarrow G'$ we have by Lemma A.0.3 that $G\{recX.F/X\} \twoheadrightarrow G'\{recX.F/X\}$ and since X is totally unguarded in G' we have $G'\{recX.F/X\} \xrightarrow{\tau} F'\{recX.F/X\}$. This is shown in Figure 4.5. We have $E'\{recX.E/X\} \mathcal{R} E'\{recX.F/X\}$ by how $\mathcal{R}$ is defined. Since $E' \underline{\leftrightarrow}_b^{\Delta} F'$ we have $E'\{recX.F/X\} \underline{\leftrightarrow}_b^{\Delta} F'\{recX.F/X\}$ by how $\underline{\leftrightarrow}_b^{\Delta}$ is lifted. Using the reflexivity of $\underline{\leftrightarrow}_b^{\Delta}$ we obtain $E'\{recX.E/X\} \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} F'\{recX.F/X\}$.

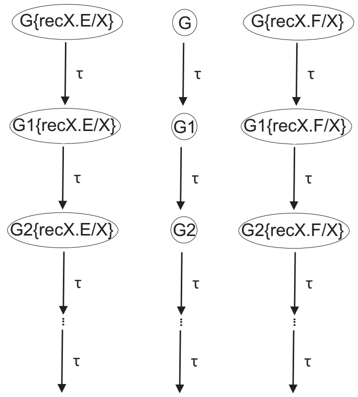

Figure 4.4: Proving Condition ($U3$)

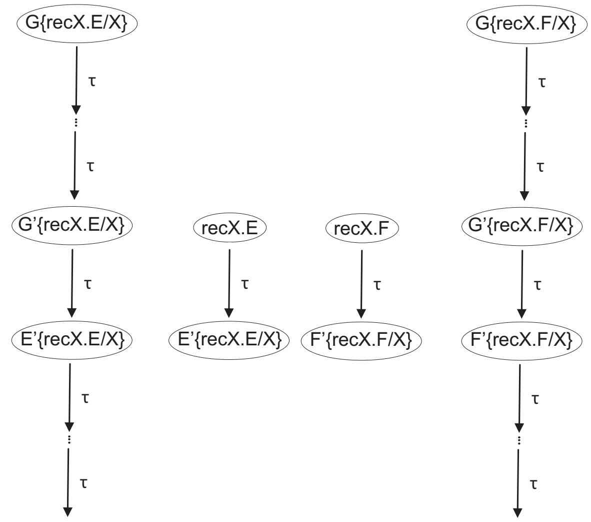

Figure 4.5: Proving Condition ($U3$)

In Lemma 4.2.4 it is shown that the relation $\mathcal{R} = \{(G\{recX.E/X\}, G\{recX.F/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$. Using this relation it follows in Lemma 4.2.5 that $E_1 \underline{\leftrightarrow}_{rb}^\Delta E_2$ implies $recX.E_1 \underline{\leftrightarrow}_{rb}^\Delta recX.E_2$ for $E_1, E_2 \in \mathbb{E}$, which is part of the proof that rooted branching bisimulation with explicit divergence is compatible with the defined grammar. When proving in Lemma 4.2.4 that Conditions ($U2$) and ($U3$) from Definition 4.2.1 hold we need to show for certain cases that it is possible to reach an expression where X is totally unguarded, in order to do this we use Lemma 4.2.2. When proving in Lemma 4.2.4 that Condition ($U3$) from Definition 4.2.1 holds we also need to show that if $E \underline{\leftrightarrow}_b^\Delta F$ and E can do zero or more τ-transitions, then F can also do zero or more τ-transitions, where each element in the sequence of τ-transitions is related. We use Lemma 4.2.3 to do this.

Lemma 4.2.2. *Let $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \underline{\leftrightarrow}_b^\Delta F$. If X is totally unguarded in E, then there exist $n \geq 0$ and $F_0, ..., F_n$ such that $F = F_0 \xrightarrow{\tau} ... \xrightarrow{\tau} F_n = F'$ where X is totally unguarded in F' and $E \underline{\leftrightarrow}_b^\Delta F_i$ for all $i \leq n$.*

Proof. Let $b \in A \backslash \{\tau\}$ and choose $n \in \omega$ such that for all derivatives G of E and F it holds that $G\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$. In Lemma B.0.9 it is proven that such n exists.
Since $E \underline{\leftrightarrow}_b^\Delta F$ also $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F\{b^n/X\}$ because of how the relation is lifted. Since X is totally unguarded in E and $b^n \xrightarrow{b} b^{n-1}$, by Lemma A.0.4 we have that $E\{b^n/X\} \xrightarrow{b} b^{n-1}$. Since $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F\{b^n/X\}$ also $F\{b^n/X\} \twoheadrightarrow F_1' \xrightarrow{b} F_2'$ such that $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F_1'$ and $b^{n-1} \underline{\leftrightarrow}_b^\Delta F_2'$. Since $b \neq \tau$ we have by Lemma A.0.5 that there exist $n \geq 0$ and $F_0, ..., F_n$ such that $F = F_0 \xrightarrow{\tau} ... \xrightarrow{\tau} F_n = F'$ and $F_1' = F'\{b^n/X\}$. Since $F'\{b^n/X\} \xrightarrow{b} F_2'$ by Lemma A.0.5 either X is totally unguarded in F' and $b^n \xrightarrow{b} F_2'$, or $F' \xrightarrow{b} F''$ and $F_2' = F''\{b^n/X\}$. Since $b^{n-1} \not\underline{\leftrightarrow}_b^\Delta F''\{b^n/X\}$ it must be the case that X is totally unguarded in F' and $b^n \xrightarrow{b} F_2'$. Since $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F'\{b^n/X\}$ by Lemma B.0.10 we have $E \underline{\leftrightarrow}_b^\Delta F'$. In Corollary 4.4 in [11] it is proven that $\underline{\leftrightarrow}_b^\Delta$ has the stuttering property. Since in Corollary 3.0.1 it is proven that the relation $\underline{\leftrightarrow}_b^\Delta$ satisfies Condition (T) from Definition 3.0.2 and Condition (D) from Definition 3.0.3 for $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ the stuttering property can be used. Since $E \underline{\leftrightarrow}_b^\Delta F$ and $E \underline{\leftrightarrow}_b^\Delta F'$ we obtain using the stuttering property that we have $E \underline{\leftrightarrow}_b^\Delta F_i$ for all $i \leq n$. So $F = F_0 \xrightarrow{\tau} ... \xrightarrow{\tau} F_n = F'$ such that X is totally unguarded in F' and $E \underline{\leftrightarrow}_b^\Delta F_i$ for all $i \leq n$. □

Lemma 4.2.3. *Let $E, F \in \mathbb{E}$. If $E \underline{\leftrightarrow}_b^\Delta F$ and $E = E_0 \xrightarrow{\tau} ... \xrightarrow{\tau} E_n$ for $n \geq 0$ then $F = F_0 \xrightarrow{\tau} ... \xrightarrow{\tau} F_m$ for $m \geq 0$, where for every i such that $0 \leq i \leq m$ there is a j such that $0 \leq j \leq n$, $E_j \underline{\leftrightarrow}_b^\Delta F_i$ and in particular $E_n \underline{\leftrightarrow}_b^\Delta F_m$.*

Proof. Assume $E \underline{\leftrightarrow}_b^\Delta F$ and $E = E_0 \xrightarrow{\tau} ... \xrightarrow{\tau} E_n$ for $n \geq 0$, then $F = F_0 \xrightarrow{\tau} ... \xrightarrow{\tau} F_m$ for $m \geq 0$ by Definition 3.0.2. We prove using induction on m that for every i such that $0 \leq i \leq m$ there is a j such that $0 \leq j \leq n$, $E_j \underline{\leftrightarrow}_b^\Delta F_i$ and $E_n \underline{\leftrightarrow}_b^\Delta F_m$.

For $m = 0$ we have $E \underline{\leftrightarrow}_b^\Delta F$.

Assume that for every i such that $0 \leq i \leq m$ there is a j such that $0 \leq j \leq n$, $E_j \underline{\leftrightarrow}_b^\Delta F_i$ and $E_n \underline{\leftrightarrow}_b^\Delta F_m$. Let $F_m \xrightarrow{\tau} F_{m+1}$. Using Definition 3.0.2 we obtain that $E_n \twoheadrightarrow E'' \xrightarrow{(\tau)} E'$, $F_m \underline{\leftrightarrow}_b^\Delta E''$ and $F_{m+1} \underline{\leftrightarrow}_b^\Delta E'$. Since $F_{m+1} \underline{\leftrightarrow}_b^\Delta E'$ we have for $i = m + 1$ that there is a j such that $E_j \underline{\leftrightarrow}_b^\Delta F_i$. □

Lemma 4.2.4. *Let $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \underline{\leftrightarrow}_{rb}^\Delta F$. Furthermore, let*

$$\mathcal{R} = \{(G\{recX.E/X\}, G\{recX.F/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$$

Then $\mathcal{R} \cup \mathcal{R}^{-1}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$.

Proof. To prove that $\mathcal{R} \cup \mathcal{R}^{-1}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$ we need to prove Conditions $(U1)$, $(U2)$ and $(U3)$ from Definition 4.2.1. Since $\mathcal{R} \cup \mathcal{R}^{-1}$ is symmetric it suffices to consider a pair $(G\{recX.E/X\}, G\{recX.F/X\})$.

In order to show Condition $(U1)$ we need to show that if $G\{recX.E/X\} \xrightarrow{a} P$, then $G\{recX.F/X\} \xrightarrow{a} Q$ and $P \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q$.

Assume $G\{recX.E/X\} \xrightarrow{a} P$, then by Lemma A.0.5 either X is totally unguarded in G and $recX.E \xrightarrow{a} P$ or $G \xrightarrow{a} G'$ and $P = G'\{recX.E/X\}$.
Case 1. Assume X is totally unguarded in G and $recX.E \xrightarrow{a} P$. Since $recX.E \xrightarrow{a} P$ we know by Lemma A.0.6 that $E \xrightarrow{a} E'$ and $P = E'\{recX.E/X\}$. Since $E \underline{\leftrightarrow}_{rb}^\Delta F$ by Lemma B.0.11 we have $F \xrightarrow{a} F'$ and $E' \underline{\leftrightarrow}_b^\Delta F'$. By Lemma A.0.3 we obtain $F\{recX.F/X\} \xrightarrow{a} F'\{recX.F/X\}$ and using the operational semantics we have $recX.F \xrightarrow{a} F'\{recX.F/X\}$. Since X is totally unguarded in G we have by Lemma A.0.4 that $G\{recX.F/X\} \xrightarrow{a} F'\{recX.F/X\}$. We have $E'\{recX.E/X\} \mathcal{R} E'\{recX.F/X\}$ by how $\mathcal{R}$ is defined. Since $E' \underline{\leftrightarrow}_b^\Delta F'$ we obtain $E'\{recX.F/X\} \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$ by how the relation is lifted. From this we have $E'\{recX.E/X\} \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$ and by the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $E'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$.
Case 2. Assume $G \xrightarrow{a} G'$ and $P = G'\{recX.E/X\}$. Since $G \xrightarrow{a} G'$ we have by Lemma A.0.3 that $G\{recX.F/X\} \xrightarrow{a} G'\{recX.F/X\}$. We obtain $G'\{recX.E/X\} \mathcal{R} G'\{recX.F/X\}$ by how $\mathcal{R}$ is defined. By the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we have $G'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta G'\{recX.F/X\}$.

To show Condition $(U2)$ we need to show that if $G\{recX.E/X\} \twoheadrightarrow P \xrightarrow{(a)} P'$ for some states P and P', then there exist states Q and Q' such that $G\{recX.F/X\} \twoheadrightarrow Q \xrightarrow{(a)} Q'$, $P \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.
Let $G\{recX.E/X\} = P'_0 \xrightarrow{\tau} ... \xrightarrow{\tau} P'_n$ for $n \geq 0$ and some states $(P'_i)_{0 \leq i \leq n}$. We prove using induction on n that $P'_n = P_n\{recX.E/X\}$ for some P_n and that there exist states $Q_0, ..., Q_m$ for $m \geq 0$ such that $G\{recX.F/X\} = Q_0\{recX.F/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} Q_m\{recX.F/X\}$ where $P_n \underline{\leftrightarrow}_b^\Delta Q_m$.
After proving this it remains to prove that if $P_n \underline{\leftrightarrow}_b^\Delta Q_m$ and $P_n\{recX.E/X\} \xrightarrow{(a)} P'$, then $Q_m\{recX.F/X\} \twoheadrightarrow Q \xrightarrow{(a)} Q'$, where $P_n\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.
From this we can conclude that Condition $(U2)$ holds.

For the base case we have $n = 0$. Take $P_0 = G = Q_0$. By the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we have $P_0 \underline{\leftrightarrow}_b^\Delta Q_0$.

Assume the condition holds for n. So $P'_n = P_n\{recX.E/X\}$, $G\{recX.F/X\} \twoheadrightarrow Q_m\{recX.F/X\}$, where $P_n \underline{\leftrightarrow}_b^\Delta Q_m$. Assume $P_n\{recX.E/X\} \xrightarrow{\tau} P'_{n+1}$, we obtain using Lemma A.0.5 that either X is totally unguarded in P_n and $recX.E \xrightarrow{\tau} P'_{n+1}$ or $P_n \xrightarrow{\tau} P_{n+1}$ and $P'_{n+1} = P_{n+1}\{recX.E/X\}$.
Case 1. Assume X is totally unguarded in P_n and $recX.E \xrightarrow{\tau} P'_{n+1}$. Since $recX.E \xrightarrow{\tau} P'_{n+1}$ using Lemma A.0.6 we obtain $E \xrightarrow{\tau} P_{n+1}$ and $P'_{n+1} = P_{n+1}\{recX.E/X\}$. We have $P_n \underline{\leftrightarrow}_b^\Delta Q_m$. Since X is totally unguarded in P_n by Lemma 4.2.2 we obtain $Q_m \twoheadrightarrow Q''$ such that X is totally unguarded in Q''. Since $Q_m \twoheadrightarrow Q''$ by Lemma A.0.3 we have $Q_m\{recX.F/X\} \twoheadrightarrow Q''\{recX.F/X\}$. Since $E \underline{\leftrightarrow}_{rb}^\Delta F$ and $E \xrightarrow{\tau} P_{n+1}$ using Lemma B.0.11 we obtain $F \xrightarrow{\tau} Q_{m+1}$ where $P_{n+1} \underline{\leftrightarrow}_b^\Delta Q_{m+1}$. Since $F \xrightarrow{\tau} Q_{m+1}$ also $F\{recX.F/X\} \xrightarrow{\tau} Q_{m+1}\{recX.F/X\}$ by Lemma A.0.3 and thus by the operational semantics $recX.F \xrightarrow{\tau} Q_{m+1}\{recX.F/X\}$. Since X is totally unguarded in Q'' we have by Lemma A.0.4 that $Q''\{recX.F/X\} \xrightarrow{\tau} Q_{m+1}\{recX.F/X\}$.
Case 2. Assume $P_n \xrightarrow{\tau} P_{n+1}$ and $P'_{n+1} = P_{n+1}\{recX.E/X\}$. Since $P_n \underline{\leftrightarrow}_b^\Delta Q_m$ we obtain $Q_m \twoheadrightarrow Q'_m \xrightarrow{(\tau)} Q_{m+1}$, where $P_n \underline{\leftrightarrow}_b^\Delta Q'_m$ and $P_{n+1} \underline{\leftrightarrow}_b^\Delta Q_{m+1}$. By Lemma A.0.3 we have $Q_m\{recX.F/X\} \twoheadrightarrow Q'_m\{recX.F/X\} \xrightarrow{(\tau)} Q_{m+1}\{recX.F/X\}$.

It remains to prove that if $P_n \underline{\leftrightarrow}_b^\Delta Q_m$ and $P_n\{recX.E/X\} \xrightarrow{(a)} P'$, then $Q_m\{recX.F/X\} \twoheadrightarrow Q \xrightarrow{(a)} Q'$, where $P_n\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$. Since $P_n\{recX.E/X\} \xrightarrow{(a)} P'$ either $a = \tau$ and $P' = P_n\{recX.E/X\}$ or $P_n\{recX.E/X\} \xrightarrow{a} P'$.
Case 1. Assume $a = \tau$ and $P' = P_n\{recX.E/X\}$. Then $P_n\{recX.E/X\}\ \mathcal{R}\ P_n\{recX.F/X\}$ because of how $\mathcal{R}$ is defined. Since $P_n \underline{\leftrightarrow}_b^\Delta Q_m$ we have $P_n\{recX.F/X\} \underline{\leftrightarrow}_b^\Delta Q_m\{recX.F/X\}$ by how the relation is lifted. So $P_n\{recX.E/X\}\ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_m\{recX.F/X\}$ and by the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $P_n\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_m\{recX.F/X\}$.
Case 2. Assume $P_n\{recX.E/X\} \xrightarrow{a} P'$. By Lemma A.0.5 we know that either X is totally unguarded in P_n and $recX.E \xrightarrow{a} P'$ or $P_n \xrightarrow{a} P_{n+1}$ and $P' = P_{n+1}\{recX.E/X\}$.

Case 2.1. Assume X is totally unguarded in P_n and $recX.E \xrightarrow{a} P'$. Since $recX.E \xrightarrow{a} P'$ by Lemma A.0.6 we have $E \xrightarrow{a} E'$ and $P' = E'\{recX.E/X\}$. So we have $P_n\{recX.E/X\} \xrightarrow{a} E'\{recX.E/X\}$. Since $E \underline{\leftrightarrow}_{rb}^\Delta F$ and $E \xrightarrow{a} E'$ by Lemma B.0.11 we have $F \xrightarrow{a} F'$ and $E' \underline{\leftrightarrow}_b^\Delta F'$. Since $F \xrightarrow{a} F'$ by Lemma A.0.3 $F\{recX.F/X\} \xrightarrow{a} F'\{recX.F/X\}$ and thus by the operational semantics $recX.F \xrightarrow{a} F'\{recX.F/X\}$. Since $P_n \underline{\leftrightarrow}_b^\Delta Q_m$ and X is totally unguarded in P_n by Lemma 4.2.2 we have that $Q_m \twoheadrightarrow Q'_m$ such that X is totally unguarded in Q'_m and $P_n \underline{\leftrightarrow}_b^\Delta Q'_m$. So by Lemma A.0.3 and Lemma A.0.4 we have $Q_m\{recX.F/X\} \twoheadrightarrow Q'_m\{recX.F/X\} \xrightarrow{a} F'\{recX.F/X\}$. Using the definition for $\mathcal{R}$ and how $\underline{\leftrightarrow}_b^\Delta$ is lifted we obtain $P_n\{recX.E/X\}\ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'_m\{recX.F/X\}$ and $E'\{recX.E/X\}\ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$ and by the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $P_n\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'_m\{recX.F/X\}$ and $E'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$.

Case 2.2. Assume $P_n \xrightarrow{a} P_{n+1}$ and $P' = P_{n+1}\{recX.E/X\}$. Since $P_n \underline{\leftrightarrow}_b^\Delta Q_m$ we know $Q_m \twoheadrightarrow Q''_m \xrightarrow{(a)} Q'_m$, where $P_n \underline{\leftrightarrow}_b^\Delta Q''_m$ and $P_{n+1} \underline{\leftrightarrow}_b^\Delta Q'_m$. By Lemma A.0.3 we have $Q_m\{recX.F/X\} \twoheadrightarrow Q''_m\{recX.F/X\} \xrightarrow{(a)} Q'_m\{recX.F/X\}$. Using the definition for $\mathcal{R}$ and how $\underline{\leftrightarrow}_b^\Delta$ is lifted we obtain $P_n\{recX.E/X\}\ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q''_m\{recX.F/X\}$ and $P_{n+1}\{recX.E/X\}\ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'_m\{recX.F/X\}$ and by the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $P_n\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q''_m\{recX.F/X\}$ and $P_{n+1}\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'_m\{recX.F/X\}$.

It remains to show that Condition $(U3)$ holds. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{recX.E/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. Then we need to show that there exists an infinite sequence of states $(Q_l)_{l\in\omega}$ such that $G\{recX.F/X\} = Q_0$, $Q_l \xrightarrow{\tau} Q_{l+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $P_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_l$ for all $l \in \omega$.

By Lemma A.0.8 we know that there exist states $(G_k)_{k\in\omega}$ such that $P_k = G_k\{recX.E/X\}$ for all $k \in \omega$. We define using induction on l an infinite sequence of states $(Q_l)_{l\in\omega}$ such that $G\{recX.F/X\} = Q_0$, $Q_l \xrightarrow{\tau} Q_{l+1}$, there exist states $(H_l)_{l\in\omega}$ such that $Q_l = H_l\{recX.E/X\}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $G_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta H_l$ and $P_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_l$ for all $l \in \omega$.

We define $H_0 = G$ and $\sigma(0) = 0$. By the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we have $G \underline{\leftrightarrow}_b^\Delta G$. By the definition of $\mathcal{R}$ and the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we have $G\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta G\{recX.F/X\}$.

Suppose the sequence $(Q_l)_{l\in\omega}$ and the mapping $\sigma : \omega \to \omega$ have been defined up-to l. Then $Q_l = H_l\{recX.E/X\}$ and $G_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta H_l$. The infinite sequence of states $(P_k)_{k\in\omega}$ contains $G_{\sigma(l)}\{recX.E/X\}$, so the infinite sequence continues from $G_{\sigma(l)}\{recX.E/X\}$. Using Lemma A.0.7 we obtain that either there is an infinite sequence of states $(R_m)_{m\in\omega}$ such that $G_{\sigma(l)} = R_0$, $R_m\{recX.E/X\} = P_{\sigma(l)+m}$ and $R_m \xrightarrow{\tau} R_{m+1}$ for all $m \in \omega$, or $G_{\sigma(l)} \twoheadrightarrow G'$ such that $P_i = G'\{recX.E/X\}$ for some $i \geq \sigma(l)$, X is totally unguarded in G' and there is an infinite sequence of states $(R_m)_{m\in\omega}$ such that $recX.E = R_0$, $R_m = P_{i+m}$ and $R_m \xrightarrow{\tau} R_{m+1}$ for all $m \in \omega$.
Case 1. Assume there is an infinite sequence of states $(R_m)_{m\in\omega}$ such that for all $m \in \omega$
$G_{\sigma(l)} = R_0$
$R_m\{recX.E/X\} = P_{\sigma(l)+m}$
$R_m \xrightarrow{\tau} R_{m+1}$
Since $G_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta H_l$, there is an infinite sequence of states $(S_n)_{n\in\omega}$ and a mapping $\rho : \omega \to \omega$ such that for all $n \in \omega$
$H_l = S_0$
$S_n \xrightarrow{\tau} S_{n+1}$
$R_{\rho(n)} \underline{\leftrightarrow}_b^\Delta S_n$
Since $S_n \xrightarrow{\tau} S_{n+1}$ by Lemma A.0.3 we have that $S_n\{recX.F/X\} \xrightarrow{\tau} S_{n+1}\{recX.F/X\}$. By the definition of $\mathcal{R}$ we have $R_{\rho(n)}\{recX.E/X\}\ \mathcal{R}\ R_{\rho(n)}\{recX.F/X\}$, since $R_{\rho(n)} \underline{\leftrightarrow}_b^\Delta S_n$ we have because of how the relation is lifted that $R_{\rho(n)}\{recX.F/X\} \underline{\leftrightarrow}_b^\Delta S_n\{recX.F/X\}$. So we have, using the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ that $R_{\rho(n)}\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta S_n\{recX.F/X\}$ for all $n \in \omega$. So for $n \geq l$ we have $\sigma(n) = \rho(n)$.
Case 2. Assume $G_{\sigma(l)} \twoheadrightarrow G'$ such that $P_i = G'\{recX.E/X\}$ for some $i \geq \sigma(l)$, X is totally unguarded in G' and there is an infinite sequence of states $(R_m)_{m\in\omega}$ such that for all $m \in \omega$
$recX.E = R_0$
$R_m = P_{i+m}$
$R_m \xrightarrow{\tau} R_{m+1}$
We will show that there will be at least one τ-step from H_l. We will do this by showing that from H_l it is possible to reach a state H_l'' such that X is totally unguarded in H_l''. From there a step from $recX.F$ can be done, since $E \underline{\leftrightarrow}_{rb}^\Delta F$ we know that at least one τ-step will be done from H_l''.

Since $G_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta H_l$ and $G_{\sigma(l)} \twoheadrightarrow G'$ we know by Lemma 4.2.3 that if $G_{\sigma(l)} = E_0' \xrightarrow{\tau} ... \xrightarrow{\tau} E_n' = G'$, then $H_l = F_0' \xrightarrow{\tau} ... \xrightarrow{\tau} F_m' = H_l'$ for $m \geq 0$ where for every i such that $0 \leq i \leq m$ there is a j such that $0 \leq j \leq n$ and $E_j' \underline{\leftrightarrow}_b^\Delta F_i'$ and in particular $G' \underline{\leftrightarrow}_b^\Delta H_l'$. We have since $P_{\sigma(l)} = G_{\sigma(l)}\{recX.E/X\}$, $P_i = G'\{recX.E/X\}$ and $\sigma(l) \leq i$ that $G_{\sigma(l)}\{recX.E/X\} \twoheadrightarrow G'\{recX.E/X\}$, since $H_l \twoheadrightarrow H_l'$ we know by Lemma A.0.3 that $H_l\{recX.F/X\} \twoheadrightarrow H_l'\{recX.F/X\}$. Since $E_j' \underline{\leftrightarrow}_b^\Delta F_i'$ we have by the definition of $\mathcal{R}$ and the way $\underline{\leftrightarrow}_b^\Delta$ is lifted that $E_j'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta F_i'\{recX.F/X\}$. Since X is totally unguarded in G' and $G' \underline{\leftrightarrow}_b^\Delta H_l'$ we have by Lemma 4.2.2 that $H_l' \twoheadrightarrow H_l''$ such that X is totally unguarded in H_l''. Let $F_0'', ..., F_o''$ be states such that $H_l' = F_0'' \xrightarrow{\tau} ... \xrightarrow{\tau} F_o'' = H_l''$, then $G' \underline{\leftrightarrow}_b^\Delta F_p''$ for all $p \leq o$. Since $H_l' \twoheadrightarrow H_l''$ we know by Lemma A.0.3 that $H_l'\{recX.F/X\} \twoheadrightarrow H_l''\{recX.F/X\}$. Since $G' \underline{\leftrightarrow}_b^\Delta F_p''$ we have by the definition of $\mathcal{R}$ and the way $\underline{\leftrightarrow}_b^\Delta$ is lifted that $G'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta F_p''\{recX.F/X\}$ for all $p \leq o$. Since X is totally unguarded in G' and $recX.E \xrightarrow{\tau} R_1$ we have by Lemma A.0.4 that $G'\{recX.E/X\} \xrightarrow{\tau} R_1$. Since $recX.E \xrightarrow{\tau} R_1$ we have by Lemma A.0.6 that $E \xrightarrow{\tau} E'$ and $R_1 = E'\{recX.E/X\}$. Since $E \underline{\leftrightarrow}_{rb}^\Delta F$ we have $F \xrightarrow{\tau} F'$ where $E' \underline{\leftrightarrow}_b^\Delta F'$. Since $F \xrightarrow{\tau} F'$

we have by Lemma A.0.3 that $F\{recX.F/X\} \xrightarrow{\tau} F'\{recX.F/X\}$ and by the operational semantics $recX.F \xrightarrow{\tau} F'\{recX.F/X\}$. Since X is totally unguarded in H_l'' we have by Lemma A.0.4 that $H_l''\{recX.F/X\} \xrightarrow{\tau} F'\{recX.F/X\}$. Since $E' \underline{\leftrightarrow}_b^\Delta F'$ we have by the definition of $\mathcal{R}$ and by how the relation $\underline{\leftrightarrow}_b^\Delta$ is lifted that $E'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta F'\{recX.F/X\}$. So $G_{\sigma(l)}\{recX.E/X\} \rightarrow^+ E'\{recX.E/X\}$ and $H_l\{recX.F/X\} \rightarrow^+ F'\{recX.F/X\}$. Furthermore, let $G_{\sigma(l)}\{recX.E/X\} = G_0'\{recX.E/X\} \xrightarrow{\tau}, ..., \xrightarrow{\tau} G_n'\{recX.E/X\} = E'\{recX.E/X\}$ and let $H_l\{recX.F/X\} = H_0'\{recX.F/X\} \xrightarrow{\tau}, ..., \xrightarrow{\tau} H_m'\{recX.F/X\} = F'\{recX.F/X\}$, then for every i such that $0 \leq i \leq m$ there is a j such that $0 \leq j \leq n$, $G_j' \underline{\leftrightarrow}_b^\Delta H_i'$ and $G_j'\{recX.E/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta H_i'\{recX.F/X\}$.

So we conclude that Conditions $(U1)$, $(U2)$ and $(U3)$ from Definition 4.2.1 all hold, so $\mathcal{R} \cup \mathcal{R}^{-1}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$. □

Lemma 4.2.5. *Rooted branching bisimulation with explicit divergence is compatible with the defined grammar.*

Proof. This is proven in Lemma C.0.13 in Appendix C. □

Corollary 4.2.1. *Rooted branching bisimulation with explicit divergence is a congruence.*

4.3 Coarsest congruence

A relation is coarser than another relation if it relates more elements. In Lemma 4.3.2 it is proven that $\underline{\leftrightarrow}_{rb}^\Delta$ is the coarsest congruence contained in $\underline{\leftrightarrow}_b^\Delta$. This means that it relates as much elements as possible without losing the congruence property, i.e., all pairs in the branching bisimulation with explicit divergence relation for which the congruence property holds are related in the rooted branching bisimulation with explicit divergence relation. To prove that $\underline{\leftrightarrow}_{rb}^\Delta$ is the coarsest congruence contained in $\underline{\leftrightarrow}_b^\Delta$ it needs to be proven that any congruence contained in $\underline{\leftrightarrow}_b^\Delta$ is a subset of $\underline{\leftrightarrow}_{rb}^\Delta$. To do this Lemma 4.3.1 is used. The proof is based on the proof for Theorem 2 in [1].

Lemma 4.3.1. *Let $P, Q \in \mathbb{P}$. If $P + R \underline{\leftrightarrow}_b^\Delta Q + R$ for all $R \in \mathbb{P}$, then $P \underline{\leftrightarrow}_{rb}^\Delta Q$.*

Proof. Assume $P + R \underline{\leftrightarrow}_b^\Delta Q + R$ for all $R \in \mathbb{P}$. In order to show that $P \underline{\leftrightarrow}_{rb}^\Delta Q$ we need to show that if $P \xrightarrow{a} P'$, then $Q \xrightarrow{a} Q'$, where $P' \underline{\leftrightarrow}_b^\Delta Q'$.

Assume $P \xrightarrow{a} P'$, then by the operational semantics we have $P + R \xrightarrow{a} P'$. Let $R = b^n$ such that $b \in A\backslash\{\tau\}$ and $n \in \omega$ such that for all derivatives G of P and Q we have $G \not\underline{\leftrightarrow}_b^\Delta b^n$ and $G \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$, such an n exists as proven in Lemma B.0.9. Since $P + R \underline{\leftrightarrow}_b^\Delta Q + R$ we have $Q + R \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, where $P + R \underline{\leftrightarrow}_b^\Delta Q''$ and $P' \underline{\leftrightarrow}_b^\Delta Q'$. We have the following cases:
Case 1. Let $a = \tau$ and $Q'' = Q'$. Then $Q + R \twoheadrightarrow Q'$ where $P + R \underline{\leftrightarrow}_b^\Delta Q'$ and $P' \underline{\leftrightarrow}_b^\Delta Q'$. By the transitivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $P + R \underline{\leftrightarrow}_b^\Delta P'$. But R is chosen in such a way that for all derivatives G of P and Q we have $G \not\underline{\leftrightarrow}_b^\Delta R$. So $P + R \not\underline{\leftrightarrow}_b^\Delta P'$. So it cannot be the case that $a = \tau$ and $Q'' = Q'$.
Case 2. Let $a = b$ and $R \twoheadrightarrow Q'' \xrightarrow{b} Q'$. Since $b \neq \tau$ we have $R \xrightarrow{b} Q'$, where $Q' = b^{n-1}$. But n is chosen in such a way that for all derivatives G of P and Q we have $G \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$. So $P' \not\underline{\leftrightarrow}_b^\Delta Q'$. So it cannot be the case that $a = b$ and $R \twoheadrightarrow Q'' \xrightarrow{b} Q'$.
Case 3. Let $Q \twoheadrightarrow Q'' \xrightarrow{a} Q'$, where $P + R \underline{\leftrightarrow}_b^\Delta Q''$ and $P' \underline{\leftrightarrow}_b^\Delta Q'$. But n is chosen in such a way that for all derivatives G of P and Q we have $G \not\underline{\leftrightarrow}_b^\Delta b^n$, so it cannot be the case that $P + R \underline{\leftrightarrow}_b^\Delta Q''$ for Q'' a derivative of Q. So $Q = Q''$ and $Q \xrightarrow{a} Q'$, where $P' \not\underline{\leftrightarrow}_b^\Delta Q'$. From this we can conclude that $P \underline{\leftrightarrow}_{rb}^\Delta Q$.

□

Lemma 4.3.2. *The relation* $\underline{\leftrightarrow}_{rb}^{\Delta}$ *is the coarsest congruence contained in* $\underline{\leftrightarrow}_{b}^{\Delta}$ *with respect to the operators of* $\mathbb{E}$*.*

Proof. Assume $\mathcal{R} \subseteq \underline{\leftrightarrow}_{b}^{\Delta}$ is a congruence with respect to the operators of $\mathbb{E}$. Let $(P, Q) \in \mathcal{R}$, thus for all $R \in \mathbb{P}$, $P + R \mathrel{\mathcal{R}} Q + R$, i.e., $P + R \mathrel{\underline{\leftrightarrow}_{b}^{\Delta}} Q + R$. By Lemma 4.3.1 $P \mathrel{\underline{\leftrightarrow}_{rb}^{\Delta}} Q$. Thus, $\mathcal{R} \subseteq \underline{\leftrightarrow}_{rb}^{\Delta}$. □

Chapter 5

Axioms

In this section we introduce the axioms for the relation $\underline{\leftrightarrow}^{\Delta}_{rb}$. In Section 6 we prove soundness of these axioms. In Section 7 completeness is proven. The axioms are listed in Table 5.1. The axioms S1–S4 and rec1–rec4 are standard [5]. Axiom rec6 is taken from [1], axioms B and rec7–rec8 are taken from [8]. Axiom D1 is a variation on axiom B from [8] and axiom rec9 is a variation on axiom R5 in [5]. Rules for reflexivity, symmetry, transitivity, and substitutivity of equality are omitted.

S1:	$E + F = F + E$
S2:	$E + (F + G) = (E + F) + G$
S3:	$E + E = E$
S4:	$E + 0 = E$
B:	$a.(\tau.(E + F) + E) = a.(E + F)$
D1:	$a.\Delta(\tau.\Delta(E + F) + F) = a.\Delta(E + F)$
rec1:	if Y is not free in $recX.E$ then $recX.E = recY.(E\{Y/X\})$
rec2:	$recX.E = E\{recX.E/X\}$
rec3:	if X is guarded in E and $F = E\{F/X\}$ then $F = recX.E$
rec4:	$recX.(X + E) = recX.E$
rec5:	$recX.(\tau.(X + E) + F) = recX.(\tau.\Delta(E + F) + F)$
rec6:	$recX.(\Delta(X + E) + F) = recX.\Delta(E + F)$
rec7:	$recX.(\tau.(X + E) + \tau.(X + F) + G) = recX.(\tau.(X + E + F) + G)$
rec8:	if X is unguarded in E then $recX.(\tau.(\tau.E + F) + G) = recX.(\tau.(E + F) + G)$
rec9:	if X is unguarded in E then $recX.(\Delta(E) + F) = recX.(\tau.X + E + F)$
rec10:	if X is unguarded in E then $recX.(\tau.(\Delta(E) + F) + G) = recX.(\tau.(X + E + F) + G)$

Table 5.1: Axioms

Part of the completeness proof consists of proving that every expression can be transformed into a guarded expression (Lemma 7.1.2), this is done using the axioms. In order to do this we need axioms to eliminate unguarded occurrences of X within a $recX$-operator (axioms rec4, rec5, rec6), we also need axioms to regroup an expression such that for instance all unguarded occurrences of X occur within the scope of the same τ-prefix (axiom rec7), lastly we need axioms for when an unguarded occurrence of X occurs inside the scope of several τ-prefixes and Δ-operators (rec8, rec9, rec10).

5.1 Axioms from axiomatising divergence preserving weak congruence

In [1] an axiomatisation for divergence preserving weak congruence is given. Axiom rec6 is taken from [1]. Here we explain why other divergence preserving axioms from [1] are not sound for rooted branching bisimulation with explicit divergence.

Axiom rec5 from [1]: $recX.(\tau.(X + E) + F) = recX.\Delta(E + F)$ is not sound for rooted branching bisimulation with explicit divergence since we use a different root condition than used in [1]. Let $P = recX.(\tau.(X + E) + F)$ and $Q = recX.\Delta(E + F)$ and let $E \to E'$ for some E'. Then $Q \to E'\{Q/X\}$, while $P \xrightarrow{\tau} P + E\{P/X\} \to E'\{P/X\}$. From P a transition from E can only be done after first doing a τ-transition, while from Q it is possible to immediately do a transition from E.

In [1] some other axioms are given which distinguish the different properties which are investigated in [1]. One axiom is sound for a divergent weak bisimulation, this axiom is however not sound for rooted branching bisimulation with explicit divergence. Here we explain why it is not sound.

$$\Delta(\Delta(E) + F) = \tau.(\Delta(E) + F)$$

Let $E = a.0$ and $F = b.0$ where $a \neq b \neq \tau$. Then by the operational semantics $\Delta(\Delta(a.0) + b.0) \xrightarrow{\tau} \Delta(\Delta(a.0)+b.0)$ so there is an infinite sequence of τ-steps starting from $\Delta(\Delta(a.0)+b.0)$. According to Theorem 3.0.1 if $\Delta(\Delta(a.0) + b.0) \underline{\leftrightarrow}_b^{\Delta} \tau(\Delta(a.0) + b.0)$ we have that $\tau(\Delta(a.0) + b.0) \xrightarrow{\tau} Q$ for some Q such that $\Delta(\Delta(a.0)+b.0) \underline{\leftrightarrow}_b^{\Delta} Q$. By the operational semantics we have $\tau(\Delta(a.0)+b.0) \xrightarrow{\tau} \Delta(a.0)+b.0$. In order for $\Delta(\Delta(a.0)+b.0) \underline{\leftrightarrow}_b^{\Delta} \Delta(a.0)+b.0$ it should be that $\Delta(a.0)+b.0 \xrightarrow{\tau} Q'$ for some Q' such that $\Delta(\Delta(a.0) + b.0) \underline{\leftrightarrow}_b^{\Delta} Q'$. By the operational semantics $\Delta(a.0) + b.0 \xrightarrow{\tau} \Delta(a.0)$, however $\Delta(\Delta(a.0) + b.0) \not\underline{\leftrightarrow}_b^{\Delta} \Delta(a.0)$ because from $\Delta(\Delta(a.0) + b.0)$ it is possible to do a b-step while this is not possible from $\Delta(a.0)$. So $\Delta(\Delta(E) + F) \not\underline{\leftrightarrow}_{rb}^{\Delta} \tau.(\Delta(E) + F)$.

5.2 Derived laws

A few useful laws can be derived using the axioms, these are listed in Table 5.2.

rec11:	$recX.(\tau.X + E) = recX.\Delta(E)$
D2:	$\Delta(E) = \tau.\Delta(E) + E$
D3:	$\Delta(\Delta(E)) = \Delta(E)$

Table 5.2: Derived laws

Using the axioms it is possible to derive $recX.(\tau.X + E) = recX.\Delta(E)$ as follows:

$$\begin{aligned} recX.(\tau.X + E) &\overset{rec4}{=} recX.(\tau.X + X + E) \\ &\overset{rec9}{=} recX.(\Delta(X) + E) \\ &\overset{S4}{=} recX.(\Delta(X + 0) + E) \\ &\overset{rec6}{=} recX.\Delta(0 + E) \\ &\overset{S4}{=} recX.\Delta(E) \end{aligned}$$

Using the axioms it is possible to derive $\Delta(E) = \tau.\Delta(E) + E$ (this is law ($\Uparrow$2) in [1]) as follows, where $X \in \mathbb{V} \backslash \mathbb{V}(E)$:

$$\begin{aligned}
\Delta(E) &\stackrel{rec2}{=} recX.\Delta(E) \\
&\stackrel{rec11}{=} recX.(\tau.X + E) \\
&\stackrel{rec2}{=} \tau.recX.(\tau.X + E) + E \\
&\stackrel{rec11}{=} \tau.recX.\Delta(E) + E \\
&\stackrel{rec2}{=} \tau.\Delta(E) + E
\end{aligned}$$

Using the axioms it is possible to derive $\Delta(\Delta(E)) = \Delta(E)$ as follows:

$$\begin{aligned}
\Delta(\Delta(E)) &\stackrel{D2}{=} \tau.\Delta(\Delta(E)) + \Delta(E) \\
&\stackrel{D2}{=} \tau.\Delta(\tau.\Delta(E) + E) + \Delta(E) \\
&\stackrel{S4}{=} \tau.\Delta(\tau.\Delta(0 + E) + E) + \Delta(E) \\
&\stackrel{D1}{=} \tau.\Delta(0 + E) + \Delta(E) \\
&\stackrel{S4}{=} \tau.\Delta(E) + \Delta(E) \\
&\stackrel{D2}{=} \tau.\Delta(E) + \tau.\Delta(E) + E \\
&\stackrel{S4}{=} \tau.\Delta(E) + E \\
&\stackrel{D2}{=} \Delta(E)
\end{aligned}$$

Chapter 6

Soundness

In order to show soundness for the axioms in Section 5 we have to show for each axiom that there exists a rooted branching bisimulation with explicit divergence between the left-hand and right-hand side. This is shown in Lemma 6.0.1.

Lemma 6.0.1. *If $E, F \in \mathbb{E}$ and $E = F$ then $E \mathrel{\underline{\leftrightarrow}}^{\Delta}_{rb} F$.*

Proof. The relation $\underline{\leftrightarrow}^{\Delta}_{rb}$ is defined for closed expressions and is lifted from $\mathbb{P}$ to $\mathbb{E}$ such that $E \mathrel{\underline{\leftrightarrow}}^{\Delta}_{rb} F$ if $E\{\overrightarrow{P}/\overrightarrow{X}\} \mathrel{\underline{\leftrightarrow}}^{\Delta}_{rb} F\{\overrightarrow{P}/\overrightarrow{X}\}$, where $\overrightarrow{P} = (P_1, ..., P_n)$ with $P_i \in \mathbb{P}$ and $\overrightarrow{X}$ contains all variables in $\mathbb{V}(E) \cup \mathbb{V}(F)$. Since $E\{\overrightarrow{P}/\overrightarrow{X}\}$ and $F\{\overrightarrow{P}/\overrightarrow{X}\}$ are both in $\mathbb{P}$ it suffices to check soundness of the axioms only for $\mathbb{P}$, this implies soundness for expressions in $\mathbb{E}$.

The axioms S1–S4 and rec1–rec4 are sound for strong bisimulation as shown in [3], it is not hard to see that $\underline{\leftrightarrow} \subseteq \underline{\leftrightarrow}^{\Delta}_{rb}$, so they are also sound for $\underline{\leftrightarrow}^{\Delta}_{rb}$, except for axiom rec3. Axiom rec3 has the form of an implication so soundness for $\underline{\leftrightarrow}$ does not imply soundness for $\underline{\leftrightarrow}^{\Delta}_{rb}$. The soundness of axiom rec3 is proven in Lemma 6.1.1.
We prove soundness for axioms B and D1 in Lemmas D.1.1, and D.1.2 respectively.
We prove soundness for axioms rec5 and rec8 in Lemmas 6.2.1 and 6.3.1 respectively. The soundness proofs for axioms rec6, rec7, rec9 and rec10 are shifted to Appendix D, they are proven in Lemmas D.2.1, D.2.3, D.2.5 and D.2.7 respectively.

So for all axioms it is proven that if $P = Q$ then $P \mathrel{\underline{\leftrightarrow}}^{\Delta}_{rb} Q$. □

We explain for the axioms D1, rec5 and rec10 why they are sound. Soundness for axiom D1 is proven in Lemma D.1.2, soundness for axiom rec5 is proven in Lemma 6.2.1 and soundness for rec10 is proven in Lemma D.2.7.

In Figure 6.1 the labelled transition systems rooted at $a.\Delta(\tau.\Delta(E+F)+F)$ and $a.\Delta(E+F)$ are shown. Here we assume $E \to E'$ and $F \to F'$. Obviously $\Delta(E+F) \mathrel{\underline{\leftrightarrow}}^{\Delta}_{b} \Delta(E+F)$. From the figure it is clear that also $\Delta(\tau.\Delta(E+F)+F) \mathrel{\underline{\leftrightarrow}}^{\Delta}_{b} \Delta(E+F)$, from both an infinite amount of τ-steps can be done, some τ-steps can be done after which E' can be reached and some τ-steps can be done after which F' can be reached. So $a.\Delta(\tau.\Delta(E+F)+F) \mathrel{\underline{\leftrightarrow}}^{\Delta}_{rb} a.\Delta(E+F)$.

Note that $\Delta(\tau.\Delta(E+F)+F) \mathrel{\not\!\underline{\leftrightarrow}}^{\Delta}_{rb} \Delta(E+F)$, since from $\Delta(\tau.\Delta(E+F)+F)$ a transition from E can only be done after first doing a τ-transition, while from $\Delta(E+F)$ it is possible to immediately do a transition from E. So the a-prefix is needed to satisfy the root condition.

Let $P = recX.(\tau.(X+E)+F)$ and $Q = recX.(\tau.\Delta(E+F)+F)$. Then Figure 6.2 shows the labelled transition systems rooted at P and Q. Here we assume $E \to E'$ and $F \to F'$. From the figure it is clear that $recX.(\tau.(X+E)+F) \mathrel{\underline{\leftrightarrow}}^{\Delta}_{rb} recX.(\tau.\Delta(E+F)+F)$, since the two labelled transition systems have the same structure.

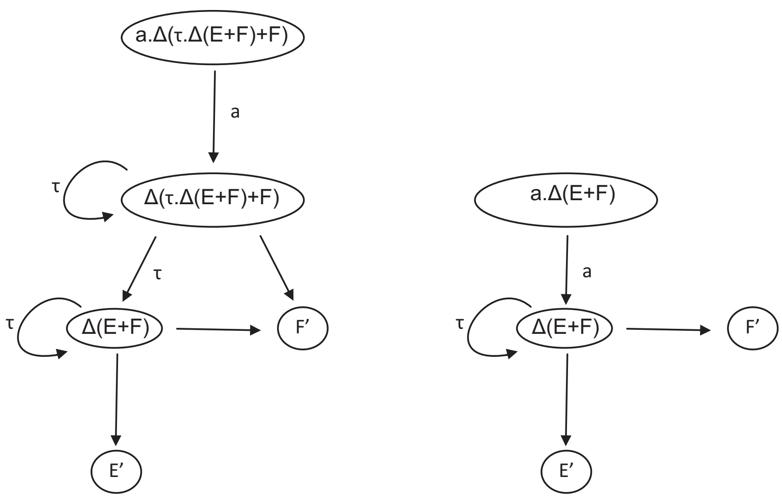

Figure 6.1: Soundness for axiom D1

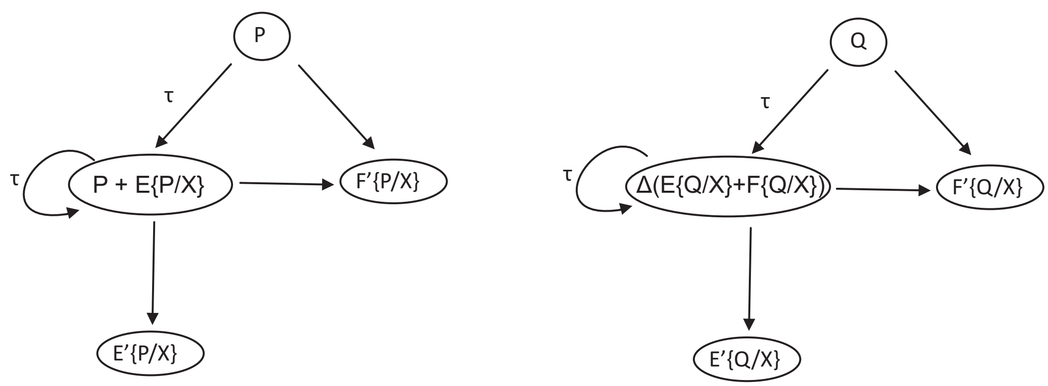

Figure 6.2: Soundness for axiom rec5

Let $P = recX.(\tau.(\Delta(E)+F)+G)$ and $Q = recX.(\tau.(X+E+F)+G)$. Then Figure 6.3 shows the labelled transition systems rooted at P and Q. Here we assume $E \rightarrow E'$, $F \rightarrow F'$ and $G \rightarrow G'$. Since X is unguarded in E we have that $E \twoheadrightarrow E''$, where X is totally unguarded in E''. So $E\{P/X\} \twoheadrightarrow E''\{P/X\} \rightarrow G'\{P/X\}$ and $E\{P/X\} \twoheadrightarrow E''\{P/X\} \rightarrow F'\{P/X\}$. So $\Delta(E\{P/X\}) + F\{P/X\} \underline{\leftrightarrow}_b^\Delta Q + E\{Q/X\} + F\{Q/X\}$, since from both an infinite amount of τ-steps can be done, some τ-steps can be done after which E' can be reached, some τ-steps can be done after which F' can be reached and some τ-steps can be done after which G' can be reached (note that if X would not be unguarded in E then G' might not be reachable from $\Delta(E\{P/X\}) + F\{P/X\})$. We also have $\Delta(E\{P/X\}) \underline{\leftrightarrow}_b^\Delta Q + E\{Q/X\} + F\{Q/X\}$, since from $E''\{P/X\}$ it is possible to reach $G'\{P/X\}$ and $F'\{P/X\}$. So $recX.(\tau.(\Delta(E)+F)+G) \underline{\leftrightarrow}_{rb}^\Delta recX.(\tau.(X+E+F)+G)$, provided X is unguarded in E.

In Sections 6.1, 6.2 and 6.3 we prove soundness for axioms rec3, rec5 and rec8 respectively. The rest of the soundness proofs are shifted to Appendix D since they are either straightforward or are similar to proving soundness for axioms rec5 or rec8.

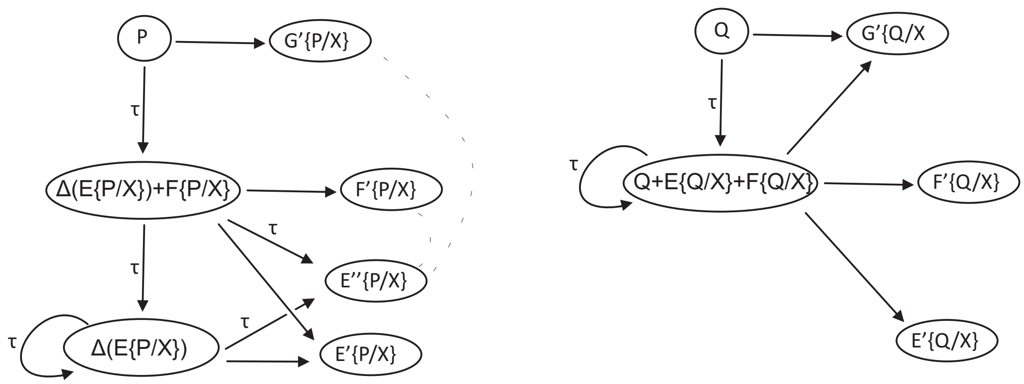

Figure 6.3: Soundness for axiom rec10

6.1 Soundness for rec3

Soundness for axiom rec3 is proven in Lemma 6.1.1. In order to prove soundness we use Lemma 6.1.2, in this lemma we use the rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^{\Delta}$ relation as defined in Definition 4.2.1. The proof is based on the proof for axiom rec3 in [1] and the proof on page 159 of [4].

Lemma 6.1.1. *Let $E \in \mathbb{E}$ and $P \in \mathbb{P}$. If X is guarded in E, $\mathbb{V}(E) \subseteq \{X\}$ and $P \underline{\leftrightarrow}_{rb}^{\Delta} E\{P/X\}$, then $P \underline{\leftrightarrow}_{rb}^{\Delta} recX.E$.*

Proof. By choosing $G = X$ in the relation in Lemma 6.1.2 we obtain $P \underline{\leftrightarrow}_{rb}^{\Delta} Q$ for $Q \in \mathbb{P}$ such that $Q \underline{\leftrightarrow}_{rb}^{\Delta} E\{Q/X\}$. Let $Q = recX.E$, then by axioma rec2 we have that $recX.E \underline{\leftrightarrow}_{rb}^{\Delta} E\{recX.E/X\}$, so we obtain $P \underline{\leftrightarrow}_{rb}^{\Delta} recX.E$. □

Lemma 6.1.2. *Let $E \in \mathbb{E}$ and $P, Q \in \mathbb{P}$. Let X be guarded in E, $\mathbb{V}(E) \subseteq \{X\}$, $P \underline{\leftrightarrow}_{rb}^{\Delta} E\{P/X\}$ and $Q \underline{\leftrightarrow}_{rb}^{\Delta} E\{Q/X\}$. Furthermore, let*

$$\mathcal{R} = \{(G\{P/X\}, G\{Q/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$$

Then $R \cup R^{-1}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^{\Delta}$.

Proof. To prove that $\mathcal{R} \cup \mathcal{R}^{-1}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^{\Delta}$ we need to prove Conditions $(U1)$, $(U2)$ and $(U3)$ from Definition 4.2.1. Since $\mathcal{R} \cup \mathcal{R}^{-1}$ is symmetric it suffices to consider a pair $(G\{P/X\}, G\{Q/X\})$.

To prove Condition $(U1)$ we have to prove that if $G\{P/X\} \xrightarrow{a} P'$, then $G\{Q/X\} \xrightarrow{a} Q'$ such that $P' \underline{\leftrightarrow}_b^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^{\Delta} Q'$.

Assume $G\{P/X\} \xrightarrow{a} P'$. Then by Lemma A.0.5 either X is totally unguarded in G and $P \xrightarrow{a} P'$ or $G \xrightarrow{a} G'$ and $P' = G'\{P/X\}$.

Case 1. Let X be totally unguarded in G and $P \xrightarrow{a} P'$. Since $P \underline{\leftrightarrow}_{rb}^{\Delta} E\{P/X\}$ we obtain $E\{P/X\} \xrightarrow{a} E'$ with $P' \underline{\leftrightarrow}_b^{\Delta} E'$. Since X is guarded in E we obtain $E \xrightarrow{a} E_1$ and $E' = E_1\{P/X\}$ using Lemma A.0.5. Since $E \xrightarrow{a} E_1$ by Lemma A.0.3 we have $E\{Q/X\} \xrightarrow{a} E_1\{Q/X\}$, where by the definition of $\mathcal{R}$ we have $E_1\{P/X\}\ \mathcal{R}\ E_1\{Q/X\}$. Since $Q \underline{\leftrightarrow}_{rb}^{\Delta} E\{Q/X\}$ also $Q \xrightarrow{a} Q'$ where

$E_1\{Q/X\} \underline{\leftrightarrow}_b^\Delta Q'$. Since X is totally unguarded in G by Lemma A.0.4 $G\{Q/X\} \xrightarrow{a} Q'$. We now have $G\{P/X\} \xrightarrow{a} P'$, $G\{Q/X\} \xrightarrow{a} Q'$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.
Case 2. Let $G \xrightarrow{a} G'$ and $P' = G'\{P/X\}$. Since $G \xrightarrow{a} G'$ by Lemma A.0.3 $G\{Q/X\} \xrightarrow{a} G'\{Q/X\}$. Since $\mathbb{V}(G) \subseteq \{X\}$ and $G \xrightarrow{a} G'$ also $\mathbb{V}(G') \subseteq \{X\}$, so $G'\{P/X\} \mathcal{R} G'\{Q/X\}$ by how $\mathcal{R}$ is defined. By the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $G'\{P/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta G'\{Q/X\}$.

To prove Condition $(U2)$ we have to prove that if $G\{P/X\} \twoheadrightarrow P'' \xrightarrow{(a)} P'$ for some states P' and P'', then there exist states Q' and Q'' such that $G\{Q/X\} \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, $P'' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q''$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.

Assume $G\{P/X\} \twoheadrightarrow P'' \xrightarrow{(a)} P'$. Then either $G \twoheadrightarrow G'' \xrightarrow{(a)} G'$, $P'' = G''\{P/X\}$ and $P' = G'\{P/X\}$ or $G \twoheadrightarrow G'$ where X is totally unguarded in G' and $P \twoheadrightarrow P'' \xrightarrow{(a)} P'$.
Case 1. Assume $G \twoheadrightarrow G'' \xrightarrow{(a)} G'$, $P'' = G''\{P/X\}$ and $P' = G'\{P/X\}$. Since $G \twoheadrightarrow G'' \xrightarrow{(a)} G'$ by Lemma A.0.3 also $G\{Q/X\} \twoheadrightarrow G''\{Q/X\} \xrightarrow{(a)} G'\{Q/X\}$. We have $G''\{P/X\} \mathcal{R} G''\{Q/X\}$ because of how $\mathcal{R}$ is defined. By the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $G''\{P/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta G''\{Q/X\}$. We have $G'\{P/X\} \mathcal{R} G'\{Q/X\}$ because of how $\mathcal{R}$ is defined. By the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ we obtain $G'\{P/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta G'\{Q/X\}$.
Case 2. Assume $G \twoheadrightarrow G'$ where X is totally unguarded in G' and $P \twoheadrightarrow P'' \xrightarrow{(a)} P'$. Since $P \underline{\leftrightarrow}_{rb}^\Delta E\{P/X\}$ we have $E\{P/X\} \twoheadrightarrow E'' \xrightarrow{(a)} E'$ where $P'' \underline{\leftrightarrow}_b^\Delta E''$ and $P' \underline{\leftrightarrow}_b^\Delta E'$. Since X is guarded in E doing a τ-step from E to E_1 will result in X also being guarded in E_1. So $E \twoheadrightarrow E_1 \xrightarrow{(a)} E_2$, where $E'' = E_1\{P/X\}$ and $E' = E_2\{P/X\}$. Since $E \twoheadrightarrow E_1 \xrightarrow{(a)} E_2$ also $E\{Q/X\} \twoheadrightarrow E_1\{Q/X\} \xrightarrow{(a)} E_2\{Q/X\}$, where $E_1\{P/X\} \mathcal{R} E_1\{Q/X\}$ and $E_2\{P/X\} \mathcal{R} E_2\{Q/X\}$ by how $\mathcal{R}$ is defined. Since $Q \underline{\leftrightarrow}_{rb}^\Delta E\{Q/X\}$ we have $Q \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, where $E_1\{Q/X\} \underline{\leftrightarrow}_b^\Delta Q''$ and $E_2\{Q/X\} \underline{\leftrightarrow}_b^\Delta Q'$. Since $G \twoheadrightarrow G'$ we have $G\{Q/X\} \twoheadrightarrow G'\{Q/X\}$ by Lemma A.0.3. Since X is totally unguarded in G' we have $G'\{Q/X\} \twoheadrightarrow Q'' \xrightarrow{(a)} Q'$, where $P'' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q''$ and $P' \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q'$.

To prove Condition $(U3)$ we have to prove that if there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{P/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists an infinite sequence of states $(Q_l)_{l\in\omega}$ such that $G\{Q/X\} = Q_0$ and $Q_l \xrightarrow{\tau} Q_{l+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $P_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_l$ for all $l \in \omega$.

Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{P/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. By Lemma A.0.7 there is an infinite sequence of states $(G_k)_{k\in\omega}$ such that $G = G_0$, $G_k\{P/X\} = P_k$ and $G_k \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$ or $G \twoheadrightarrow H$ such that $P_i = H\{E/X\}$ for some $i \geq 0$, X is totally unguarded in H and there is an infinite sequence of states $(R_m)_{m\in\omega}$ such that $P = R_0$, $R_l = P_{i+l}$ and $R_m \xrightarrow{\tau} R_{m+1}$ for all $m \in \omega$.
Case 1. Let there be an infinite sequence of states $(G_k)_{k\in\omega}$ such that $G = G_0$, $G_k\{P/X\} = P_k$ and $G_k \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$. Since $G_k \xrightarrow{\tau} G_{k+1}$, by Lemma A.0.3 $G_k\{Q/X\} \xrightarrow{\tau} G_{k+1}\{Q/X\}$. Since $G = G_0$ and $\mathbb{V}(G) \subseteq \{X\}$ also $\mathbb{V}(G_k) \subseteq \{X\}$ for all $k \in \omega$. So for all $k \in \omega$ we have $G_k\{P/X\} \mathcal{R} G_k\{Q/X\}$ by how $\mathcal{R}$ is defined. By the reflexivity of $\underline{\leftrightarrow}_b^\Delta$ also $G_k\{P/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta G_k\{Q/X\}$ for all $k \in \omega$. So there is an infinite sequence of states $(Q_l)_{l\in\omega}$ such that $G\{Q/X\} = Q_0$ and $Q_l \xrightarrow{\tau} Q_{l+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $P_{\sigma(l)} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta Q_l$ for all $l \in \omega$.
Case 2. Let $G \twoheadrightarrow H$ such that $P_i = H\{P/X\}$ for some $i \geq 0$, X is totally unguarded in H and there is an infinite sequence of states $(R_m)_{m\in\omega}$ such that $P = R_0$, $R_l = P_{i+l}$ and $R_m \xrightarrow{\tau} R_{m+1}$ for all $m \in \omega$. Since $G \twoheadrightarrow H$ we have $G\{Q/X\} \twoheadrightarrow H\{Q/X\}$. We have for $j \leq i$ that $P_j = G_j\{P/X\}$, for some G_j. Since $G_j\{P/X\} \mathcal{R} G_j\{Q/X\}$ also $G_j\{P/X\} \underline{\leftrightarrow}_b^\Delta \circ \mathcal{R} \circ \underline{\leftrightarrow}_b^\Delta G_j\{Q/X\}$. So the map-

ping is defined until i. Since $H\{P/X\} \xrightarrow{\tau} P'$ and X is totally unguarded in H by Lemma A.0.5 we obtain that $P \xrightarrow{\tau} P'$. Since $P \underline{\leftrightarrow}_{rb}^{\Delta} E\{P/X\}$ and X is guarded in E, also $E\{P/X\} \xrightarrow{\tau} E_1\{P/X\}$, for $E \xrightarrow{\tau} E_1$ and $P' \underline{\leftrightarrow}_{b}^{\Delta} E_1\{P/X\}$. Since $P' = R_1$ there is an infinite sequence of τ-steps starting from P'. So there is an infinite sequence of states $(S_n)_{n\in\omega}$ such that $E_1\{P/X\} = S_0$ and $S_n \xrightarrow{\tau} S_{n+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $R_{\sigma(n)} \underline{\leftrightarrow}_{b}^{\Delta} S_n$ for all $n \in \omega$. Since X is guarded in E doing a τ-step from E to E_1 will result in X also being guarded in E_1. So there is an infinite sequence $(E_o)_{o\in\omega}$ such that $E = E_0$ and $E_o \xrightarrow{\tau} E_{o+1}$. So all states S_n are of the form $E_o\{P/X\}$, and by Lemma A.0.3 we have $E_o\{Q/X\} \xrightarrow{\tau} E_{o+1}\{Q/X\}$ for $o \in \omega$. By the definition of $\mathcal{R}$ we obtain $E_o\{P/X\}\ \mathcal{R}\ E_o\{Q/X\}$ for all $o \in \omega$. Since $Q \underline{\leftrightarrow}_{rb}^{\Delta} E\{Q/X\}$ and $E\{Q/X\} \xrightarrow{\tau} E_1\{Q/X\}$, also $Q \xrightarrow{\tau} Q'$ and $E_1\{Q/X\} \underline{\leftrightarrow}_{b}^{\Delta} Q'$. Since from $E_1\{Q/X\}$ it is possible to do an infinite sequence of τ-steps there is an infinite sequence of states $(T_p)_{p\in\omega}$ such that $Q = T_0$ and $T_p \xrightarrow{\tau} T_{p+1}$ and there exists a mapping $\rho : \omega \to \omega$ such that $E_{\rho(p)}\{Q/X\} \underline{\leftrightarrow}_{b}^{\Delta} T_p$ for all $p \in \omega$. So for all $p \in \omega$ we have $R_{\sigma(\rho(p))} \underline{\leftrightarrow}_{b}^{\Delta} E_{\rho(p)}\{P/X\}\ \mathcal{R}\ E_{\rho(p)}\{Q/X\} \underline{\leftrightarrow}_{b}^{\Delta} T_p$. Since X is totally unguarded in H and $Q \xrightarrow{\tau} Q'$, by Lemma A.0.4 also $H\{Q/X\} \xrightarrow{\tau} Q'$. So there exists an infinite sequence of states $(Q_l)_{l\in\omega}$ such that $G\{Q/X\} = Q_0$ and $Q_l \xrightarrow{\tau} Q_{l+1}$ and there exists a mapping $\sigma : \omega \to \omega$ such that $P_{\sigma(l)} \underline{\leftrightarrow}_{b}^{\Delta} \circ \mathcal{R} \circ \underline{\leftrightarrow}_{b}^{\Delta} Q_l$ for all $l \in \omega$.

So we conclude that Conditions $(U1)$, $(U2)$ and $(U3)$ from Definition 4.2.1 hold, so $\mathcal{R} \cup \mathcal{R}^{-1}$ is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_{b}^{\Delta}$. □

6.2 Soundness for rec5

Soundness for axiom rec5 is proven in Lemma 6.2.1. In order to show soundness a relation is defined of which it is proven that it is a rooted branching bisimulation with explicit divergence. This relation helps in finding a rooted branching bisimulation with explicit divergence for rec5. The proof is similar to the proof for axioms rec5 and rec6 in [1].

In order to show that a relation $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence we need to prove Condition (T) from Definition 3.0.2, Condition (D) from Theorem 3.0.1 and Condition (R) from Definition 3.0.4.

Since Condition (R) implies Condition (T) and Condition (D) we only need to prove Condition (R). Let $\mathcal{R}$ be a rooted branching bisimulation with explicit divergence. Assume $P\ \mathcal{R}\ Q$ and $P \xrightarrow{a} P'$ for some state $P' \in \mathbb{P}$. By Condition (R), $Q \xrightarrow{a} Q'$ and $P'\ \mathcal{R}\ Q'$.

Since $Q \twoheadrightarrow Q \xrightarrow{(a)} Q'$, $P\ \mathcal{R}\ Q$ and $P'\mathcal{R}Q'$ Condition (T) holds.
Let $a = \tau$ and let there be an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. Since $Q \xrightarrow{\tau} Q'$ and $P_1 = P'\ \mathcal{R}\ Q'$ Condition (D) holds.

Lemma 6.2.1. *For $E, F \in \mathbb{E}$, $recX.(\tau.(X + E) + F) \underline{\leftrightarrow}_{rb}^{\Delta} recX.(\tau.\Delta(E + F) + F)$, where $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$.*

Proof. In Lemma 6.2.2 it is proven that $\mathcal{R} \subseteq \underline{\leftrightarrow}_{rb}^{\Delta}$, where $\mathcal{R}$ is the relation appearing in Lemma 6.2.2. Choosing $G = X$ in $\mathcal{R}_0$ implies $recX.(\tau.(X + E) + F) \underline{\leftrightarrow}_{rb}^{\Delta} recX.(\tau.\Delta(E + F) + F)$. □

Lemma 6.2.2. *Let $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. Let:*

$$P = recX.(\tau.\Delta(E + F) + F)$$
$$Q = recX.(\tau.(X + E) + F)$$

Define the following relations:

$$\mathcal{R}_0 = \{(G\{P/X\}, G\{Q/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$$
$$\mathcal{R}_1 = \{(\Delta(E\{P/X\} + F\{P/X\}), Q + E\{Q/X\})\}$$

$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1}$

Then $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence.

Proof. In order to prove that $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence we need to prove Condition (R) from Definition 3.0.4. If $R_1 \ \mathcal{R} \ R_2$ and $R_1 \xrightarrow{a} R_1'$ then $R_2 \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'.
First we consider the case where $R_1 \ \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \ R_2$, so we only consider pairs $\{(G\{P/X\}, G\{Q/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$ and $\{(G\{Q/X\}, G\{P/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$. We use induction on the height of the derivation tree for the transition $R_1 \xrightarrow{a} R_1'$.

Case $G = 0$

This case is trivial.

Case $G = a.H$

Case 1. Assume $R_1 = a.H\{P/X\} \xrightarrow{a} R_1'$ and $R_2 = a.H\{Q/X\}$. By the operational semantics $R_1' = H\{P/X\}$. By the operational semantics $a.H\{Q/X\} \xrightarrow{a} H\{Q/X\}$ and $H\{P/X\} \ \mathcal{R} \ H\{Q/X\}$.
Case 2. Assume $R_1 = a.H\{Q/X\} \xrightarrow{a} R_1'$ and $R_2 = a.H\{P/X\}$. This case is similar to the above case.

Case $G = X$

Case 1. Assume $R_1 = P \xrightarrow{a} R_1'$ and $R_2 = Q$. Thus $recX.(\tau.\Delta(E+F)+F) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\tau.\Delta(E\{P/X\}+F\{P/X\})+F\{P/X\} \xrightarrow{a} R_1'$. Then we have the following two cases:

Case 1.1. Assume $\tau.\Delta(E\{P/X\}+F\{P/X\}) \xrightarrow{a} R_1'$. By the operational semantics we have $a = \tau$ and $R_1' = \Delta(E\{P/X\}+F\{P/X\})$. Using Lemma A.0.6 we have $Q = recX.(\tau.(X+E)+F) \xrightarrow{\tau} Q + E\{Q/X\}$ and $\Delta(E\{P/X\}+F\{P/X\}) \ \mathcal{R} \ Q + E\{Q/X\}$.

Case 1.2. Assume $F\{P/X\} \xrightarrow{a} R_1'$. By induction $F\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'. By the operational semantics $\tau.(Q+E\{Q/X\})+F\{Q/X\} \xrightarrow{a} R_2'$ and thus $Q \xrightarrow{a} R_2'$.
Case 2. Assume $R_1 = Q \xrightarrow{a} R_1'$ and $R_2 = P$. Thus $recX.(\tau.(X+E)+F) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\tau.(Q+E\{Q/X\})+F\{Q/X\} \xrightarrow{a} R_1'$. Then we have the following two cases:

Case 2.1. Assume $\tau.(Q+E\{Q/X\}) \xrightarrow{a} R_1'$. By the operational semantics we have $a = \tau$ and $R_1' = Q + E\{Q/X\}$. Using Lemma A.0.6 we have $P = recX.(\tau.\Delta(E+F)+F) \xrightarrow{\tau} \Delta(E\{P/X\} + F\{P/X\})$, where $Q + E\{Q/X\} \ \mathcal{R} \ \Delta(E\{P/X\}+F\{P/X\})$.

Case 2.2. Assume $F\{Q/X\} \xrightarrow{a} R_1'$. By induction $F\{P/X\} \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'. By the operational semantics $\tau.\Delta(E\{P/X\}+F\{P/X\})+F\{P/X\} \xrightarrow{a} R_2'$ and thus $P \xrightarrow{a} R_2'$.

Case $G = \Delta(H)$

Case 1. Assume $R_1 = \Delta(H\{P/X\}) \xrightarrow{a} R_1'$ and $R_2 = \Delta(H\{Q/X\})$. Then we have the following two cases:

Case 1.1. Assume $a = \tau$ and $R_1' = \Delta(H\{P/X\})$, by the operational semantics $\Delta(H\{Q/X\}) \xrightarrow{\tau} \Delta(H\{Q/X\})$ and $\Delta(H\{P/X\}) \ \mathcal{R} \ \Delta(H\{Q/X\})$.

Case 1.2. By a smaller derivation tree $H\{P/X\} \xrightarrow{a} R_1'$. By induction $H\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'. Since $H\{Q/X\} \xrightarrow{a} R_2'$, by the operational semantics $\Delta(H\{Q/X\}) \xrightarrow{a} R_2'$.
Case 2. Assume $R_1 = \Delta(H\{Q/X\}) \xrightarrow{a} R_1'$ and $R_2 = \Delta(H\{P/X\})$. This case is similar to the above case.

Case $G = H_1 + H_2$

Case 1. Assume $R_1 = H_1\{P/X\} + H_2\{P/X\} \xrightarrow{a} R_1'$ and $R_2 = H_1\{Q/X\} + H_2\{Q/X\}$. Without loss of generality we have that $H_1\{P/X\} \xrightarrow{a} R_1'$ by a smaller derivation tree. By induction $H_1\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'. Since $H_1\{Q/X\} \xrightarrow{a} R_2'$, by the operational semantics $H_1\{Q/X\} + H_2\{Q/X\} \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = H_1\{Q/X\} + H_2\{Q/X\} \xrightarrow{a} R_1'$ and $R_2 = H_1\{P/X\} + H_2\{P/X\}$. This case is similar to the above case.

Case $G = recY.H$

If $X = Y$ there are no variables which occur free in G, so $G\{P/X\} = G = G\{Q/X\}$. So assume $X \neq Y$.
Case 1. Assume $R_1 = (recY.H)\{P/X\} \xrightarrow{a} R_1'$ and $R_2 = (recY.H)\{Q/X\}$. Since $P, Q \in \mathbb{P}$ it follows that

$$(recY.H)\{P/X\} = recY.H\{P/X\}$$
$$(recY.H)\{Q/X\} = recY.H\{Q/X\}$$

Since $recY.H\{P/X\} \xrightarrow{a} R_1'$ we know that $(H\{P/X\})\{recY.H\{P/X\}/Y\} \xrightarrow{a} R_1'$ by a smaller derivation tree. In $(H\{P/X\})\{recY.H\{P/X\}/Y\}$ first all X are replaced in H as described by $(H\{P/X\})$, after which all Y in $(H\{P/X\})$ are replaced by $recY.H\{P/X\}$. Since in $recY.H\{P/X\}$ all X are replaced by P, it is unnecessary to replace all X in H as described by $(H\{P/X\})$. So

$$(H\{P/X\})\{recY.H\{P/X\}/Y\} = (H\{recY.H/Y\})\{P/X\}$$

So $(H\{recY.H/Y\})\{P/X\} \xrightarrow{a} R_1'$. By induction $(H\{recY.H/Y\})\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \mathcal{R} R_2'$ for some R_2'. By the operational semantics $(recY.H)\{Q/X\} \xrightarrow{a} R_2'$.
Case 2. Assume $R_1 = (recY.H)\{Q/X\} \xrightarrow{a} R_1'$ and $R_2 = (recY.H)\{P/X\}$. This case is similar to the above case.

Now the case where $R_1 \ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \ R_2$ is considered. So we consider the pairs $\{(\Delta(E\{P/X\} + F\{P/X\}), Q + E\{Q/X\})\}$ and $\{(Q + E\{Q/X\}, \Delta(E\{P/X\} + F\{P/X\}))\}$. For this we will make use of the case $R_1 \ \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \ R_2$.
Case 1. Assume $R_1 = \Delta(E\{P/X\} + F\{P/X\}) \xrightarrow{a} R_1'$ and $R_2 = Q + E\{Q/X\}$. Then we have the following cases:

Case 1.1. Assume $a = \tau$ and $R_1' = \Delta(E\{P/X\} + F\{P/X\})$. By the operational semantics $\tau.(X+E)+F \xrightarrow{\tau} X+E$, thus by Lemma A.0.6 we have $Q = recX.(\tau.(X+E)+F) \xrightarrow{\tau} Q+E\{Q/X\}$. So $Q + E\{Q/X\} \xrightarrow{\tau} Q + E\{Q/X\}$, where $\Delta(E\{P/X\} + F\{P/X\}) \ \mathcal{R} \ Q + E\{Q/X\}$.

Case 1.2. Assume $E\{P/X\} \xrightarrow{a} R_1'$, since $E\{P/X\} \ \mathcal{R}_0 \ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'. Since $E\{Q/X\} \xrightarrow{a} R_2'$ also $Q + E\{Q/X\} \xrightarrow{a} R_2'$.

Case 1.3. Assume $F\{P/X\} \xrightarrow{a} R_1'$, since $F\{P/X\} \ \mathcal{R}_0 \ F\{Q/X\}$ we have $F\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'. Since $F\{Q/X\} \xrightarrow{a} R_2'$ also $\tau.(Q + E\{Q/X\}) + F\{Q/X\} \xrightarrow{a} R_2'$ and thus $Q = recX.(\tau.(X + E) + F) \xrightarrow{a} R_2'$, so $Q + E\{Q/X\} \xrightarrow{a} R_2'$.
Case 2. Assume $R_1 = Q + E\{Q/X\} \xrightarrow{a} R_1'$ and $R_2 = \Delta(E\{P/X\} + F\{P/X\})$. Then we have the following cases:

Case 2.1 Assume $Q \xrightarrow{a} R_1'$, then by Lemma A.0.6 and the operational semantics either $\tau.(X + E) \xrightarrow{\tau} X + E$ and $R_1' = Q + E\{Q/X\}$ or $F \xrightarrow{a} F'$ and $R_1' = F'\{Q/X\}$.

Case 2.1.1. Assume $\tau.(X+E) \xrightarrow{\tau} X+E$ and $R_1' = Q+E\{Q/X\}$. By the operational semantics we have $\Delta(E\{P/X\}+F\{P/X\}) \xrightarrow{\tau} \Delta(E\{P/X\}+F\{P/X\})$, where $Q+E\{Q/X\} \ \mathcal{R} \ \Delta(E\{P/X\}+F\{P/X\})$.

Case 2.1.2. Assume $F \xrightarrow{a} F'$ and $R_1' = F'\{Q/X\}$. By Lemma A.0.3 we have $F\{P/X\} \xrightarrow{a} F'\{P/X\}$, and thus $\Delta(E\{P/X\} + F\{P/X\}) \xrightarrow{a} F'\{P/X\}$, where $F'\{Q/X\} \ \mathcal{R} \ F'\{P/X\}$.

Case 2.2 Assume $E\{Q/X\} \xrightarrow{a} R_1'$, since $E\{Q/X\} \ \mathcal{R}_0^{-1} \ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R_2'$ and $R_1' \ \mathcal{R} \ R_2'$ for some R_2'. Since $E\{P/X\} \xrightarrow{a} R_2'$ also $\Delta(E\{P/X\} + F\{P/X\}) \xrightarrow{a} R_2'$.

So $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence. □

6.3 Soundness for rec8

Soundness for axiom rec8 is proven in Lemma 6.3.1. In order to show soundness a relation is defined of which it is proven that it is a (rooted) branching bisimulation with explicit divergence. This relation helps in finding a rooted branching bisimulation with explicit divergence for rec8. The proof is similar to the proof for axioms rec5 and rec6 in [1] and the proof of axiom R4 in [8].

Lemma 6.3.1. *For $E, F, G \in \mathbb{E}$, let X be unguarded in E, then $recX.(\tau.(\tau.E + F) + G) \underline{\leftrightarrow}_{rb}^{\Delta}$ $recX.(\tau.(E + F) + G)$, where $\mathbb{V}(E) \cup \mathbb{V}(F) \cup \mathbb{V}(G) \subseteq \{X\}$.*

Proof. In Lemma 6.3.2 it is proven that $\mathcal{R} \subseteq \underline{\leftrightarrow}_{b}^{\Delta}$, where $\mathcal{R}$ is the relation appearing in Lemma 6.3.2. Furthermore, $\mathcal{R}$ is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. Choosing $H = X$ in $\mathcal{R}_0$ implies $recX.(\tau.(\tau.E + F) + G) \underline{\leftrightarrow}_{rb}^{\Delta} recX.(\tau.(E + F) + G)$. □

Lemma 6.3.2. *Let $\mathbb{V}(E) \cup \mathbb{V}(F) \cup \mathbb{V}(G) \subseteq \{X\}$ and let X be unguarded in E. Let:*

$$P = recX.(\tau.(\tau.E + F) + G)$$
$$Q = recX.(\tau.(E + F) + G)$$

Let $E'_{-1} = \tau.E + F$ and $E''_0 = E + F$ and for $n \in \omega$ let there be $E_0...E_n$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} ... \xrightarrow{\tau} E_n$, where X is totally unguarded in E_n. Let $E'_i = E_i$ and $E''_j = E_j$.

Define the following relations:

$$\mathcal{R}_0 = \{(H\{P/X\}, H\{Q/X\}) | \mathbb{V}(H) \subseteq \{X\}\}$$
$$\mathcal{R}_1 = \{(E'_i\{P/X\}, E''_j\{Q/X\} | -1 \leq i \leq n, 0 \leq j \leq n)\}$$
$$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1}$$

Then $\mathcal{R}$ is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$.

Proof. Since X is unguarded in E it does not lie within a subexpression $a.E'$ with $a \in A \backslash \{\tau\}$. So for $n \in \omega$ there are $E_0...E_n$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} ... \xrightarrow{\tau} E_n$, where X is totally unguarded in E_n.

By Lemma A.0.6 we have $P \xrightarrow{\tau} E'_{-1}\{P/X\}$. So $P \xrightarrow{\tau} E'_{-1}\{P/X\} \xrightarrow{\tau} E'_0\{P/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E'_n\{P/X\} \xrightarrow{\tau} E'_{-1}\{P/X\}$ for $E'_i = E_i$. By Lemma A.0.6 we have $Q \xrightarrow{\tau} E''_0\{Q/X\}$. So $Q \xrightarrow{\tau} E''_0\{Q/X\} \xrightarrow{\tau} E''_1\{Q/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E''_n\{Q/X\} \xrightarrow{\tau} E''_0\{Q/X\}$ for $E''_j = E_j$.

In order to prove that $\mathcal{R}$ is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (T) from Definition 3.0.2, Condition (D) from Theorem 3.0.1 and for the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (R) from Definition 3.0.4. First we consider the case where $R_1\ \mathcal{R}_0 \cup \mathcal{R}_0^{-1}\ R_2$. For this case we only prove Condition (R) from Definition 3.0.4, since this implies Condition (T) from Definition 3.0.2 and Condition (D) from Theorem 3.0.1. We prove that if $R_1\ \mathcal{R}\ R_2$ and $R_1 \xrightarrow{a} R'_1$ then $R_2 \xrightarrow{a} R'_2$ and $R'_1\ \mathcal{R}\ R'_2$ for some R'_2. We use induction on the height of the derivation tree for the transition $R_1 \xrightarrow{a} R'_1$. The only case which differs from the proof in Lemma 6.2.2 is the case where $G = X$, so only this case is considered here.

Case $G = X$

Case 1. Assume $R_1 = P \xrightarrow{a} R'_1$ and $R_2 = Q$. Thus $recX.(\tau.(\tau.E + F) + G) \xrightarrow{a} R'_1$. By a smaller derivation tree also $\tau.(\tau.E\{P/X\} + F\{P/X\}) + G\{P/X\} \xrightarrow{a} R'_1$. Then we have the following cases:

Case 1.1. Assume $a = \tau$ and $R'_1 = \tau.E\{P/X\} + F\{P/X\} = E'_{-1}\{P/X\}$. By the operational semantics $\tau.(E+F) \xrightarrow{\tau} E + F = E''_0$, using Lemma A.0.6 we have $Q = recX.(\tau.(E+F)+G) \xrightarrow{\tau} E''_0\{Q/X\}$, where $E'_{-1}\{P/X\} \mathcal{R} E''_0\{Q/X\}$.

Case 1.2. Assume $G\{P/X\} \xrightarrow{a} R'_1$, by induction $G\{Q/X\} \xrightarrow{a} R'_2$ and $R'_1 \mathcal{R} R'_2$ for some R'_2. Since $G\{Q/X\} \xrightarrow{a} R'_2$ by the operational semantics we get that $\tau.(E\{Q/X\} + F\{Q/X\}) + G\{Q/X\} \xrightarrow{a} R'_2$, thus $Q \xrightarrow{a} R'_2$.

Case 2. Assume $R_1 = Q \xrightarrow{a} R'_1$ and $R_2 = P$. Thus $recX.(\tau.(E+F)+G) \xrightarrow{a} R'_1$. By a smaller derivation tree also $\tau.(E\{Q/X\}+F\{Q/X\})+G\{Q/X\} \xrightarrow{a} R'_1$. Then we have the following cases:

Case 2.1. Assume $a = \tau$ and $R'_1 = E\{Q/X\} + F\{Q/X\} = E''_0\{Q/X\}$. By the operational semantics $\tau.(\tau.E+F)+G) \xrightarrow{\tau} \tau.E + F = E'_{-1}$, using Lemma A.0.6 we have $P = recX.(\tau.(\tau.E + F) + G) \xrightarrow{\tau} E'_{-1}\{P/X\}$ where $E''_0\{Q/X\} \mathcal{R} E'_{-1}\{P/X\}$.

Case 2.2. Assume $G\{Q/X\} \xrightarrow{a} R'_1$, by induction $G\{P/X\} \xrightarrow{a} R'_2$ and $R'_1 \mathcal{R} R'_2$ for some R'_2. Since $G\{P/X\} \xrightarrow{a} R'_2$ by the operational semantics we get that $\tau.(\tau.E\{P/X\} + F\{P/X\}) + G\{P/X\} \xrightarrow{a} R'_2$, thus $P \xrightarrow{a} R'_2$.

Now the case where $R_1 \ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \ R_2$ is considered. For this case we will make use of the case $R_1 \ \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \ R_2$. In order to prove Condition (T) from Definition 3.0.2 we need to prove that if $R_1 \ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \ R_2$ and $R_1 \xrightarrow{a} R'_1$ then $R_2 \twoheadrightarrow R' \xrightarrow{(a)} R'_2$, $R_1 \mathcal{R} R'$ and $R'_1 \mathcal{R} R'_2$ for some R' and R'_2.
Case 1. Assume for some i that $R_1 = E'_i\{P/X\} \xrightarrow{a} R'_1$ and for some j $R_2 = E''_j\{Q/X\}$. Then we have the following cases:

Case 1.1. Let $E'_i = E'_{-1} = \tau.E + F$, since $E''_0\{Q/X\} \xrightarrow{\tau} E''_1\{Q/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E''_n\{Q/X\} \xrightarrow{\tau} E''_0\{Q/X\}$ we have $E''_j\{Q/X\} \twoheadrightarrow E''_0\{Q/X\} = E\{Q/X\}+F\{Q/X\}$, where $E'_{-1}\{P/X\} \mathcal{R} E''_0\{Q/X\}$. We have the following cases:

Case 1.1.1. By the operational semantics $\tau.E + F \xrightarrow{\tau} E$. We have $E'_0\{P/X\} \mathcal{R} E''_0\{Q/X\}$, where $E'_0\{P/X\} = E\{P/X\}$.

Case 1.1.2. Let $F\{P/X\} \xrightarrow{a} R'_1$. Since $F\{P/X\} \mathcal{R}_0 F\{Q/X\}$ we have $F\{Q/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. Since $F\{Q/X\} \xrightarrow{a} R'_2$ we have $E''_0\{Q/X\} \xrightarrow{a} R'_2$. So $E''_j\{Q/X\} \twoheadrightarrow E''_0\{Q/X\} \xrightarrow{a} R'_2$, where $E'_i\{P/X\} \mathcal{R} E''_0\{Q/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 1.2. Let $E'_i = E'_0 = E$, since $E''_0\{Q/X\} \xrightarrow{\tau} E''_1\{Q/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E''_n\{Q/X\} \xrightarrow{\tau} E''_0\{Q/X\}$ we have $E''_j\{Q/X\} \twoheadrightarrow E''_0\{Q/X\} = E\{Q/X\} + F\{Q/X\}$, where $E'_0\{P/X\} \mathcal{R} E''_0\{Q/X\}$. Since $E\{P/X\} \mathcal{R}_0 E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. So $E''_j\{Q/X\} \twoheadrightarrow E''_0\{Q/X\} \xrightarrow{a} R'_2$, where $E'_0\{P/X\} \mathcal{R} E''_0\{Q/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 1.3. Otherwise, since $E''_0\{Q/X\} \xrightarrow{\tau} E''_1\{Q/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E''_n\{Q/X\} \xrightarrow{\tau} E''_0\{Q/X\}$, $E'_i = E_i$ and $E''_j = E_j$ we have $E''_j\{Q/X\} \twoheadrightarrow E''\{Q/X\}$, where $E'' = E'_i$. Since $E''\{P/X\} \mathcal{R}_0 E''\{Q/X\}$ we have $E''\{Q/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. So $E''_j\{Q/X\} \twoheadrightarrow E''\{Q/X\} \xrightarrow{a} R'_2$, where $E'_i\{P/X\} \mathcal{R} E''\{Q/X\}$ and $R'_1 \mathcal{R} R'_2$.
Case 2. Assume for some j that $R_1 = E''_j\{Q/X\} \xrightarrow{a} R'_1$ and for some i $R_2 = E'_i\{P/X\}$. Then we have the following cases:

Case 2.1. Let $E''_j = E''_0 = E + F$, since $E'_{-1}\{P/X\} \xrightarrow{\tau} E'_0\{P/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E'_n\{P/X\} \xrightarrow{\tau} E'_{-1}\{P/X\}$ we have $E'_i\{P/X\} \twoheadrightarrow E'_{-1}\{P/X\} = \tau.E\{P/X\} + F\{P/X\}$, where $E''_0\{Q/X\} \mathcal{R} E'_{-1}\{P/X\}$. We have the following cases:

Case 2.1.1. Let $E\{Q/X\} \xrightarrow{a} R'_1$. Since $E\{Q/X\} \mathcal{R}_0^{-1} E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. So $E'_i\{P/X\} \twoheadrightarrow E'_0\{P/X\} \xrightarrow{a} R'_2$, where $E''_j\{Q/X\} \mathcal{R} E'_0\{P/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 2.1.2. Let $F\{Q/X\} \xrightarrow{a} R'_1$. Since $F\{Q/X\} \mathcal{R}_0^{-1} F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. So $E'_i\{P/X\} \twoheadrightarrow E'_{-1}\{P/X\} \xrightarrow{a} R'_2$, where $E''_0\{Q/X\} \mathcal{R} E'_{-1}\{P/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 2.2. Otherwise, since $E'_{-1}\{P/X\} \xrightarrow{\tau} E'_0\{P/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E'_n\{P/X\} \xrightarrow{\tau} E'_{-1}\{P/X\}$, $E''_j = E_j$ and $E'_i = E_i$ we have $E'_i\{P/X\} \twoheadrightarrow E''\{P/X\}$, where $E'' = E''_j$. Since $E''\{Q/X\} \mathcal{R}_0^{-1} E''\{P/X\}$ we have $E''\{P/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. So $E'_i\{P/X\} \twoheadrightarrow E''\{P/X\} \xrightarrow{a} R'_2$, where $E''_j\{Q/X\} \mathcal{R}$

$E''\{P/X\}$ and $R_1' \; \mathcal{R} \; R_2'$.

Here we prove Condition (D) from Theorem 3.0.2. In order to prove this we need to prove that if $R_1 \; \mathcal{R}_1 \cup \mathcal{R}_1^{-1} R_2$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $R_1 = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k \; \mathcal{R} \; R_2$ for all $k \in \omega$, then there exists a state R_2' such that $R_2 \to^+ R_2'$ and $P_k \; \mathcal{R} \; R_2'$ for some $k \in \omega$.

Case 1. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that for some i we have $E_i'\{P/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and for some j $P_k \; \mathcal{R} \; E_j''\{Q/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 1.1. Let $E_i' = E_{-1}' = \tau.E + F$, since $E_0''\{Q/X\} \xrightarrow{\tau} E_1''\{Q/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E_n''\{Q/X\} \xrightarrow{\tau} E_0''\{Q/X\}$ we have $E_j''\{Q/X\} \twoheadrightarrow E_0''\{Q/X\} = E\{Q/X\} + F\{Q/X\}$. We have the following cases:

Case 1.1.1. Let $\tau.E\{P/X\} \xrightarrow{\tau} E\{P/X\}$, where $E\{P/X\} = P_1$. Since $E\{P/X\} \; \mathcal{R}_0 \; E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{\tau} R_2'$, where $P_2 \; \mathcal{R} \; R_2'$. So $E_j''\{Q/X\} \to^+ R_2'$, where $P_2 \; \mathcal{R} \; R_2'$.

Case 1.1.2. Let $F\{P/X\} \xrightarrow{\tau} P_1$. Since $F\{P/X\} \; \mathcal{R}_0 \; F\{Q/X\}$ we have $F\{Q/X\} \xrightarrow{\tau} R_2'$, where $P_1 \; \mathcal{R} \; R_2'$. So $E_j''\{Q/X\} \to^+ R_2'$, where $P_2 \; \mathcal{R} \; R_2'$.

Case 1.2. Let $E_i' = E_0' = E$, since $E_0''\{Q/X\} \xrightarrow{\tau} E_1''\{Q/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E_n''\{Q/X\} \xrightarrow{\tau} E_0''\{Q/X\}$ we have $E_j''\{Q/X\} \twoheadrightarrow E_0''\{Q/X\} = E\{Q/X\} + F\{Q/X\}$. Since $E\{P/X\} \; \mathcal{R}_0 \; E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{\tau} R_2'$, where $P_1 \; \mathcal{R} \; R_2'$. So $E_j''\{Q/X\} \to^+ R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

Case 1.3. Otherwise, since $E_0''\{Q/X\} \xrightarrow{\tau} E_1''\{Q/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E_n''\{Q/X\} \xrightarrow{\tau} E_0''\{Q/X\}$, $E_i' = E_i$ and $E_j'' = E_j$ we have $E_j''\{Q/X\} \twoheadrightarrow E''\{Q/X\}$, where $E'' = E_i'$. Since $E''\{P/X\} \; \mathcal{R}_0 \; E''\{Q/X\}$ we have $E''\{Q/X\} \xrightarrow{\tau} R_2'$, where $R_1' \; \mathcal{R} \; R_2'$. So $E_j''\{Q/X\} \to^+ R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

Case 2. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that for some j we have $E_j''\{Q/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and for some i $P_k \; \mathcal{R} \; E_i'\{P/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 2.1. Let $E_j'' = E_0'' = E + F$, since $E_{-1}'\{P/X\} \xrightarrow{\tau} E_0'\{P/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E_n'\{P/X\} \xrightarrow{\tau} E_{-1}'\{P/X\}$, $E_j'' = E_j$ and $E_i' = E_i$ we have $E_i'\{P/X\} \twoheadrightarrow E_{-1}'\{P/X\} = \tau.E\{P/X\} + F\{P/X\}$. We have the following cases:

Case 2.1.1. Let $E\{Q/X\} \xrightarrow{\tau} P_1$. Since $E\{Q/X\} \; \mathcal{R}_0^{-1} \; E\{P/X\}$ we have $E\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1 \; \mathcal{R} \; R_2'$. So $E_i'\{P/X\} \to^+ R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

Case 2.1.2. Let $F\{Q/X\} \xrightarrow{\tau} P_1$. Since $F\{Q/X\} \; \mathcal{R}_0^{-1} \; F\{P/X\}$ we have $F\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1 \; \mathcal{R} \; R_2'$. So $E_i'\{P/X\} \to^+ R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

Case 2.2. Otherwise, since $E_{-1}'\{P/X\} \xrightarrow{\tau} E_0'\{P/X\} \xrightarrow{\tau} ... \xrightarrow{\tau} E_n'\{P/X\} \xrightarrow{\tau} E_{-1}'\{P/X\}$, $E_j'' = E_j$ and $E_i' = E_i$ we have $E_i'\{P/X\} \twoheadrightarrow E''\{P/X\}$, where $E'' = E_j''$. Since $E''\{Q/X\} \; \mathcal{R}_0^{-1} \; E''\{P/X\}$ we have $E''\{P/X\} \xrightarrow{\tau} R_2'$, where $R_1' \; \mathcal{R} \; R_2'$. So $E_i'\{P/X\} \to^+ R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

□

Chapter 7

Completeness

In this section we prove completeness for the axioms in Section 5. The completeness proof is based on the completeness proof in [5], [8] and [1] and it consists of the following steps:

- Every expression can be transformed into a guarded expression by using the axioms (Section 7.1).
- For every guarded expression E there is a guarded standard equation system $\mathcal{E}$ such that E provably satisfies $\mathcal{E}$ (Section 7.2).
- If two expressions E and F both provably satisfy the same guarded equation system then $E = F$ (Section 7.2).
- If E and F are guarded expressions such that $E \underline{\leftrightarrow}^{\Delta}_{rb} F$ and both E and F provably satisfy a guarded standard equation system, then there is a single guarded equation system $\mathcal{E}$ such that both E and F provably satisfy $\mathcal{E}$ (Section 7.3).

In [5] and [1] a technical condition on standard equation systems called saturatedness is used. Saturation is however unsound in branching bisimulation semantics so we cannot use it here. In Section 7.2 we explain why saturation is unsound in branching bisimulation semantics.

7.1 Reducing to guarded expressions

The first step in the completeness proof consists of proving that every expression can be transformed into a guarded expression. As stated in Section 2 a guarded expression F is an expression where for every subexpression $recX.G$ of F the variable X is guarded in G. A variable X is guarded in G if every free occurrence of X in G lies within a subexpression of the form $a.H$ with $a \in A\backslash\{\tau\}$. In [5] and [8] unguarded expressions are rewritten into guarded expressions by eliminating free occurrences of X within a τ-prefix. Since we preserve divergence we cannot do this since divergence occurs when from a state a sequence of τ-transitions can be done after which the initial state is reached again. In [8] for instance $recX.(\tau.(X + E)) = recX.(\tau.E)$, but when preserving divergence this is unsound. An unguarded occurrence of X which lies in the scope of a τ-prefix describes divergent behaviour, so that X can not be eliminated. In [1] the Δ-operator was added in order to describe divergent behaviour using a guarded expression. If we have for instance the expression $recX.(\tau.X + E)$ we can rewrite this expression using law rec11 into $recX.\Delta(E)$. Without the Δ-operator we would need to have a different, more complex definition of guardedness.

In Lemma 7.1.2 we prove that every expression can be transformed into a guarded expression, we prove this using structural induction. For the case $E = recX.E'$ we need to prove that there exists a guarded F' where X is guarded in F' and $recX.E' = recX.F'$. In order to prove this we prove a stronger statement which is proven in Lemma 7.1.1. In order to prove Lemma 7.1.1 we use induction on the nesting depth $d(E')$ of recursions in E', this is inductively defined as follows:

$d(0) = 0$
$d(a.E) = d(E)$
$d(E + F) = \max(d(E), d(F))$
$d(recX.E) = 1 + d(E)$
$d(X) = 0$
$d(\Delta(E)) = d(E)$

Let $E \in \mathbb{E}$ and $X \in \mathbb{V}(E)$, we prove in Lemma 7.1.1 that there exists a guarded F such that:

- X is guarded in F
- There does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(F)$ which lies within a subexpression $recZ.G$ of F
- $recX.E = recX.F$

We prove this by induction on the nesting depth $d(E)$ of recursions in E, the proof has the same structure as the proof of Theorem 3 in [8]. We however also have to eliminate Δ-operators which are in front of unguarded occurrences of X, which is not necessary in [8]. For $d(E) = 0$ we rewrite E into an expression F such that X is guarded in F and $recX.E = recX.F$. Since $d(E) = 0$ we know that no unguarded occurrence of X lies within the scope of a $recX$-operator, so each unguarded occurrence can only lie within the scope of τ-prefixes, $+$-operators and Δ-operators. We first distinguish the case where some X lie within several τ-prefixes and Δ-operators. Using the axioms we reduce the number of τ-prefixes and Δ-operators until each X lies within the scope of at most one τ-prefix or Δ-operator. If we have for instance an expression E with an unguarded occurrence of X which lies within several τ-prefixes and Δ-operators and the outer two operators are τ-prefixes, we can use axiom rec8 to eliminate a τ-prefix.

Once we have an expression where each unguarded occurrence of X lies within at most one τ-prefix or Δ-operator we can rewrite this expression further where we eliminate unguarded occurrences of X to obtain an expression with only one unguarded occurrence of X. If we have for instance several unguarded occurrences of X which lie within the scope of a τ-prefix we can use axiom rec7 to reduce the number of unguarded occurrences of X.

Finally we obtain an expression with only one unguarded occurrence of X which lies within the scope of at most one τ-prefix or Δ-operator. We can eliminate the unguarded occurrences of X by using axiom rec4, rec5 or rec6.

Lemma 7.1.1. *Let $E \in \mathbb{E}$ and $X \in \mathbb{V}(E)$, then there exists a guarded F such that:*

- *X is guarded in F*
- *There does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(F)$ which lies within a subexpression $recZ.G$ of F*
- *$recX.E = recX.F$*

Proof. We prove this by induction on the nesting depth $d(E)$ of recursions in E. Let $d(E) = 0$, then since E does not have subexpressions of the form $recZ.G$ we only need to prove that there exists a guarded F such that X is guarded in F and $recX.E = recX.F$. If X is guarded in E we are done since E can be taken for F and $recX.E = recX.E$. So assume X is unguarded in E. Since $d(E) = 0$ we know that no unguarded occurrence of X lies within the $recX$-operator, so it can only lie within the scope of τ-prefixes, $+$-operators or Δ-operators. We first distinguish the case where at least one unguarded occurrence of X lies within the scope of several τ-prefixes and Δ-operators. Using induction on the number of τ-prefixes and Δ-operators it is possible to rewrite the expression so that all unguarded occurrences of X lie in the scope of at most one τ-prefix or Δ-operator. We identify the following cases:

- $E = \tau.(\tau.E' + E'') + F'$, where X is unguarded in E'
- $E = \tau.(\Delta(E') + E'') + F'$, where X is unguarded in E'
- $E = \Delta(E') + F'$, where X is unguarded in E'

Case 1. Let $E = \tau.(\tau.E' + E'') + F'$, where X is unguarded in E'. Then we can use axiom rec8 which gives $recX.(\tau.(\tau.E'+E'')+F') = recX.(\tau.(E'+E'')+F')$. So we have eliminated a τ-prefix and we can continue with the expression $\tau.(E' + E'') + F'$.
Case 2. Let $E = \tau.(\Delta(E')+E'')+F'$, where X is unguarded in E'. Then we can use axiom rec10 which gives $recX.(\tau.(\Delta(E') + E'') + F') = recX.(\tau.(X + E' + E'') + F')$. So we have eliminated a Δ-operator and we can continue with the expression $\tau.(X + E' + E'') + F'$.
Case 3. Let $E = \Delta(E') + F'$, where X is unguarded in E'. Since X is unguarded in E' we can use axiom rec9 which gives $recX.(\Delta(E') + F') = recX.(\tau.X + E' + F')$. So we have eliminated a Δ-operator and we can continue with the expression $\tau.X + E' + F'$.

When each unguarded occurrence of X lies within at most one τ-prefix or Δ-operator we remove unguarded occurrences of X until we obtain an expression with only one unguarded occurrence of X. We have the following cases:

- $E = \tau.(X + E') + \tau.(X + E'') + F$
- $E = \tau.(X + E') + \Delta.(X + E'') + F$
- $E = \Delta.(X + E') + \Delta.(X + E'') + F$
- $E = X + E'$, where X is unguarded in E'

Case 1. Let $E = \tau.(X + E') + \tau.(X + E'') + F$. Then we can use axiom rec7 which gives $recX.(\tau.(X + E') + \tau.(X + E'') + F) = recX.(\tau.(X + E' + E'') + F)$, so we have eliminated an unguarded occurrence of X.
Case 2. Let $E = \tau.(X+E')+\Delta.(X+E'')+F$. Then we can eliminate an occurrence of X in the following way:

$$\begin{aligned} recX.(\tau.(X + E') + \Delta.(X + E'') + F) &\overset{rec6}{=} recX.\Delta(E'' + \tau.(X + E') + F) \\ &\overset{rec9}{=} recX.(\tau.X + E'' + \tau.(X + E') + F) \\ &\overset{rec7}{=} recX.(\tau.(X + E') + E'' + F) \end{aligned}$$

Case 3. Let $E = \Delta.(X + E') + \Delta.(X + E'') + F$. Then we can eliminate an occurrence of X in the following way:

$$\begin{aligned} recX.(\Delta.(X + E') + \Delta.(X + E'') + F) &\overset{rec9}{=} recX.(\tau.X + X + E' + \Delta.(X + E'') + F) \\ &\overset{rec9}{=} recX.(\tau.X + X + E'' + \tau.X + X + E' + F) \\ &\overset{S3}{=} recX.(\tau.X + X + E'' + E' + F) \\ &\overset{rec4}{=} recX.(\tau.X + E'' + E' + F) \end{aligned}$$

Case 4. Let $E = X + E'$. Then we can use axiom rec4 which gives $recX.(X + E') = recX.E'$.

When there is one unguarded occurrence of X which lies within at most one τ-prefix or Δ-operator we eliminate this X using the following cases:

- $E = X + E'$, where X is guarded in E'

- $E = \tau.(X + E') + F$, where X is guarded in E' and F
- $E = \Delta(X + E') + F$, where X is guarded in E' and F

Case 1. Let $E = X + E'$, where X is guarded in E'. Then we can use axiom rec4 which gives $recX.(X + E') = recX.E'$.
Case 2. Let $E = \tau.(X + E') + F$, where X is guarded in E' and F. Then we can use axiom rec5 which gives $recX.(\tau.(X + E') + F) = recX.(\tau.\Delta(E' + F) + F)$.
Case 3. Let $E = \Delta(X + E') + F$, where X is guarded in E' and F. Then we can use axiom rec6 which gives $recX.(\Delta(X + E') + F) = recX.\Delta(E' + F)$.

The result of the described procedure is an expression without unguarded occurrences of X. This completes the proof for the case where $d(E) = 0$.

Let $recX'.E'$ be a subexpression of E such that this subexpression does not lie within another recursion, thus $recX'.E'$ is an outermost recursion. Since $d(E') < d(E)$ the induction hypothesis implies that there exists an expression F' such that:

- X' is guarded in F'
- There does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(F')$ which lies within a subexpression $recZ.G$ of F'
- $recX'.E' = recX'.F'$

Let $Y \in \mathbb{V}(F')$ be a free and unguarded occurrence in F' which does not lie within a subexpression $recZ.G$ of F'. Then it will lie in the scope of $recX'$ in the expression $recX'.F'$. However, in the expression $F'\{recX'.F'/X'\}$ there does not exist an unguarded occurrence of any variable which lies within a recursion. By axiom rec2 we have $recX'.F' = F'\{recX'.F'/X'\}$, so $recX'.E' = F'\{recX'.F'/X'\}$. By performing this step for every outermost recursion of E and using the transformation as described in the base case for $d(E) = 0$ to remove free unguarded occurrences of X we obtain a guarded F such that:

- X is guarded in F
- There does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(F)$ which lies within a subexpression $recZ.G$ of F
- $recX.E = recX.F$

□

Lemma 7.1.2. *Let $E \in \mathbb{E}$. There exists a guarded F with $E = F$.*

Proof. We prove this by induction on the structure of E.

Case $E = 0$

Since 0 does not have a subexpression $recX.G$ this case is trivial.

Case $E = a.E'$

The induction hypothesis implies that there exists a guarded F' such that $E' = F'$. We can substitute F' for E' in $a.E'$. So take $F = a.F'$.

Case $E = E_1 + E_2$

The induction hypothesis implies that there exists a guarded F_1 such that $E_1 = F_1$ and there exists a guarded F_2 such that $E_2 = F_2$. We can substitute F_1 for E_1 and F_2 for E_2 in $E_1 + E_2$. So take $F = F_1 + F_2$.

Case $E = recX.E'$

In Lemma 7.1.1 it is proven that for $E' \in \mathbb{E}$ there exists a guarded F' such that:

- X is guarded in F'
- There does not exist a free and unguarded occurrence of a variable $Y \in \mathbb{V}(F')$ which lies within a subexpression $recZ.G$ of F'
- $recX.E' = recX.F'$

So let $F = recX.F'$ to obtain a guarded F such that $E = F$.

Case $E = X$

Since X does not have a subexpression $recX.G$ this case is trivial.

Case $E = \Delta(E')$

The induction hypothesis implies that there exists a guarded F' such that $E' = F'$. We can substitute F' for E' in $\Delta(E')$. So take $F = \Delta(F')$. □

7.2 Equation systems

In this section we consider the next two steps of the completeness proof. We prove that for every guarded expression E there is a guarded standard equation system $\mathcal{E}$ such that E provably satisfies $\mathcal{E}$ and we prove that if two expressions E and F both provably satisfy the same guarded equation system then $E = F$. We start by giving the definition for a standard equation system. This is followed by defining when an equation system is guarded. Finally we define when an expression provably satisfies an equation system.

Definitions 7.2.1 and 7.2.2 together hold the definition for a standard equation system. The definitions are equal to the definitions in [1].

Definition 7.2.1. (Equation system) Let $V \subseteq \mathbb{V}$ be a set of variables and let $\vec{X} = (X_1, ..., X_n)$ be a sequence of variables, where $X_i \notin V$. An equation system over the free variables V and the formal variables $\vec{X}$ is a set of equations

$$\mathcal{E} = \{X_i = E_i | 1 \leq i \leq n\}$$

such that $E_i \in \mathbb{E}$ and $\mathbb{V}(E_i) \subseteq \{X_1, ..., X_n\} \cup V$ for $1 \leq i \leq n$.

Definition 7.2.2. (Standard equation system) The equation system $\mathcal{E}$ is called a standard equation system over the free variables V and the formal variables $(X_1, ..., X_n)$ if there exists a partition $\{X_1, ..., X_n\} = \Omega^\Sigma \cup \Omega^\Delta$ such that for every $1 \leq i \leq n$:

- if $X_i \in \Omega^\Sigma$, then E_i is a sum of expressions $a.X_j$ and variables $Y \in V$, where $a \in A$ and $1 \leq j \leq n$.
- if $X_i \in \Omega^\Delta$, then $E_i = \Delta(E_i')$, where E_i' is a sum of expressions $a.X_j$ and variables $Y \in V$, where $a \in A$ and $1 \leq j \leq n$.

Here $\Omega^\Sigma \cap \Omega^\Delta = \emptyset$.

W.l.o.g. we may assume that $X_1 \in \Omega^\Sigma$. Let $X_1 \in \Omega^\Delta$, so $E_1 = \Delta(E')$, where E' is a sum of expressions $a.X_j$ and variables $Y \in V$, where $a \in A$ and $1 \leq j \leq n$. Law D2 gives $\Delta(E) = \tau.\Delta(E) + E$. So we can introduce a new formal variable X_0 and add the equation $X_0 = \tau.X_1 + E'$, since E' is a sum of expressions $a.X_j$ and variables $Y \in V$ we have $X_0 \in \Omega^\Sigma$.

Let $\mathcal{E}$ be an equation system. The relation $\prec \subseteq \vec{X} \times \vec{X}$ is defined as follows:

- $X_j \prec X_i$ if the variable X_j is unguarded in the expression E_i

The equation system $\mathcal{E}$ is guarded if $\prec$ is well-founded on $\vec{X}$. So an equation system is guarded if it does not contain a cycle of τ-steps. We write $X_i \xrightarrow{a} X_j$ if $E_i \xrightarrow{a} X_j$.

Let $\vec{F} = (F_1, ..., F_n)$ be an ordered sequence of expressions. Then $\vec{F}$ provably satisfies the equation system $\mathcal{E}$ if $F_i = E_i\{\vec{F}/\vec{X}\}$ for all $1 \leq i \leq n$. An expression F provably satisfies $\mathcal{E}$ if there exists a sequence of expressions $(F_1, ..., F_n)$, which provably satisfies $\mathcal{E}$ and such that $F = F_1$.

Now we can explain why saturation is unsound in branching bisimulation semantics. A standard equation system $\mathcal{E} = \{X_i = E_i | 1 \leq i \leq n\}$ is saturated if for all $1 \leq i, j, k, l \leq n$ and $Y \in V$ we have:

- if $X_i \twoheadrightarrow X_k \xrightarrow{a} X_l \twoheadrightarrow X_j$, then also $X_i \xrightarrow{a} X_j$.
- if $X_i \twoheadrightarrow X_j$ and Y occurs in E_j, then Y occurs already in E_i.

Consider the expression $E = \tau.(a.0 + b.0) + c.0$, then E provably satisfies the following equation system:

$$
\begin{aligned}
X_1 &= \tau.X_2 + c.X_3 \\
X_2 &= a.X_3 + b.X_3 \\
X_3 &= 0
\end{aligned}
$$

We have $X_1 \xrightarrow{\tau} X_2 \xrightarrow{a} X_3$, so in order to have a saturated equation system the equation system should be rewritten into an equation system where $X_1 \xrightarrow{a} X_3$. It is however not possible to rewrite the expression $\tau.(a.0+b.0)+c.0$ into an expression where it is possible to do an a-transition without first doing the τ-transition, since after doing the τ-transition the possibility to do the c-transition is lost.

Before we continue we consider an example of a guarded standard equation system for the expression $a.(\tau.\Delta(b.0) + b.0)$:

Example 7.2.1. Let $P = a.(\tau.\Delta(b.0)+b.0)$. Figure 7.1 shows the labelled transition system that is rooted at P. By introducing a formal variable for each expression we obtain the labelled transition system in Figure 7.2. We now have the following equation system $\mathcal{E} = \{X_i = E_i | 1 \leq i \leq 4\}$ which provably satisfies P:

$$
\begin{aligned}
X_1 &= a.X_2 \\
X_2 &= \tau.X_3 + b.X_4 \\
X_3 &= \Delta(b.X_4) \\
X_4 &= 0
\end{aligned}
$$

Let $\vec{P} = (P_1, ..., P_4)$, where $P_1 = a.P_2$, $P_2 = \tau.P_3+b.P_4$, $P_3 = \Delta(b.P_4)$ and $P_4 = 0$. This shows that P provably satisfies $\mathcal{E}$. Note that $\mathcal{E}$ is standard, since for each i either $X_i \in \Omega^{\Sigma}$ or $X_i \in \Omega^{\Delta}$. The standard equation system $\mathcal{E}$ is also guarded, since the only occurrence of an unguarded variable is in E_2. In the expression E_2 the variable X_3 is unguarded. It is possible to create a linear ordering $\prec$ on the variables such that $X_3 \prec X_2$, so $\mathcal{E}$ is guarded.

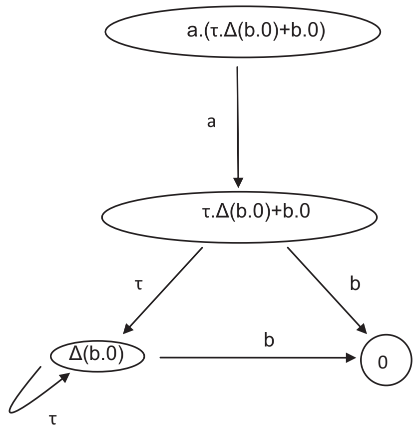

Figure 7.1: The labelled transition system rooted at $a.(\tau.\Delta(b.0) + b.0)$

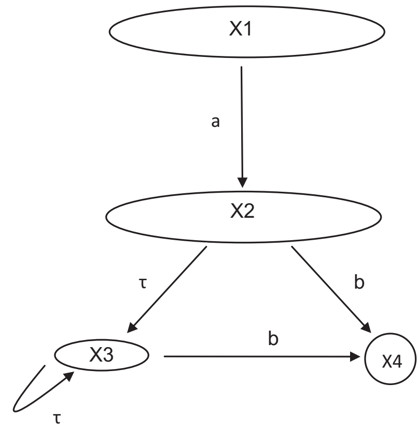

Figure 7.2: The labelled transition system from Figure 7.1 with for each expression a formal variable

In Lemma 7.2.1 we prove that every guarded $E \in \mathbb{E}$ provably satisfies a guarded standard equation system over the free variables $\mathbb{V}(E)$. In [8] this step is incorporated in the proof that if E and F are guarded expressions such that $E \underline{\leftrightarrow}^{\Delta}_{rb} F$ and both E and F provably satisfy a guarded standard equation system, then there is a single guarded equation system $\mathcal{E}$ such that both E and F provably satisfy $\mathcal{E}$. However since we distinguish a partition as described in Definition 7.2.2 it is more convenient to prove these steps separately.

The proof that for every guarded expression E there is a guarded standard equation system $\mathcal{E}$ such that E provably satisfies $\mathcal{E}$ is similar to Theorem 5 in [1], except that we do not transform the guarded standard equation system into a guarded and saturated standard equation system since saturation is unsound for branching bisimulation semantics. The structure of the proof that if two expressions E and F both provably satisfy the same guarded equation system then $E = F$ is analogous to Theorem 4.2 in [5], except that we use a different axiom. We have shifted the proof to Lemma E.0.9 in the appendix.

Lemma 7.2.1. *Every guarded $E \in \mathbb{E}$ provably satisfies a guarded standard equation system over the free variables $\mathbb{V}(E)$.*

Proof. This is Theorem 5 in [1] without the last step where the guarded standard equation system is transformed into a guarded and saturated standard equation system. □

Lemma 7.2.2. *Let $E, F \in \mathbb{E}$ and let $\mathcal{E}$ be a guarded equation system such that both E and F provably satisfy $\mathcal{E}$. Then $E = F$.*

Proof. Using axiom rec3 this can be shown analogously to Theorem 4.2 in [5]. We have given the proof in the appendix in Lemma E.0.9. □

7.3 Joining two equation systems

Here we prove the last step of our completeness proof. A guarded equation system $\mathcal{E}$ is constructed such that if E and F are guarded expressions such that $E \underline{\leftrightarrow}^{\Delta}_{rb} F$ and both E and F provably satisfy a guarded standard equation system, then both E and F provably satisfy $\mathcal{E}$. This is proven in Lemma 7.3.2. The construction of $\mathcal{E}$ is based on [1] and [8]. In [1] the saturatedness of the equation systems is used when proving that E and F both provably satisfy $\mathcal{E}$. Saturation is however unsound for branching bisimulation so the proof that E and F both provably satisfy $\mathcal{E}$ is mainly based on the proof in [8]. In [8] the Δ-operator is however not necessary since there divergence is not preserved, concerning the Δ-operator our proof is therefore different from the proof in [8]. In Example 7.3.1 we show how $\mathcal{E}$ is constructed for $E = a.(\tau.\Delta(b.0) + b.0)$ and $F = a.\Delta(b.0)$. We also show why E and F provably satisfy $\mathcal{E}$.

Example 7.3.1. Let $E = a.(\tau.\Delta(b.0)+b.0)$ and $F = a.\Delta(b.0)$, then $E \underline{\leftrightarrow}_{rb}^{\Delta} F$. The expression E provably satisfies the guarded standard equation system $\mathcal{E}_1 = \{X_i = E_i | 1 \leq i \leq 4\}$:

$$\begin{array}{rcl} X_1 & = & a.X_2 \\ X_2 & = & \tau.X_3 + b.X_4 \\ X_3 & = & \Delta(b.X_4) \\ X_4 & = & 0 \end{array}$$

The expression F provably satisfies the guarded standard equation system $\mathcal{E}_2 = \{Y_j = F_j | 1 \leq j \leq 3\}$:

$$\begin{array}{rcl} Y_1 & = & a.Y_2 \\ Y_2 & = & \Delta(b.Y_3) \\ Y_3 & = & 0 \end{array}$$

Let $\vec{E} = (E'_1, ..., E'_4)$, where $E'_1 = a.E'_2$, $E'_2 = \tau.E'_3 + b.E'_4$, $E'_3 = \Delta(b.E'_4)$ and $E'_4 = 0$. This shows that E provably satisfies $\mathcal{E}_1$. Let $\vec{F} = (F'_1, ..., F'_3)$, where $F'_1 = a.F'_2$, $F'_2 = \Delta(b.F'_3)$ and $F'_3 = 0$. This shows that F provably satisfies $\mathcal{E}_2$.

We now construct an equation system $\mathcal{E} = \{Z_{i,j} = G_{i,j} | (i,j) \in I\}$ after which we prove that both E and F provably satisfy $\mathcal{E}$. Let $I = \{(i,j) | 1 \leq i \leq m, 1 \leq j \leq n, E'_i \underline{\leftrightarrow}_b^{\Delta} F'_j\}$, then we have $I = \{(1,1), (2,2), (3,2)(4,3)\}$. For every $(i,j) \in I$ let $Z_{i,j}$ be a new variable and let $\vec{Z} = (Z_{i,j})_{(i,j) \in I}$. Furthermore, for $(i,j) \in I$ we define

$$\begin{array}{rcl} H_{i,j} & = & \sum\{a.Z_{k,l} | X_i \xrightarrow{a} X_k, Y_j \xrightarrow{a} Y_l, E'_k \underline{\leftrightarrow}_b^{\Delta} F'_l\} + \\ & & \sum\{\tau.Z_{k,j} | (i,j) \neq (1,1) \wedge X_i \xrightarrow{\tau} X_k \wedge E'_k \underline{\leftrightarrow}_b^{\Delta} F'_j\} + \\ & & \sum\{\tau.Z_{i,l} | (i,j) \neq (1,1) \wedge Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}_b^{\Delta} F'_l\} + \\ & & \sum\{W | W \text{ is totally unguarded in } E'_i \text{ and in } F'_j)\} \end{array}$$

$$G_{i,j} = \begin{cases} \Delta(H_{i,j}) & \text{if } X_i, Y_j \in \Omega^{\Delta} \\ H_{i,j} & \text{otherwise} \end{cases}$$

Then we obtain $\mathcal{E} = \{Z_{i,j} = G_{i,j} | (i,j) \in I\}$:

$$\begin{array}{rcl} Z_{1,1} & = & a.Z_{2,2} \\ Z_{2,2} & = & \tau.Z_{3,2} + b.Z_{4,3} \\ Z_{3,2} & = & \Delta(b.Z_{4,3}) \\ Z_{4,3} & = & 0 \end{array}$$

In order to show that E provably satisfies $\mathcal{E}$ we construct an ordered sequence $\vec{R} = (R_{i,j})_{(i,j) \in I}$. First we define for $(i,j) \in I$ $S_{i,j}$ by

$$\begin{array}{rcl} S_{i,j} & = & \sum\{a.E'_k | X_i \xrightarrow{a} X_k, Y_j \xrightarrow{a} Y_l, E'_k \underline{\leftrightarrow}_b^{\Delta} F'_l\} + \\ & & \sum\{\tau.E'_k | (i,j) \neq (1,1) \wedge X_i \xrightarrow{\tau} X_k \wedge E'_k \underline{\leftrightarrow}_b^{\Delta} F'_j\} + \\ & & \sum\{W | W \text{ is totally unguarded in } E'_i \text{ and in } F'_j)\} \end{array}$$

We define $R_{i,j}$ as follows:

$$R_{i,j} = \begin{cases} E'_i & \text{if } (X_i \in \Omega^\Sigma \text{ or } Y_j \in \Omega^\Sigma) \text{ and } ((i,j) = (1,1) \text{ or } \neg\exists l\{Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}_b^\Delta F'_l\}) \\ \Delta(\tau.E'_i + S_{i,j}) & \text{if } X_i, Y_j \in \Omega^\Delta \\ \tau.E'_i + S_{i,j} & \text{otherwise} \end{cases}$$

We obtain for $\vec{R}$:

$$\begin{aligned} R_{1,1} &= E'_1 = a.E'_2 \\ R_{2,2} &= \tau.E'_2 + S_{2,2} = \tau.E'_2 + \tau.E'_3 + b.E'_4 \\ R_{3,2} &= \Delta(\tau.E'_3 + S_{3,2}) = \Delta(\tau.E'_3 + b.E'_4) \\ R_{4,3} &= E'_4 = 0 \end{aligned}$$

For $G_{i,j}\{\vec{R}/\vec{Z}\}$ we obtain:

$$\begin{aligned} G_{1,1}\{\vec{R}/\vec{Z}\} &= a.R_{2,2} = a.(\tau.E'_2 + \tau.E'_3 + b.E'_4) \\ G_{2,2}\{\vec{R}/\vec{Z}\} &= \tau.R_{3,2} + b.R_{4,3} + \tau.R_{2,2} = \tau.\Delta(\tau.E'_3 + b.E'_4) + b.E'_4 + \tau.(\tau.E'_2 + \tau.E'_3 + b.E'_4) \\ G_{3,2}\{\vec{R}/\vec{Z}\} &= \Delta(b.R_{4,3}) = \Delta(b.E'_4) \\ G_{4,3}\{\vec{R}/\vec{Z}\} &= 0 \end{aligned}$$

So $R_{1,1} = E$. We have that $R_{1,1} = G_{1,1}\{\vec{R}/\vec{Z}\}$:

$$\begin{aligned} a.E'_2 &\stackrel{B}{=} a.(\tau.E'_2 + E'_2) \\ &= a.(\tau.E'_2 + \tau.E'_3 + b.E'_4) \end{aligned}$$

We have that $R_{2,2} = G_{2,2}\{\vec{R}/\vec{Z}\}$:

$$\begin{aligned} \tau.E'_2 + \tau.E'_3 + b.E'_4 &= \tau.(\tau.E'_3 + b.E'_4) + \tau.\Delta(b.E'_4) + b.E'_4 \\ &\stackrel{B}{=} \tau.(\tau.(\tau.E'_3 + b.E'_4) + \tau.E'_3 + b.E'_4) + \tau.\Delta(b.E'_4) + b.E'_4 \\ &= \tau.(\tau.E'_2 + \tau.E'_3 + b.E'_4) + \tau.\Delta(b.E'_4) + b.E'_4 \\ &\stackrel{D3}{=} \tau.(\tau.E'_2 + \tau.E'_3 + b.E'_4) + \tau.\Delta(\Delta(b.E'_4)) + b.E'_4 \\ &\stackrel{D2}{=} \tau.(\tau.E'_2 + \tau.E'_3 + b.E'_4) + \tau.\Delta(\tau.\Delta(b.E'_4) + b.E'_4) + b.E'_4 \\ &= \tau.\Delta(\tau.E'_3 + b.E'_4) + b.E'_4 + \tau.(\tau.E'_2 + \tau.E'_3 + b.E'_4) \end{aligned}$$

We have that $R_{3,2} = G_{3,2}\{\vec{R}/\vec{Z}\}$:

$$\begin{aligned} \Delta(\tau.E'_3 + b.E'_4) &= \Delta(\tau.\Delta(b.E'_4) + b.E'_4) \\ &\stackrel{D2}{=} \Delta(\Delta(b.E'_4)) \\ &\stackrel{D3}{=} \Delta(b.E'_4) \end{aligned}$$

So we have $R_{i,j} = G_{i,j}\{\vec{R}/\vec{Z}\}$, so E provably satisfies $\mathcal{E}$.

In order to show that F provably satisfies $\mathcal{E}$ we construct a different ordered sequence $\vec{R} = (R_{i,j})_{(i,j)\in I}$. First we define for $(i,j) \in I$ $S_{i,j}$ by

$$\begin{aligned} S_{i,j} = &\sum \{a.F'_l | X_i \xrightarrow{a} X_k, Y_j \xrightarrow{a} Y_l, E'_k \underline{\leftrightarrow}_b^\Delta F'_l\} + \\ &\sum \{\tau.F'_l | (i,j) \neq (1,1) \wedge Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}_b^\Delta F'_l\} + \\ &\sum \{W | W \text{ is totally unguarded in } E'_i \text{ and in } F'_j)\} \end{aligned}$$

We define $R_{i,j}$ as follows:

$$R_{i,j} = \begin{cases} F'_j & \text{if } (X_i \in \Omega^\Sigma \text{ or } Y_j \in \Omega^\Sigma) \text{ and } ((i,j) = (1,1) \text{ or } \neg\exists k\{X_i \xrightarrow{\tau} X_k \wedge E'_k \underline{\leftrightarrow}_b^\Delta F'_j\}) \\ \Delta(\tau.F'_j + S_{i,j}) & \text{if } X_i, Y_j \in \Omega^\Delta \\ \tau.F'_j + S_{i,j} & \text{otherwise} \end{cases}$$

We obtain for $\vec{R}$:

$$\begin{aligned} R_{1,1} &= F'_1 = a.F'_2 \\ R_{2,2} &= \tau.F'_2 + S_{2,2} = \tau.F'_2 + b.F'_3 \\ R_{3,2} &= \Delta(\tau.F'_2 + S_{3,2}) = \Delta(\tau.F'_2 + b.F'_3) \\ R_{4,3} &= F'_3 = 0 \end{aligned}$$

For $G_{i,j}\{\vec{R}/\vec{Z}\}$ we obtain:

$$\begin{aligned} G_{1,1}\{\vec{R}/\vec{Z}\} &= a.R_{2,2} = a.(\tau.F'_2 + b.F'_3) \\ G_{2,2}\{\vec{R}/\vec{Z}\} &= \tau.R_{3,2} + b.R_{4,3} = \Delta(\tau.F'_2 + b.F'_3) + b.F'_3 \\ G_{3,2}\{\vec{R}/\vec{Z}\} &= \Delta(b.R_{4,3}) = \Delta(b.F'_3) \\ G_{4,3}\{\vec{R}/\vec{Z}\} &= 0 \end{aligned}$$

So $R_{1,1} = F$. We have that $R_{1,1} = G_{1,1}\{\vec{R}/\vec{Z}\}$:

$$\begin{aligned} a.F'_2 &= a.(\Delta(b.F'_3)) \\ &\overset{D2}{=} a.(\tau.\Delta(b.F'_3) + b.F'_3) \\ &= a.(\tau.F'_2) + b.F'_3) \end{aligned}$$

We have that $R_{2,2} = G_{2,2}\{\vec{R}/\vec{Z}\}$:

$$\begin{aligned} \tau.F'_2 + b.F'_3 &= \tau.\Delta(b.F'_3) + b.F'_3 \\ &\overset{S3}{=} \tau.\Delta(b.F'_3) + b.F'_3 + b.F'_3 \\ &\overset{D2}{=} \Delta(b.F'_3) + b.F'_3 \\ &\overset{D3}{=} \Delta(\Delta(b.F'_3)) + b.F'_3 \\ &\overset{D2}{=} \Delta(\tau.\Delta(b.F'_3) + b.F'_3) + b.F'_3 \\ &= \Delta(\tau.F'_2 + b.F'_3) + b.F'_3 \end{aligned}$$

We have that $R_{3,2} = G_{3,2}\{\vec{R}/\vec{Z}\}$:

$$\begin{aligned} \Delta(\tau.F'_2 + b.F'_3) &= \Delta(\tau.\Delta(b.F'_3) + b.F'_3) \\ &\overset{D2}{=} \Delta(\Delta(b.F'_3)) \\ &\overset{D3}{=} \Delta(b.F'_3) \end{aligned}$$

So we have $R_{i,j} = G_{i,j}\{\vec{R}/\vec{Z}\}$, so F provably satisfies $\mathcal{E}$.

In Lemma 7.3.1 we prove that a guarded expression can be rewritten such that this expression is a summation over action-prefixes and free variables. This lemma is used in Lemma 7.3.2.

Lemma 7.3.1. *Let $E \in \mathbb{E}$ be guarded, then*

$$E = \sum\{a.E' | E \xrightarrow{a} E'\} + \sum\{Y | Y \textit{ is totally unguarded in } E\}$$

Proof. We prove this using induction on the structure of E, the only interesting cases are when $E = recX.E'$ and when $E = \Delta(E')$, the other cases are trivial.

Case $E = recX.E'$

By induction we have that $E' = \sum\{a.E''|E' \xrightarrow{a} E''\} + \sum\{Y|Y \text{ is totally unguarded in } E'\}$. By axiom rec2 we have $recX.E' = E'\{recX.E'/X\}$. Since E is guarded we know the variable X is guarded in E', so $E'\{recX.E'/X\}$ has the required form.

Case $E = \Delta(E')$

By induction we have that $E' = \sum\{a.E''|E' \xrightarrow{a} E''\} + \sum\{Y|Y \text{ is totally unguarded in } E'\}$. By axiom D2 we have $\Delta(E') = \tau.\Delta(E') + E'$, where $\tau.\Delta(E') + E'$ has the required form. □

Lemma 7.3.2. *Let $E, F \in \mathbb{E}$ such that $E \underline{\leftrightarrow}^{\Delta}_{rb} F$. Furthermore, E provably satisfies the guarded standard equation system $\mathcal{E}_1 = \{X_i = E_i | 1 \leq i \leq m\}$ and F provably satisfies the guarded standard equation system $\mathcal{E}_2 = \{Y_j = F_j | 1 \leq j \leq n\}$. Then there exists a guarded equation system $\mathcal{E}$ such that both E and F provably satisfy $\mathcal{E}$.*

Proof. Since E provably satisfies the guarded standard equation system $\mathcal{E}_1 = \{X_i = E_i | 1 \leq i \leq m\}$ there exists a sequence of expressions $\vec{E} = (E'_1, ..., E'_m)$, such that $E'_i = E_i\{\vec{E}/\vec{X}\}$ for all $1 \leq i \leq m$ and $E = E'_1$. Since F provably satisfies the guarded standard equation system $\mathcal{E}_2 = \{Y_j = F_j | 1 \leq j \leq n\}$ there exists a sequence of expressions $\vec{F} = (F'_1, ..., F'_n)$, such that $F'_j = F_j\{\vec{F}/\vec{Y}\}$ for all $1 \leq j \leq n$ and $F = F'_1$. W.l.o.g. we may assume $X_1, Y_1 \in \Omega^{\Sigma}$.

Let $I = \{(i,j) | 1 \leq i \leq m, 1 \leq j \leq n, E'_i \underline{\leftrightarrow}^{\Delta}_b F'_j\}$. For every $(i,j) \in I$ let $Z_{i,j}$ be a new variable and let $\vec{Z} = (Z_{i,j})_{(i,j)\in I}$. Furthermore, for $(i,j) \in I$ we define

$$\begin{aligned} H_{i,j} \quad = \quad & \sum\{a.Z_{k,l} | X_i \xrightarrow{a} X_k, Y_j \xrightarrow{a} Y_l, E'_k \underline{\leftrightarrow}^{\Delta}_b F'_l\} + \\ & \sum\{\tau.Z_{k,j} | (i,j) \neq (1,1) \wedge X_i \xrightarrow{\tau} X_k \wedge E'_k \underline{\leftrightarrow}^{\Delta}_b F'_j\} + \\ & \sum\{\tau.Z_{i,l} | (i,j) \neq (1,1) \wedge Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}^{\Delta}_b F'_l\} + \\ & \sum\{W | W \text{ is totally unguarded in } E'_i \text{ and in } F'_j)\} \end{aligned}$$

$$G_{i,j} = \begin{cases} \Delta(H_{i,j}) & \text{if } X_i, Y_j \in \Omega^{\Delta} \\ H_{i,j} & \text{otherwise} \end{cases}$$

The equation system $\mathcal{E}$ over the formal variables $\vec{Z}$ contains for every $(i,j) \in I$ the equation $Z_{i,j} = G_{i,j}$. In order to show that $\mathcal{E}$ is guarded we have to show that $\mathcal{E}$ does not contain a cycle of τ-steps. So it is should not be the case that $Z_{i,j} \to^+ Z_{k,l} \to^+ Z_{i,j}$, where $(k,l) \neq (i,j)$. If $Z_{i,j} \to^+ Z_{k,l} \to^+ Z_{i,j}$ then $X_i \to^+ X_k \to^+ X_i$, where $k \neq i$ or $Y_j \to^+ Y_l \to^+ Y_j$, where $l \neq j$, but since both $\mathcal{E}_1$ and $\mathcal{E}_2$ are guarded this is not the case. Note that $\mathcal{E}$ is also standard, since $H_{i,j}$ is a sum of expressions $a.Z_{k,l}$ and W. So if $G_{i,j} = H_{i,j}$, then $Z_{i,j} \in \Omega^{\Sigma}$ and if $G_{i,j} = \Delta(H_{i,j})$, then $Z_{i,j} \in \Omega^{\Delta}$.

We will show that E provably satisfies $\mathcal{E}$; proving that F also provably satisfies $\mathcal{E}$ can be done analogously. In order to prove that E provably satisfies $\mathcal{E}$ we have to show that there exists a sequence of expressions which provably satisfies $\mathcal{E}$ and E is equal to the first expression in the sequence. We define for every $(i,j) \in I$ an expression $R_{i,j}$. Let $\vec{R} = (R_{i,j})_{(i,j)\in I}$, we will construct $\vec{R}$ such that $\vec{R}$ provably satisfies $\mathcal{E}$ and that $R_{1,1} = E$.

For $(i,j) \in I$ we define $S_{i,j}$ by

$$\begin{array}{rcl} S_{i,j} & = & \sum\{a.E'_k | X_i \xrightarrow{a} X_k, Y_j \xrightarrow{a} Y_l, E'_k \underline{\leftrightarrow}_b^{\Delta} F'_l\} + \\ & & \sum\{\tau.E'_k | (i,j) \neq (1,1) \wedge X_i \xrightarrow{\tau} X_k \wedge E'_k \underline{\leftrightarrow}_b^{\Delta} F'_j\} + \\ & & \sum\{W | W \text{ is totally unguarded in } E'_i \text{ and in } F'_j)\} \end{array}$$

We define $R_{i,j}$ as follows. We let $R_{i,j} = E'_i$ if $X_i \in \Omega^{\Sigma}$ or $Y_j \in \Omega^{\Sigma}$ and we are at the root or there are no inert τ-transitions from Y_j. A τ-transition from E to E' is inert if E and E' are branching bisimilar with explicit divergence. We let $R_{i,j} = \Delta(\tau.E'_i + S_{i,j})$ if $X_i, Y_j \in \Omega^{\Delta}$ and $R_{i,j} = \tau.E'_i + S_{i,j}$ otherwise.

$$R_{i,j} = \begin{cases} E'_i & \text{if } (X_i \in \Omega^{\Sigma} \text{ or } Y_j \in \Omega^{\Sigma}) \text{ and } ((i,j) = (1,1) \text{ or } \neg\exists l\{Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}_b^{\Delta} F'_l\}) \\ \Delta(\tau.E'_i + S_{i,j}) & \text{if } X_i, Y_j \in \Omega^{\Delta} \\ \tau.E'_i + S_{i,j} & \text{otherwise} \end{cases}$$

We have $R_{1,1} = E'_1 = E$. It remains to prove that $R_{i,j} = G_{i,j}\{\vec{R}/\vec{Z}\}$.

We have:

$$\begin{array}{rcl} H_{i,j}\{\vec{R}/\vec{Z}\} & = & \sum\{a.R_{k,l} | X_i \xrightarrow{a} X_k, Y_j \xrightarrow{a} Y_l, E'_k \underline{\leftrightarrow}_b^{\Delta} F'_l\} + \\ & & \sum\{\tau.R_{k,j} | (i,j) \neq (1,1) \wedge X_i \xrightarrow{\tau} X_k \wedge E'_k \underline{\leftrightarrow}_b^{\Delta} F'_j\} + \\ & & \sum\{\tau.R_{i,l} | (i,j) \neq (1,1) \wedge Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}_b^{\Delta} F'_l\} + \\ & & \sum\{W | W \text{ is totally unguarded in } E'_i \text{ and in } F'_j)\} \end{array}$$

We prove $a.R_{i,j} = a.E'_i$ in order to obtain

$$H_{i,j}\{\vec{R}/\vec{Z}\} \quad = \quad S_{i,j} + \sum\{\tau.E'_i | (i,j) \neq (1,1) \wedge Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}_b^{\Delta} F'_l\}$$

If $R_{i,j} = E'_i$ we have $a.R_{i,j} = a.E'_i$.

From Lemma 7.3.1 it follows that $E'_i = E'_i + S_{i,j}$. From this we can derive $a.(\Delta(\tau.E'_i + S_{i,j})) = a.E'_i$, for $E'_i = \Delta(E')$ as follows:

$$\begin{array}{rcl} a.(\Delta(\tau.E'_i + S_{i,j})) & = & a.(\Delta(\tau.\Delta(E') + S_{i,j})) \\ & \overset{D3}{=} & a.(\Delta(\tau.\Delta(\Delta(E')) + S_{i,j})) \\ & = & a.(\Delta(\tau.\Delta(E'_i) + S_{i,j})) \\ & = & a.(\Delta(\tau.\Delta(E'_i + S_{i,j}) + S_{i,j})) \\ & \overset{D1}{=} & a.(\Delta(E'_i + S_{i,j})) \\ & = & a.(\Delta(E'_i)) \\ & = & a.(\Delta(\Delta(E'))) \\ & \overset{D3}{=} & a.(\Delta(E')) \\ & = & a.E'_i \end{array}$$

Using $E'_i = E'_i + S_{i,j}$ we can derive $a.(\tau.E'_i + S_{i,j}) = a.E'_i$ as follows:

$$\begin{array}{rcl} a.(\tau.E'_i + S_{i,j}) & = & a.(\tau.(E'_i + S_{i,j}) + S_{i,j}) \\ & \overset{B}{=} & a.(E'_i + S_{i,j}) \\ & = & a.E'_i \end{array}$$

We distinguish two cases. In case 1 $X_i \in \Omega^{\Sigma}$ or $Y_j \in \Omega^{\Sigma}$ and in case 2 $X_i, Y_j \in \Omega^{\Delta}$.

Case 1. Assume $X_i \in \Omega^\Sigma$ or $Y_j \in \Omega^\Sigma$. Then we have $G_{i,j} = H_{i,j}$.

Let $(i,j) = (1,1)$, then $R_{1,1} = E'_1$ and $H_{1,1}\{\vec{R}/\vec{Z}\} = S_{1,1}$. In order to show that $R_{1,1} = G_{1,1}\{\vec{R}/\vec{Z}\}$ we have to show that $E'_1 = S_{1,1}$, by Lemma 7.3.1 it suffices to show that:

If $E'_1 \xrightarrow{a} E'_k$ then $\exists l\{F'_1 \xrightarrow{a} F'_l \wedge E'_k \underline{\leftrightarrow}^\Delta_b F'_l\}$ and
if W is totally unguarded in E'_1 then W is totally unguarded in F'_1

This follows from $E'_1 \underline{\leftrightarrow}^\Delta_{rb} F'_1$ and Corollary 3.0.2.

Suppose $\neg\exists l\{Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}^\Delta_b F'_l\}$, then $R_{i,j} = E'_i$ and $H_{i,j}\{\vec{R}/\vec{Z}\} = S_{i,j}$. Using Lemma 7.3.1 we can show that $E'_i = S_{i,j}$ by showing that:

If $E'_i \xrightarrow{a} E'_k$ then $\exists l\{F'_j \xrightarrow{a} F'_l \wedge E'_k \underline{\leftrightarrow}^\Delta_b F'_l\}$ or $a = \tau$ and $E'_k \underline{\leftrightarrow}^\Delta_b F'_j$ and
if W is totally unguarded in E'_i then W is totally unguarded in F'_j

Let $E'_i \xrightarrow{a} E'_k$, since $E'_i \underline{\leftrightarrow}^\Delta_b F'_j$ we have by Corollary 3.0.1 that $F'_j \twoheadrightarrow F'_p \xrightarrow{(a)} F'_l$, where $E'_i \underline{\leftrightarrow}^\Delta_b F'_p$ and $E'_k \underline{\leftrightarrow}^\Delta_b F'_l$. If $F'_j \xrightarrow{(a)} F'_l$ then $\exists l\{F'_j \xrightarrow{a} F'_l \wedge E'_k \underline{\leftrightarrow}^\Delta_b F'_l\}$ or $a = \tau$ and $E'_k \underline{\leftrightarrow}^\Delta_b F'_j$. So let $F'_j \xrightarrow{\tau} F'_o \twoheadrightarrow F'_p \xrightarrow{(a)} F'_l$. Then $\exists l\{Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}^\Delta_b F'_l\}$, which is not the case since we assumed $\neg\exists l\{Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}^\Delta_b F'_l\}$.
If W is totally unguarded in E'_i then since $E'_i \underline{\leftrightarrow}^\Delta_b F'_j$ we have $F'_j \twoheadrightarrow F'_p$, where W is totally unguarded in F'_p. However we can not have $F'_j \twoheadrightarrow F'_p$ for the same reason as above. So if W is totally unguarded in E'_i then W is totally unguarded in F'_j.

Otherwise we have $R_{i,j} = \tau.E'_i + S_{i,j} = H_{i,j}\{\vec{R}/\vec{Z}\}$.

Case 2. Assume $X_i, Y_j \in \Omega^\Delta$. Then we have:

$$\begin{aligned} G_{i,j}\{\vec{R}/\vec{Z}\} &= \Delta(H_{i,j})\{\vec{R}/\vec{Z}\} \\ &= \Delta(S_{i,j} + \sum\{\tau.E'_i \mid (i,j) \neq (1,1) \wedge Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}^\Delta_b F'_l\}) \end{aligned}$$

Since $X_i, Y_j \in \Omega^\Delta$ we know $(i,j) \neq (1,1)$ and $Y_j \xrightarrow{\tau} Y_l \wedge E'_i \underline{\leftrightarrow}^\Delta_b F'_l$. So

$$G_{i,j}\{\vec{R}/\vec{Z}\} = \Delta(S_{i,j} + \tau.E'_i) = R_{i,j}$$

So $R_{1,1} = E$ and $R_{i,j} = G_{i,j}\{\vec{R}/\vec{Z}\}$. From this we can conclude that E provably satisfies $\mathcal{E}$.

□

Chapter 8

Conclusions and future work

We have shown that rooted branching bisimulation with explicit divergence is a congruence and that it is the coarsest congruence contained in the branching bisimulation with explicit divergence relation. We have also given a sound and complete axiomatisation for rooted branching bisimulation with explicit divergence, where we have proven completeness for closed and open expressions.

In the axiomatisation we have given, the number of axioms concerning recursion is quite large. For future work it is possible to investigate whether certain axioms can be merged or whether a set of axioms can be replaced by one axiom to create a smaller axiomatisation. The language we have considered contains a constant nil, action prefix, the choice operator and recursion. The language can however be extended, for instance with operators for sequential composition or parallelism.

Appendix A

Properties of the transition relation of a labelled transition system

The following lemmas state some useful properties about the transition relation of a labelled transition system.

Lemma A.0.3. *For $G, G', E \in \mathbb{E}$, if $G \xrightarrow{a} G'$, then $G\{E/X\} \xrightarrow{a} G'\{E/X\}$.*

Proof. This is Lemma 1 in [1]. □

Lemma A.0.4. *For $G, E, E' \in \mathbb{E}$, if $E \xrightarrow{a} E'$ and the variable X is totally unguarded in G then $G\{E/X\} \xrightarrow{a} E'$.*

Proof. This is Lemma 2 in [1]. □

Lemma A.0.5. *Let $G\{E/X\} \xrightarrow{a} H$ be derivable by a derivation tree of height n. Then one of the following two cases holds:*

1. *X is totally unguarded in G and $E \xrightarrow{a} H$, which can be derived by a derivation tree of height at most n.*
2. *$G \xrightarrow{a} G'$ and $H = G'\{E/X\}$. Furthermore, if X is guarded in G and $a = \tau$ then X is also guarded in G'.*

Proof. This is Lemma 3 in [1]. □

Lemma A.0.6. *Let $E, E' \in \mathbb{E}$, then $recX.E \xrightarrow{a} E'$ if and only if there exists an $E_1 \in \mathbb{E}$ with $E \xrightarrow{a} E_1$ and $E' = E_1\{recX.E/X\}$.*

Proof. This is Lemma 6 in [1]. □

Lemma A.0.7. *Let $G, E \in \mathbb{E}$. There exists an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{E/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$ if and only if either there is an infinite sequence of states $(G_k)_{k\in\omega}$ such that $G = G_0$, $G_k\{E/X\} = P_k$ and $G_k \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$, or $G \twoheadrightarrow H$ such that $P_i = H\{E/X\}$ for some $i \geq 0$, X is totally unguarded in H and there is an infinite sequence of states $(E_l)_{l\in\omega}$ such that $E = E_0$, $E_l = P_{i+l}$ and $E_l \xrightarrow{\tau} E_{l+1}$ for all $l \in \omega$.*

Proof. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{E/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. By Lemma A.0.5 we obtain that either $P_k = G_k\{E/X\}$, where $G_k \xrightarrow{\tau} G_{k+1}$ or $G \twoheadrightarrow H$ such that $P_i = H\{E/X\}$ for some $i \geq 0$, where X is totally unguarded in

H and $H\{E/X\} \xrightarrow{\tau} E_1$, where $E \xrightarrow{\tau} E_1$.
In case $P_k = G_k\{E/X\}$, where $G_k \xrightarrow{\tau} G_{k+1}$ there is an infinite sequence of states $(G_k)_{k\in\omega}$ such that $G = G_0$, $G_k\{E/X\} = P_k$ and $G_k \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$.
In case $G \twoheadrightarrow H$ such that $P_i = H\{E/X\}$ for some $i \geq 0$, where X is totally unguarded in H and $H\{E/X\} \xrightarrow{\tau} E_1$, where $E \xrightarrow{\tau} E_1$ it must be the case that there is an infinite sequence of τ-steps starting from E, since there is an infinite sequence starting from $G\{E/X\}$. We have $E_1 = P_{i+1}$, so in general $E_l = P_{i+l}$. So $G \twoheadrightarrow H$ such that $P_i = H\{E/X\}$ for some $i \geq 0$, X is totally unguarded in H and there is an infinite sequence of states $(E_l)_{l\in\omega}$ such that $E = E_0$, $E_l = P_{i+l}$ and $E_l \xrightarrow{\tau} E_{l+1}$ for all $l \in \omega$.

Assume there is an infinite sequence of states $(G_k)_{k\in\omega}$ such that $G = G_0$, $G_k\{E/X\} = P_k$ and $G_k \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$. Since $G_k \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$, by Lemma A.0.3 $G_k\{E/X\} \xrightarrow{\tau} G_{k+1}\{E/X\}$ where $G = G_0$. So there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{E/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$.
Assume $G \twoheadrightarrow H$ such that $P_i = H\{E/X\}$ for some $i \geq 0$, X is totally unguarded in H and there is an infinite sequence of states $(E_l)_{l\in\omega}$ such that $E = E_0$, $E_l = P_{i+l}$ and $E_l \xrightarrow{\tau} E_{l+1}$ for all $l \in \omega$. Since $G \twoheadrightarrow H$ by Lemma A.0.3 $G\{E/X\} \twoheadrightarrow H\{E/X\}$. Since $E \xrightarrow{\tau} E_1$ by Lemma A.0.4 $H\{E/X\} \xrightarrow{\tau} E_1$. So there is an infinite sequence of τ-steps starting from $G\{E/X\}$. □

Lemma A.0.8. *Let $G, E \in \mathbb{E}$, if there exists an infinite sequence of states $(P_k)_{k\in\omega}$ such that $G\{recX.E/X\} = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$ then there exist states $(G_k)_{k\in\omega}$ such that $P_k = G_k\{recX.E/X\}$ for all $k \in \omega$.*

Proof. We prove this using induction on k.

For $k = 0$ we have $P_0 = G\{recX.E/X\}$, so choose $G_0 = G$.

Suppose the sequence has been defined up to k. Then $P_k = G_k\{recX.E/X\}$ and $G_k\{recX.E/X\} \xrightarrow{\tau} P_{k+1}$. Using Lemma A.0.5 we have that either X is totally unguarded in G_k and $recX.E \xrightarrow{\tau} P_{k+1}$ or $G_k \xrightarrow{\tau} G_{k+1}$ and $P_{k+1} = G_{k+1}\{recX.E/X\}$.
Case 1. Let X be totally unguarded in G_k and $recX.E \xrightarrow{\tau} P_{k+1}$. By Lemma A.0.6 we have that $E \xrightarrow{\tau} E'$ and $P_{k+1} = E'\{recX.E/X\}$. So choose $G_{k+1} = E'$.
Case 2. If $G_k \xrightarrow{\tau} G_{k+1}$ and $P_{k+1} = G_{k+1}\{recX.E/X\}$ we are done.

□

Appendix B

Rooted branching bisimulation with explicit divergence for open expressions

In Lemma B.0.10 it is proven that the relation $\underline{\leftrightarrow}_b^\Delta$ satisfies Condition (T) from Definition 3.0.2 and Condition (D) from Definition 3.0.3 for expressions $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. In Lemma B.0.11 it is proven that the relation $\underline{\leftrightarrow}_{rb}^\Delta$ satisfies Condition (R) from Definition 3.0.4 for expressions $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. In order to prove this Lemma B.0.9 is needed.

Lemma B.0.9. *Let $E, F \in \mathbb{E}$ and $b \in A \backslash \{\tau\}$. There exists an $n \in \omega$ such that for all derivatives G of E and F it holds that $G\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$.*

Proof. Let $\mathbb{G}$ be the set of all derivatives of E and F and define $\mathbb{W}$ such that for all $G \in \mathbb{G}$ we have

$$\mathbb{W} = \{m \in \omega \cup \infty | G \underline{\leftrightarrow}_b^\Delta b^m\}$$

Let $k \in \omega$ be the highest number such that $k \in \mathbb{W} \backslash \infty$, since $\mathbb{W} \backslash \infty$ is a set of natural numbers such a number exists. Let $l \in \omega$ such that $l = k + 1$. Then $l \notin \mathbb{W} \backslash \infty$. Since $l \neq \infty$ also $l \notin \mathbb{W}$. So we have an l such that $G \not\underline{\leftrightarrow}_b^\Delta b^l$. Since $b \neq \tau$ we have $b^i \not\underline{\leftrightarrow}_b^\Delta b^j$ for $i \neq j$, so $b^{l+1} \not\underline{\leftrightarrow}_b^\Delta b^l$. Let $n \in \omega$ such that $n = l + 1$. Then since $G \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$ and $b^n \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$ we have $G\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$. □

Since there exists an $n \in \omega$ such that for all derivatives G of E and F it holds that $G\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$, we can conclude Corollary B.0.1 after proving Lemma B.0.10.

Lemma B.0.10. *Let $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. Let $b \in A \backslash \{\tau\}$ and $n \in \omega$ such that for all derivatives G of E and F it holds that $G\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$. Furthermore, let*

$$\mathcal{R} = \{(E, F) | E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F\{b^n/X\}\}$$

Then $\mathcal{R}$ satisfies Condition (T) from Definition 3.0.2 and Condition (D) from Definition 3.0.3.

Proof. In order to show that $\mathcal{R}$ satisfies Condition (T) from Definition 3.0.2 and Condition (D) from Theorem 3.0.1 it needs to be shown that if $E \xrightarrow{a} E_1$, then $F \twoheadrightarrow F_1 \xrightarrow{(a)} F_2$ such that $E \mathrel{\mathcal{R}} F_1$ and $E_1 \mathrel{\mathcal{R}} F_2$. It also needs to be shown that if there is an infinite sequence of states $(P_k)_{k \in \omega}$ such that $E = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there exists a state F_1 such that $F \xrightarrow{\tau} F_1$ and $P_k \mathrel{\mathcal{R}} F_1$ for some $k \in \omega$. Since $\underline{\leftrightarrow}_b^\Delta$ is symmetric we have that $\mathcal{R} \cup \mathcal{R}^{-1} = \mathcal{R}$, so it suffices to consider a pair (E, F).

Assume $E \xrightarrow{a} E_1$. Then by Lemma A.0.3, $E\{b^n/X\} \xrightarrow{a} E_1\{b^n/X\}$. Since $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F\{b^n/X\}$ by Definition 3.0.2 $F\{b^n/X\} \twoheadrightarrow F' \xrightarrow{(a)} F''$ where $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F'$ and $E_1\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F''$. By Lemma A.0.5 and since $b \neq \tau$ we have $F \twoheadrightarrow F_1$ and $F' = F_1\{b^n/X\}$. Since $F_1\{b^n/X\} \xrightarrow{(a)} F''$ either $F_1\{b^n/X\} \xrightarrow{a} F''$ or ($a = \tau$ and $F'' = F_1\{b^n/X\}$). If $F_1\{b^n/X\} \xrightarrow{a} F''$ then by Lemma A.0.5 either X is totally unguarded in F_1 and $b^n \xrightarrow{a} F''$ or $F_1 \xrightarrow{a} F_2$ and $F'' = F_2\{b^n/X\}$. Assume X is totally unguarded in F_1 and $b^n \xrightarrow{a} F''$, then $a = b$ and $F'' = b^{n-1}$. But $E_1\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$, so it must be the case that $F_1 \xrightarrow{a} F_2$ and $F'' = F_2\{b^n/X\}$. Since $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F_1\{b^n/X\}$ we have $E \mathcal{R} F_1$ by the definition of $\mathcal{R}$, since $E_1\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F_2\{b^n/X\}$ we have $E_1 \mathcal{R} F_2$ by the definition of $\mathcal{R}$. If $a = \tau$ and $F'' = F_1\{b^n/X\}$ then since $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F_1\{b^n/X\}$ we have $E \mathcal{R} F_1$ by the definition of $\mathcal{R}$. Since $E_1\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F_2\{b^n/X\}$ we have $E_1 \mathcal{R} F_2$ by the definition of $\mathcal{R}$. So Condition (T) from Definition 3.0.2 holds.
Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $E = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. Since $P_k \xrightarrow{\tau} P_{k+1}$, by Lemma A.0.3 $P_k\{b^n/X\} \xrightarrow{\tau} P_{k+1}\{b^n/X\}$. Since $E\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F\{b^n/X\}$ and $E = P_0$ there exists a state F' such that $F\{b^n/X\} \xrightarrow{\tau} F'$ and $P_k\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F'$ for some $k \in \omega$. Since $F\{b^n/X\} \xrightarrow{\tau} F'$ and $b \neq \tau$, by Lemma A.0.5 $F \xrightarrow{\tau} F_1$ and $F' = F_1\{b^n/X\}$. Since $P_k\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F_1\{b^n/X\}$ for some $k \in \omega$, by the definition of $\mathcal{R}$ we have $P_k \mathcal{R} F_1$ for some $k \in \omega$. So Condition (D) from Theorem 3.0.1 holds. Since Condition (D) from Theorem 3.0.1 implies Condition (D) from Definition 3.0.3 as shown in [11], we have that Condition (D) from Definition 3.0.3 holds. □

Corollary B.0.1. *The relation $\underline{\leftrightarrow}_b^\Delta$ satisfies Condition (T) from Definition 3.0.2 and Condition (D) from Definition 3.0.3 for $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$.*

Lemma B.0.11. *Let $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$ and $E \underline{\leftrightarrow}_{rb}^\Delta F$. If $E \xrightarrow{a} E_1$, then $F \xrightarrow{a} F_1$ and $E_1 \underline{\leftrightarrow}_b^\Delta F_1$.*

Proof. Let $b \in A\backslash\{\tau\}$ and choose $n \in \omega$ such that for all derivatives G of E and F it holds that $G\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$. In Lemma B.0.9 it is proven that such an n exists.
Assume $E \underline{\leftrightarrow}_{rb}^\Delta F$ and $E \xrightarrow{a} E_1$. Then $E\{b^n/X\} \underline{\leftrightarrow}_{rb}^\Delta F\{b^n/X\}$ by how the relation is lifted. Since $E \xrightarrow{a} E_1$, by Lemma A.0.3 $E\{b^n/X\} \xrightarrow{a} E_1\{b^n/X\}$. By Definition 3.0.4 $F\{b^n/X\} \xrightarrow{a} F'$ such that $E_1\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F'$. Since $F\{b^n/X\} \xrightarrow{a} F'$, by Lemma A.0.5 either X is totally unguarded in F and $b^n \xrightarrow{a} F'$ or $F \xrightarrow{a} F_1$ and $F' = F_1\{b^n/X\}$. Assume X is totally unguarded in F and $b^n \xrightarrow{a} F'$, then $a = b$ and $F' = b^{n-1}$. But $E_1\{b^n/X\} \not\underline{\leftrightarrow}_b^\Delta b^{n-1}$, so it must be the case that $F \xrightarrow{a} F_1$ and $F' = F_1\{b^n/X\}$. It remains to prove that $E_1 \underline{\leftrightarrow}_b^\Delta F_1$. Since $E_1\{b^n/X\} \underline{\leftrightarrow}_b^\Delta F_1\{b^n/X\}$, by Lemma B.0.10 we obtain $E_1 \underline{\leftrightarrow}_b^\Delta F_1$.

From this we conclude that if $E \xrightarrow{a} E_1$, then $F \xrightarrow{a} F_1$ and $E_1 \underline{\leftrightarrow}_b^\Delta F_1$. □

Corollary B.0.2. *The relation $\underline{\leftrightarrow}_{rb}^\Delta$ satisfies Condition (R) from Definition 3.0.4 for $E, F \in \mathbb{E}$ such that $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$.*

Appendix C

Congruence proofs

The section holds the actual proofs that rooted branching bisimulation with explicit divergence is an equivalence and that it is compatible with the defined grammar.

Lemma C.0.12. *Rooted branching bisimulation with explicit divergence is an equivalence*

Proof. In order to prove that rooted branching bisimulation with explicit divergence is an equivalence we need to show that it is reflexive, symmetric and transitive.

In order to show reflexivity we need to show that for all states P we have $P \underline{\leftrightarrow}^{\Delta}_{rb} P$, so if $P \xrightarrow{a} P'$ for some state P', then there is a P'' such that $P \xrightarrow{a} P''$ and $P' \underline{\leftrightarrow}^{\Delta}_{b} P''$. By Lemma 3.0.1 we know $\underline{\leftrightarrow}^{\Delta}_{b}$ is an equivalence, so it is reflexive. Take $P'' = P'$, then $P \xrightarrow{a} P''$ and $P' \underline{\leftrightarrow}^{\Delta}_{b} P''$ by the reflexivity of $\underline{\leftrightarrow}^{\Delta}_{b}$. So $P \underline{\leftrightarrow}^{\Delta}_{rb} P$.

In order to show symmetry we need to show that if for some states P and Q we have $P \underline{\leftrightarrow}^{\Delta}_{rb} Q$, then $Q \underline{\leftrightarrow}^{\Delta}_{rb} P$. Assume $P \underline{\leftrightarrow}^{\Delta}_{rb} Q$ and $P \xrightarrow{a} P'$. By Definition 3.0.4 there exists a state Q' such that $Q \xrightarrow{a} Q'$ and $P' \underline{\leftrightarrow}^{\Delta}_{b} Q'$. By Lemma 3.0.1 we know $\underline{\leftrightarrow}^{\Delta}_{b}$ is an equivalence, so it is symmetric. So $P' \underline{\leftrightarrow}^{\Delta}_{b} Q'$ implies $Q' \underline{\leftrightarrow}^{\Delta}_{b} P'$. Since $Q \xrightarrow{a} Q'$ and $P \xrightarrow{a} P'$ the root condition is satisfied. So $Q \underline{\leftrightarrow}^{\Delta}_{rb} P$.

In order to show transitivity we need to show that if for some states P, Q and R we have $P \underline{\leftrightarrow}^{\Delta}_{rb} Q$ and $Q \underline{\leftrightarrow}^{\Delta}_{rb} R$, then $P \underline{\leftrightarrow}^{\Delta}_{rb} R$. Assume $P \xrightarrow{a} P'$ for some state P'. By Definition 3.0.4 we know $Q \xrightarrow{a} Q'$ and $P' \underline{\leftrightarrow}^{\Delta}_{b} Q'$. Since $Q \xrightarrow{a} Q'$ we know $R \xrightarrow{a} R'$ and $Q' \underline{\leftrightarrow}^{\Delta}_{b} R'$. By Lemma 3.0.1 we know $\underline{\leftrightarrow}^{\Delta}_{b}$ is an equivalence, so it is transitive. So $P' \underline{\leftrightarrow}^{\Delta}_{b} Q'$ and $Q' \underline{\leftrightarrow}^{\Delta}_{b} R'$ implies $P' \underline{\leftrightarrow}^{\Delta}_{b} R'$. Since $P \xrightarrow{a} P'$ and $R \xrightarrow{a} R'$ the root condition is satisfied. So $P \underline{\leftrightarrow}^{\Delta}_{rb} R$.

We conclude that rooted branching bisimulation with explicit divergence is reflexive, symmetric and transitive, so it is an equivalence. □

Lemma C.0.13. *Rooted branching bisimulation with explicit divergence is compatible with the defined grammar.*

Proof. Take $P_1, P_2, P_3, P_4 \in \mathbb{P}$, $E_1, E_2 \in \mathbb{E}$ and $a \in A$

Case *0*

This case is trivial.

Case $a.P$

Assume $P_1 \underline{\leftrightarrow}^{\Delta}_{rb} P_2$, to prove: $a.P_1 \underline{\leftrightarrow}^{\Delta}_{rb} a.P_2$. From the operational semantics it follows that if $a.P_1 \xrightarrow{b} P'$, then $b = a$ and $P' = P_1$, so $a.P_1 \xrightarrow{a} P_1$. Also, $a.P_2 \xrightarrow{a} P_2$. So Condition (T) from

Definition 3.0.2 holds. If $a = \tau$ it could be that there is an infinite sequence of states $(Q_k)_{k\in\omega}$ such that $a.P_1 = Q_0, Q_k \xrightarrow{\tau} Q_{k+1}$ and $Q_k \underline{\leftrightarrow}_b^\Delta a.P_2$ for all $k \in \omega$. Since $a.P_2 \xrightarrow{a} P_2$, $Q_1 = P_1$ and $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$, there exists an infinite sequence of states $(Q'_l)_{l\in\omega}$ such that $a.P_2 = Q'_0, Q'_k \xrightarrow{\tau} Q'_{k+1}$ for all $l \in \omega$, and $Q_k \underline{\leftrightarrow}_b^\Delta Q'_l$ for all $k, l \in \omega$. So Condition (D) of Definition 3.0.3 holds. So $a.P_1 \underline{\leftrightarrow}_b^\Delta a.P_2$. To prove $a.P_1 \underline{\leftrightarrow}_{rb}^\Delta a.P_2$ it needs to be shown that the root condition (Condition (R) from Definition 3.0.4) holds. Since $a.P_1 \xrightarrow{a} P_1$, $a.P_2 \xrightarrow{a} P_2$ and $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$ this is the case.

Case $P_1 + P_3$

Assume $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$ and $P_3 \underline{\leftrightarrow}_{rb}^\Delta P_4$, to prove: $P_1 + P_3 \underline{\leftrightarrow}_{rb}^\Delta P_2 + P_4$. Let $P_1 + P_3 \xrightarrow{a} P'$. Then by the operational semantics either $P_1 \xrightarrow{a} P'$ or $P_3 \xrightarrow{a} P'$. Since $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$ and by Condition (R) of Definition 3.0.4, if $P_1 \xrightarrow{a} P'$, then there exists a state P'_2 such that $P_2 \xrightarrow{a} P'_2$, with $P' \underline{\leftrightarrow}_b^\Delta P'_2$. For the same reason, if $P_3 \xrightarrow{a} P'$, then there exists a state P'_4 such that $P_4 \xrightarrow{a} P'_4$ and $P' \underline{\leftrightarrow}_b^\Delta P'_4$. So if $P_1 + P_3 \xrightarrow{a} P'$ then there exists a P'' such that $P_2 + P_4 \xrightarrow{a} P''$ with $P' \underline{\leftrightarrow}_b^\Delta P''$. So both Condition (T) from Definition 3.0.2 and Condition (R) from Definition 3.0.4 hold. Assume there is an infinite sequence of states $(Q_k)_{k\in\omega}$ such that $P_1 + P_3 = Q_0, Q_k \xrightarrow{\tau} Q_{k+1}$ and $Q_k \underline{\leftrightarrow}_b^\Delta P_2 + P_4$ for all $k \in \omega$. It can be assumed without loss of generality that $P_1 \xrightarrow{\tau} Q_1$. Hence since $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$, there exists an infinite sequence of states $(Q'_l)_{l\in\omega}$ such that $P_2 = Q'_0, Q'_l \xrightarrow{\tau} Q'_{l+1}$ for all $l \in \omega$, and $Q_k \underline{\leftrightarrow}_b^\Delta Q'_l$ for all $k, l \in \omega$. By the operational semantics $P_2 + P_4 \xrightarrow{\tau} Q'_1$, so Condition (D) of Definition 3.0.3 holds.

Case $recX.E$

Assume $E_1 \underline{\leftrightarrow}_{rb}^\Delta E_2$, to prove: $recX.E_1 \underline{\leftrightarrow}_{rb}^\Delta recX.E_2$. The relation $\underline{\leftrightarrow}_{rb}^\Delta$ is lifted such that $E \underline{\leftrightarrow}_{rb}^\Delta F$ if for all $\overrightarrow{P} = (P_1, ..., P_n)$ with $P_i \in \mathbb{P}$ we have $E\{\overrightarrow{P}/\overrightarrow{X}\} \underline{\leftrightarrow}_{rb}^\Delta F\{\overrightarrow{P}/\overrightarrow{X}\}$. Here $\overrightarrow{X} = (X_1, ..., X_n)$ is a sequence of variables that contains all variables from $\mathbb{V}(E) \cup \mathbb{V}(F)$. Due to the way the relation $\underline{\leftrightarrow}_{rb}^\Delta$ is lifted, it suffices to consider only those $E_1, E_2 \in \mathbb{E}$, where $\mathbb{V}(E_1) \cup \mathbb{V}(E_2) \subseteq \{X\}$. We will first show why this is the case:

Let $Y \in \mathbb{V}(E_1) \cup \mathbb{V}(E_2)$ and let $\mathbb{V}(recY.E_1) \cup \mathbb{V}(recY.E_2) \subseteq \{Z\}$. Then we have to prove that for all $\overrightarrow{P} = (P_1, ..., P_n)$ with $P_i \in \mathbb{P}$ we have

$$(recY.E_1)\{\overrightarrow{P}/Z\} \underline{\leftrightarrow}_{rb}^\Delta (recY.E_2)\{\overrightarrow{P}/Z\}$$

Since from this we can conclude that $recY.E_1 \underline{\leftrightarrow}_{rb}^\Delta recY.E_2$ because of how the relation is lifted. From $E_1 \underline{\leftrightarrow}_{rb}^\Delta E_2$ we have $E_1\{\overrightarrow{P}/Z\} \underline{\leftrightarrow}_{rb}^\Delta E_2\{\overrightarrow{P}/Z\}$ because of how the relation is lifted. Assume $E_1 \underline{\leftrightarrow}_{rb}^\Delta E_2$ implies $recX.E_1 \underline{\leftrightarrow}_{rb}^\Delta recX.E_2$ for $\mathbb{V}(E_1) \cup \mathbb{V}(E_2) \subseteq \{X\}$, then $E_1\{\overrightarrow{P}/Z\} \underline{\leftrightarrow}_{rb}^\Delta E_2\{\overrightarrow{P}/Z\}$ implies $(recY.E_1)\{\overrightarrow{P}/Z\} \underline{\leftrightarrow}_{rb}^\Delta (recY.E_2)\{\overrightarrow{P}/Z\}$. So we only need to consider those $E_1, E_2 \in \mathbb{E}$, where $\mathbb{V}(E_1) \cup \mathbb{V}(E_2) \subseteq \{X\}$.

Assume $E_1 \underline{\leftrightarrow}_{rb}^\Delta E_2$ holds. Then since the relation $\mathcal{R} \cup \mathcal{R}^{-1}$ appearing in Lemma 4.2.4 is a rooted branching bisimulation with explicit divergence up-to $\underline{\leftrightarrow}_b^\Delta$, by Lemma 4.2.1 it is also a rooted branching bisimulation with explicit divergence. Choosing $G = X$ implies $(recX.E_1, recX.E_2) \in \mathcal{R}$ and thus $recX.E_1 \underline{\leftrightarrow}_{rb}^\Delta recX.E_2$.

Case $\Delta(P)$

Assume $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$, to prove: $\Delta(P_1) \underline{\leftrightarrow}_{rb}^\Delta \Delta(P_2)$. From the operational semantics it follows that $\Delta(P_1) \xrightarrow{a} P'_1$ if $P_1 \xrightarrow{a} P'_1$ and $\Delta(P_2) \xrightarrow{a} P'_2$ if $P_2 \xrightarrow{a} P'_2$. Since for this rule it means that $\Delta(P_1)$ has the same behavior as P_1 and $\Delta(P_2)$ has the same behavior as P_2 and since $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$ only the case where a τ-loop is done needs to be considered. By the operational semantics $\Delta(P_1) \xrightarrow{\tau} \Delta(P_1)$. For Condition (T) from Definition 3.0.2 to hold there must exist states P' and P'' such that $\Delta(P_2) \twoheadrightarrow P'' \xrightarrow{(a)} P'$, $\Delta(P_1) \underline{\leftrightarrow}_b P''$ and $\Delta(P_1) \underline{\leftrightarrow}_b P'$. Take $P' = \Delta(P_2)$ and $P'' = \Delta(P_2)$. By the operational semantics $\Delta(P_2) \xrightarrow{\tau} \Delta(P_2)$, so $\Delta(P_2) \twoheadrightarrow \Delta(P_2) \xrightarrow{(a)} \Delta(P_2)$ and $\Delta(P_1) \underline{\leftrightarrow}_b \Delta(P_2)$. A τ-loop creates an infinite sequence of states $(Q_k)_{k\in\omega}$ such that $P_1 = Q_0$, $Q_k \xrightarrow{\tau} Q_{k+1}$ for all $k \in \omega$ and $Q_k \underline{\leftrightarrow}_b^\Delta P_2$, since $Q_k = P_1$ for all $k \in \omega$. For Condition (D) of Definition 3.0.3 to hold there must exist an infinite sequence of states $(Q'_l)_{l\in\omega}$, such that $P_2 = Q'_0$, $Q'_l \xrightarrow{\tau} Q'_{l+1}$ for all $l \in \omega$ and $Q_k \underline{\leftrightarrow}_b^\Delta Q'_l$ for all $k, l \in \omega$. Take as $Q'_l = P_2$ for all $l \in \omega$. Since the τ-loop, $Q'_l \xrightarrow{\tau} Q'_{l+1}$ and $Q_k \underline{\leftrightarrow}_b^\Delta Q'_l$ since $P_1 \underline{\leftrightarrow}_{rb}^\Delta P_2$. Since $\Delta(P_1) \xrightarrow{\tau} \Delta(P_1)$ and $\Delta(P_2) \xrightarrow{\tau} \Delta(P_2)$ and since $\Delta(P_1) \underline{\leftrightarrow}_b^\Delta \Delta(P_2)$ the root condition holds, so $\Delta(P_1) \underline{\leftrightarrow}_{rb}^\Delta \Delta(P_2)$. □

Appendix D

Soundness proofs

Here we prove soundness for the axioms, we do this by proving that there exists a rooted branching bisimulation with explicit divergence between the left-hand and right-hand side. In order to show that a relation $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence we need to prove Condition (T) from Definition 3.0.2, Condition (D) from Theorem 3.0.1 and Condition (R) from Definition 3.0.4.

Since Condition (R) implies Condition (T) and Condition (D) we only need to prove Condition (R). Let $\mathcal{R}$ be a rooted branching bisimulation with explicit divergence. Assume $P \mathrel{\mathcal{R}} Q$ and $P \xrightarrow{a} P'$ for some state $P' \in \mathbb{P}$. By Condition (R), $Q \xrightarrow{a} Q'$ and $P' \mathrel{\mathcal{R}} Q'$.
Since $Q \twoheadrightarrow Q \xrightarrow{(a)} Q'$, $P \mathrel{\mathcal{R}} Q$ and $P'\mathcal{R}Q'$ Condition (T) holds.
Let $a = \tau$ and let there be an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. Since $Q \xrightarrow{\tau} Q'$ and $P_1 = P' \mathrel{\mathcal{R}} Q'$ Condition (D) holds.

D.1 Axioms B and D1

In Lemma D.1.1 we prove soundness for axiom B and in Lemma D.1.2 we prove soundness for axiom D1.

Lemma D.1.1. *For $P, Q \in \mathbb{P}$, $a.(\tau.(P + Q) + P) \leftrightarrow^{\Delta}_{rb} a.(P + Q)$.*

Proof. According to Condition (R) from Definition 3.0.4 in order to prove that $a.(\tau.(P + Q) + P) \leftrightarrow^{\Delta}_{rb} a.(P + Q)$ we need to prove that if $a.(\tau.(P + Q) + P) \xrightarrow{b} S$, then $a.(P + Q) \xrightarrow{b} T$ where $S \leftrightarrow^{\Delta}_{b} T$ and if $a.(P + Q) \xrightarrow{b} T$ then $a.(\tau.(P + Q) + P) \xrightarrow{b} S$ and $T \leftrightarrow^{\Delta}_{b} S$. By the operational semantics if $a.(\tau.(P + Q) + P) \xrightarrow{b} S$, then $b = a$ and $S = \tau.(P + Q) + P$. Also by the operational semantics if $a.(P + Q) \xrightarrow{b} T$, then $b = a$ and $T = P + Q$. So if $\tau.(P + Q) + P \leftrightarrow^{\Delta}_{b} P + Q$ then $a.(\tau.(P + Q) + P) \leftrightarrow^{\Delta}_{rb} a.(P + Q)$.

Define $\mathcal{R}_0 = \{(\tau.(P + Q) + P, P + Q)|P, Q \in \mathbb{P}\} \cup \{(G, G)|G \in \mathbb{P}\}$. Then by proving that $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ is a branching bisimulation with explicit divergence we prove that $\tau.(P+Q)+P \leftrightarrow^{\Delta}_{b} P+Q$. We start by proving the conditions for $\{(\tau.(P + Q) + P, P + Q)|P, Q \in \mathbb{P}\}$. We need to prove Condition (T) from Definition 3.0.2 and Condition (D) from Theorem 3.0.2.

For Condition (T) we need to prove that if $\tau.(P + Q) + P \xrightarrow{a} S$ then there exist states T and T' such that $P + Q \twoheadrightarrow T' \xrightarrow{(a)} T$, $\tau.(P + Q) + P \mathrel{\mathcal{R}} T'$ and $S \mathrel{\mathcal{R}} T$. We also need to prove that if $P + Q \xrightarrow{a} T$ then there exist states S and S' such that $\tau.(P + Q) + P \twoheadrightarrow S' \xrightarrow{(a)} S$, $P + Q \mathrel{\mathcal{R}} S'$ and $T \mathrel{\mathcal{R}} S$.
Assume $\tau.(P + Q) + P \xrightarrow{a} S$, by the operational semantics either $\tau.(P + Q) \xrightarrow{\tau} P + Q$ or $P \xrightarrow{a} S$.

Let $\tau.(P+Q) \xrightarrow{\tau} P+Q$, we have $P+Q \twoheadrightarrow P+Q \xrightarrow{(\tau)} P+Q$, where $\tau.(P+Q)+P \ \mathcal{R}\ P+Q$ and $P+Q \ \mathcal{R}\ P+Q$ by the definition of $\mathcal{R}$. Let $P \xrightarrow{a} S$, then by the operational semantics $P+Q \twoheadrightarrow P+Q \xrightarrow{a} S$, where $\tau.(P+Q)+P \ \mathcal{R}\ P+Q$ and $S \ \mathcal{R}\ S$ by the definition of $\mathcal{R}$.
Assume $P+Q \xrightarrow{a} T$, then by the operational semantics $\tau.(P+Q)+P \xrightarrow{\tau} P+Q \xrightarrow{a} T$, so $\tau.(P+Q)+P \twoheadrightarrow P+Q \xrightarrow{a} T$, where $P+Q \ \mathcal{R}\ P+Q$ and $T \ \mathcal{R}\ T$ by how $\mathcal{R}$ is defined.

For Condition (D) we need to prove that if there is an infinite sequence of states $(S_k)_{k\in\omega}$ such that $\tau.(P+Q)+P = S_0$ and $S_k \xrightarrow{\tau} S_{k+1}$ for all $k \in \omega$, then there exists an infinite sequence of states $(T_l)_{l\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $P+Q = T_0$, $T_l \xrightarrow{\tau} T_{l+1}$ and $S_{\sigma(l)} \ \mathcal{R}\ T_l$ for all $l \in \omega$. We also need to prove that if there is an infinite sequence of states $(T_l)_{l\in\omega}$ such that $P+Q = T_0$ and $T_l \xrightarrow{\tau} T_{l+1}$ for all $l \in \omega$, then there exists an infinite sequence of states $(S_k)_{k\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $\tau.(P+Q)+P = S_0$, $S_k \xrightarrow{\tau} S_{k+1}$ and $T_{\sigma(k)} \ \mathcal{R}\ S_k$ for all $k \in \omega$.
Assume there is an infinite sequence of states $(S_k)_{k\in\omega}$ such that $\tau.(P+Q)+P = S_0$ and $S_k \xrightarrow{\tau} S_{k+1}$ for all $k \in \omega$. Then either $\tau.(P+Q) \xrightarrow{\tau} P+Q$, where $S_1 = P+Q$ or $P \xrightarrow{\tau} S_1$. Let $\tau.(P+Q) \xrightarrow{\tau} P+Q$, then there is an infinite sequence of τ-steps starting from $P+Q$. So there exists an infinite sequence of states $(T_l)_{l\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $P+Q = T_0$, $T_l \xrightarrow{\tau} T_{l+1}$ and $S_{\sigma(l)} \ \mathcal{R}\ T_l$ for all $l \in \omega$. Since $G \ \mathcal{R}\ G$ for any $G \in \mathbb{P}$ the mapping can be defined as $\sigma(l) = l+1$. Let $P \xrightarrow{\tau} S_1$, by the operational semantics also $P+Q \xrightarrow{\tau} S_1$. So there exists an infinite sequence of states $(T_l)_{l\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $P+Q = T_0$, $T_l \xrightarrow{\tau} T_{l+1}$ and $S_{\sigma(l)} \ \mathcal{R}\ T_l$ for all $l \in \omega$. Since $G \ \mathcal{R}\ G$ for any $G \in \mathbb{P}$ the mapping can be defined as $\sigma(l) = l$.
Now assume there is an infinite sequence of states $(T_l)_{l\in\omega}$ such that $P+Q = T_0$ and $T_l \xrightarrow{\tau} T_{l+1}$ for all $l \in \omega$. Since $\tau.(P+Q)+P \xrightarrow{\tau} P+Q$ there is an infinite sequence of states $(S_k)_{k\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $\tau.(P+Q)+P = S_0$, $S_k \xrightarrow{\tau} S_{k+1}$ and $T_{\sigma(k)} \ \mathcal{R}\ S_k$ for all $k \in \omega$. Here the mapping is defined for $k = 0$ as $\sigma(0) = 0$ since $P+Q \ \mathcal{R}\ \tau.(P+Q)+P$ by the definition of $\mathcal{R}$. For $k > 0$ the mapping is defined as $\sigma(k) = k+1$ since $G \ \mathcal{R}\ G$ for any $G \in \mathbb{P}$ by the definition of $\mathcal{R}$.

Proving Conditions (T) and (D) for $\{(G,G)|G \in \mathbb{P}\}$ is straightforward.

Since $\mathcal{R}$ is a branching bisimulation with explicit divergence we conclude that $\tau.(P+Q)+P \ \underline{\leftrightarrow}_b^{\Delta}\ P+Q$ which implies $a.(\tau.(P+Q)+P) \ \underline{\leftrightarrow}_{rb}^{\Delta}\ a.(P+Q)$. □

Lemma D.1.2. *For $P,Q \in \mathbb{P}$, $a.\Delta(\tau.\Delta(P+Q)+Q) \ \underline{\leftrightarrow}_{rb}^{\Delta}\ a.\Delta(P+Q)$.*

Proof. According to Condition (R) from Definition 3.0.4 in order to prove that $a.\Delta(\tau.\Delta(P+Q)+Q) \ \underline{\leftrightarrow}_{rb}^{\Delta}\ a.\Delta(P+Q)$ we need to prove that if $a.\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{b} S$, then $a.\Delta(P+Q) \xrightarrow{b} T$ where $S \ \underline{\leftrightarrow}_b^{\Delta}\ T$ and if $a.\Delta(P+Q) \xrightarrow{b} T$ then $a.\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{b} S$ and $T \ \underline{\leftrightarrow}_b^{\Delta}\ S$. By the operational semantics if $a.\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{b} S$, then $b = a$ and $S = \Delta(\tau.\Delta(P+Q)+Q)$. Also by the operational semantics if $a.\Delta(P+Q) \xrightarrow{b} T$, then $b = a$ and $T = \Delta(P+Q)$. So if $\Delta(\tau.\Delta(P+Q)+Q) \ \underline{\leftrightarrow}_b^{\Delta}\ \Delta(P+Q)$ then $a.\Delta(\tau.\Delta(P+Q)+Q) \ \underline{\leftrightarrow}_{rb}^{\Delta}\ a.\Delta(P+Q)$.

Define $\mathcal{R}_0 = \{(\Delta(\tau.\Delta(P+Q)+Q), \Delta(P+Q))|P,Q \in \mathbb{P}\} \cup \{(G,G)|G \in \mathbb{P}\}$. Then by proving that $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ is a branching bisimulation with explicit divergence we prove that $\Delta(\tau.\Delta(P+Q)+Q) \ \underline{\leftrightarrow}_b^{\Delta}\ \Delta(P+Q)$. We start by proving the conditions for $\{(\Delta(\tau.\Delta(P+Q)+Q), \Delta(P+Q))|P,Q \in \mathbb{P}\}$. We need to prove Condition (T) from Definition 3.0.2 and Condition (D) from Theorem 3.0.1.

For Condition (T) we need to prove that if $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{a} S$ then there exist states T and T' such that $\Delta(P+Q) \twoheadrightarrow T' \xrightarrow{(a)} T$, $\Delta(\tau.\Delta(P+Q)+Q) \ \mathcal{R}\ T'$ and $S \ \mathcal{R}\ T$. We also need to prove that if $\Delta(P+Q) \xrightarrow{a} T$ then there exist states S and S' such that $\Delta(\tau.\Delta(P+Q)+Q) \twoheadrightarrow S' \xrightarrow{(a)} S$, $\Delta(P+Q) \ \mathcal{R}\ S'$ and $T \ \mathcal{R}\ S$.
Assume $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{a} S$, by the operational semantics either $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{\tau}$

$\Delta(\tau.\Delta(P+Q)+Q)$, $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{\tau} \Delta(P+Q)$ or $Q \xrightarrow{a} S$. Let $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{\tau} \Delta(\tau.\Delta(P+Q)+Q)$, we have $\Delta(P+Q) \twoheadrightarrow \Delta(P+Q) \xrightarrow{(\tau)} \Delta(P+Q)$, where $\Delta(\tau.\Delta(P+Q)+Q) \mathrel{\mathcal{R}} \Delta(P+Q)$ by the definition of $\mathcal{R}$. Let $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{\tau} \Delta(P+Q)$, we have $\Delta(P+Q) \twoheadrightarrow \Delta(P+Q) \xrightarrow{(\tau)} \Delta(P+Q)$, where $\Delta(\tau.\Delta(P+Q)+Q) \mathrel{\mathcal{R}} \Delta(P+Q)$ and $\Delta(P+Q) \mathrel{\mathcal{R}} \Delta(P+Q)$ by the definition of $\mathcal{R}$. Let $Q \xrightarrow{a} S$, we have $\Delta(P+Q) \twoheadrightarrow \Delta(P+Q) \xrightarrow{a} S$, where $\Delta(\tau.\Delta(P+Q)+Q) \mathrel{\mathcal{R}} \Delta(P+Q)$ and $S \mathrel{\mathcal{R}} S$ by the definition of $\mathcal{R}$.
Assume $\Delta(P+Q) \xrightarrow{a} T$, by the operational semantics either $\Delta(P+Q) \xrightarrow{\tau} \Delta(P+Q)$, $P \xrightarrow{a} T$ or $Q \xrightarrow{a} T$. Let $\Delta(P+Q) \xrightarrow{\tau} \Delta(P+Q)$, we have $\Delta(\tau.\Delta(P+Q)+Q) \twoheadrightarrow \Delta(\tau.\Delta(P+Q)+Q) \overset{(\tau)}{\Delta}(\tau.\Delta(P+Q)+Q)$, where $\Delta(P+Q) \mathrel{\mathcal{R}} \Delta(\tau.\Delta(P+Q)+Q)$ by the definition of $\mathcal{R}$. Let $P \xrightarrow{a} T$, we have $\Delta(\tau.\Delta(P+Q)+Q) \twoheadrightarrow \Delta(P+Q) \xrightarrow{a} T$, where $\Delta(P+Q) \mathrel{\mathcal{R}} \Delta(P+Q)$ and $T \mathrel{\mathcal{R}} T$ by the definition of $\mathcal{R}$. Let $Q \xrightarrow{a} T$, we have $\Delta(\tau.\Delta(P+Q)+Q) \twoheadrightarrow \Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{a} T$, where $\Delta(P+Q) \mathrel{\mathcal{R}} \Delta(\tau.\Delta(P+Q)+Q)$ and $T \mathrel{\mathcal{R}} T$ by the definition of $\mathcal{R}$.

For Condition (D) we need to prove that if there is an infinite sequence of states $(S_k)_{k\in\omega}$ such that $\Delta(\tau.\Delta(P+Q)+Q) = S_0$ and $S_k \xrightarrow{\tau} S_{k+1}$ for all $k \in \omega$, then there exists a state T such that $\Delta(P+Q) \xrightarrow{\tau} T$, and $S_k \mathrel{\mathcal{R}} T$ for some $k \in \omega$. We also need to prove that if there is an infinite sequence of states $(T_l)_{l\in\omega}$ such that $\Delta(P+Q) = T_0$ and $T_l \xrightarrow{\tau} T_{l+1}$ for all $l \in \omega$, then there exists a state S such that $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{\tau} S$, and $T_l \mathrel{\mathcal{R}} S$ for some $l \in \omega$.
Assume there is an infinite sequence of states $(S_k)_{k\in\omega}$ such that $\Delta(\tau.\Delta(P+Q)+Q) = S_0$ and $S_k \xrightarrow{\tau} S_{k+1}$ for all $k \in \omega$. By the operational semantics we have $\Delta(P+Q) \xrightarrow{\tau} \Delta(P+Q)$, where $\Delta(\tau.\Delta(P+Q)+Q) \mathrel{\mathcal{R}} \Delta(P+Q)$ by the definition of $\mathcal{R}$.
Now assume there is an infinite sequence of states $(T_l)_{l\in\omega}$ such that $\Delta(P+Q) = T_0$ and $T_l \xrightarrow{\tau} T_{l+1}$ for all $l \in \omega$. By the operational semantics we have $\Delta(\tau.\Delta(P+Q)+Q) \xrightarrow{\tau} \Delta(\tau.\Delta(P+Q)+Q)$, where $\Delta(P+Q) \mathrel{\mathcal{R}} \Delta(\tau.\Delta(P+Q)+Q)$ by the definition of $\mathcal{R}$.

Proving Conditions (T) and (D) for $\{(G,G) | G \in \mathbb{P}\}$ is straightforward.

Since $\mathcal{R}$ is a branching bisimulation with explicit divergence we conclude that $\Delta(\tau.\Delta(P+Q)+Q) \leftrightarroweq_b^{\Delta} \Delta(P+Q)$ which implies $a.\Delta(\tau.\Delta(P+Q)+Q) \leftrightarroweq_{rb}^{\Delta} a.\Delta(P+Q)$. □

D.2 Axioms rec6–rec10

Soundness for axioms rec6–rec10 is proven in Lemma D.2.1 until Lemma D.2.7. Soundness for these axioms is proven in the same manner for each axiom. A relation is defined of which it is proven that it is a (rooted) branching bisimulation with explicit divergence. These relations help in finding a rooted branching bisimulation with explicit divergence for the axioms. The proofs are similar to the proofs for axioms rec5 and rec6 in [1]. The proofs for axioms rec9 and rec10 are also similar to the proof of axiom R4 in [8].

Lemma D.2.1. *For $E, F \in \mathbb{E}$, $recX.(\Delta(X+E)+F) \leftrightarroweq_{rb}^{\Delta} recX.\Delta(E+F)$, where $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$.*

Proof. In Lemma D.2.2 it is proven that $\mathcal{R} \subseteq \leftrightarroweq_{rb}^{\Delta}$, where $\mathcal{R}$ is the relation appearing in Lemma D.2.2. Choosing $G = X$ in $\mathcal{R}_0$ implies $recX.(\Delta(X+E)+F) \leftrightarroweq_{rb}^{\Delta} recX.\Delta(E+F)$. □

Lemma D.2.2. *Let $\mathbb{V}(E) \cup \mathbb{V}(F) \subseteq \{X\}$. Let:*

$$P = recX.(\Delta(X+E)+F)$$
$$Q = recX.\Delta(E+F)$$

Define the following relations:

$\mathcal{R}_0 = \{(G\{P/X\}, G\{Q/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$
$\mathcal{R}_1 = \{(\Delta(P + E\{P/X\}), \Delta(E\{Q/X\} + F\{Q/X\}))\}$
$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1}$

Then $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence.

Proof. In order to prove that $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence we need to prove Condition (R) from Definition 3.0.4. If $R_1 \; \mathcal{R} \; R_2$ and $R_1 \xrightarrow{a} R_1'$ then $R_2 \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'.
First we consider the case where $R_1 \; \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \; R_2$. We use induction on the height of the derivation tree for the transition $R_1 \xrightarrow{a} R_1'$. The only case which differs from the proof in Lemma 6.2.2 is the case where $G = X$, so only this case is considered here.

Case $G = X$

Case 1. Assume $R_1 = P \xrightarrow{a} R_1'$ and $R_2 = Q$. Thus $recX.(\Delta(X + E) + F) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\Delta(P + E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_1'$. Then we have the following cases:

Case 1.1. Assume $a = \tau$ and $R_1' = \Delta(P+E\{P/X\})$. By the operational semantics $\Delta(E+F) \xrightarrow{\tau} \Delta(E + F)$, using Lemma A.0.6 we have $Q = recX.\Delta(E + F) \xrightarrow{\tau} \Delta(E\{Q/X\} + F\{Q/X\})$ and $\Delta(P + E\{P/X\}) \; \mathcal{R} \; \Delta(E\{Q/X\} + F\{Q/X\})$.

Case 1.2. Assume $P \xrightarrow{a} R_1'$, by induction $Q \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'.

Case 1.3. Assume $E\{P/X\} \xrightarrow{a} R_1'$, by induction $E\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'. Since $E\{Q/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\Delta(E\{Q/X\} + F\{Q/X\}) \xrightarrow{a} R_2'$, thus $Q \xrightarrow{a} R_2'$.

Case 1.4. Assume $F\{P/X\} \xrightarrow{a} R_1'$, by induction $F\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'. Since $F\{Q/X\} \xrightarrow{a} R_2'$ also $\Delta(E\{Q/X\} + F\{Q/X\}) \xrightarrow{a} R_2'$, thus $Q \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = Q \xrightarrow{a} R_1'$ and $R_2 = P$. Thus $recX.\Delta(E+F) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\Delta(E\{Q/X\} + F\{Q/X\}) \xrightarrow{a} R_1'$. Then we have the following cases:

Case 2.1. Assume $a = \tau$ and $R_1' = \Delta(E\{Q/X\} + F\{Q/X\})$. By the operational semantics $\Delta(X + E) + F \xrightarrow{\tau} \Delta(X + E)$, using Lemma A.0.6 we have $P = recX.(\Delta(X + E) + F) \xrightarrow{\tau} \Delta(P + E\{P/X\})$ and $\Delta(E\{Q/X\} + F\{Q/X\}) \; \mathcal{R} \; \Delta(P + E\{P/X\})$.

Case 2.2. Assume $E\{Q/X\} \xrightarrow{a} R_1'$, by induction $E\{P/X\} \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'. Since $E\{P/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\Delta(P+E\{P/X\})+F\{P/X\} \xrightarrow{a} R_2'$, thus $P \xrightarrow{a} R_2'$.

Case 2.3. Assume $F\{Q/X\} \xrightarrow{a} R_1'$, by induction $F\{P/X\} \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'. Since $F\{P/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\Delta(P+E\{P/X\})+F\{P/X\} \xrightarrow{a} R_2'$, thus $P \xrightarrow{a} R_2'$.

Now the case where $R_1 \; \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \; R_2$ is considered. For this we will make use of the case $R_1 \; \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \; R_2$.
Case 1. Assume $R_1 = \Delta(P + E\{P/X\}) \xrightarrow{a} R_1'$ and $R_2 = \Delta(E\{Q/X\} + F\{Q/X\})$. Then we have the following cases:

Case 1.1. Assume $a = \tau$ and $R_1' = \Delta(P+E\{P/X\})$. By the operational semantics $\Delta(E\{Q/X\} + F\{Q/X\}) \xrightarrow{\tau} \Delta(E\{Q/X\} + F\{Q/X\})$ and $\Delta(P + E\{P/X\}) \; \mathcal{R} \; \Delta(E\{Q/X\} + F\{Q/X\})$.

Case 1.2. Assume $P \xrightarrow{a} R_1'$, since $P \; \mathcal{R}_0 \; Q$ we have $Q \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'. Since $Q = recX.\Delta(E+F) \xrightarrow{a} R_2'$ by the operational semantics we get that $\Delta(E\{Q/X\}+F\{Q/X\}) \xrightarrow{a} R_2'$.

Case 1.3. Assume $E\{P/X\} \xrightarrow{a} R_1'$, since $E\{P/X\} \; \mathcal{R}_0 \; E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{a} R_2'$ and $R_1' \; \mathcal{R} \; R_2'$ for some R_2'. Since $E\{Q/X\} \xrightarrow{a} R_2'$ also $\Delta(E\{Q/X\} + F\{Q/X\}) \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = \Delta(E\{Q/X\} + F\{Q/X\}) \xrightarrow{a} R_1'$ and $R_2 = \Delta(P + E\{P/X\})$. Then we have the following cases:

Case 2.1 Assume $a = \tau$ and $R_1' = \Delta(E\{Q/X\} + F\{Q/X\})$. By the operational semantics $\Delta(P + E\{P/X\}) \xrightarrow{\tau} \Delta(P + E\{P/X\})$ and $\Delta(E\{Q/X\} + F\{Q/X\}) \; \mathcal{R} \; \Delta(P + E\{P/X\})$.

Case 2.2 Assume $E\{Q/X\} \xrightarrow{a} R_1'$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. Since $E\{P/X\} \xrightarrow{a} R_2'$ also $\Delta(P + E\{P/X\}) \xrightarrow{a} R_2'$.

Case 2.3. Assume $F\{Q/X\} \xrightarrow{a} R_1'$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. Since $F\{P/X\} \xrightarrow{a} R_2'$, also $\Delta(P + E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_2'$, thus $P \xrightarrow{a} R_2'$ and $\Delta(P + E\{P/X\}) \xrightarrow{a} R_2'$.

So $\mathcal{R}$ is a rooted branching bisimulation with explicit divergence. □

Lemma D.2.3. *For* $E, F, G \in \mathbb{E}$, $recX.(\tau.(X+E)+\tau.(X+F)+G) \underline{\leftrightarrow}_{rb}^{\Delta} recX.(\tau.(X+E+F)+G)$, *where* $\mathbb{V}(E) \cup \mathbb{V}(F) \cup \mathbb{V}(G) \subseteq \{X\}$.

Proof. In Lemma D.2.4 it is proven that $\mathcal{R} \subseteq \underline{\leftrightarrow}_b^{\Delta}$, where $\mathcal{R}$ is the relation appearing in Lemma D.2.4. Furthermore, $\mathcal{R}$ is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. Choosing $H = X$ in $\mathcal{R}_0$ implies $recX.(\tau.(X + E) + \tau.(X + F) + G) \underline{\leftrightarrow}_{rb}^{\Delta} recX.(\tau.(X + E + F) + G)$. □

Lemma D.2.4. *Let* $\mathbb{V}(E) \cup \mathbb{V}(F) \cup \mathbb{V}(G) \subseteq \{X\}$. *Let:*

$$P = recX.(\tau.(X + E) + \tau.(X + F) + G)$$
$$Q = recX.(\tau.(X + E + F) + G)$$

Define the following relations:

$$\mathcal{R}_0 = \{(H\{P/X\}, H\{Q/X\}) | \mathbb{V}(H) \subseteq \{X\}\}$$
$$\mathcal{R}_1 = \{(P + E\{P/X\}, Q + E\{Q/X\} + F\{Q/X\})\}$$
$$\mathcal{R}_2 = \{(P + F\{P/X\}, Q + E\{Q/X\} + F\{Q/X\})\}$$
$$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}$$

Then $\mathcal{R}$ *is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from* $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$.

Proof. In order to prove that $\mathcal{R}$ is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (T) from Definition 3.0.2, Condition (D) from Theorem 3.0.1 and for the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (R) from Definition 3.0.4. First we consider the case where $R_1\ \mathcal{R}_0 \cup \mathcal{R}_0^{-1}\ R_2$. For this case we only prove Condition (R) from Definition 3.0.4, since this implies Condition (T) from Definition 3.0.2 and Condition (D) from Theorem 3.0.1. We prove that if $R_1\ \mathcal{R}\ R_2$ and $R_1 \xrightarrow{a} R_1'$ then $R_2 \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We use induction on the height of the derivation tree for the transition $R_1 \xrightarrow{a} R_1'$. The only case which differs from the proof in Lemma 6.2.2 is the case where $G = X$, so only this case is considered here.

Case $G = X$

Case 1. Assume $R_1 = P \xrightarrow{a} R_1'$ and $R_2 = Q$. Thus $recX.(\tau.(X + E) + \tau.(X + F) + G) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\tau.(P + E\{P/X\}) + \tau.(P + F\{P/X\}) + G\{P/X\} \xrightarrow{a} R_1'$. Then we have the following cases:

Case 1.1. Assume $a = \tau$ and $R_1' = P+E\{P/X\}$. By the operational semantics $\tau.(X+E+F) \xrightarrow{\tau} X+E+F$, using Lemma A.0.6 we have $Q = recX.(\tau.(X+E+F)+G) \xrightarrow{\tau} Q+E\{Q/X\}+F\{Q/X\}$, where $P + E\{P/X\}\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$.

Case 1.2. Assume $a = \tau$ and $R_1' = P+F\{P/X\}$. By the operational semantics $\tau.(X+E+F) \xrightarrow{\tau} X+E+F$, using Lemma A.0.6 we have $Q = recX.(\tau.(X+E+F)+G) \xrightarrow{\tau} Q+E\{Q/X\}+F\{Q/X\}$, where $P + F\{P/X\}\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$.

Case 1.3. Assume $G\{P/X\} \xrightarrow{a} R_1'$, by induction $G\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. Since $G\{Q/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\tau.(Q + E\{Q/X\} + F\{Q/X\}) + G\{Q/X\} \xrightarrow{a} R_2'$, thus $Q \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = Q \xrightarrow{a} R'_1$ and $R_2 = P$. Thus $recX.(\tau.(X+E+F)+G) \xrightarrow{a} R'_1$. By a smaller derivation tree also $\tau.(Q+E\{Q/X\}+F\{Q/X\})+G\{Q/X\} \xrightarrow{a} R'_1$. Then we have the following cases:

Case 2.1. Assume $a=\tau$ and $R'_1 = Q+E\{Q/X\}+F\{Q/X\}$. By the operational semantics $\tau.(X+E)+\tau.(X+F)+G \xrightarrow{\tau} X+E$, using Lemma A.0.6 we have $P = recX.(\tau.(X+E)+\tau.(X+F)+G) \xrightarrow{\tau} P+E\{P/X\}$ where $Q+E\{Q/X\}+F\{Q/X\} \mathcal{R} P+E\{P/X\}$.

Case 2.2. Assume $G\{Q/X\} \xrightarrow{a} R'_1$, by induction $G\{P/X\} \xrightarrow{a} R'_2$ and $R'_1 \mathcal{R} R'_2$ for some R'_2. Since $G\{P/X\} \xrightarrow{a} R'_2$ by the operational semantics we get that $\tau.(P+E\{P/X\})+\tau.(P+F\{P/X\})+G\{P/X\} \xrightarrow{a} R'_2$, thus $P \xrightarrow{a} R'_2$.

Now the cases where $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1}\ R_2$ and $R_1\ \mathcal{R}_2 \cup \mathcal{R}_2^{-1}\ R_2$ are considered. For this we will make use of the case $R_1\ \mathcal{R}_0 \cup \mathcal{R}_0^{-1}\ R_2$. In order to prove Condition (T) from Definition 3.0.2 we need to prove that if $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}\ R_2$ and $R_1 \xrightarrow{a} R'_1$ then $R_2 \twoheadrightarrow R' \xrightarrow{(a)} R'_2$, $R_1 \mathcal{R} R'$ and $R'_1 \mathcal{R} R'_2$ for some R' and R'_2.

Case 1. Assume $R_1 = P+E\{P/X\} \xrightarrow{a} R'_1$ and $R_2 = Q+E\{Q/X\}+F\{Q/X\}$. Then we have the following cases:

Case 1.1. Let $P \xrightarrow{a} R'_1$, since $P\ \mathcal{R}_0\ Q$ we have $Q \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. Since $Q \xrightarrow{a} R'_2$ we have $Q+E\{Q/X\}+F\{Q/X\} \twoheadrightarrow Q+E\{Q/X\}+F\{Q/X\} \xrightarrow{a} R'_2$, where $P+E\{P/X\} \mathcal{R} Q+E\{Q/X\}+F\{Q/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 1.2. Let $E\{P/X\} \xrightarrow{a} R'_1$, since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. Since $E\{Q/X\} \xrightarrow{a} R'_2$ we have $Q+E\{Q/X\}+F\{Q/X\} \twoheadrightarrow Q+E\{Q/X\}+F\{Q/X\} \xrightarrow{a} R'_2$, where $P+E\{P/X\} \mathcal{R} Q+E\{Q/X\}+F\{Q/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 2. Assume $R_1 = Q+E\{Q/X\}+F\{Q/X\} \xrightarrow{a} R'_1$ and $R_2 = P+E\{P/X\}$. Then we have the following cases:

Case 2.1. Let $Q \xrightarrow{a} R'_1$, since $Q\ \mathcal{R}_0^{-1}\ P$ we have $P \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. Since $P \xrightarrow{a} R'_2$ we have $P+E\{P/X\} \twoheadrightarrow P+E\{P/X\} \xrightarrow{a} R'_2$, where $Q+E\{Q/X\}+F\{Q/X\} \mathcal{R} P+E\{P/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 2.2. Let $E\{Q/X\} \xrightarrow{a} R'_1$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. Since $E\{P/X\} \xrightarrow{a} R'_2$ we have $P+E\{P/X\} \twoheadrightarrow P+E\{P/X\} \xrightarrow{a} R'_2$, where $Q+E\{Q/X\}+F\{Q/X\} \mathcal{R} P+E\{P/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 2.3. Let $F\{Q/X\} \xrightarrow{a} R'_1$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R'_2$, where $R'_1 \mathcal{R} R'_2$. We have by Lemma A.0.6 that $P = recX.(\tau.(X+E)+\tau.(X+F)+G) \xrightarrow{\tau} P+F\{P/X\}$. So $P+E\{P/X\} \twoheadrightarrow P+F\{P/X\} \xrightarrow{a} R'_2$, where $Q+E\{Q/X\}+F\{Q/X\} \mathcal{R} P+F\{P/X\}$ and $R'_1 \mathcal{R} R'_2$.

Case 3. Assume $R_1 = P+F\{P/X\} \xrightarrow{a} R'_1$ and $R_2 = Q+E\{Q/X\}+F\{Q/X\}$. This case is similar to case 1.

Case 4. Assume $R_1 = Q+E\{Q/X\}+F\{Q/X\} \xrightarrow{a} R'_1$ and $R_2 = P+F\{P/X\}$. This case is similar to case 2.

Here we prove Condition (D) from Theorem 3.0.2. In order to prove this we need to prove that if $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}\ R_2$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $R_1 = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k \mathcal{R} R_2$ for all $k \in \omega$, then there exists a state R'_2 such that $R_2 \rightarrow^+ R'_2$ and $P_k \mathcal{R} R'_2$ for some $k \in \omega$.

Case 1. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P+E\{P/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k \mathcal{R} Q+E\{Q/X\}+F\{Q/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 1.1. Let $P \xrightarrow{\tau} P_1$, since $P\ \mathcal{R}_0\ Q$ we have $Q \xrightarrow{\tau} R'_2$, where $P_1 \mathcal{R} R'_2$. So $Q+E\{Q/X\}+F\{Q/X\} \rightarrow^+ R'_2$, where $P_1 \mathcal{R} R'_2$.

Case 1.2. Let $E\{P/X\} \xrightarrow{\tau} P_1$, since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{\tau} R'_2$, where $P_1 \mathcal{R} R'_2$. So $Q+E\{Q/X\}+F\{Q/X\} \rightarrow^+ R'_2$, where $P_1 \mathcal{R} R'_2$.

Case 2. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $Q+E\{Q/X\}+F\{Q/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ P+E\{P/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 2.1. Let $Q \xrightarrow{\tau} P_1$, since $Q\ \mathcal{R}_0^{-1}\ P$ we have $P \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. So $P+E\{P/X\} \rightarrow^+ R_2'$, where $P_1\ \mathcal{R}\ R_2'$.

Case 2.2. Let $E\{Q/X\} \xrightarrow{\tau} P_1$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. So $P+E\{P/X\} \rightarrow^+ R_2'$, where $P_1\ \mathcal{R}\ R_2'$.

Case 2.3. Let $F\{Q/X\} \xrightarrow{\tau} P_1$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. We have by Lemma A.0.6 that $P = recX.(\tau.(X+E)+\tau.(X+F)+G) \xrightarrow{\tau} P+F\{P/X\}$. So $P+E\{P/X\} \rightarrow^+ R_2'$, where $P_1\ \mathcal{R}\ R_2'$.

Case 3. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $P+F\{P/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ Q+E\{Q/X\}+F\{Q/X\}$ for all $k \in \omega$. This case is similar to case 1.

Case 4. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $Q+E\{Q/X\}+F\{Q/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ P+F\{P/X\}$ for all $k \in \omega$. This case is similar to case 2.

□

Lemma D.2.5. *For $E, F \in \mathbb{E}$, let X be unguarded in E, then $recX.(\Delta(E)+F) \underline{\leftrightarrow}^{\Delta}_{rb} recX.(\tau.X+E+F)$, where $\mathbb{V}(E)\cup\mathbb{V}(F) \subseteq \{X\}$.*

Proof. In Lemma D.2.6 it is proven that $\mathcal{R} \subseteq \underline{\leftrightarrow}^{\Delta}_{b}$, where $\mathcal{R}$ is the relation appearing in Lemma D.2.6. Furthermore, $\mathcal{R}$ is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. Choosing $G = X$ in $\mathcal{R}_0$ implies $recX.(\Delta(E)+F) \underline{\leftrightarrow}^{\Delta}_{rb} recX.(\tau.X+E+F)$. □

Lemma D.2.6. *Let $\mathbb{V}(E)\cup\mathbb{V}(F) \subseteq \{X\}$ and let X be unguarded in E. Let:*

$$P = recX.(\Delta(E)+F)$$
$$Q = recX.(\tau.X+E+F)$$

For $n \in \omega$ let there be $E_0...E_n$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} ... \xrightarrow{\tau} E_n$, where X is totally unguarded in E_n.

Define the following relations:

$$\mathcal{R}_0 = \{(G\{P/X\}, G\{Q/X\}) | \mathbb{V}(G) \subseteq \{X\}\}$$
$$\mathcal{R}_1 = \{(\Delta(E\{P/X\}), Q)\}$$
$$\mathcal{R}_2 = \{(E_i\{P/X\}, Q | 0 \le i \le n)\}$$
$$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}$$

Then $\mathcal{R}$ is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$.

Proof. In order to prove that $\mathcal{R}$ is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (T) from Definition 3.0.2, Condition (D) from Theorem 3.0.1 and for the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (R) from Definition 3.0.4. First we consider the case where $R_1\ \mathcal{R}_0 \cup \mathcal{R}_0^{-1}\ R_2$. For this case we only prove Condition (R) from Definition 3.0.4, since this implies Condition (T) from Definition 3.0.2 and Condition (D) from Theorem 3.0.1. We prove that if $R_1\ \mathcal{R}\ R_2$ and $R_1 \xrightarrow{a} R_1'$ then $R_2 \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We use induction on the height of the derivation tree for the transition $R_1 \xrightarrow{a} R_1'$. The only case which differs from the proof in Lemma 6.2.2 is the case where $G = X$, so only this case is considered here.

Case $G = X$

Case 1. Assume $R_1 = P \xrightarrow{a} R_1'$ and $R_2 = Q$. Thus $recX.(\Delta(E)+F) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_1'$. Then we have the following cases:

Case 1.1. Assume $a = \tau$ and $R_1' = \Delta(E\{P/X\})$. By the operational semantics $\tau.X \xrightarrow{\tau} X$, so by Lemma A.0.6 we have $Q \xrightarrow{\tau} Q$, where $\Delta(E\{P/X\})\ \mathcal{R}\ Q$.

Case 1.2. Assume $E\{P/X\} \xrightarrow{a} R_1'$, by induction $E\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. Since $E\{Q/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\tau.Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_2'$, thus $Q \xrightarrow{a} R_2'$.

Case 1.3. Assume $F\{P/X\} \xrightarrow{a} R_1'$, by induction $F\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. Since $F\{Q/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\tau.Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_2'$, thus $Q \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = Q \xrightarrow{a} R_1'$ and $R_2 = P$. Thus $recX.(\tau.X + E + F) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\tau.Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_1'$. Then we have the following cases:

Case 2.1. Assume $a = \tau$ and $R_1' = Q$. By the operational semantics $\Delta(E) \xrightarrow{\tau} \Delta(E)$, using Lemma A.0.6 we have $P = recX.(\Delta(E) + F) \xrightarrow{\tau} \Delta(E\{P/X\})$ where $Q\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 2.2. Assume $E\{Q/X\} \xrightarrow{a} R_1'$, by induction $E\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. Since $E\{P/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_2'$, thus $P \xrightarrow{a} R_2'$.

Case 2.3. Assume $F\{Q/X\} \xrightarrow{a} R_1'$, by induction $F\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. Since $F\{P/X\} \xrightarrow{a} R_2'$ by the operational semantics we get that $\Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_2'$, thus $P \xrightarrow{a} R_2'$.

Now the case where $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}\ R_2$ is considered. For this we will make use of the case $R_1\ \mathcal{R}_0 \cup \mathcal{R}_0^{-1}\ R_2$. In order to prove Condition (T) from Definition 3.0.2 we need to prove that if $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}\ R_2$ and $R_1 \xrightarrow{a} R_1'$ then $R_2 \twoheadrightarrow R' \xrightarrow{(a)} R_2'$, $R_1\ \mathcal{R}\ R'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R' and R_2'.

Case 1. Assume $R_1 = \Delta(E\{P/X\}) \xrightarrow{a} R_1'$ and $R_2 = Q$. Then we have the following cases:

Case 1.1. Let $a = \tau$ and $R_1' = \Delta(E\{P/X\})$. We have $Q \twoheadrightarrow Q$, where $\Delta(E\{P/X\})\ \mathcal{R}\ Q$.

Case 1.2. Let $E\{P/X\} \xrightarrow{a} R_1'$, since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{a} R_2'$, where $R_1'\ \mathcal{R}\ R_2'$. Since $E\{Q/X\} \xrightarrow{a} R_2'$ we have $Q = recX.(\tau.X + E + F) \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = Q \xrightarrow{a} R_1'$ and $R_2 = \Delta(E\{P/X\})$. Thus $recX.(\tau.X + E + F) \xrightarrow{a} R_1'$. Then we have the following cases:

Case 2.1. Let $a = \tau$ and $R_1' = Q$. We have $\Delta(E\{P/X\}) \twoheadrightarrow \Delta(E\{P/X\})$, where $Q\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 2.2. Let $E\{Q/X\} \xrightarrow{a} R_1'$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R_2'$, where $R_1'\ \mathcal{R}\ R_2'$. Since $E\{P/X\} \xrightarrow{a} R_2'$ we have $\Delta(E\{P/X\}) \xrightarrow{a} R_2'$.

Case 2.3. Let $F\{Q/X\} \xrightarrow{a} R_1'$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R_2'$, where $R_1'\ \mathcal{R}\ R_2'$. Since X is unguarded in E it does not lie within a subexpression $a.E'$ with $a \in A\backslash\{\tau\}$. So $E\{P/X\} \twoheadrightarrow E'\{P/X\}$, where X is totally unguarded in $E'\{P/X\}$. So by Lemma A.0.5 we have $E'\{P/X\} \xrightarrow{a} R_2'$. So $E\{P/X\} \twoheadrightarrow E'\{P/X\} \xrightarrow{a} R_2'$, where $Q\ \mathcal{R}\ E'\{P/X\}$ and $R_1'\ \mathcal{R}\ R_2'$.

Case 3. Assume for some i that $R_1 = E_i\{P/X\} \xrightarrow{a} R_1'$ and $R_2 = Q$. Since $E \twoheadrightarrow E_i$ we have $Q \twoheadrightarrow E_i\{Q/X\}$. Since $E_i\{P/X\}\ \mathcal{R}_0\ E_i\{Q/X\}$ we have $E_i\{Q/X\} \xrightarrow{a} R_2'$, where $R_1'\ \mathcal{R}\ R_2'$. So $Q \twoheadrightarrow E_i\{Q/X\} \xrightarrow{a} R_2'$, where $E_i\{P/X\}\ \mathcal{R}\ E_i\{Q/X\}$ and $R_1'\ \mathcal{R}\ R_2'$.

Case 4. Assume $R_1 = Q \xrightarrow{a} R_1'$ and for some i we have $R_2 = E_i\{P/X\}$. So $recX.(\tau.X + E + F) \xrightarrow{a} R_1'$. Then we have the following cases:

Case 4.1. Let $a = \tau$ and $R_1' = Q$. We have $E_i\{P/X\} \twoheadrightarrow E_i\{P/X\}$ and $Q\ \mathcal{R}\ E_i\{P/X\}$.

Case 4.2. Let $E\{Q/X\} \xrightarrow{a} R_1'$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R_2'$, where $R_1'\ \mathcal{R}\ R_2'$. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{a} R_2'$. So $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\} \xrightarrow{a} R_2'$, where $Q\ \mathcal{R}\ E_n\{P/X\}$ and $R_1'\ \mathcal{R}\ R_2'$.

Case 4.3. Let $F\{Q/X\} \xrightarrow{a} R_1'$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R_2'$, where $R_1'\ \mathcal{R}\ R_2'$. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{a} R_2'$. So $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\} \xrightarrow{a} R_2'$, where $Q\ \mathcal{R}\ E_n\{P/X\}$ and $R_1'\ \mathcal{R}\ R_2'$.

Here we prove Condition (D) from Theorem 3.0.2. In order to prove this we need to prove that if $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}\ R_2$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $R_1 = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ R_2$ for all $k \in \omega$, then there exists a state R_2' such that $R_2 \to^+ R_2'$ and $P_k\ \mathcal{R}\ R_2'$ for some $k \in \omega$.
Case 1. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $\Delta(E\{P/X\}) = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ Q$ for all $k \in \omega$. Then we have the following cases:

Case 1.1. Let $P_1 = \Delta(E\{P/X\})$. We have $Q \xrightarrow{\tau} Q$, where $\Delta(E\{P/X\})\ \mathcal{R}\ Q$.

Case 1.1. Let $E\{P/X\} \xrightarrow{\tau} P_1$. Since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. Since $E\{Q/X\} \xrightarrow{\tau} R_2'$ we have $Q \xrightarrow{\tau} R_2'$.

Case 2. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $Q = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ \Delta(E\{P/X\})$ for all $k \in \omega$. So $recX.(\tau.X + E + F) \xrightarrow{\tau} P_1$. Then we have the following cases:

Case 2.1. Let $P_1 = Q$. By the operational semantics we have $\Delta(E\{P/X\}) \to^+ \Delta(E\{P/X\})$, where $Q\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 2.2. Let $E\{Q/X\} \xrightarrow{\tau} P_1$. Since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. By the operational semantics $\Delta(E\{P/X\}) \to^+ R_2'$.

Case 2.3. Let $F\{Q/X\} \xrightarrow{\tau} P_1$. Since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. Since X is unguarded in E it does not lie within a subexpression $a.E'$ with $a \in A\backslash\{\tau\}$. So $E\{P/X\} \twoheadrightarrow E'\{P/X\}$, where X is totally unguarded in $E'\{P/X\}$. So by Lemma A.0.5 we have $E'\{P/X\} \xrightarrow{\tau} R_2'$. So $E\{P/X\} \to^+ R_2'$, where $P_1\ \mathcal{R}\ R_2'$.

Case 3. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that for some i we have $E_i\{P/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ Q$ for all $k \in \omega$. We have $Q \to^+ Q$ and $P_0\ \mathcal{R}\ Q$.

Case 4. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $Q = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and for some i we have $P_k\ \mathcal{R}\ E_i\{P/X\}$ for all $k \in \omega$. So $recX.(\tau.X + E + F) \xrightarrow{\tau} P_1$ and we have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. Then we have the following cases:

Case 4.1. Let $P_1 = Q$, we have $E_n\{P/X\} \xrightarrow{\tau} \Delta(E\{P/X\})$, so $E_i\{P/X\} \to^+ \Delta(E\{P/X\})$, where $Q\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 4.2. Let $E\{Q/X\} \xrightarrow{\tau} P_1$. Since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. So $E_i\{P/X\} \to^+ R_2'$, where $P_1\ \mathcal{R}\ R_2'$.

Case 4.3. Let $F\{Q/X\} \xrightarrow{\tau} P_1$. Since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1\ \mathcal{R}\ R_2'$. So $E_i\{P/X\} \to^+ R_2'$, where $P_1\ \mathcal{R}\ R_2'$.

□

Lemma D.2.7. *For $E, F, G \in \mathbb{E}$, let X be unguarded in E, then $recX.(\tau.(\Delta(E) + F) + G) \underline{\leftrightarrow}_{rb}^{\Delta}$ $recX.(\tau.(X + E + F) + G)$, where $\mathbb{V}(E) \cup \mathbb{V}(F) \cup \mathbb{V}(G) \subseteq \{X\}$.*

Proof. In Lemma D.2.8 it is proven that $\mathcal{R} \subseteq \underline{\leftrightarrow}_b^{\Delta}$, where $\mathcal{R}$ is the relation appearing in Lemma D.2.8. Furthermore, $\mathcal{R}$ is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$. Choosing $H = X$ in $\mathcal{R}_0$ implies $recX.(\tau.(\Delta(E) + F) + G) \underline{\leftrightarrow}_{rb}^{\Delta} recX.(\tau.(X + E + F) + G)$. □

Lemma D.2.8. *Let $\mathbb{V}(E) \cup \mathbb{V}(F) \cup \mathbb{V}(G) \subseteq \{X\}$ and let X be unguarded in E. Let:*

$$P = recX.(\tau.(\Delta(E) + F) + G)$$
$$Q = recX.(\tau.(X + E + F) + G)$$

For $n \in \omega$ let there be $E_0 ... E_n$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} ... \xrightarrow{\tau} E_n$, where X is totally unguarded in E_n.

Define the following relations:

$$\mathcal{R}_0 = \{(H\{P/X\}, H\{Q/X\}) | \mathbb{V}(H) \subseteq \{X\}\}$$
$$\mathcal{R}_1 = \{(\Delta(E\{P/X\}) + F\{P/X\}, Q + E\{Q/X\} + F\{Q/X\})\}$$
$$\mathcal{R}_2 = \{(\Delta(E\{P/X\}), Q + E\{Q/X\} + F\{Q/X\})\}$$
$$\mathcal{R}_3 = \{(E_i\{P/X\}, Q + E\{Q/X\} + F\{Q/X\} | 0 \leq i \leq n)\}$$

$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0^{-1} \cup \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1} \cup \mathcal{R}_3 \cup \mathcal{R}_3^{-1}$

Then $\mathcal{R}$ is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$.

Proof. In order to prove that $\mathcal{R}$ is a branching bisimulation with explicit divergence which is rooted with respect to the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (T) from Definition 3.0.2, Condition (D) from Theorem 3.0.1 and for the pairs from $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$ we need to prove Condition (R) from Definition 3.0.4. First we consider the case where $R_1\ \mathcal{R}_0 \cup \mathcal{R}_0^{-1}\ R_2$. For this case we only prove Condition (R) from Definition 3.0.4, since this implies Condition (T) from Definition 3.0.2 and Condition (D) from Theorem 3.0.1. We prove that if $R_1\ \mathcal{R}\ R_2$ and $R_1 \xrightarrow{a} R_1'$ then $R_2 \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We use induction on the height of the derivation tree for the transition $R_1 \xrightarrow{a} R_1'$. The only case which differs from the proof in Lemma 6.2.2 is the case where $G = X$, so only this case is considered here.

Case $G = X$

Case 1. Assume $R_1 = P \xrightarrow{a} R_1'$ and $R_2 = Q$. Thus $recX.(\tau.(\Delta(E) + F) + G) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\tau.(\Delta(E\{P/X\}) + F\{P/X\}) + G\{P/X\} \xrightarrow{a} R_1'$. Then we have the following cases:

Case 1.1. Assume $a = \tau$ and $R_1' = \Delta(E\{P/X\}) + F\{P/X\}$. By the operational semantics $\tau.(X + E + F) \xrightarrow{\tau} X + E + F$, so by Lemma A.0.6 we have $Q \xrightarrow{\tau} Q + E\{Q/X\} + F\{Q/X\}$. We have $\Delta(E\{P/X\}) + F\{P/X\}\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$.

Case 1.2. Assume $G\{P/X\} \xrightarrow{a} R_1'$, by induction $G\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. By the operational semantics we have $Q \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = Q \xrightarrow{a} R_1'$ and $R_2 = P$. Thus $recX.(\tau.(X + E + F) + G) \xrightarrow{a} R_1'$. By a smaller derivation tree also $\tau.(Q + E\{Q/X\} + F\{Q/X\}) + G\{Q/X\} \xrightarrow{a} R_1'$. Then we have the following cases:

Case 2.1. Assume $a = \tau$ and $R_1' = Q + E\{Q/X\} + F\{Q/X\}$. By the operational semantics $\tau.(\Delta(E) + F) \xrightarrow{\tau} \Delta(E) + F$, so by Lemma A.0.6 we have $P \xrightarrow{\tau} \Delta(E\{P/X\}) + F\{P/X\}$, where $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ \Delta(E\{P/X\}) + F\{P/X\}$.

Case 2.2. Assume $G\{Q/X\} \xrightarrow{a} R_1'$, by induction $G\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. By the operational semantics we have $P \xrightarrow{a} R_2'$.

Now the case where $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1} \cup \mathcal{R}_3 \cup \mathcal{R}_3^{-1}\ R_2$ is considered. For this we will make use of the case $R_1\ \mathcal{R}_0 \cup \mathcal{R}_0^{-1}\ R_2$. In order to prove Condition (T) from Definition 3.0.2 we need to prove that if $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1}\ R_2$ and $R_1 \xrightarrow{a} R_1'$ then $R_2 \twoheadrightarrow R' \xrightarrow{(a)} R_2'$, $R_1\ \mathcal{R}\ R'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R' and R_2'.

Case 1. Assume $R_1 = \Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_1'$ and $R_2 = Q + E\{Q/X\} + F\{Q/X\}$. Then we have the following cases:

Case 1.1. Let $a = \tau$ and $R_1' = \Delta(E\{P/X\})$. We have $Q + E\{Q/X\} + F\{Q/X\} \twoheadrightarrow Q + E\{Q/X\} + F\{Q/X\}$, where $\Delta(E\{P/X\})\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$.

Case 1.2. Let $E\{P/X\} \xrightarrow{a} R_1'$, since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \twoheadrightarrow Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_2'$.

Case 1.3. Let $F\{P/X\} \xrightarrow{a} R_1'$, since $F\{P/X\}\ \mathcal{R}_0\ F\{Q/X\}$ we have $F\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \twoheadrightarrow Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_2'$.

Case 2. Assume $R_1 = Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_1'$ and $R_2 = \Delta(E\{P/X\}) + F\{P/X\}$. Then we have the following cases:

Case 2.1. Let $Q \xrightarrow{a} R_1'$, so $recX.(\tau.(X + E + F) + G) \xrightarrow{a} R_1'$. Then we have the following cases:

Case 2.1.1. Let $a = \tau$ and $R_1' = Q + E\{Q/X\} + F\{Q/X\}$. We have $\Delta(E\{P/X\}) \twoheadrightarrow \Delta(E\{P/X\})$ and $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 2.1.2. Let $G\{Q/X\} \xrightarrow{a} R_1'$, since $G\{Q/X\}\ \mathcal{R}_0^{-1}\ G\{P/X\}$ we have $G\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We have $E\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{a} R_2'$. We have $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ E_n\{P/X\}$.

Case 2.2. Let $E\{Q/X\} \xrightarrow{a} R_1'$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. So $\Delta(E\{P/X\}) + F\{P/X\} \twoheadrightarrow \Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_2'$.

Case 2.3. Let $F\{Q/X\} \xrightarrow{a} R_1'$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. So $\Delta(E\{P/X\}) + F\{P/X\} \twoheadrightarrow \Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_2'$.

Case 3. Assume $R_1 = \Delta(E\{P/X\}) \xrightarrow{a} R_1'$ and $R_2 = Q + E\{Q/X\} + F\{Q/X\}$. Then we have the following cases:

Case 3.1. Let $a = \tau$ and $R_1' = \Delta(E\{P/X\})$. We have $Q + E\{Q/X\} + F\{Q/X\} \twoheadrightarrow Q + E\{Q/X\} + F\{Q/X\})$, where $\Delta(E\{P/X\})\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\})$.

Case 3.2. Let $E\{P/X\} \xrightarrow{a} R_1'$, since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \twoheadrightarrow Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_2'$.

Case 4. Assume $R_1 = Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_1'$ and $R_2 = \Delta(E\{P/X\})$. Then we have the following cases:

Case 4.1. Let $Q \xrightarrow{a} R_1'$, so $recX.(\tau.(X + E + F) + G) \xrightarrow{a} R_1'$. Then we have the following cases:

Case 4.1.1. Let $a = \tau$ and $R_1' = Q + E\{Q/X\} + F\{Q/X\}$. We have $\Delta(E\{P/X\}) \twoheadrightarrow \Delta(E\{P/X\})$ and $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 4.1.2. Let $G\{Q/X\} \xrightarrow{a} R_1'$, since $G\{Q/X\}\ \mathcal{R}_0^{-1}\ G\{P/X\}$ we have $G\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We have $E\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{a} R_2'$. We have $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ E_n\{P/X\}$.

Case 4.2. Let $E\{Q/X\} \xrightarrow{a} R_1'$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. So $\Delta(E\{P/X\}) \twoheadrightarrow \Delta(E\{P/X\}) \xrightarrow{a} R_2'$.

Case 4.3. Let $F\{Q/X\} \xrightarrow{a} R_1'$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We have $E\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{a} R_2'$. We have $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ E_n\{P/X\}$.

Case 5. Assume for some i we have $R_1 = E_i\{P/X\} \xrightarrow{a} R_1'$ and $R_2 = Q + E\{Q/X\} + F\{Q/X\}$. We have $E\{Q/X\} \twoheadrightarrow E_i\{Q/X\}$. Since $E_i\{P/X\}\ \mathcal{R}_0\ E_i\{Q/X\}$ we have $E_i\{Q/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \twoheadrightarrow E_i\{Q/X\} \xrightarrow{a} R_2'$, where $E_i\{P/X\}\ \mathcal{R}\ E_i\{Q/X\}$ and $R_1'\ \mathcal{R}\ R_2'$.

Case 6. Assume $R_1 = Q + E\{Q/X\} + F\{Q/X\} \xrightarrow{a} R_1'$ and for some i we have $R_2 = E_i\{P/X\}$. Then we have the following cases:

Case 6.1. Let $Q \xrightarrow{a} R_1'$, so $recX.(\tau.(X + E + F) + G) \xrightarrow{a} R_1'$. Then we have the following cases:

Case 6.1.1. Let $a = \tau$ and $R_1' = Q + E\{Q/X\} + F\{Q/X\}$. We have $E_i\{P/X\} \twoheadrightarrow E_i\{P/X\}$ and $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ E_i\{P/X\}$.

Case 6.1.2. Let $G\{Q/X\} \xrightarrow{a} R_1'$, since $G\{Q/X\}\ \mathcal{R}_0^{-1}\ G\{P/X\}$ we have $G\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{a} R_2'$. We have $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ E_n\{P/X\}$.

Case 6.2. Let $E\{Q/X\} \xrightarrow{a} R_1'$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{\tau} \Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_2'$, where $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ \Delta(E\{P/X\}) + F\{P/X\}$.

Case 6.3. Let $F\{Q/X\} \xrightarrow{a} R_1'$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{a} R_2'$ and $R_1'\ \mathcal{R}\ R_2'$ for some R_2'. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{\tau} \Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{a} R_2'$. We have $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ E_n\{P/X\}$.

Here we prove Condition (D) from Theorem 3.0.2. In order to prove this we need to prove that if $R_1\ \mathcal{R}_1 \cup \mathcal{R}_1^{-1} \cup \mathcal{R}_2 \cup \mathcal{R}_2^{-1} \cup \mathcal{R}_3 \cup \mathcal{R}_3^{-1}\ R_2$ and there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $R_1 = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ R_2$ for all $k \in \omega$, then there exists a state R_2' such that $R_2 \rightarrow^+ R_2'$ and $P_k\ \mathcal{R}\ R_2'$ for some $k \in \omega$.

Case 1. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $\Delta(E\{P/X\})+F\{P/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 1.1. Let $P_1 = \Delta(E\{P/X\})$, we have $Q \xrightarrow{\tau} Q + E\{Q/X\} + F\{Q/X\}$, so $Q + E\{Q/X\} + F\{Q/X\} \rightarrow^{+} Q + E\{Q/X\} + F\{Q/X\}$, where $P_1\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$.

Case 1.2. Let $E\{P/X\} \xrightarrow{\tau} P_1$, since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \rightarrow^{+} R_2'$.

Case 1.3. Let $F\{P/X\} \xrightarrow{\tau} P_1$, since $F\{P/X\}\ \mathcal{R}_0\ F\{Q/X\}$ we have $F\{Q/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \rightarrow^{+} R_2'$.

Case 2. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $Q+E\{Q/X\}+F\{Q/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ \Delta(E\{P/X\})+F\{P/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 2.1. Let $Q \xrightarrow{\tau} P_1$, so $recX.(\tau.(X+E+F)+G) \xrightarrow{\tau} P_1$. Then we have the following cases:

Case 2.1.1. Let $P_1 = Q+E\{Q/X\}+F\{Q/X\}$. By the operational semantics $\Delta(E\{P/X\}) \rightarrow^{+} \Delta(E\{P/X\})$ and $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 2.1.2. Let $G\{Q/X\} \xrightarrow{\tau} P_1$, since $G\{Q/X\}\ \mathcal{R}_0^{-1}\ G\{P/X\}$ we have $G\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. We have $E\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{\tau} R_2'$.

Case 2.2. Let $E\{Q/X\} \xrightarrow{\tau} P_1$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. So $\Delta(E\{P/X\}) + F\{P/X\} \rightarrow^{+} R_2'$.

Case 2.3. Let $F\{Q/X\} \xrightarrow{\tau} P_1$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. So $\Delta(E\{P/X\}) + F\{P/X\} \rightarrow^{+} R_2'$.

Case 3. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $\Delta(E\{P/X\}) = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 3.1. Let $P_1 = \Delta(E\{P/X\})$. By the operational semantics $\tau.(X + E + F) \xrightarrow{\tau} X + E + F$. So $Q+E\{Q/X\}+F\{Q/X\} \rightarrow^{+} Q+E\{Q/X\}+F\{Q/X\})$, where $\Delta(E\{P/X\})\ \mathcal{R}\ Q+E\{Q/X\}+F\{Q/X\})$.

Case 3.2. Let $E\{P/X\} \xrightarrow{\tau} P_1$, since $E\{P/X\}\ \mathcal{R}_0\ E\{Q/X\}$ we have $E\{Q/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \rightarrow^{+} R_2'$.

Case 4. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $Q+E\{Q/X\}+F\{Q/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ \Delta(E\{P/X\})$ for all $k \in \omega$. Then we have the following cases:

Case 4.1. Let $Q \xrightarrow{\tau} P_1$, so $recX.(\tau.(X+E+F)+G) \xrightarrow{\tau} P_1$. Then we have the following cases:

Case 4.1.1. Let $P_1 = Q+E\{Q/X\}+F\{Q/X\}$. By the operational semantics $\Delta(E\{P/X\}) \rightarrow^{+} \Delta(E\{P/X\})$ and $Q + E\{Q/X\} + F\{Q/X\}\ \mathcal{R}\ \Delta(E\{P/X\})$.

Case 4.1.2. Let $G\{Q/X\} \xrightarrow{\tau} P_1$, since $G\{Q/X\}\ \mathcal{R}_0^{-1}\ G\{P/X\}$ we have $G\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. We have $E\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{\tau} R_2'$.

Case 4.2. Let $E\{Q/X\} \xrightarrow{\tau} P_1$, since $E\{Q/X\}\ \mathcal{R}_0^{-1}\ E\{P/X\}$ we have $E\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. So $\Delta(E\{P/X\}) \rightarrow^{+} R_2'$.

Case 4.3. Let $F\{Q/X\} \xrightarrow{\tau} P_1$, since $F\{Q/X\}\ \mathcal{R}_0^{-1}\ F\{P/X\}$ we have $F\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. We have $E\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{\tau} R_2'$.

Case 5. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that for some i we have $E_i\{P/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k\ \mathcal{R}\ Q + E\{Q/X\} + F\{Q/X\}$ for all $k \in \omega$. We have $E\{Q/X\} \twoheadrightarrow E_i\{Q/X\}$. Since $E_i\{P/X\}\ \mathcal{R}_0\ E_i\{Q/X\}$ we have $E_i\{Q/X\} \xrightarrow{\tau} R_2'$ and $P_1\ \mathcal{R}\ R_2'$ for some R_2'. So $Q + E\{Q/X\} + F\{Q/X\} \rightarrow^{+} R_2'$, where $P_1\ \mathcal{R}\ R_2'$.

Case 6. Assume there is an infinite sequence of states $(P_k)_{k\in\omega}$ such that $Q+E\{Q/X\}+F\{Q/X\} = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and for some i we have $P_k\ \mathcal{R}\ E_i\{P/X\}$ for all $k \in \omega$. Then we have the following cases:

Case 6.1. Let $Q \xrightarrow{\tau} P_1$, so $recX.(\tau.(X + E + F) + G) \xrightarrow{\tau} P_1$. Then we have the following cases:

Case 6.1.1. Let $P_1 = Q + E\{Q/X\} + F\{Q/X\}$. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n, so $E_n\{P/X\} \xrightarrow{\tau} \Delta(\{P/X\}) + F\{P/X\}$, where $Q + E\{Q/X\} + F\{Q/X\} \; \mathcal{R} \; \Delta(\{P/X\}) + F\{P/X\}$.

Case 6.1.2. Let $G\{Q/X\} \xrightarrow{\tau} P_1$, since $G\{Q/X\} \; \mathcal{R}_0^{-1} \; G\{P/X\}$ we have $G\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1 \; \mathcal{R} \; R_2'$ for some R_2'. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

Case 6.2. Let $E\{Q/X\} \xrightarrow{\tau} P_1$, since $E\{Q/X\} \; \mathcal{R}_0^{-1} \; E\{P/X\}$ we have $E\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1 \; \mathcal{R} \; R_2'$ for some R_2'. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \xrightarrow{\tau} \Delta(E\{P/X\}) + F\{P/X\} \xrightarrow{\tau} R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

Case 6.3. Let $F\{Q/X\} \xrightarrow{\tau} P_1$, since $F\{Q/X\} \; \mathcal{R}_0^{-1} \; F\{P/X\}$ we have $F\{P/X\} \xrightarrow{\tau} R_2'$ and $P_1 \; \mathcal{R} \; R_2'$ for some R_2'. We have $E_i\{P/X\} \twoheadrightarrow E_n\{P/X\}$, where X is totally unguarded in E_n. So $E_n\{P/X\} \rightarrow^+ R_2'$, where $P_1 \; \mathcal{R} \; R_2'$.

□

Appendix E

Unique solution of equations

In Lemma E.0.9 we prove that if $\mathcal{E}$ is a guarded equation system such that both E and F provably satisfy $\mathcal{E}$, then $E = F$. We prove this by induction on the number of equations in $\mathcal{E}$, the proof is analogous to the proof of theorem 4.2 in [5].

Lemma E.0.9. *Let $E, F \in \mathbb{E}$ and let $\mathcal{E}$ be a guarded equation system such that both E and F provably satisfy $\mathcal{E}$. Then $E = F$.*

Proof. Since E provably satisfies $\mathcal{E} = \{X_i = G_i | 1 \leq i \leq m\}$ there exists an ordered sequence $\vec{E} = (E_1, ..., E_m)$, such that $E = E_1$ and $E_i = G_i\{\vec{E}/\vec{X}\}$ for $1 \leq i \leq m$. Since F provably satisfies $\mathcal{E} = \{X_i = G_i | 1 \leq i \leq m\}$ there exists an ordered sequence $\vec{F} = (F_1, ..., F_m)$, such that $F = F_1$ and $F_i = G_i\{\vec{F}/\vec{X}\}$ for $1 \leq i \leq m$. We prove that $\vec{E} = \vec{F}$ using induction on the number of equations in $\mathcal{E}$. If $\vec{E} = \vec{F}$ we can conclude that $E = F$, since $E = E_1 = F_1 = F$.

If $m = 1$ we have that $\mathcal{E}$ contains only one equation, so $\mathcal{E} = \{X_1 = G_1\}$. Since $\mathcal{E}$ is guarded there exists a linear order $\prec$ such that whenever X_1 is unguarded in G_1 then $X_1 \prec X_1$, but since $X_1 \prec X_1$ cannot be the case we know that X_1 is guarded in G_1. Since $E_1 = G_1\{E_1/X_1\}$ and X_1 is guarded in G_1 we can use axiom rec3 to obtain $E_1 = recX_1.G_1$. Analogously we have $F_1 = recX_1.G_1$, so $E_1 = F_1$.

Assume that for $\vec{E} = (E_1, ..., E_m)$ and $\vec{F} = (F_1, ..., F_m)$ we have $\vec{E} = \vec{F}$. Let $\mathcal{E} = \{X_i = G_i | 1 \leq i \leq m + 1\}$. Then we have for $1 \leq i \leq m$:

$$E_{m+1} = G_{m+1}\{\vec{E}/\vec{X}\}\{E_{m+1}/X_{m+1}\}$$

$$E_i = G_i\{\vec{E}/\vec{X}\}\{E_{m+1}/X_{m+1}\}$$

$$F_{m+1} = G_{m+1}\{\vec{F}/\vec{X}\}\{F_{m+1}/X_{m+1}\}$$

$$F_i = G_i\{\vec{F}/\vec{X}\}\{F_{m+1}/X_{m+1}\}$$

Since $\mathcal{E}$ is guarded and $X_{m+1} \not\prec X_{m+1}$ we have that X_{m+1} is guarded in $G_{m+1}\{\vec{E}/\vec{X}\}$ and $G_{m+1}\{\vec{F}/\vec{X}\}$. Since $E_{m+1} = G_{m+1}\{\vec{E}/\vec{X}\}\{E_{m+1}/X_{m+1}\}$ and $F_{m+1} = G_{m+1}\{\vec{F}/\vec{X}\}\{F_{m+1}/X_{m+1}\}$ we can use axiom rec3 to obtain:

$$E_{m+1} = recX_{m+1}.G_{m+1}\{\vec{E}/\vec{X}\}$$

$$F_{m+1} = recX_{m+1}.G_{m+1}\{\vec{F}/\vec{X}\}$$

Since $\vec{E} = \vec{F}$ we obtain $E_{m+1} = F_{m+1}$, so $(E_1, ..., E_{m+1}) = (F_1, ..., F_{m+1})$. □

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