Performance analysis of production lines at Heineken

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PERFORMANCE ANALYSIS OF
PRODUCTION LINES AT HEINEKEN

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DEN DOLECH 2, EINDHOVEN
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To My family
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Abstract

The design and improvement of manufacturing flow lines are very important. To be able to design and improve the flow lines, a fast and accurate method is needed. This method should be able to estimate the performance characteristics. In this thesis we use an approximation method which is fast and accurate to analyze unreliable single-machine tandem lines with finite buffers; each machine can have different speeds, with exponentially distributed up-and-down times behavior. The main idea of this method is that the original line is decomposed into subsystems. Each subsystem consists of two machines and one buffer in between. To link the decomposed two-machine subsystems together, a set of decomposition equations is derived. We use an extended version of The David-Dallery-Xie algorithm to solve these decomposition equations. Some performance measures are evaluated such as throughput, buffer contents, blocking and starvation probabilities. At the end, we implement our method to analyze Heineken’s production lines. We also implement the cross-entropy method to improve and design their production lines by finding optimal speed configurations and buffer allocations.

Key words: tandem lines, decomposition, finite buffers, blocking, unreliable, exponential, optimization, cross-entropy
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A special thanks to my lovely girlfriend, Jian Jingxian, who always supports me all the time. Because of her, my life is so meaningful in Netherlands. And last, but not least, I would like to thank my father, my mother and my sister for their continuous support and encouragement.

Eindhoven, Noord Brabant

Grant Patrizio Kesuma

July 31, 2009
Chapter 1

Introduction

In this chapter we present the background of this thesis, the problem description as well as the goal of the thesis. To be able to develop an analytical model, some assumptions are also made. The framework and the outline of this thesis are given at the end of this chapter.

1.1 Background

Since 2003 Heineken ’s-Hertogenbosch has implemented a method, developed by the Japan Institute of Plant Maintenance (JIPM), which is called Total Productive Maintenance (TPM) to minimize production losses. Heineken ’s-Hertogenbosch applied for JIPM certification and it was noted that the conveyor belt of production line 15A should be reduced. At the end of 2008, Heineken reduced the conveyor belt of production line 15A. The effect of this reduction was investigated by Ellen Weerts [19] and Edward Christian [6]. Ellen Weerts used an analytical model to investigate it, while Edward used a simulation model. Both models have advantages and disadvantages. Table 1.1 shows the difference between an analytical model and a simulation model.

In their project, the approximation using the analytical model developed by Ellen was not as good as the simulation model. Ellen used an analytical model based on an aggregation method developed by De Koster (see [12] and [11]). In this thesis, we use a different

<table>
<thead>
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<td>limited</td>
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<tr>
<td>Run time</td>
<td>long</td>
<td>short</td>
</tr>
<tr>
<td>Data requirement</td>
<td>large</td>
<td>small</td>
</tr>
<tr>
<td>Flexibility</td>
<td>high</td>
<td>low</td>
</tr>
</tbody>
</table>

Table 1.1: Comparison between simulation and analytical model
analytical model based on a decomposition method which turns out to be better. The decomposition method that we will use is based on the decomposition method developed by Burman [2]. The motivation why we choose Burman’s method is that this method has successfully been implemented to the Hewlett Packard, Johnson and Johnson production lines and gave an immense financial benefit to them [3]. Using this analytical model we analyze not only production line 15A, but also production lines 8A, 8B, 16A and 16B. The performance of these lines is measured using the Operational Performance Indicator No Order No Activity (OPI NONA). OPI NONA is a ratio between the theoretical production time and the effective working time (see Figure 2.1 and Figure 2.2 in Chapter 2). No Order No Activity means that the times when there are no orders available are not taken into account in the OPI calculation. In this thesis, we want to investigate the line performance by calculating OPI. For the rest of this thesis we refer to OPI NONA as OPI.

1.2 Problem Description

Heineken ’s-Hertogenbosch has many production lines. In this thesis, we mainly focus on production lines 15A, 8A, 8B, 16A and 16B. These lines consist of several machines in a sequence with a conveyor belt as a buffer in between two machines. Since the conveyor belt has a certain width and length, the capacity of these conveyor belts are different. The term ’buffer’ will be used in the rest of this thesis which refers to conveyor belt. As an example, the map of the line 15A can be seen in Figure 1.1. Each machine has a different task. As an example, in production line 15A, the depalletizer (A) takes the crates with empty bottles out from the pallets. The bottle unpacker (B) takes out empty bottles from a crate and puts them in the bottle washer (D). Machine C which is the crate-logo detector, selects crates with the correct logo before being washed. After being washed, the EBI (E) checks the quality of the washed bottles. EBI stands for ’empty bottle inspector’, so this machine looks whether there is a crack in the bottle or whether there is still something inside the bottle. After being inspected the bottles go to the filler machine (F) to be filled with beer. After that, in the pasteurizer (G), the bottles are heated up to 60°. The next machine, labeler (H), puts a label on the bottles. At the labeler, the bottles get two big labels and one small label on the neck. After that the bottles is either packed directly into a crate by machine I1 or six-packed by machine I2 and then place it on a crate. The machine before Palletizer will check whether a crate is completely filled or not. Finally the filled crates go to the Palletizer (J) to be placed on the pallets. It is noted that Production line 15A has also a possibility to put in new glass. The new glass comes directly into the line before EBI. Even though it does not need to be washed, it does have to be inspected by EBI.

To see the dynamic behavior of this line, an illustration is given by Figure 1.2. Suppose that machine 2 breaks down, but the other machines are still operational. This causes
Figure 1.1: Map of Production Line 15A
buffer 2 to decrease while buffer 1 increases. Once buffer 2 is empty, machine 3 will be
starved. On the other hand, once buffer 1 is full, machine 1 will be blocked. Moreover,
it is also possible that machines slow down. As an example, suppose that machine 1 and
machine 2 are operational. We assume that machine 2 works faster than machine 1. When
the buffer between these machines is empty, machine 2 will slow down and run at the same
speed as machine 1. The case machine 1 works faster than machine 2 is analogous. Since
all machines can breakdown and work at many speeds, the throughput of this production
system depends on the breakdown rate, the breakdown duration and processing rate/speed.

1.3 Project Goal

The goal of this project is to predict and investigate the performance of Heineken’s pro-
duction lines by using an analytical model. The performance measures include throughput,
buffer contents, blocking and starvation probabilities. An optimization method is also de-
veloped to find the optimal speed configurations as well as buffer allocation. Furthermore,
user friendly graphical user interface software is built to help Heineken to investigate the
performance of their other lines in the future.

1.4 Assumptions

Some important assumptions are made in order to build the analytical model.
• The transport time between two machines is zero which means that the product that comes out of the first machine can go immediately to the next machine if the buffer is empty.

• The system input (Line 15A) only consists of the bottles that are going to the depalletizer. The new glass which comes into the line before EBI is not taken into account.

• Up-and-down times of machines are exponentially distributed.

• Breakdown can only occur when the machines are producing, which means that we have operational dependent failures instead of time dependent failures.

1.5 Framework

The framework of this thesis is the following. First, we find the analytic solution of the two-machine model. This solution is based on the Gershwin and Schick model (see [9]). In their model, they assume that the failure rate is proportional to the speed while we assume that the failure rate is a function of the speed. Burman [2] derives a new set of decomposition equations for a longer line. He also assumes that the failure rate is proportional to speed. Based on this, we derive a set of decomposition equations which are almost similar to his. The difference is that in our decomposition equations we assume that the failure rate is a function of speed. To solve the decomposition equations, Burman adapted the David-Dallery-Xie algorithm [4]. In his dissertation, Burman also proposed a method to accelerate this algorithm. We use the same method as Burman’s to accelerate the algorithm. We call our algorithm the General Accelerated David-Dallery-Xie (GADDX) algorithm. The analytical model mentioned before needs input to approximate the throughput of the lines. The required inputs are failure rate, repair rate, processing rate, and buffer capacity. We obtain these inputs from Heineken’s database. Some statistical methods are used to extract this data from their database system. The next step is to analyze Heineken’s current production lines and optimize it. Finally, we develop an independent user-friendly GUI software in which three features are integrated such as performance evaluation, sensitivity analysis and optimization. This software can help Heineken to analyze their production lines in the future.

1.6 Thesis Outline

In this chapter we have introduced the background, problem description, the goal of the project and some assumptions to develop the model and the framework of this thesis. In
Figure 1.3: Framework of the thesis
Chapter 2 we explain the data system, the performance measure used at Heineken as well as the investigation of the relationship between machine speed and breakdown rate. Chapter 3 deals with a continuous model of a two-machine system which is used to analyze the performance of a longer line. Chapter 4 introduces a decomposition method to approximate the behavior of a longer line as well as the algorithm to solve a set of decomposition equations. In Chapter 5, we analyze Heineken’s production lines. In this Chapter we also present the optimal speed configurations and the optimal buffer allocations of their production lines. Finally, we conclude this thesis by giving suggestions and recommendations to Heineken in Chapter 6.
Chapter 2

Data Analysis

This chapter introduces the database systems at Heineken. The formula to calculate OPI is also given. Some parameters for calculating the OPI values of production lines 8A, 8B, 15A, 16A and 16B are presented. These parameters will be used as inputs of the analytical model which is developed in Chapter 4. In the last section we investigate the relation between speed and breakdown rate as well as the relation between speed and breakdown duration. These relations are very important for the input of the analytical model.

2.1 Data System

In Heineken, there are 2 database systems. The first one is the Manufacturing Execution System (MES). In this system, all the data of the last few years are stored. This system records every action of each machine of the production line such as producing, blockage, starvation, cleaning, changeover, internal storing, operator stop. Date and time of the action are also stored. The second data system is V-Online. In this system, the speed of a machines is recorded every minute. However, this system only records the speed of some machines.

2.2 Line Parameters

The parameters given in the tables below will be used, together with the throughput, to calculate OPI. These parameters are Average Change Over Time (ACOT), Average Planned Down Time (APDT) and filler nominal speed. Table 2.1 presents the parameters for production lines 8A, 8B, 15A, 16A and 16B. Others parameters such as average speed, failure rate (or mean up-time), repair rate (or mean down-time) and buffer capacity will be given in Appendix.
Production Lines | ACOT | APDT | Filler Nominal Speed |
<table>
<thead>
<tr>
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<th></th>
<th></th>
</tr>
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<tr>
<td>Line 15A</td>
<td>2.817</td>
<td>1.700</td>
<td>36000</td>
</tr>
<tr>
<td>Line 8A</td>
<td>2.876</td>
<td>0.775</td>
<td>40000</td>
</tr>
<tr>
<td>Line 8B</td>
<td>3.137</td>
<td>1.198</td>
<td>36000</td>
</tr>
<tr>
<td>Line 16A</td>
<td>1.927</td>
<td>2.537</td>
<td>45000</td>
</tr>
<tr>
<td>Line 16B</td>
<td>2.859</td>
<td>2.443</td>
<td>45000</td>
</tr>
</tbody>
</table>

Table 2.1: Line Parameters

2.3 OPI NONA

As stated before, to measure the performance of the production line, Heineken uses OPI NONA. That means that the times that there are no orders available are not taken into account. The diagram of OPI calculation is shown in Figure 2.1. In the model, we use the following formula to calculate the OPI:

\[ OPI = \frac{TH}{v_n} \times \frac{24 - APDT - ACOT}{24} \times 100 \]  

(2.3.1)

The nominal speed of the filler \(v_n\) is the installed filler speed in bottles per hour. APDT is the average planned down time of the filler and ACOT is the average change-over time of the filler.

2.4 Relation between Speed and Breakdown

Based on the experienced people at Heineken, it is believed that there is a relation between speed and breakdown. The breakdown is defined as an operation stop and an internal storing. In this section we investigate the existence of this relation. First we investigate the relation between speed and breakdown rate. After that, we also investigate the relation between speed and breakdown duration.

2.4.1 Speed and Breakdown Rate

To investigate the relation between speed and breakdown rate, we extract the data from Heineken’s database systems, MES and V-Online. The interesting production lines to investigate are production line 8A, 8B, 15A, 16A and 16B. Samples were taken from July 2008 until December 2008. It is noted that not all samples are available, for instance production line 15A consists of 10 machines, but the available samples are only from 7 machines. Before investigating the relation between speed and breakdown rate, the first thing to do is to screen the data. Heineken is interested to find the relation in the operating range only
Table 2.2: Operating Range of some machines

<table>
<thead>
<tr>
<th>Machine</th>
<th>Operating Range</th>
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<tr>
<td>ETIMA 8B</td>
<td>[22000,28000] and [37000,46000]</td>
</tr>
<tr>
<td>Valinpakker 8B</td>
<td>[22000,29000] and [44000,54000]</td>
</tr>
<tr>
<td>Inpakker 15A</td>
<td>[29000,47000]</td>
</tr>
<tr>
<td>ETIMA 16B</td>
<td>[44000,51000]</td>
</tr>
<tr>
<td>Valinpakker 16B</td>
<td>[37000,59000]</td>
</tr>
</tbody>
</table>
Figure 2.2: OPI Calculation

(see Table 2.2). The complete scatter plots of all production lines are in Appendix E. We use SPSS software to find the relation. We use the regression feature to obtain the linear or nonlinear regression. See [8], [16], and [18] for some references related to regression model and see [7] for the reference related to SPSS. Here, we present some results. We found a linear relation for Valinpakker 16B and ETIMA 16B. See Figure 2.4 and Figure 2.5. The regression model of Valinpakker 16B is given by

\[ Y = 2.023 \cdot 10^{-4} X + 1.344 \]  \hspace{1cm} (2.4.1)

where \( X \) is speed and \( Y \) is breakdown rate. We use this equation to make a prediction. The red line is the mean response of our regression model and the blue lines are 95% confidence interval. The coefficient determination \( R^2 \) of the regression model is 0.614. This means that our model can explain around 61.4% of the variance. We can do the same thing to ETIMA 16B. The regression model of ETIMA 16B is given by

\[ Y = 4.209 \cdot 10^{-4} X - 14.557 \]  \hspace{1cm} (2.4.2)

where \( X \) is speed and \( Y \) is breakdown rate. It is noted for this machine that the coefficient determination \( R^2 \) is 0.501.

The next result is that there is a quadratic relation between speed and breakdown rate for Inpakker 15A. See Figure 2.6.
The regression model of Inpakker 15A is given by

$$ Y = 1.068 \cdot 10^{-8} X^2 - 8.764 \cdot 10^{-4} X + 19.397 $$  \hspace{1cm} (2.4.3)

where $X$ is speed and $Y$ is breakdown rate. The coefficient determination ($R^2$) is 0.617. A special treatment is needed for production line 8B since this production line is used by two different types of bottles. When it is producing a big bottle, its speed will be slow. On the other hand, when it is producing a small bottle, its speed will be fast. For instance, the operating range of ETIMA 8B is between 22000 and 28000 for big bottles, and between 37000 and 46000 for small bottles (see Figure 2.7). Heineken is interested in investigating whether the breakdown rate between these two groups of operating ranges are different. In order to do that, we use a t-test method to compare the means of these two groups.

$$ H_0 : \mu_1 = \mu_2 $$

$$ H_1 : \mu_1 \neq \mu_2 $$ \hspace{1cm} (2.4.4)

Equation (2.4.4) shows the hypothesis of the t-test. The term $\mu_i$ is the mean of $i$th group. The results are shown in Table 2.3. The table shows the mean, variance and number of observations of each group as well as the degree of freedom (df). Since the $p$-value is less than 0.05 (95% confidence interval), we can not accept $H_0$, hence we conclude that for ETIMA
**Figure 2.4:** Linear relation: Valinpakker 16B

**Figure 2.5:** Linear relation: ETIMA 16B
Figure 2.6: Quadratic relation: Inpakker 15A

Figure 2.7: Two different means: ETIMA 8B
### Table 2.3: T-test ETIMA 8B

<table>
<thead>
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<th></th>
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<th>Low Speeds</th>
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<tr>
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<td>3.02</td>
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<tr>
<td>Variance</td>
<td>0.3935</td>
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</tr>
<tr>
<td>Observation</td>
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<tr>
<td>df</td>
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</tr>
<tr>
<td>p-value</td>
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Table 2.3: T-test ETIMA 8B

8B, the operating range for small bottles is statistically different from the operating range for big bottles. We did the same thing to Valinpakker 8B. As shown in Table 2.2, for producing small bottles Heineken runs the machine at speed 22000 to 29000 while for big bottles from 44000 to 54000. The result of our investigation for this machine is that the breakdown rates of these operating ranges are statistically equal. It means that the breakdown rate does not depend on the machine speeds. For other machines, we did the same investigation using regression methods or t-test (if applicable). All scatter plots in Appendix E indicate that there is no relation except for Valinpakker 16A which has a positive linear relation. However, its small coefficient of determination \( R^2 = 0.182 \) indicates that the model is not good enough to be used for prediction. All these relations will be implemented to analyze Heineken’s production lines in Chapter 5.

#### 2.4.2 Speed and Breakdown Duration

Another relation we would like to investigate is the relation between speed and breakdown duration. We took samples from Heineken’s database systems, MES and V-Online, from July 2008 until December 2008. Samples were taken for production lines 8A, 8B, 15A, 16A and 16B. Here, we did the same thing as before. We used a regression model to investigate this relation. We implemented this method to all production lines and fortunately we did not find any relation. This results imply that if the Heineken’s operators run machines at high speed, the breakdown duration or down-time (if breakdown happens) will be the same as the breakdown duration when the machines run at any other speeds. However, we believe that the machine speed is not the only factor which causes the length of the breakdown. In the beginning, we mentioned that the breakdown can be caused by an internal storing or an operator stop. As an example, when the breakdown happens the machine will not be repaired immediately. It takes sometime for the ‘long stop team’ or ‘short stop team’ at Heineken to fix the problem. The breakdown caused by internal storing cannot be predicted by these teams while the operator stop can be predicted since it is the operator who stops the machine. Another interesting feature that we can see from the scatter plots is that the operators have their favorite speeds. See Figure 2.8 as an example. In this example, we can see that there are some favorite speeds which are used by the operators. For this machine,
Figure 2.8: Speed vs Breakdown duration

Filler 16A, the favorite speeds are around 45000 and 18000. This causes that most of the breakdowns, including long breakdowns occur at these speeds. The complete lists of speed usages are presented in Appendix F.
Chapter 3

Continuous Model

In this chapter, a continuous model is presented. The first section begins with an introduction, some assumptions and terminologies of the model. The two-stage model is given in Section 3.2. Section 3.3 gives the solution of the two-stage model. Finally, the last section presents the numerical results of the model.

3.1 Introduction

We model the flow line as a continuous time, mixed state Markov process. In this model, the material is treated as though it is a continuous fluid. Each machine has a processing rate ($\mu$), failure rate ($p$), and repair rate ($r$). We assume that the failure rate and the repair rate are exponentially distributed. Furthermore, there is also a finite buffer with capacity $N$ between the two machines. The machine produces material at these processing rates unless impeded by a failure, blockage or a starvation of raw material. Unlike discrete material flow line models, the machine holds no material. The machine functions like a random water faucet. It allows the liquid to pass, but all the liquid is stored either before or after the faucet itself.

![Two-stage model](image.png)

Figure 3.1: Two-stage model
3.2 Continuous Two-Machine Line

In this section, we determine the analytic solution of the two-stage model as shown in Figure 3.1. The line consists of two machines (\(M_1\) and \(M_2\)) and one finite buffer (\(B\)) in between. We assume that the first machine is never starved which means that there are always raw materials available to be processed. This machine has failure rate \(p_1\), repair rate \(r_1\) and processing rate \(\mu_1\). The buffer capacity of the buffer \(B\) between machine 1 and machine 2 is \(N\). Moreover, we also assume that the second machine cannot be blocked. The parameters of machine 2 are failure rate \(p_2\), repair rate \(r_2\), and processing rate \(\mu_2\).

3.2.1 Model Assumption and Terminology

Let \(x\) be the buffer content. We also define \(\alpha_1(t)\) and \(\alpha_2(t)\) as the status of machine 1 and machine 2, respectively, at time \(t\). This value equals zero if the machine is under repair and equals 1 if the machine is operational.

Define the system state as

\[
s = (x, \alpha_1(t), \alpha_2(t))
\] (3.2.1)

Further, we define also the failure rate functions, \(g_1(\cdot)\) respectively \(g_2(\cdot)\) as the failure rate function of machine 1 respectively machine 2. During time interval \((t, t + \delta t)\),

**When** \(0 < x < N\)

1. the change in \(x\) is \((\alpha_1 \mu_1 - \alpha_2 \mu_2) \delta t\)

2. the probability of repair of machine \(i\), that is, the probability that \(\alpha_i(t + \delta t) = 1\) given that \(\alpha_i(t) = 0\) is \(r_i \delta t\)

3. the probability of failure of machine \(i\), that is, the probability that \(\alpha_i(t + \delta t) = 0\) given that \(\alpha_i(t) = 1\) is \(p_i \delta t\)

**When** \(x = 0\)

1. the change in \(x\) is \((\alpha_1 \mu_1 - \alpha_2 \mu_2)^+\). It means that when the buffer is zero, the value of \(x\) can only increase.

2. the probability of repair is \(r_1 \delta t\)

3. if machine 1 is down, machine 2 cannot fail. If machine 1 is up, the probability of failure of machine 2 is \(g_2(\mu) \delta t\), where \(\mu = \min(\mu_1, \mu_2)\) and \(g_2(\cdot)\) is the failure rate function of machine 2

4. the probability failure of machine 1 is \(p_1 \delta t\)
Since we have an operation dependent failure, the machines cannot break down when they are not operational. If the buffer is empty, the second machine cannot operate faster than the first machine. If the first machine speed is faster than the second one, the second operates as it does normally, and the buffer is immediately not empty.

**When** $x = N$

1. the change in $x$ is $(\alpha_1 \mu_1 - \alpha_2 \mu_2)^-$. It means that when the buffer is full, the value of $x$ can only decrease.

2. the probability of repair is $r_i \delta t$

3. if machine 2 is down, machine 1 cannot fail since it is not operational. If machine 2 is up, the probability of failure of machine 1 is $g_1(\mu) \delta t$, where $\mu = \min(\mu_1, \mu_2)$ and $g_1(\cdot)$ is the failure rate function of machine 1

4. the probability of failure of machine 2 is $p_2 \delta t$

### 3.2.2 Performance Measures

First we define:

1. $p(0, \alpha_1, \alpha_2)$ as the probability that the system is in the state where there is 0 material in the buffer, and machine 1, respectively machine 2, have status $\alpha_1$, respectively $\alpha_2$

2. $p(N, \alpha_1, \alpha_2)$ as the probability that the system is in the state where there is $N$ material in the buffer, and machine 1, respectively machine 2, have status $\alpha_1$, respectively $\alpha_2$

3. $f_x^{x+\delta x} f(y, \alpha_1, \alpha_2) dy$ as the probability of finding machine 1 in state $\alpha_1$ and machine 2 in state $\alpha_2$ and finding between $x$ and $x + \delta x$ unit of material in Buffer B.

Next, the efficiency of machine $M_i$ is defined as the probability that machine $M_i$ is processing a workpiece. We use notation $E_i$ for the efficiency of machine $M_i$. It is given by

$$E_i = E_i^{(\max)} + E_i^{(\min)}$$  \hspace{1cm} (3.2.2)

where

$$E_1^{(\max)} = \text{prob}[\alpha_1 = 1, x < N]$$  \hspace{1cm} (3.2.3)

$$E_2^{(\max)} = \text{prob}[\alpha_2 = 1, x > 0]$$  \hspace{1cm} (3.2.4)

$$E_1^{(\min)} = p(N, 1, 1) \frac{\mu_2}{\mu_1}$$  \hspace{1cm} (3.2.5)

$$E_2^{(\min)} = p(0, 1, 1) \frac{\mu_1}{\mu_2}$$  \hspace{1cm} (3.2.6)
$E_i^{(\text{max})}$ can be interpreted as the probability that machine $i$ is processing material at speed $\mu_i$ while $E_i^{(\text{min})}$ is the probability that machine $i$ is processing material at speed lower than $\mu_i$. The value of $E_i^{(\text{max})}$ can be computed as follows.

\[
E_1^{(\text{max})} = \text{prob}[\alpha_1 = 1, x < N] = \int_0^N (f(x, 1, 0) + f(x, 1, 1))dx + p(0, 1, 1) \quad (3.2.7)
\]

\[
E_2^{(\text{max})} = \text{prob}[\alpha_2 = 1, x > 0] = \int_0^N (f(x, 0, 1) + f(x, 1, 1))dx + p(N, 1, 1) \quad (3.2.8)
\]

The most important performance measure is the production rate (throughput rate) of machine $M_i$ which is defined as

\[
P_i = \mu_i E_i, \quad i = 1, 2
\]

Since there is no creation or destruction of material, flow is conserved. It is called conservation of flow (COF). Here, we also assume that the yield of the system is 100% which means no defect products. COF of the two-stage model is defined as

\[
P = P_1 = P_2
\]

The isolated efficiency $e_i$ of machine $M_i$ is

\[
e_i = \frac{r_i}{r_i + p_i}, \quad i = 1, 2 \quad (3.2.10)
\]

and it represents the fraction of time that $M_i$ is operational. The isolated production rate, $\rho_i$ is given by

\[
\rho_i = \mu_i e_i, \quad i = 1, 2 \quad (3.2.11)
\]

and it represents what the production rate of $M_i$ would be if it were never impeded by other machines or buffers. In fact, the actual production rate $P_i$ is less because of blocking and starvation. This actual production rate is given by (3.2.57) or (3.2.58) below. It is noted that in the continuous model, the machines can be starved and blocked simultaneously, while in the discrete model, it is not possible.
3.2.3 Transition Equations

To describe the system behaviors, differential equations are used. In the continuous model, the change of state is a very small amount during a very short time interval. As an example, the buffer content can rise or fall depending on the states of adjacent machines. Below, we will derive 4 equations that describe the internal storage behavior.

**Both machines are up**

First we define the probability that both machines are operational with a storage level between \( x \) and \( x + \delta x \) at time \( t + \delta t \) as \( f(x, 1, 1, t + \delta t) \). This probability can be determined by

\[
f(x, 1, 1, t + \delta t) = (1 - (p_1 + p_2)\delta t) f(x - \mu_1 \delta t + \mu_2 \delta t, 1, 1, t) + r_1 \delta t f(x + \mu_2 \delta t, 0, 1, t) + r_2 \delta t f(x - \mu_1 \delta t, 1, 0, t)
\]

The explanation of this equation is as follows.

- If both machines are operational at time \( t \) and the storage level is between \( x - \mu_1 \delta t + \mu_2 \delta t \) and \( x - \mu_1 \delta t + \mu_2 \delta t + \delta x \) then there will be no failures between \( t \) and \( t + \delta t \) with probability
  
  \[
  (1 - p_1 \delta t)(1 - p_2 \delta t)
  \]

  and moreover, the storage level will be between \( x \) and \( x + \delta x \) at time \( t + \delta t \).

- If, at time \( t \), machine 1 is under repair and machine 2 is operational, then the probability that both machines will be operational at time \( t + \delta t \) is \( (1 - p_2 \delta t)r_1 \delta t \). In addition, during that interval, the buffer loses \( \mu_2 \delta t \).

- Similarly, if, at time \( t \), machine 2 is under repair and machine 1 is operational, then the probability that both machines will be operational at time \( t + \delta t \) is \( (1 - p_1 \delta t)r_2 \delta t \). In addition, the buffer increases \( \mu_1 \delta t \) during that interval.

By letting \( \delta t \rightarrow 0 \), this equation becomes

\[
\frac{\partial f}{\partial t}(x, 1, 1, t) = -(p_1 + p_2)f(x, 1, 1, t) + (\mu_2 - \mu_1)\frac{\partial f}{\partial x}(x, 1, 1, t) + r_1 f(x, 0, 1, t) + r_2 f(x, 1, 0, t)
\]

(3.2.12)

**Machine 1 is up and Machine 2 is down**

In this case the probability of finding machine 1 up and machine 2 down with storage level between \( x \) and \( x + \delta x \) at time \( t + \delta t \) is given by \( f(x, 1, 0, t + \delta t) \delta x \), where

\[
f(x, 1, 0, t + \delta t) = (1 - (p_1 + r_2)\delta t) f(x - \mu_1 \delta t, 1, 0, t) + p_2 \delta t f(x - \mu_1 \delta t + \mu_2 \delta t, 1, 1, t) + r_1 \delta t f(x, 0, 0, t)
\]

The explanation of this equation is as follows.
• If machine 1 is operational and machine 2 is down, the storage level is between \(x - \mu_1 \delta t\) and \(x - \mu_1 \delta t + \delta x\) then there will be no failure of machine 1 and no finished repair of machine 2 before \(t + \delta t\) with probability

\[
(1 - p_1 \delta t)(1 - r_2 \delta t)
\]

and at time \(t + \delta t\), the storage level will be between \(x\) and \(x + \delta x\).

• If at time \(t\) both machines are operational, the probability that machine 1 will remain operational at time \(t + \delta t\) is \((1 - p_1 \delta t)\). In the meantime, the second machine will break down with probability \(p_2 \delta t\). During that time period, the storage changes \(-\mu_1 \delta t + \mu_2 \delta t\).

• If both machines are down at time \(t\), the probability that machine 1 will be operational and machine 2 will remain under repair is \(r_1 \delta t(1 - r_2 \delta t)\). During this period, the storage level does not change.

By letting \(\delta t \to 0\), this equation becomes

\[
\frac{\partial f}{\partial t}(x, 1, 0, t) = -\mu_1 \frac{\partial f}{\partial x}(x, 1, 0, t) - (p_1 + r_2)f(x, 1, 0, t) + p_2 f(x, 1, 1, t) + r_1 f(x, 0, 0, t)
\]

(3.2.13)

**Machine 1 is down and Machine 2 is up**

The probability that machine 1 is down and machine 2 is up at time \(t + \delta t\) with storage level between \(x\) and \(x + \delta x\) is given by \(f(x, 0, 1, t + \delta t) \delta x\), where

\[
f(x, 0, 1, t + \delta t) = (1 - (r_1 + p_2) \delta t)f(x + \mu_2 \delta t, 0, 1, t) + p_1 \delta tf(x - \mu_1 \delta t + \mu_2 \delta t, 1, 1, t) + r_2 \delta tf(x, 0, 0, t)
\]

This is because

• If machine 1 is down and machine 2 is up at time \(t\) and the storage level is between \(x + \mu_2 \delta t\) and \(x + \mu_2 \delta t + \delta x\) then nothing will happen with probability

\[
(1 - p_2 \delta t)(1 - r_1 \delta t)
\]

and at time \(t + \delta t\), the storage level will be between \(x\) and \(x + \delta x\).

• If at time \(t\), both machines are operational, the probability that machine 1 will be down and machine 2 will remain up is \(p_1 \delta t(1 - p_2)\delta t\). During that time interval, the storage changes \(-\mu_1 \delta t + \mu_2 \delta t\).

• If both machines are down at time \(t\), the probability that machine 2 will be operational and machine 1 will remain under repair is \(r_2 \delta t(1 - r_1 \delta t)\). During this period, the storage level does not change.
By letting $\delta t \to 0$, this equation becomes
\[
\frac{\partial f}{\partial t}(x, 0, 1, t) = \mu_2 \frac{\partial f}{\partial x}(x, 0, 1, t) - (p_2 + r_1)f(x, 0, 1, t) + p_1 f(x, 1, 1, t) + r_2 f(x, 0, 0, t) \tag{3.2.14}
\]

**Both machines are down**

In this case, both machines are not operational at time $t + \delta t$. The probability of finding this situation at time $t + \delta t$ with a storage level between $x$ and $x + \delta x$ is $f(x, 0, 0, t + \delta t)\delta x$, where
\[
f(x, 0, 0, t + \delta t) = (1 - (r_1 + r_2)\delta t)f(x, 0, 0, t) + p_1 \delta tf(x - \mu_1\delta t, 1, 0, t) + p_2 \delta tf(x + \mu_2\delta t, 0, 1, t)
\]

Almost similar explanation as before,

- If both machines are down at time $t$ and the storage level is $x$, then the probability that nothing happens before $t + \delta t$ is
  \[
  (1 - r_1\delta t)(1 - r_2\delta t),
  \]
  and at time $t + \delta t$, the storage level remains the same.

- If, at time $t$, the first machine is up and the second one is down, the probability that both machines will be down at time $t + \delta t$ is $p_1\delta t(1 - r_2\delta t)$. During this time interval, the storage level increases $\mu_1\delta t$.

- Same argument as before, if at time $t$, the second machine is up and the first one is down, the probability that both machines will be down at time $t + \delta t$ is $p_2\delta t(1 - p_1\delta t)$. During this time interval, the storage level decreases $\mu_2\delta t$.

By letting $\delta t \to 0$, this equation becomes
\[
\frac{\partial f}{\partial t}(x, 0, 0, t) = -(r_1 + r_2)f(x, 0, 0, t) + p_1 f(x, 1, 0, t) + p_2 f(x, 0, 1, t) \tag{3.2.15}
\]

If we can solve these equations, (3.2.12), (3.2.13),(3.2.14),(3.2.15), we can determine the performance measures such as production rate, average buffer content and others. Before we solve these equations, we need some boundary conditions of the model.

### 3.2.4 Boundary Behavior

**Lower Boundary**

Here we have $x = 0$. 

• Boundary-to-Boundary Equations
Suppose the system is in state \( (0, 0, 0) \) at time \( t + \delta t \), then the possible previous states at time \( t \) are state \( (0, 0, 0) \) in which nothing happens or state \( (0, 1, 0) \) in which machine 1 will be down before \( t + \delta t \). The transition probability from \( (0, 0, 0) \) to \( (0, 0, 0) \) is 
\[
(1 - (r_1 + r_2)\delta t)
\]
and the transition probability from state \( (0, 1, 0) \) to state \( (0, 0, 0) \) is 
\[
(1 - r_2\delta t)p_1\delta t.
\]
Thus, we have 
\[
p(0, 0, 0, t + \delta t) = (1 - (r_1 + r_2)\delta t)p(0, 0, 0, t) + (1 - r_2\delta t)p_1\delta t)p(0, 1, 0, t) \quad (3.2.16)
\]
As \( \delta t \to 0 \), \(3.2.16\) becomes 
\[
\frac{d}{dt}p(0, 0, 0, t) = -(r_1 + r_2)p(0, 0, 0, t) + p_1p(0, 1, 0, t) \quad (3.2.17)
\]
Moreover, we can also conclude that 
\[
p(0, 1, 0, t) = 0 \quad (3.2.18)
\]
The reason is that if the system ever reaches that state, it leaves instantly because the buffer immediately accumulates material and the state changes instantly.

• Interior-to-Boundary Equations
If the system is in state \( (0, 0, 1) \) at time \( t + \delta t \), the system could have been in one of four states at time \( t \). It could have been in state \( (0, 0, 0) \) with a repair of the second machine. It could have been in state \( (0, 0, 1) \) if nothing happens. It could have been in state \( (0, 1, 1) \) with the first machine failing and the second one not failing. It could have been in any state \( (x, 0, 1) \) where \( 0 < x < \mu_2\delta t \) if no repair of the first machine or failure of the second occurred. It can be written as 
\[
p(0, 0, 1, t + \delta t) = (1 - r_1\delta t)r_2\delta t)p(0, 0, 0, t) + (1 - r_1\delta t)p(0, 0, 1, t) + p_1\delta t(1 - g_2(\mu_1)\delta t)p(0, 1, 1, t) + \int_0^{\mu_2\delta t} f(x, 0, 1, t)dx \quad (3.2.19)
\]
or, 
\[
\frac{d}{dt}p(0, 0, 1, t) = r_2p(0, 0, 0, t) - r_1p(0, 0, 1, t) + p_1p(0, 1, 1, t) + \mu_2f(0, 0, 1, t) \quad (3.2.20)
\]
Now, we consider two cases. The first case is \( \mu_1 > \mu_2 \). In this case, it is impossible to get from \( (0, 1, 1) \) at \( t \) to \( (0, 1, 1) \) at \( t + \delta t \) if \( \delta t \) is small. The same reason as before, when both machines are operational, material accumulates in the buffer instantly. The same things happen if the system is in state \( (0, 0, 1) \) at time \( t \) and the first machine is repaired during \( (t, t + \delta t) \), there will be some material in the buffer at time \( t + \delta t \).
Finally, if at \( t \) the state is \((x, 1, 1)\), and no failure occurs in \((t, t+\delta t)\) the material in the buffer will increase. Thus, there is no way of getting to state \((0, 1, 1)\). In conclusion, if \( \mu_1 > \mu_2 \)

\[
p(0, 1, 1, t) = 0 \quad (3.2.21)
\]

The second case is \( \mu_1 \leq \mu_2 \). Here, it is possible to get to \((0, 1, 1)\) from \((0, 0, 1)\) if machine 1 is repaired, and from \((x, 1, 1)\) where \( 0 \leq x \leq (\mu_2 - \mu_1)\delta t \) if no failures occur. It is also possible to get to state \((0,1,1)\) from \((0,1,1)\) if no failures occur. Hence, the probability that the system is in state \((0,1,1)\) can be written as

\[
p(0, 1, 1, t + \delta t) = (1 - p_1\delta t)(1 - g_2(\mu_1)\delta t)p(0, 1, 1, t) + r_1\delta t p(0, 0, 1, t) + \int_0^{(\mu_2-\mu_1)\delta t} f(x, 1, 1, t) dx, \quad (3.2.22)
\]

or,

\[
\frac{d}{dt} p(0, 1, 1, t) = -(p_1+g_2(\mu_1))p(0, 0, 1, t)+r_1 p(0, 0, 1, t)+(\mu_2-\mu_1)f(0, 1, 1, t) \quad (3.2.23)
\]

**Boundary-to-Interior Equations**

Consider the state \((x, 1, 0)\) at time \( t+\delta t \), where \( 0 < x < \mu_1\delta t \); the system could either have been in an internal state at time \( t \) or at a boundary state sometime during the time interval \((t, t+\delta t)\). The only possible internal states are \((x, 1, 1)\) or \((x, 0, 0)\), but their probabilities would contribute second order terms in the following equation.

\[
\int_0^{\mu_1\delta t} f(x, 1, 0, t + \delta t) dx = \int_t^{t+\delta t} (r_1 p(0, 0, 0, s) + g_2(\mu_1)p(0, 1, 1, s)) ds + (1 - p_1\delta t)(1 - r_2\delta t)p(0, 1, 0, t) \quad (3.2.24)
\]

As stated before, the zero’th order term in the equation above is \( p(0,1,0, t) = 0 \) Hence, the equation \((3.2.17)\) can be rewritten as

\[
\frac{d}{dt} p(0, 0, 0, t) = -(r_1 + r_2)p(0, 0, 0, t) \quad (3.2.25)
\]

The first order terms in \((3.2.24)\) are

\[
\mu_1 f(0, 1, 0, t) = r_1 p(0, 0, 0, t) + g_2(\mu_1)p(0, 1, 1, t) \quad (3.2.26)
\]

We consider two cases. The first case is \( \mu_1 > \mu_2 \). In this case, it is possible to reach state \((x, 1, 1)\) where \( 0 < x \leq (\mu_1 - \mu_2)\delta t \) from the boundary. Due to equation \((3.2.21)\) and equation \((3.2.18)\), the balance equation is (first order)

\[
\int_0^{(\mu_1-\mu_2)\delta t} f(x, 1, 1, t + \delta t) dx = r_1 \int_t^{t+\delta t} p(0, 0, 1, s) ds \quad (3.2.27)
\]
or, 

\[(\mu_1 - \mu_2)f(0,1,1,t) = r_1p(0,0,1,t)\]  \hspace{1cm} (3.2.28)

The next case is \(\mu_1 \leq \mu_2\). In this case, state \((x,1,1)\) can only be reached from state\((x',\alpha_1,\alpha_2)\) where \(x' \geq x\). Consequently, there is no counterpart to (3.2.28).

**Upper Boundary**

Below the derivation of the upper boundary is given. In this case we have \(x = N\). Analogous phenomena occur on the upper boundary.

- **Boundary-to-Boundary Equations**

  Suppose the system is in state \((N,0,0)\) at time \(t + \delta t\). The only possible previous state at time \(t\) is state \((N,0,0)\) itself. Since we assume that the system has operation dependent failure, it is not possible to get to state \((N,0,0)\) from state \((N,1,0)\). If at time \(t\) the system is in state \((N,0,1)\), the material will instantaneously leave the buffer which means that \(p(N,1,0) = 0\). Hence, it can be written as

\[p(N,0,0,t+\delta t) = (1-r_1\delta t)(1-r_2\delta t)p(N,0,0,t)\]

By letting \(\delta t \rightarrow 0\), the above equation becomes

\[\frac{d}{dt}p(N,0,0,t) = -(r_1 + r_2)p(N,0,0,t)\]  \hspace{1cm} (3.2.29)

- **Interior-to-Boundary Equations**

  To arrive at state \((N,1,0)\) at time \(t + \delta t\), the system could have been in one of four states at time \(t\). It could have been in state \((N,1,0)\) at time \(t\) with no repair of the second machine. (The first machine could not have failed since it was not running due to blocking). It could have been in state \((N,0,0)\) with a repair of the first machine. It could have been in state \((N,1,1)\) with a failure of second machine. Finally, it could have been in any state \((x,1,0)\) where \(N - \mu_1\delta t \leq x \leq N\). Symbolically

\[p(N,1,0,t+\delta t) = (1-r_2\delta t)p(N,1,0,t) + r_1\delta tp(N,0,0,t) + (1-r_1\delta t)p_2\delta tp(N,1,1,t) + \int_{N-\mu_1\delta t}^{N} f(x,1,0,t)dx\]

As \(\delta t \rightarrow 0\), it becomes

\[\frac{d}{dt}p(N,1,0,t) = -r_2p(N,1,0,t) + r_1p(N,0,0,t) + p_2p(N,1,1,t) + \mu_1f(N,1,0,t)\]  \hspace{1cm} (3.2.30)

Consider the case when \(\mu_2 > \mu_1\).

In this case, it is impossible to get from \((N,1,1)\) at \(t\) to \((N,1,1)\) at time \(t + \delta t\) if \(\delta t\) is small. The reason is that when both machines are operational, material leaves the
buffer instantly. The same thing happens if the system is in state \((N, 1, 0)\) at time \(t\) and the second machine is repaired during \((t, t + \delta t)\), there will be some material that leaves the buffer at time \(t + \delta t\). Finally, if at \(t\) the state is \((x, 1, 1)\), the material in the buffer will decrease, so there is no way of getting to state \((N, 1, 1)\). Thus, we can conclude that, if \(\mu_2 > \mu_1\)

\[
p(N, 1, 1, t) = 0 \tag{3.2.31}
\]

Consider the other case when \(\mu_1 \geq \mu_2\).

\[
p(N, 1, 1, t + \delta t) = (1 - g_1(\mu_2)\delta t)(1 - p_2\delta t)p(N, 1, 1, t) + \int_{N-(\mu_1-\mu_2)\delta t}^{N} f(x, 1, 1, t)dx + r_2p(N, 1, 0, t)
\]

By letting \(\delta t \to 0\), the previous equation becomes

\[
\frac{d}{dt}p(N, 1, 1, t) = -(g_1(\mu_2) + p_2)p(N, 1, 1, t) + (\mu_1 - \mu_2)f(N, 1, 1, t) + r_2p(N, 1, 0, t) \tag{3.2.32}
\]

**Boundary-to-Interior Equations**

Suppose that at time \(t + \delta t\) the system is in state \((x, 0, 1)\) where \(N - \mu_2\delta t < x < N\). The system could have been in an internal state at time \(t\) or at a boundary state some time during the time interval \((t, t + \delta t)\). The only possible internal states are \((x, 1, 1)\) or \((x, 0, 0)\), but their probabilities would contribute second order terms in the following equation.

\[
\int_{N - \mu_2\delta t}^{N} f(x, 0, 1, t + \delta t)dx = \int_{t}^{t + \delta t} (r_2p(N, 0, 0, s) + g_1(\mu_2)p(N, 1, 1, s))ds + (1 - r_1\delta t)(1 - p_2\delta t)p(N, 0, 1, t) \tag{3.2.33}
\]

Since we have

\[
p(N, 0, 1, t) = 0 \tag{3.2.34}
\]

the last equation becomes (the first order)

\[
\mu_2f(N, 0, 1, t) = r_2p(N, 0, 0, t) + g_1(\mu_2)p(N, 1, 1, t) \tag{3.2.35}
\]

Consider the case when \(\mu_2 > \mu_1\). In this case, it is possible to reach state \((x, 1, 1)\) where \(N - (\mu_2 - \mu_1)\delta t < x < N\) from the boundary. Due to (3.2.34) and (3.2.31), the balance equation is

\[
\int_{N - (\mu_2 - \mu_1)\delta t}^{N} f(x, 1, 1, t + \delta t)dx = r_2 \int_{t}^{t + \delta t} p(N, 1, 0, s)ds \tag{3.2.36}
\]
or,
\[(\mu_2 - \mu_1)f(N, 1, 1, t) = r_2p(N, 1, 0, t) \tag{3.2.37}\]

If \(\mu_1 \geq \mu_2\), state \((x, 1, 1)\) can only be reached from state \((x', \alpha_1, \alpha_2)\) where \(x' < x\). Consequently, there is no counterpart to (3.2.37).

**Normalization**

Since the sum of all probabilities is always 1, we have
\[
\sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} \left[ \int_{0}^{N} f(x, \alpha_1, \alpha_2)dx + p(0, \alpha_1, \alpha_2) + p(N, \alpha_1, \alpha_2) \right] = 1 \tag{3.2.38}\]

### 3.2.5 Identities

Some important identities are given below. These identities are applied in the rest of the thesis.

\[
p(0, 0, 0) = 0
\]
\[
p(0, 1, 0) = 0
\]
\[
p(N, 0, 0) = 0
\]
\[
p(N, 0, 1) = 0
\]
\[
p(0, 1, 1) = 0 \text{ if } \mu_1 > \mu_2
\]
\[
p(N, 1, 1) = 0 \text{ if } \mu_2 > \mu_1
\]

Another identity which is used in establishing conservation flow
\[
(\mu_2 - \mu_1)f(x, 1, 1) + \mu_2f(x, 0, 1) - \mu_1f(x, 1, 0) = 0 \quad , 0 \leq x \leq N \tag{3.2.39}\]

**Proof.** In steady state equations (3.2.12),(3.2.13),(3.2.14) and (3.2.15) become

\[
0 = -(p_1 + p_2)f(x, 1, 1) + (\mu_2 - \mu_1) \frac{\delta f}{\delta x}(x, 1, 1) + r_1f(x, 0, 1) + r_2f(x, 1, 0)
\]
\[
0 = -\mu_1 \frac{\delta f}{\delta x}(x, 1, 0) - (p_1 + r_2)f(x, 1, 0) + p_2f(x, 1, 1) + r_1f(x, 0, 0)
\]
\[
0 = \mu_2 \frac{\delta f}{\delta x}(x, 0, 1) - (r_1 + p_2)f(x, 0, 1) + p_1f(x, 1, 1) + r_2f(x, 0, 0)
\]
\[
0 = -(r_1 + r_2)f(x, 0, 0) + p_1f(x, 1, 0) + p_2f(x, 0, 1)
\]

If we add these equations, we obtain
\[
0 = \frac{d}{dx} [(\mu_2 - \mu_1)f(x, 1, 1) + \mu_2f(x, 0, 1) - \mu_1f(x, 1, 0)] \tag{3.2.40}\]

Therefore,
\[
(\mu_2 - \mu_1)f(x, 1, 1) + \mu_2f(x, 0, 1) - \mu_1f(x, 1, 0) = K
\]
where $K$ is some constant to be determined. The proof is complete when $K = 0$. The value of $K$ can be obtained by evaluating the last equation for some particular value of $x$, e.g. $x = 0$ or $x = N$. Let $x = 0$. If $\mu_1 > \mu_2$, equation (3.2.20) is

$$ r_1 p(0, 0, 1) = \mu_2 f(0, 0, 1) \quad (3.2.41) $$

Combining this with equation (3.2.28),

$$ (\mu_2 - \mu_1) f(0, 1, 1) + \mu_2 f(0, 0, 1) = 0 \quad (3.2.42) $$

Furthermore, when $\mu_1 > \mu_2$, equation (3.2.26) implies $\mu_1 f(0, 1, 0) = 0$. Thus,

$$ (\mu_2 - \mu_1) f(x, 1, 1) + \mu_2 f(x, 1, 0) - \mu_1 f(x, 1, 0) = 0 \quad (3.2.43) $$

If $\mu_2 < \mu_1$, equation (3.2.42) can be obtained by combining (3.2.20) and (3.2.23). The proof is complete by combining it with equation (3.2.26). The corresponding result can be obtained at $x = N$ in a similar manner.

\[ \square \]

**Production Rate**

$$ P_1 = \mu_1 \left[ \int_0^N (f(x, 1, 0) + f(x, 1, 1)) dx + p(0, 1, 1) \right] + \mu_2 p(N, 1, 1) \quad (3.2.44) $$

$$ P_2 = \mu_2 \left[ \int_0^N (f(x, 0, 1) + f(x, 1, 1)) dx + p(N, 1, 1) \right] + \mu_1 p(0, 1, 1) \quad (3.2.45) $$

**Average Buffer Content**

$$ \bar{x} = \sum_{\alpha_1=0}^1 \sum_{\alpha_2=0}^1 \left[ \int_0^N x f(x, \alpha_1, \alpha_2) dx + N p(N, \alpha_1, \alpha_2) \right] \quad (3.2.46) $$

**Conservation of Flow**

$$ P_1 = P_2 \quad (3.2.47) $$

As mentioned before, the conservation of flow is applicable for two-stage model. Below, we give the proof.
Proof. \( P_1 \) and \( P_2 \) are given by equation (3.2.44) and (3.2.45), respectively. Subtract these to obtain

\[
P_2 - P_1 = \int_0^N \left[ (\mu_2 - \mu_1)f(x,1,1) + \mu_2 f(x,1,1) - \mu_1 f(x,1,0) \right] dx
\]

By equation (3.2.39), the proof is complete.

**Blocking and Starvation**

Since we assume that the first machine is saturated, it will never be starved. The blocking is only applicable to machine 1.

\[
p_b = \frac{p(N,1,0) + (1 - \frac{\mu_2}{\mu_1})p(N,1,1)}{\mu_1}
\]

(3.2.48)

where \( p_b \) is the probability that machine 1 is unable to operate. This can happen because

- machine 2 is down and the buffer becomes full
- both machines are operational and the buffer is full

The starvation is only applicable to machine 2.

\[
p_s = \frac{p(0,0,1) + (1 - \frac{\mu_1}{\mu_2})p(0,1,1)}{\mu_2}
\]

(3.2.49)

where \( p_s \) is the probability that machine 2 is unable to operate. This happens because

- machine 1 is down and the buffer becomes empty
- both machines are operational and the buffer is empty

**Repair Frequency and Failure Frequency**

Here, we derive an important relation between repair frequency and failure frequency. It will be proved that these frequencies are the same. First, we define \( D_i \) as the probability that machine \( i \) is under repair.

\[
D_1 = \frac{p(0,0,1) + \int_0^N (f(x,0,0) + f(x,0,1))dx}{\mu_1}
\]

\[
D_2 = \frac{p(N,1,0) + \int_0^N (f(x,0,0) + f(x,1,0))dx}{\mu_2}
\]

(3.2.50)

(3.2.51)

Then

\[
FF_i = r_i D_i, \ i = 1, 2
\]

(3.2.52)
where $FF_i$ is a failure frequency defined as follows.

\[
FF_1 = p_1E_1^{(max)} + g_1(\mu_2)p(N, 1, 1) \\
FF_2 = p_2E_2^{(max)} + g_2(\mu_1)p(0, 1, 1)
\]

Note that, if $\mu_1 > \mu_2$, then $g_2(\mu_1) = p_2$ and also if $\mu_2 > \mu_1$, then $g_1(\mu_2) = p_1$.

**Proof.** Here, we only present for $i = 1$. A similar proof holds for $i = 2$.

\[
FF_1 - r_1D_1 = p_1p(0, 1, 1) - r_1p(0, 0, 1) \\
+ \int_0^N [p_1(f(x, 1, 0) + f(x, 1, 1)) - r_1(f(x, 0, 0) + f(x, 0, 1))] \, dx \\
+ g_1(\mu_2)p(N, 1, 1)
\]

Combining equation (3.2.14) and (3.2.15), the terms inside the integral in the previous equation can be transformed to $-\mu_2 \frac{d}{dx} f(x, 0, 1)$. By the steady state version of equation (3.2.20), this becomes

\[
FF_1 - r_1D_1 = -\mu_2 f(N, 0, 1) + g_1(\mu_2)
\]

In the steady state, the equation (3.2.35) implies $g_1(\mu_2) = \mu_2 f(N, 0, 1)$ since $p(N, 0, 0) = 0$. Thus,

\[
FF_1 = r_1D_1
\]

**Flow Rate-Idle Time (FR-IT)**

\[
P_1 = \mu_1e_1(1 - p_b) + \frac{p(N, 1, 1)(p_1\mu_2 - g_1(\mu_2)\mu_1)}{(r_1 + p_1)}
\]

\[
P_2 = \mu_2e_2(1 - p_s) + \frac{p(0, 1, 1)(p_2\mu_1 - g_2(\mu_1)\mu_2)}{(r_2 + p_2)}
\]

**Proof.** We only prove equation (3.2.57). Equation (3.2.58) can be proved in a similar way. First we note that

\[
E_1 + D_1 + p_b + p_s = 1
\]
Since the first machine can not be starved, then \( p_s = 0 \). Multiplying both sides by \( \mu_1 \) and using equation (3.2.52) gives

\[
P_1 + \frac{\mu_1 \left[p_1 E_1^{(max)} + g_1(\mu_2) p(N, 1, 1)\right]}{r_1} = \mu_1 (1 - p_b)
\]

By equation (3.2.9), last equation can be rewritten as

\[
\frac{r_1 P_1 + p_1 P_1}{r_1} = \mu_1 (1 - p_b) + \frac{p(N, 1, 1) [p_1 \mu_2 - g_1(\mu_2) \mu_1]}{r_1}
\]

The proof is complete by using the definition from equation (3.2.10).

3.3 Analytic Solution

In this section, the solution of the equations (3.2.12)-(3.2.15) is given. Here we distinguish between 2 cases. The equations (3.2.12)-(3.2.15) are ordinary differential equations. It is well-known that the solution of (3.2.12)-(3.2.15) is in the exponential form:

\[
f(x, \alpha_1, \alpha_2) = Ce^{\lambda x}Y_1^{\alpha_1}Y_2^{\alpha_2} \quad (3.3.1)
\]

Note that \( C, \lambda, Y_1 \) and \( Y_2 \) are parameters to be determined. Substituting (3.3.1) into the steady state version of equation (3.2.12)-(3.2.15) requires

\[
\sum_{i=1}^{2} (p_i Y_i - r_i) = 0 \quad (3.3.2)
\]

\[
(p_1 Y_1 - r_1) \frac{1 + Y_1}{Y_1} = -\mu_1 \lambda \quad (3.3.3)
\]

\[
(p_2 Y_2 - r_2) \frac{1 + Y_2}{Y_2} = \mu_2 \lambda \quad (3.3.4)
\]

One solution of these equations is

\[
Y_1 = \frac{r_1}{p_1}, \quad Y_2 = \frac{r_2}{p_2}, \quad \lambda = 0
\]

The equations (3.3.2)-(3.3.4) can be reduced to a single quadratic equation in \( Y_1 \):

\[
-(\mu_2 - \mu_1)p_1 Y_1^2 + [(\mu_2 - \mu_1)(r_1 + r_2) - (\mu_2 p_1 + \mu_1 p_2)] Y_1 + \mu_2 (r_1 + r_2) = 0 \quad (3.3.5)
\]
3.3.1 General Case

In this case, the processing rates of machines are different, i.e. $\mu_1 \neq \mu_2$. We replace (3.3.1) by a sum of two terms corresponding to the two solutions $Y_{1j}$ of (3.3.5):

$$f(x, \alpha_1, \alpha_2) = \sum_{j=1}^{2} C_j e^{\lambda_j x} Y_{1j}^{\alpha_1} Y_{2j}^{\alpha_2}$$  \hspace{1cm} (3.3.6)

We consider two cases.

**Case 1: $\mu_1 > \mu_2$**

In the steady state, we have from (3.2.21) or identities

$$p(0, 1, 1) = 0$$  \hspace{1cm} (3.3.7)

Equation (3.2.28) implies that

$$p(0, 0, 1) = \frac{(\mu_1 - \mu_2)f(0, 1, 1)}{r_1} = \frac{\mu_1 - \mu_2}{r_1} \sum_{j=1}^{2} C_j Y_{1j} Y_{2j}$$  \hspace{1cm} (3.3.8)

Since $p(0, 1, 1) = 0$ and $p(0, 0, 0) = 0$, equation (3.2.26) requires that $f(0, 1, 0) = 0$. It gives

$$\sum_{j=1}^{2} C_j Y_{1j} = 0$$  \hspace{1cm} (3.3.9)

To obtain $p(N, 1, 1)$, equation (3.2.35) is required. Since $p(N, 0, 0) = 0$, it gives

$$p(N, 1, 1) = \frac{\mu_2}{g_1(\mu_2)} f(N, 0, 1) = \frac{\mu_2}{g_1(\mu_2)} \sum_{j=1}^{2} C_j e^{\lambda_j N} Y_{2j}$$  \hspace{1cm} (3.3.10)

The steady state version of equation(3.2.30) can be used to determine $p(N, 1, 0)$. It gives

$$p(N, 1, 0) = \frac{p_2}{r_2} p(N, 1, 1) + \frac{\mu_1}{r_2} f(N, 1, 0) = \frac{p_2 \mu_2}{r_2 g_1(\mu_2)} \sum_{j=1}^{2} C_j e^{\lambda_j N} Y_{2j} + \frac{\mu_1}{r_2} \sum_{j=1}^{2} C_j e^{\lambda_j N} Y_{1j}$$  \hspace{1cm} (3.3.11)

Now, the constants $C_1$ and $C_2$ will be determined. It requires two equations to solve it.
The first one is equation (3.3.9) and the other one is the normalization equation which is given by (3.2.38). The normalization equation can be reduced in the form of

$$A_1C_1 + A_2C_2 = 1 \quad (3.3.12)$$

where \(A_j\) depends on the data. If we already determined \(A_1\) and \(A_2\), the constants \(C_1\) and \(C_2\) are easy to solve from (3.3.12) and (3.3.9).

**Case 2: \(\mu_2 > \mu_1\)**

From the steady state version of (3.2.20) and (3.2.23), we obtain

$$r_1p(0,0,1) = p_1p(0,1,1) + \mu_2f(0,0,1) \quad (3.3.13)$$

$$(p_1 + g_2(\mu_1))p(0,1,1) = r_1p(0,0,1) + (\mu_2 - \mu_1)f(0,1,1) \quad (3.3.14)$$

Combining (3.3.13) and (3.3.14) gives

$$p(0,1,1) = \frac{C_1Y_{21}}{g_2(\mu_1)}(\mu_2 + (\mu_2 - \mu_1)Y_{11}) + \frac{C_2Y_{22}}{g_2(\mu_1)}(\mu_2 + (\mu_2 - \mu_1)Y_{12}) \quad (3.3.15)$$

$$p(0,0,1) = \frac{p_1}{r_1}p(0,1,1) + \frac{\mu_2}{r_1}(C_1Y_{21} + C_2Y_{22}) \quad (3.3.16)$$

The value of \(p(N,1,0)\) can be determined from equation (3.2.37). It gives

$$p(N,1,0) = \frac{(\mu_2 - \mu_1)}{r_2}f(N,1,1)$$

$$= \frac{(\mu_2 - \mu_1)}{r_2} \left[ \sum_{j=1}^{2} C_j e^{\lambda_j N} Y_{2j} \right] \quad (3.3.17)$$

It is noted that \(p(N,1,1) = 0\) in this case. The coefficients \(C_1\) and \(C_2\) are determined from (3.2.38) and (3.3.35). Since \(p(N,1,1) = 0\) and \(p(N,0,0) = 0\), equation (3.2.35) requires \(f(N,0,1) = 0\). It gives

$$\sum_{j=1}^{2} C_j e^{\lambda_j N} Y_{2j} = 0 \quad (3.3.18)$$

Two linear equations (3.2.38) and (3.3.18) is sufficient to solve \(C_1\) and \(C_2\).

### 3.3.2 Special Case

Here, we have \(\mu_1 = \mu_2\). Equation (3.3.5) gives

$$Y_1 = \frac{r_1 + r_2}{p_1 + p_2} \quad (3.3.19)$$
From (3.3.2), $Y_2$ can be determined.

$$Y_2 = \frac{r_1 + r_2}{p_1 + p_2} \tag{3.3.20}$$

Then, equation (3.3.3) or (3.3.4) gives

$$\lambda = \frac{1}{\mu}(r_1 p_2 - r_2 p_1) \left( \frac{1}{p_1 + p_2} + \frac{1}{r_1 + r_2} \right) \tag{3.3.21}$$

In the same way as before, the boundary conditions yield

$$p(0, 1, 1) = C \frac{\mu}{g_2(\mu)} \left( p_1 + p_2 \right) + p(N, 1, 1) (3.3.22)$$

$$p(0, 0, 1) = \frac{p_1}{r_1} p(0, 1, 1) + C \frac{\mu}{g_1(\mu)} \left( r_1 p_2 - r_2 p_1 \right) \tag{3.3.23}$$

$$p(N, 1, 1) = C e^{\lambda N} \frac{\mu}{g_1(\mu)} \left( p_1 + p_2 \right) \tag{3.3.24}$$

$$p(N, 1, 0) = C \mu e^{\lambda N} \frac{r_1 + r_2}{p_1 + p_2} \left[ \frac{p_2}{r_2 g_1(\mu)} + \frac{1}{r_2} \right] \tag{3.3.25}$$

The constant $C$ can be obtained easily from normalization equation (3.2.38).

### 3.3.3 Performance Measures

**Production Rate**

Production rate can be computed from (3.2.44) or (3.2.45) but it is easier to use equation (3.2.57) or (3.2.58). Blocking and starvation probabilities can be computed using equations (3.2.48) and (3.2.49).

**Average Buffer Content**

If $\mu_1 = \mu_2$ and $\lambda \neq 0$, (3.2.46) becomes

$$\bar{x} = C \frac{1 + Y_1}{\lambda} \left[ e^{\lambda N} (\lambda N - 1) + 1 \right] + N (p(N, 1, 0) + p(N, 1, 1)) \tag{3.3.26}$$

If $\mu_1 = \mu_2$ and $\lambda = 0$, this must be replaced by

$$\bar{x} = C \frac{1}{2} (N(1 + Y))^2 + N (p(N, 1, 0) + p(N, 1, 1)) \tag{3.3.27}$$

If $\mu_1 \neq \mu_2$, the average buffer content is

$$\bar{x} = \sum_{j=1}^{2} C_j \left( \frac{1 + Y_{1j}}{\lambda_j} \right) \left( \frac{1 + Y_{2j}}{\lambda_j} \right) \left[ e^{\lambda_j N} (\lambda_j N - 1) + 1 \right] + N (p(N, 1, 0) + p(N, 1, 1)) \tag{3.3.28}$$
as long as both $\lambda_j \neq 0$. If $\lambda_1$ or $\lambda_2$ is zero, the corresponding term in the summation must be replaced by

$$\frac{C_j}{2} N^2 (1 + Y_{1j})(1 + Y_{2j})$$

\[ (3.3.29) \]

### 3.4 Numerical Results

In this section we present some numerical results of two-stage model. The first experiment shows the importance of considering the failure rate as a function of speed. In the second experiment, we can see the effect of changing the failure rate parameters while in third experiment we can see the effect of changing repair rate parameters. The last experiment we vary the speed of machine 2. It is noted that for these three experiments, we assume that the failure rate is constant at any speeds.

**Experiment 1**

Suppose that the first machine is a reliable machine with $\mu_1 = 1$. The other parameters are $p_2 = 0.05, r_2 = 0.09, \mu_2 = 2$ and $N = 5$. We compare three different models. The first model is 'Linear' where the second machine has a linear failure rate function given by

$$g_2(\mu) = 0.035\mu - 0.02$$

The second model is 'Proportional' where the failure rate of the machine 2 is proportional to its speed. The third model is 'Constant' where the failure rate of machine 2 is constant at any speeds. The results are shown in Table 3.1.

In this experiment, the two-stage model with constant failure rate has a lower throughput than the other two models with a linear failure rate function or proportional failure rate. This happens because in the constant model, when machine 2 slows down (in the case of an empty buffer), its failure rate is constant while in the proportional model its failure rate is lower.

**Experiment 2**

In this experiment we use the following parameters.

- $p_1 = 0.01$ and $p_2$ varying
- $r_1 = r_2 = 0.09$
From Table 3.2, it can conclude that if the failure rate of the machine increases then the throughput of the system will decrease. As an illustration see Figure 3.2.

**Experiment 3**

In this experiment we use the following parameters.

- $p_1 = p_2 = 0.01$
- $r_1 = 0.09$ and $r_2$ varying
- $\mu_1 = \mu_2 = 1$

Table (3.3) shows that if the repair rate decreases then the throughput of the system will decrease as well.

**Experiment 4**

In this experiment we use the following parameters.

|   | $p_2$ | $N = 0$ | 10  | 20  | 30  | 40  | 50  | $\infty$
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<td>0.5857</td>
<td>0.5943</td>
<td>0.5977</td>
<td>0.5991</td>
<td>0.5996</td>
<td>0.600</td>
</tr>
</tbody>
</table>

Table 3.2: Throughput for changing $p_2$ and $N$

Figure 3.2: Experiment 2 with $N = 10$
Table 3.3: Throughput for changing $r_2$ and $N$

<table>
<thead>
<tr>
<th></th>
<th>$r_2$</th>
<th>$N = 0$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.0817</td>
<td>0.8107</td>
<td>0.8363</td>
<td>0.8500</td>
<td>0.8585</td>
<td>0.8643</td>
<td>0.8685</td>
<td>0.891</td>
</tr>
<tr>
<td>b</td>
<td>0.0590</td>
<td>0.7809</td>
<td>0.8059</td>
<td>0.8200</td>
<td>0.8290</td>
<td>0.8350</td>
<td>0.8394</td>
<td>0.855</td>
</tr>
<tr>
<td>c</td>
<td>0.0300</td>
<td>0.6923</td>
<td>0.7135</td>
<td>0.7260</td>
<td>0.7338</td>
<td>0.7390</td>
<td>0.7424</td>
<td>0.750</td>
</tr>
<tr>
<td>d</td>
<td>0.0015</td>
<td>0.5625</td>
<td>0.5770</td>
<td>0.5857</td>
<td>0.5910</td>
<td>0.5943</td>
<td>0.5964</td>
<td>0.600</td>
</tr>
</tbody>
</table>

Table 3.4: Throughput of changing $\mu_2$ and $N$

<table>
<thead>
<tr>
<th></th>
<th>$\mu_2$</th>
<th>$N = 0$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.9900</td>
<td>0.8100</td>
<td>0.8384</td>
<td>0.8522</td>
<td>0.8605</td>
<td>0.8662</td>
<td>0.8762</td>
<td>0.891</td>
</tr>
<tr>
<td>b</td>
<td>0.9500</td>
<td>0.7773</td>
<td>0.8134</td>
<td>0.8281</td>
<td>0.8365</td>
<td>0.8418</td>
<td>0.8454</td>
<td>0.855</td>
</tr>
<tr>
<td>c</td>
<td>0.8333</td>
<td>0.6791</td>
<td>0.7208</td>
<td>0.7352</td>
<td>0.7414</td>
<td>0.7443</td>
<td>0.7457</td>
<td>0.750</td>
</tr>
<tr>
<td>d</td>
<td>0.6667</td>
<td>0.5482</td>
<td>0.5879</td>
<td>0.5984</td>
<td>0.6016</td>
<td>0.6026</td>
<td>0.6029</td>
<td>0.600</td>
</tr>
</tbody>
</table>

- $p_1 = p_2 = 0.01$
- $r_1 = r_2 = 0.09$
- $\mu_1 = 1$ and $\mu_2$ vary

From Table 3.4, it can be concluded that if the speed of machine 2 decreases, then the throughput will also decrease. Furthermore, all experiments show that increasing the buffer capacity will increase the throughput of the system. As an example we plot the result of experiment 4c (see Figure 3.3).
Figure 3.3: Experiment 4c
Chapter 4

Decomposition of Long Flow Lines

In the previous chapter, we have seen how to find the analytic solution of a two-stage model. For a longer line, we can only approximate the throughput or other performance measures. In order to do that a decomposition method is introduced here. The first section gives a general picture of the decomposition method. Section 4.2 presents the detail derivation of decomposition equations which are Interruption of Flow equations (IOF), Resumption of Flow equations (ROF) and Processing Rate equations. After that an algorithm, so-called General Accelerated David-Dallery-Xie (GADDX) is given to solve these decomposition equations in Section 4.3. Some numerical results are presented in the last section to demonstrate the accuracy and speed of this algorithm.

4.1 Introduction

A decomposition method to approximate a long flow line is given here. This decomposition method consists of three important set of equations which are Interruption of Flow equations (IOF), Resumption of Flow equations (ROF) and Processing Rate equations. The decomposition equations which will be derived here are proposed for analyzing an unreliable $k$-stage continuous material flow line. It is assumed that the flow line is saturated, asynchronous, and has finite buffer. In general, the decomposition method works as follows. The original line is broken into $k - 1$ two-stage lines ($L(i)$ for $i = 1, \ldots, k - 1$) as shown in figure 4.1. The equations that link these decomposed subsystem will be derived. The purpose of this chapter is to present the derivation these equations, while next chapter presents the algorithm to solve these equations. Some notations are defined first.

- $M_i =$ machine $i$ of original line
- $p_i =$ failure rate of machine $i$ in original line
- $r_i =$ repair rate of machine $i$ in original line
Figure 4.1: Decomposition Method

- $\mu_i$ = processing rate of machine $i$ in original line
- $N_i$ = buffer capacity of Buffer $i$
- $M_u(i)$ = upstream machine of subsystem $i$
- $M_d(i)$ = downstream machine of subsystem $i$
- $p_u(i)$ = failure rate of $M_u$ in subsystem $i$ while $p_d(i)$ for $M_d$
- $r_u(i)$ = repair rate of $M_u$ in subsystem $i$ while $r_d(i)$ for $M_d$
- $\mu_u(i)$ = effective average production rate of $M_u$ in subsystem $i$
- $\mu_d(i)$ = effective average production rate of $M_d$ in subsystem $i$
- $n_i(t)$ = the amount of material in buffer $i$ at time $t$ for $i = 1, \ldots, k - 1$

By convention, for all $t$, let

$$n_0(t) = \infty$$
$$n_k(t) = 0$$

(4.1.1)
We introduce the definition of $M_u(i)$ and $M_d(i)$ being up or down.

$M_u(i)$ is down = material is not flowing into $B_i$ because of failure of upstream machine

\[
M_u(i) = \begin{cases} 
M_i \text{ is down,} & i = 1 \\
M_i \text{ is down or } (M_u(i-1) \text{ is down and } n_{i-1} = 0), & i = 2, \ldots, k-1 
\end{cases}
\]

$M_u(i)$ is up = $M_u(i)$ is not down, $i = 1, \ldots, k-1$ \hspace{1cm} (4.1.2)

$M_d(i)$ is down = material is not flowing out of $B_i$ because of failure of downstream machine

\[
M_d(i) = \begin{cases} 
M_{i+1} \text{ is down,} & i = k-1 \\
M_{i+1} \text{ is down or } (M_d(i+1) \text{ is down and } n_{i+1} = N_{i+1}), & i = 1, \ldots, k-2 
\end{cases}
\]

$M_d(i)$ is up = $M_d(i)$ is not down, $i = 1, \ldots, k-1$ \hspace{1cm} (4.1.3)

The probability of finding $M_u(i)$ in state $\alpha_u(i)$, $M_d(i)$ in state $\alpha_d(i)$ and finding between $x$ and $x + \delta x$ units of material in Buffer $B_i$ is defined as

\[
\int_x^{x+\delta x} f_i(y, \alpha_u(i), \alpha_d(i)) \, dy \hspace{1cm} (4.1.4)
\]

Buffer $B_i$ can hold a maximum amount of material up to $N_i$. It is noted that in the flow line model, when the buffer becomes full, the upstream machine immediately cannot process material faster than the downstream machine of the buffer. Similarly, if the buffer becomes empty, the downstream machine immediately cannot process faster than the upstream machine of the buffer. We also define

\[
\begin{align*}
j_u^* (i, t) &= \max \{ j : j \leq i \text{ and } n_{j-1}(t) > 0 \} \quad i = 1, \ldots, k-1 \hspace{1cm} (4.1.5) \\
\end{align*}
\]

\[
\begin{align*}
\begin{cases} 
\end{cases} 
\end{align*}
\]

Equation (4.1.5) states that all buffers between $M_{j_u^*(i,t)}$ and $M_i$ are empty. It means that $M_i$ cannot be producing at a rate faster than $M_{j_u^*(i,t)}$. Analogously, equation (4.1.6) states that all buffers between $M_{j_d^*(i,t)}$ and $M_i$ are full. It means that $M_i$ cannot be producing at a rate faster than $M_{j_d^*(i,t)}$. The actual production rate of $M_i$ can be determined by finding the minimum of $\mu_i$, $\mu_{j_u^*(i,t)}$, $\mu_{j_d^*(i,t)}$. Formally, for $i = 1, \ldots, k-1$

\[
\begin{align*}
\mu_i(t) &= \text{the instantaneous production rate of Machine } i \text{ at time } t \\
&= \min \{ \mu_i, \mu_{j_u^*(i,t)}, \mu_{j_d^*(i,t)} \} \hspace{1cm} (4.1.7) \\
\mu_i(t)\delta t &= \text{the amount of material processed by } M_i \text{ in } (t, t + \delta t) \hspace{1cm} (4.1.8)
\end{align*}
\]

We also define:
• $p_i \delta t$ as the probability that $M_i$ goes from up to down during $(t, t + \delta t)$ when $n_i < N$ for small $\delta t$

• $r_i \delta t$ as the probability that $M_i$ goes from down to up during $(t, t + \delta t)$ for small $\delta t$

It is noted that the failure rate is dependent on whether the machine is blocked or starved, while the repair rate is independent of such external activity. In the Chapter 2, we found the failure rate function of Heineken’s machines. Therefore, we define the probability that $M_i$ goes from up to down in $(t, t + \delta t)$ for small $\delta t$ at $t$ as

$$p_i(t) \delta t = g_i(\cdot) \delta t$$

where $g_i(\cdot)$ is the failure rate function of machine $i$. The dynamics of the buffer level is defined as follows:

$$n_i(t) = \mu_i(t) - \mu_{i+1}(t)$$ (4.1.9)

The efficiency of $M_u(i)$ and $M_d(i)$ are $E_u(i)$ and $E_d(i)$ which are defined (for $i = 1, \ldots, k$):

$$E_u(i) = E_u(i)^{(\text{max})} + E_u(i)^{(\text{min})}$$ (4.1.10)

$$E_d(i) = E_d(i)^{(\text{max})} + E_d(i)^{(\text{min})}$$ (4.1.11)

where

$$E_u(i)^{(\text{max})} = \text{prob}\{\alpha_u(i) = 1, n_i < N_i\}$$ (4.1.12)

$$E_d(i)^{(\text{max})} = \text{prob}\{\alpha_d(i) = 1, n_i > 0\}$$ (4.1.13)

$$E_u(i)^{(\text{min})} = \frac{p_i(N, 1, 1)}{\mu_u(i)}$$ (4.1.14)

$$E_d(i)^{(\text{min})} = \frac{p_i(0, 1, 1)}{\mu_d(i)}$$ (4.1.15)

where $\mu_u(i)$ and $\mu_d(i)$ are defined as follows

• $\mu_u(i) \delta t$ = the amount of material produced by $M_u(i)$ in $(t, t + \delta t)$ for small $\delta t$ when $\alpha_u(i) = 1, n_{i-1} > 0$, and $N_i < n_i$.

• $\mu_d(i-1) \delta t$ = the amount of material produced by $M_d(i-1)$ in $(t, t + \delta t)$ for small $\delta t$ when $\alpha_d(i-1) = 1, n_{i-1} > 0$, and $N_i < n_i$.

The second terms of (4.1.10) and (4.1.11) are required for the continuous material model because a machine can be partially blocked or starved and producing at reduced rate. The non-isolated production rate of machine $i$ is the effective production rate of the machine
when it is part of a large system. We define

\[ P_u(i) = \text{the steady state non-isolated production rate of } M_u(i) \]

\[ = E_u(i)\mu_u(i), \quad i = 1, \ldots, k - 1 \]

\[ P_d(i) = \text{the steady state non-isolated production rate of } M_d(i) \]

\[ = E_d(i)\mu_d(i), \quad i = 1, \ldots, k - 1 \]

\[ P(i) = \text{the steady state non-isolated production rate of } L(i) \]

\[ = P_u(i) = P_d(i), \quad i = 1, \ldots, k - 1 \]

\[ P = \text{the steady state production rate of a } k \text{-stage line (4.1.16)} \]

Similar to the two-stage model, since there is no creation and destruction in the entire line, flow is conserved. \( P_u(i) \) and \( P_d(j) \) for all \( i \) and \( j \) from 1 to \( k \) should be equal. Therefore, the calculation of \( P_u(i) \) or \( P_d(i) \) for any \( L(i) \) should provide the throughput of the entire flow line (\( P \)). In principle, we have to show that the decomposition equations derived in this chapter satisfy this condition.

**Average Buffer Level**

Since \( B_i \) is assumed to have identical behavior in both the long line and in \( L(i) \), the average buffer level of \( B_i \), \( \bar{n}_i \) is defined by (for \( i = 1, \ldots, k - 1 \))

\[ \bar{n}_i = \sum_{\alpha_u(i)=0}^{1} \sum_{\alpha_d(i)=0}^{1} \left[ \int_0^{N_i} n_i f_i(n_i, \alpha_u(i), \alpha_d(i))dn_i + N_i p(N_i, \alpha_u(i), \alpha_d(i)) \right] \]  

\[ (4.1.17) \]

**Conservation of Flow**

The conservation of flow equations (COF) are identical to those of the two-stage model.

\[ P(i) = P(1) \quad i = 2, \ldots, k - 1 \]  

\[ (4.1.18) \]

We also define an isolated efficiency as follows

\[ e_i = \text{the steady state isolated efficiency of machine } i \]

\[ = \frac{r_i}{r_i + p_i}, \quad i = 1, \ldots, k \]  

\[ (4.1.19) \]

**Flow-Rate Idle-Time (FR-IT)**

\[ P(i) = e_i\mu_i(1 - p_b - p_s) + \frac{p_s(N, 1, 1)(p_u(i)\mu_d(i) - g_u(\mu_d(i))\mu_u(i))}{(r_u(i) + p_u(i))} + \frac{p_b(0, 1, 1)(p_d(i)\mu_u(i) - g_d(\mu_u(i))\mu_d(i))}{(r_d(i) + p_d(i))} \]  

\[ (4.1.20) \]

where \( p_s \) respectively \( p_b \) is the probability that Stage \( i \) is being starved respectively being blocked.
4.2 Decomposition Equations

In this section we derive a set of decomposition equations. There are 3 decomposition equations which are Interruption of Flow equations (IOF), Resumption of Flow equations (ROF) and Processing Rate equations. Here, we only derive the decomposition equations for the upstream machines while the downstream machines derivations are analogous.

4.2.1 Interruption of Flow Equations

This subsection presents the IOF equations which are related to the failure rates of up-and-down stream machines. We use the terms $M_u(i)$ and $M_d(i)$ for upstream and downstream machines of subsystem $i$, respectively. The failure of the upstream machine, say $M_u(i)$, from the perspective of buffer $B_i$ could be caused by either a failure of machine $M_i$ or starvation caused by $B_{i-1}$ being empty simultaneously with $M_u(i-1)$ being down. Our IOF equations take into account these two cases. Since we consider a continuous flow line, it is possible that when the buffer $B_{i-1}$ is empty, the upstream machine $M_u(i)$ is running at slower speed due to a starvation. In this case machine $M_u(i)$ is running slower than $\mu_i$. On the other hand, the failure rate of this machine changes corresponding to its failure rate function. It is also noted that we consider the flow lines under the assumption of operation dependent failure (ODF) in which a machine cannot breakdown when it is not operational.

First we define the upstream failure rate of $L(i)$.

\[
p_{u}(i)\delta t = \text{prob} [M_u(i) \text{ is down at } t + \delta t \mid M_u(i) \text{ up and } n_i < N_i \text{ at } t] \\
\qquad \quad i = 2, \ldots, k - 1 \quad (4.2.1)
\]

Using the definition of $M_u(i)$ is down from (4.1.2), we obtain

\[
p_{u}(i)\delta t = \text{prob} [M_i \text{ is down or } (n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down}) \text{ at } t + \delta t] \\
\qquad \quad \mid M_u(i) \text{ is up and } n_i < N_i \text{ at } t] \quad (4.2.2)
\]

Since \{M_i \text{ is down}\} and \{M_u(i-1) \text{ is down and } n_{i-1} = 0\} are mutually exclusive, we obtain

\[
p_{u}(i)\delta t = \text{prob}[M_i \text{ is down at } t + \delta t \mid M_u(i) \text{ is up and } n_i < N_i \text{ at } t] + \\
\quad \text{prob}[n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down at } t + \delta t] \\
\qquad \quad \mid M_u(i) \text{ is up and } n_i < N_i \text{ at } t] \quad (4.2.3)
\]
We define the mutually exclusive events

\[ a = M_i \text{ is up at } t + \delta t \text{ and } M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_{i-1} = 0 \text{ at } t \]

\[ b = M_i \text{ is down at } t + \delta t \text{ and } M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_{i-1} = 0 \text{ at } t \]

\[ c = M_i \text{ is down at } t + \delta t \text{ and } M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_{i-1} \neq 0 \text{ at } t \]

\[ d = M_i \text{ is up at } t + \delta t \text{ and } M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_{i-1} \neq 0 \text{ at } t \]

From probability theory, we know that

\[ \text{prob}(A \text{ and } B|B) = \frac{\text{prob}(A \text{ and } B)}{\text{prob}(B)} = \text{prob}(A|B) \]  

Using this, the first term of equation (4.2.3) is equal to

\[ \text{prob}(b \text{ or } c \mid M_i \text{ is down at } t + \delta t, M_u(i) \text{ is up, } n_i < N_i \text{ at } t, \text{ and } n_i - 1 = 0 \text{ at } t) \]

\[ \times \left( \frac{\text{prob}(M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_i - 1 = 0 \text{ at } t)}{\text{prob}(M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_i - 1 = 0 \text{ at } t)} \right) \]

We use proposition (A.0.2) from Appendix A to analyze equation (4.2.5). The first term of (4.2.3) can be written as

\[
\begin{bmatrix}
\frac{\text{prob}(b \text{ or } c \mid M_i \text{ is down at } t + \delta t, M_u(i) \text{ is up, } n_i < N_i \text{ and } n_{i-1} = 0 \text{ at } t)}{\text{prob}(b \text{ or } c \text{ or } d \mid M_i \text{ is down at } t + \delta t, M_u(i) \text{ is up, } n_i < N_i \text{ and } n_{i-1} = 0 \text{ at } t)} \times \\
\frac{\text{prob}(M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_i - 1 = 0 \text{ at } t)}{\text{prob}(M_u(i) \text{ is up, } n_i < N_i, \text{ and } n_i - 1 = 0 \text{ at } t)}
\end{bmatrix}
\]
We use (4.2.4) to obtain

\[
\left( \frac{\text{prob}[M_i \text{ is down at } t + \delta t | M_u(i) \text{ is up} \text{, } n_i < N_i \text{ and } n_{i-1} = 0 \text{ at } t]}{\text{prob}(M_u(i) \text{ is up} \text{, } n_i < N_i \text{ at } t)} \right) \times 
\left( \frac{\text{prob}[M_i \text{ is down at } t + \delta t | M_u(i) \text{ is up} \text{, } n_i < N_i \text{ and } n_{i-1} \neq 0 \text{ at } t]}{\text{prob}(M_u(i) \text{ is up} \text{, } n_i < N_i \text{ at } t)} \right)
\] 

(4.2.6)

From (4.1.2), if \( M_u(i) \) is up and \( n_{i-1} = 0 \) at \( t \), then \( M_i \) and \( M_{i-1} \) must be up at \( t \). Same argument as a two-stage model, to reach the state where \( n_{i-1} = 0, M_i \) is up and \( M_{i-1} \) is up at \( t \), the upstream machine must be slower than the downstream machine. Moreover, the probability of \( M_i \) failing at time \( t + \delta t \) is reduced corresponding to failure rate function \( g_i(\mu_u(i-1)) \). Therefore, when \( \mu_u(i-1) \leq \mu_d(i-1) \), we have

\[
\text{prob}[M_i \text{ is down at } t + \delta t | M_u(i) \text{ is up} \text{, } n_i < N_i \text{, and } n_{i-1} = 0 \text{ at } t] = g_i(\mu_u(i-1))\delta t
\]

(4.2.7)

and zero, otherwise.

Since blocking and starvation cannot occur at the same time in the continuous flow line model, we have

\[
\text{prob}(n_{i-1} = 0 \text{, and } n_i = N_i \text{ at time } t) = 0 \quad i = 2, \ldots, k - 1
\]

(4.2.8)

By the definition of \( M_u \) up and (4.2.8), we have

\[
P_{i-1}(0, 1, 1) = \text{prob}(M_u(i-1) \text{ is up} \text{, and } n_{i-1} = 0, M_d(i-1) \text{ is up} \text{, at } t)
\]

\[
= \text{prob}(M_u(i-1) \text{ is up} \text{, } M_i \text{ is up} \text{, (} n_i < N_i \text{ or } n_i = N_i \text{ and } M_d(i) \text{ is up})), \text{and } n_{i-1} = 0 \text{ at } t
\]

\[
= \text{prob}(M_u(i-1) \text{ is up}, M_i \text{ is up}, (n_i < N_i \text{ and } n_{i-1} = 0 \text{ at } t)
\]

\[
= \text{prob}(M_u(i) \text{ is up}, n_i < N_i \text{ and } n_{i-1} = 0 \text{ at } t)
\]

(4.2.9)

By (4.1.10), the second factor of the first term of (4.2.6) is

\[
\frac{P_{i-1}(0, 1, 1)}{E_u(i) - p_i(N_i, 1, 1) \frac{\mu_d(i)}{\mu_u(i)}}
\]

(4.2.10)

The first factor of the second term of (4.2.6) is \( p_i \delta t \).

The second factor of the second term of (4.2.6) can be written as

\[
\frac{1 - \text{prob}(M_u(i) \text{ is up} \text{, } n_i < N_i \text{, and } n_{i-1} = 0 \text{ at } t)}{\text{prob}(M_u(i) \text{ is up} \text{, } n_i < N_i \text{ at } t)}
\]

\[
= 1 - \frac{P_{i-1}(0, 1, 1)}{E_u(i) - p_i(N_i, 1, 1) \frac{\mu_d(i)}{\mu_u(i)}}
\]

(4.2.11)
We can conclude that the first term of (4.2.3) is
\[ g_i(\mu_u(i - 1)) \left( \frac{p_{i-1}(0,1,1)\mu_u(i)}{P(i) - p_i(N,1,1)\mu_d(i)} \right) + p_i \left( 1 - \frac{p_{i-1}(0,1,1)\mu_u(i)}{P(i) - p_i(N,1,1)\mu_d(i)} \right) \tag{4.2.12} \]

Now, we investigate the second term of (4.2.3).
\[
\text{prob}[n_{i-1} = 0 \text{ and } M_u(i - 1) \text{ is down at } t + \delta t | M_u(i) \text{ up and } n_i < N_i \text{ at } t]
\]
\[
= \text{prob}[n_{i-1} = 0 \text{ and } M_u(i - 1) \text{ is down and } M_d(i - 1) \text{ is up at } t + \delta t]
\]
\[
\text{prob}[n_{i-1} = 0 \text{ and } M_u(i - 1) \text{ is down and } M_d(i - 1) \text{ is down at } t + \delta t]
\]
\[
= \text{prob}[n_{i-1} = 0 \text{ and } M_u(i - 1) \text{ is down at } t + \delta t | M_u(i) \text{ is up and } n_i < N_i \text{ at } t]
\]
\[
\tag{4.2.13}
\]

The second term of (4.2.13) is zero. The probability of the first term of (4.2.13) can be divided into 3.

- \( n_{i-1} = 0, M_u(i - 1) \text{ is down and } M_d(i - 1) \text{ is up at } t, \text{ and nothing happens in } (t, t + \delta t), \) with probability \( 1 - r_u(i - 1)\delta t \)
- \( 0 < n_{i-1} < \mu_d(i - 1)\delta t, M_u(i - 1) \text{ is down and } M_d(i - 1) \text{ is up at } t, \) the buffer empties in \( (t, t + \delta t), \) with probability \( 1 - r_u(i - 1)\delta t - p_d(i - 1)\delta t \)
- \( n_{i-1} = 0, M_u(i - 1) \text{ is up and } M_d(i - 1) \text{ is up at } t, \) and \( M_u(i - 1) \) fails in \( (t, t + \delta t), \) with probability \( p_u(i - 1)\delta t \)

By definition (4.1.2), if \( M_u(i) \) is up at \( t, \) then \( M_i \) is up. If \( M_i \) is up and \( n_i < N_i \) at \( t, \) then \( M_u(i - 1) \) must also be up. Thus, the first term is zero. The second term can be written as, by the definition of conditional probability,
\[
\text{prob} \left( 0 < n_{i-1} < \mu_d(i - 1)\delta t \text{ and } M_u(i - 1) \text{ is down and } M_d(i - 1) \text{ is up and } n_i < N_i \text{ at } t \right) \times \frac{\text{prob } [M_u(i) \text{ is up and } n_i < N_i \text{ at } t]}{(1 - r_u(i - 1)\delta t - p_d(i - 1)\delta t)} \tag{4.2.14}
\]

As mentioned before, the probability of stage \( i \) being simultaneously blocked and starved is zero. We can approximate the previous equation by assuming that the probability that the stage \( i \) is blocked and almost starved \( (0 < n_{i-1} \leq \mu_d(i - 1)\delta t) \) simultaneously is also zero. Therefore, the statement that \( M_u(i) \) is up and \( n_i < N_i \) at \( t \) is not required in the
given that the machine equation is the probability that the machine given that it is now operational and the buffer behind it is not full. The third term of this numerator. By definition (4.1.2), if $M_d(i-1)$ is up and $n_{i-1} > 0$, then $M_i$ must be up and therefore $M_u(i)$ must also be up. Hence, we can simplify (4.2.14) to

$$
\text{prob} \left( \begin{array}{c}
0 < n_{i-1} \leq \mu_d(i-1)\delta t \\
M_u(i-1) \text{ is down and } M_d(i-1) \text{ up at } t
\end{array} \right) \times
\frac{\text{prob} [M_u(i) \text{ is up and } n_i < N_i \text{ at } t]}{(1 - r_u(i-1)\delta t - p_d(i-1)\delta t)}
$$

$$
= \int_0^{\mu_d(i-1)\delta t} f_{i-1}(x, 0, 1)dx
\times
\frac{\text{prob} [M_u(i) \text{ is up and } n_i < N_i \text{ at } t]}{(1 - r_u(i-1)\delta t - p_d(i-1)\delta t)}
$$

$$
= \left( \frac{f_{i-1}(0, 0, 1)}{\text{prob} [M_u(i) \text{ is up and } n_i < N_i \text{ at } t]} \right) \mu_d(i-1)\delta t
$$

The third term is

$$
\left( \frac{p_{i-1}(0, 1, 1)}{E_u(i) - p_i(N, 1, 1)\frac{\mu_d(i)}{\mu_u(i)}} \right) p_u(i-1)\delta t
$$

Similar to the two-stage model (see equation (3.2.20)) we have

$$
p_i(0, 0, 1)r_u(i) = f_i(0, 0, 1)\mu_d(i) + p_i(0, 1, 1)p_u(i)
$$

By using this, we can combine (4.2.14) and (4.2.15). Hence the second term of (4.2.3) becomes

$$
r_u(i-1) \left( \frac{p_{i-1}(0, 0, 1)\mu_u(i)}{P(i) - p_i(N, 1, 1)\mu_d(i)} \right)
$$

Equations (4.2.12) and (4.2.18) give the solution of equation (4.2.3) which is the IOF equation for upstream.

$$
p_u(i) = g_i(\mu_u(i-1)) \left( \frac{p_{i-1}(0, 1, 1)\mu_u(i)}{P(i) - p_i(N, 1, 1)\mu_d(i)} \right) + p_i \left( 1 - \frac{p_{i-1}(0, 1, 1)\mu_u(i)}{P(i) - p_i(N, 1, 1)\mu_d(i)} \right) + r_u(i-1) \left( \frac{p_{i-1}(0, 0, 1)\mu_u(i)}{P(i) - p_i(N, 1, 1)\mu_d(i)} \right)
$$

The first two terms of this equation is the probability that the machine $M_i$ will be down given that it is now operational and the buffer behind it is not full. The third term of this equation is the probability that the machine $M_{i-1}$ is down and the buffer behind it is empty given that the machine $M_i$ is operational and the buffer behind it is not full. It is noted
that when $\mu_u(i-1) > \mu_d(i-1)$, the value of the first two terms of this equation reduces to $p_i$, as in Gershwin (1989). An almost identical derivation for downstream equations results:

$$p_d(i) = g_{i+1}(\mu_d(i+1)) \left( \frac{p_{i+1}(N,1,1)\mu_d(i)}{P(i) - p_i(0,1,1)\mu_u(i)} \right) + p_{i+1} \left( 1 - \frac{p_{i+1}(N,1,1)\mu_d(i)}{P(i) - p_i(0,1,1)\mu_u(i)} \right)$$

$$+ r_d(i+1) \left( \frac{p_{i+1}(N,1,0)\mu_d(i)}{P(i) - p_i(0,1,1)\mu_u(i)} \right)$$

(4.2.20)

When the failure rate function is proportional to machine speed, the decomposition equations $p_u(i)$ and $p_d(i)$ reduce to the Burman (1995) decomposition equations.

### 4.2.2 Resumption of Flow Equations

Here we derive the decomposition equations that provide the repair rate of $M_u(i)$ and $M_d(i)$. These equations are called Resumption of Flow equations (ROF). As mentioned before the failure of machine $i$ from the perspective of Buffer $i$ could be caused by either Machine $i$ failing or a starvation caused by Buffer $B_{i-1}$ being empty simultaneously with $M_u(i-1)$ being down. The ROF equations derived here account for the recovery from both of these conditions. The repair rate of $M_u(i)$ is approximated as a weighted average of the rates of repair as follows:

$$r_u(i)\delta t = A(i-1)X(i) + B(i)(1-X(i)) \quad i = 2, \ldots, k-1$$

(4.2.21)

where

$$A(i-1) = \text{prob}[M_i \text{ is up and } \neg(n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down at } t + \delta t) \mid n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down at } t]$$

$$= r_u(i-1)\delta t$$

(4.2.22)

$$B(i) = \text{prob}[M_i \text{ up and } \neg(n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down}) \text{ at } t + \delta t \mid M_i \text{ is down at } t]$$

$$= r_i\delta t$$

(4.2.23)

$$X(i) = \text{prob}[n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down at } t \mid M_i \text{ is down or } (n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down}) \text{ at } t]$$

$$= \frac{\text{prob}[M_i \text{ is down or } (n_{i-1} = 0 \text{ and } M_u(i-1) \text{ is down}) \text{ at } t]}{\text{prob}[M_u(i) \text{ is down}]}$$

(4.2.24)
To determine $X(i)$, we use the fact that the rate of transitions $M_i$ makes in and out of the failed state must be equal (see equation (3.2.52) for a two-stage model). In the case of a longer line, we have the following balance equation

$$r_u(i) \text{prob}[M_u(i) \text{ is down}] = p_u(i) F_u^{(max)}(i) + g_u(\mu_d(i)) p_{i-1}(N, 1, 1) \quad (4.2.25)$$

Combining this with (4.1.16), $X(i)$ can be rewritten as

$$X(i) = \frac{p_{i-1}(0, 0, 1)r_u(i)\mu_u(i)}{p_u(i)P(i) + p_{i-1}(N, 1, 1)(\mu_u(i)g_u(\mu_d(i)) - p_u(i)\mu_d(i))} \quad (4.2.26)$$

Finally, after substituting (4.2.22), (4.2.23) and (4.2.26) into (4.2.21), we obtain the ROF equation for the upstream machine:

$$r_u(i) = r_u(i - 1) \left( \frac{p_{i-1}(0, 0, 1)r_u(i)\mu_u(i)}{p_u(i)P(i) + p_{i-1}(N, 1, 1)(\mu_u(i)g_u(\mu_d(i)) - p_u(i)\mu_d(i))} \right)$$

$$r_i - \frac{r_u(i)P(i) + p_{i-1}(N, 1, 1)(\mu_u(i)g_u(\mu_d(i)) - p_u(i)\mu_d(i))}{p_u(i)P(i) + p_{i-1}(N, 1, 1)(\mu_u(i)g_u(\mu_d(i)) - p_u(i)\mu_d(i))} \quad (4.2.27)$$

By a similar derivation, we can obtain the ROF equation for the downstream machine:

$$r_d(i) = r_d(i + 1) \left( \frac{p_{i+1}(N, 1, 0)r_d(i)\mu_d(i)}{p_d(i)P(i) + p_{i+1}(0, 1, 1)(\mu_d(i)g_d(\mu_u(i)) - p_d(i)\mu_u(i))} \right)$$

$$r_{i+1} = \frac{r_d(i + 1)P(i) + p_{i+1}(0, 1, 1)(\mu_d(i)g_d(\mu_u(i)) - p_d(i)\mu_u(i))}{p_d(i)P(i) + p_{i+1}(0, 1, 1)(\mu_d(i)g_d(\mu_u(i)) - p_d(i)\mu_u(i))} \quad (4.2.28)$$

### 4.2.3 Processing Rate Equations

The last decomposition equations are derived for $\mu_u(i)$ (processing rate of the first stage in $L(i)$) and $\mu_d(i)$ (processing rate of the second stage in $L(i)$). These equations are derived directly from the FR-IT equation (4.1.20) and the COF equation (4.1.18). It is noted that in the case of a longer line equations (3.2.57) and (3.2.58) can be rewritten as

$$P_s = 1 - \frac{P(i)}{\mu_d(i - 1) e_d(i - 1)} + \frac{A_u}{\mu_d(i - 1)} \quad (4.2.29)$$

$$P_b = 1 - \frac{P(i)}{\mu_u(i) e_u(i)} + \frac{B_u}{\mu_u(i)} \quad (4.2.30)$$

where

$$A_u = \frac{p(0, 1, 1)[\mu_d(i - 1)\mu_u(i - 1) - g_d(\mu_u(i - 1))\mu_d(i - 1)]}{r_d(i)} \quad (4.2.31)$$
By a little algebraic manipulation, the COF equation (4.1.18) results in

\[
\mu_u(i) = \left( \frac{1}{e_{i+1} \mu_{i+1}} - \frac{(e_d(i-1)A_u + e_u(i)B_u)}{P(i) e_{i+1} \mu_{i+1}} \right) \times \left( \frac{P(i) - e_u(i) B_u}{e_u(i) P(i)} \right) \quad (4.2.33)
\]

By a similar derivation, we can obtain the processing rate equation for the downstream machine:

\[
\mu_d(i) = \left( \frac{1}{e_{i+1} \mu_{i+1}} - \frac{(e_u(i+1)B_d + e_d(i)A_d)}{P(i) e_{i+1} \mu_{i+1}} \right) \times \left( \frac{P(i) - e_d(i) A_d}{e_d(i) P(i)} \right) \quad (4.2.34)
\]

where

\[
A_d = \frac{p(0, 1, 1) [p_d(i) \mu_u(i) - g_d(\mu_u(i)) \mu_d(i)]}{r_d(i+1)} \quad (4.2.35)
\]

\[
B_d = \frac{p(N, 1, 1) [p_u(i+1) \mu_d(i+1) - g_u(\mu_d(i)) \mu_u(i+1)]}{r_u(i+1)} \quad (4.2.36)
\]

### 4.2.4 Summary

In this section we summarize all the derived decomposition equations.

**Upstream equations**

\[
p_u(i) = g_u(\mu_u(i-1)) \left( \frac{p_{i-1}(0, 1, 1) \mu_u(i)}{P(i) - p_u(N, 1, 1) \mu_d(i)} \right) + p_i \left( 1 - \frac{p_{i-1}(0, 1, 1) \mu_u(i)}{P(i) - p_u(N, 1, 1) \mu_d(i)} \right)
\]

\[
r_u(i) = r_u(i-1) \left( \frac{p_{i-1}(0, 0, 1) \mu_u(i)}{p_u(i) P(i) + p_{i-1}(N, 1, 1) (\mu_u(i) g_u(\mu_u(i)) - p_u(i) \mu_d(i))} \right)
\]

\[
\mu_u(i) = \left( \frac{1}{e_{i+1} \mu_{i+1}} - \frac{(e_d(i-1)A_u + e_u(i)B_u)}{P(i) e_{i+1} \mu_{i+1}} \right) \times \left( \frac{P(i) - e_u(i) B_u}{e_u(i) P(i)} \right) \quad (4.2.39)
\]
\[ A_u = \frac{p(0, 1, 1) [p_d(i - 1)u(i - 1) - g_d(u(i - 1))u_d(i - 1)]}{r_d(i)} \quad (4.2.40) \]
\[ B_u = \frac{p(N, 1, 1) [p_u(i)u_d(i) - g_u(u_d(i))u(i)]}{r_u(i)} \quad (4.2.41) \]

**Downstream equations**

\[ p_d(i) = g_{i+1}(\mu_d(i + 1)) \left( \frac{p_{i+1}(N, 1, 1)\mu_d(i)}{P(i) - p_i(0, 1, 1)\mu_u(i)} \right) + p_{i+1} \left( 1 - \frac{p_{i+1}(N, 1, 1)\mu_d(i)}{P(i) - p_i(0, 1, 1)\mu_u(i)} \right) \]
\[ + r_d(i + 1) \left( \frac{p_{i+1}(N, 1, 0)\mu_d(i)}{P(i) - p_i(0, 1, 1)\mu_u(i)} \right) \quad (4.2.42) \]

\[ r_d(i) = r_d(i + 1) \left( \frac{p_{i+1}(N, 1, 0)r_d(i)\mu_d(i)}{p_d(i)P(i) + p_{i+1}(0, 1, 1)(\mu_d(i)g_d(u(i)) - p_d(i)\mu_u(i))} \right) \]
\[ - \frac{r_{i+1}p_{i+1}(0, 0, 1)r_d(i)\mu_u(i)}{p_d(i)P(i) + p_{i+1}(0, 1, 1)(\mu_d(i)g_d(u(i)) - p_d(i)\mu_u(i))} \quad (4.2.43) \]

\[ \mu_d(i) = \left( \frac{1}{\tau_{i+1}p_{i+1}} - \frac{(e_u(i+1)B_d + e_d(i)A_d)}{P(i)\tau_{i+1}p_{i+1}} + \frac{1}{P(i)} + \frac{B_d}{P(i)\mu_u(i+1)} - \frac{1}{\mu_u(i+1)e_u(i+1)} \right) \times \]
\[ \left( \frac{P(i) - e_d(i)A_d}{e_d(i)P(i)} \right) \quad (4.2.44) \]

where

\[ A_d = \frac{p(0, 1, 1) [p_d(i)\mu_u(i) - g_d(u(i))\mu_d(i)]}{r_d(i + 1)} \quad (4.2.45) \]
\[ B_d = \frac{p(N, 1, 1) [p_u(i + 1)\mu_d(i + 1) - g_u(u_d(i + 1))\mu_u(i + 1)]}{r_u(i + 1)} \quad (4.2.46) \]

### 4.3 The Dallery-David-Xie Algorithm

In this section we present the algorithm, the so-called General Accelerated David-Dallery-Xie (GADDX), to solve the decomposition equations which were developed in the previous section. The following algorithm is almost identical to the algorithm presented in Burman (1995). The difference is that our decomposition equations are more general than Burman’s. In our decomposition equations, we express the failure rate as a function of the speed while Burman only assumes that the failure rate is proportional to the speed. Furthermore, we also use the same substitutions as Burman proposed to improve the algorithm. The first one is based on the Conservation of Flow equation (COF). \(P(i - 1)\) is substituted for \(P(i)\) in all the upstream equations and \(P(i + 1)\) for \(P(i)\) in all of the downstream equations.
Without this substitution in the processing rate equations (4.2.39) and (4.2.44) there would not be enough equations to solve for all the unknowns because each upstream processing rate equation would have an equivalent downstream equation. Burman also proposed the following adjustments:

\[
P(i) - p_i(N_i, 1, 1)\mu_d(i) \Rightarrow P(i)
\]
\[
P(i) - p_i(0, 1, 1)\mu_u(i) \Rightarrow P(i)
\]

These substitutions are made to achieve a high rate of convergence of the algorithm. The motivation for these substitutions is that the current values of \(p_i(N_i, 1, 1)\) and \(p_i(0, 1, 1)\) are used to estimate the values of \(p_u(i)\) at each step of the algorithm. However, \(p_u(i)\) and \(p_d(i)\) are also required to estimate \(p_i(N_i, 1, 1)\) and \(p_i(0, 1, 1)\). The effect of this is that one is forced to use values \(p_i(N_i, 1, 1)\) and \(p_i(0, 1, 1)\) from a previous iteration of the algorithm to obtain the current estimate of \(p_u(i)\) and \(p_d(i)\), while most of the other values used to estimate \(p_u(i)\) and \(p_d(i)\) are from the current iteration. He did some spot checks about the omission of \(p_i(N_i, 1, 1)\) and \(p_i(0, 1, 1)\), the results still appeared to be accurate even in case with machines with significantly different production rates. We believe that the biases in the experiment on long lines in the next section are at least partially due to these substitutions. However, from the experiments performed later, it still shows that the algorithm performs very well. Moreover, we also tried the algorithm with or without these substitutions and these substitutions are really substantial to achieve a high rate of convergence of the algorithm.

After all substitutions mentioned above, we can rewrite the equations (4.2.37)-(4.2.39) into the following form:

\[
p_u(i) = \mu_u(i)K_1 + p_i
\]
\[
r_u(i) = \frac{r_u(i)\mu_u(i)K_2}{p_u(i)} + r_i
\]
\[
\mu_u(i) = \left(1 + \frac{p_u(i)}{r_u(i)}\right)
\]

where

\[
K_1 = \frac{1}{P(i-1)}[p_{i-1}(0,1,1)(g_i(\mu_u(i-1)) - p_i)
+ r_u(i-1)p_{i-1}(0,0,1)]
\]
\[
K_2 = \frac{p_{i-1}(0,0,1)(r_u(i-1) - r_i)}{P(i-1)}
\]
\[
K_3 = \frac{1}{\varepsilon \mu_i + \frac{1}{P(i-1)} + \frac{A_u}{P(i-1)\mu_d(i-1)} - \frac{1}{\mu_d(i-1)\varepsilon_d(i-1)}} - \frac{r_u(i-1)A_u}{P(i-1)\varepsilon \mu_i}
\]
A similar procedure can be used to generate the decomposition equations for the downstream
machine from equation (4.2.42)-(4.2.44).

\[ p_d(i) = \mu_d(i)K_4 + p_{i+1} \] (4.3.9)

\[ r_d(i) = \frac{r_d(i)\mu_d(i)K_5}{p_d(i)} + r_{i+1} \] (4.3.10)

\[ \mu_d(i) = \left( \frac{1}{K_6} \right) \left( 1 + \frac{p_d(i)}{r_d(i)} \right) \] (4.3.11)

where

\[ K_4 = \frac{1}{P(i+1)} \left[ p_{i+1}(N, 1, 1)(\mu_d(i+1)) - p_{i+1} \right] + r_d(i+1)p_{i+1}(N, 1, 0) \] (4.3.12)

\[ K_5 = \frac{p_{i+1}(N, 1, 1)(r_d(i+1) - r_{i+1})}{P(i+1)} \] (4.3.13)

\[ K_6 = \frac{1}{e_{i+1}\mu_{i+1}} + \frac{1}{P(i+1)} + \frac{B_d}{P(i+1)\mu_{i+1}} - \frac{1}{\mu_{i+1}} - \frac{e_u(i+1)B_d}{P(i+1)e_{i+1}\mu_{i+1}} \] (4.3.14)

If we analyze carefully, we see a dependency among equations (4.3.3)-(4.3.5), e.g. the right
side of (4.3.3) contains a \( \mu_u(i) \). In other words, the upstream equations (4.3.3)-(4.3.5) are
a set of three non-linear equations in three unknowns which can be represented as

\[ p_u(i) = f(p_u(i), r_u(i), \mu_u(i)) \]

\[ r_u(i) = f'(p_u(i), r_u(i), \mu_u(i)) \]

\[ \mu_u(i) = f''(p_u(i), r_u(i), \mu_u(i)) \] (4.3.15)

Equations (4.3.9)-(4.3.11) have an equivalent representation for the downstream equations

\[ p_d(i) = f^*(p_d(i), r_d(i), \mu_d(i)) \]

\[ r_d(i) = f'^*(p_d(i), r_d(i), \mu_d(i)) \]

\[ \mu_d(i) = f''^*(p_d(i), r_d(i), \mu_d(i)) \] (4.3.16)

To find the solution of (4.3.15) and (4.3.16), we can use a solver software such as Maple or
others. The closed form solutions of (4.3.15) are the following

\[ p_u(i) = \frac{r_i p_i K_3 + p_i K_2 + r_i K_1}{r_i K_3 + K_2 - K_1} \] (4.3.17)

\[ r_u(i) = \frac{r_i p_i K_3 + p_i K_2 + r_i K_1}{p_i K_3 + K_1 - K_2} \] (4.3.18)

\[ \mu_u(i) = \frac{r_i + p_i}{r_i K_3 + K_2 - K_1} \] (4.3.19)
The closed form solutions of (4.3.16) are the following

\[ p_d(i) = \frac{r_ip_iK_6 + p_iK_5 + r_iK_4}{r_iK_6 + K_5 - K_4} \quad (4.3.20) \]

\[ r_d(i) = \frac{r_ip_iK_6 + p_iK_5 + r_iK_4}{p_iK_6 + K_4 - K_5} \quad (4.3.21) \]

\[ \mu_d(i) = \frac{r_i + p_i}{r_iK_6 + K_5 - K_4} \quad (4.3.22) \]

Finally, below we present the General Accelerated David-Dallery-Xie (GADDX) to solve our decomposition equations.

**Algorithm (GADDX)**

1. **Initialization.** Provide the following initial guesses for the parameters of each two-stage line:

\[ p_u(i) = p_i \]
\[ r_u(i) = r_i \]
\[ \mu_u(i) = \mu_i \]
\[ p_d(i) = p_{i+1} \]
\[ r_d(i) = r_{i+1} \]
\[ \mu_d(i) = \mu_{i+1} \quad i = 1, \ldots, k - 1 \quad (4.3.23) \]

2. Perform Step 1 and Step 2 until the termination condition is satisfied.

   **Step 1.** Let \( i \) range over values from 2 to \( k - 1 \). Evaluate \( L(i - 1) \) using the two-stage model with the most recent values of \( r_d(i - 1), p_d(i - 1), \mu_d(i - 1), r_u(i - 1), p_u(i - 1), \mu_u(i - 1) \). Then substitute these parameters and the resulting \( P(i - 1) \) into the upstream decomposition equations (4.3.17)-(4.3.19).

   **Step 2.** Let \( i \) range over values from \( k - 2 \) to 1. Evaluate \( L(i + 1) \) using the two-stage model with the most recent values of \( r_d(i + 1), p_d(i + 1), \mu_d(i + 1), r_u(i + 1), p_u(i + 1), \mu_u(i + 1) \). Then substitute these parameters and the resulting \( P(i + 1) \) into the downstream decomposition equations (4.3.20)-(4.3.22).

3. **Termination Condition.** Terminate the algorithm when the greatest value of

\[ \|P(i) - P(1)\| \quad (4.3.24) \]

is less than some specified value, for \( i = 2, \ldots, k - 1 \)

It is noted that each equation is evaluated with the most recent values of each parameter, and this makes the order of evaluation very important. Similarly, in Step 2, each equation is also evaluated with the most recent value of each parameter.
4.4 Numerical Results

In this section we present some numerical results to investigate the accuracy and the efficiency of the algorithm described before. The first ten numerical experiments, case 1-10,

<table>
<thead>
<tr>
<th>Case</th>
<th>Throughput</th>
<th>Buffer Content</th>
<th>$#$ call</th>
<th>Throughput</th>
<th>Buffer Content</th>
<th>$#$ call</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4680</td>
<td>$N_1 = 3.1723$</td>
<td>20</td>
<td>0.4680</td>
<td>$N_1 = 3.1723$</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N_2 = 1.8277$</td>
<td></td>
<td></td>
<td>$N_2 = 1.8277$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.9026</td>
<td>$N_1 = 3.2530$</td>
<td>20</td>
<td>0.9026</td>
<td>$N_1 = 3.2521$</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N_2 = 1.7470$</td>
<td></td>
<td></td>
<td>$N_2 = 1.7479$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.3207</td>
<td>$N_1 = 3.1494$</td>
<td>34</td>
<td>0.3207</td>
<td>$N_1 = 3.1495$</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N_2 = 1.8506$</td>
<td></td>
<td></td>
<td>$N_2 = 1.8505$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.3558</td>
<td>$N_1 = 6.1220$</td>
<td>22</td>
<td>0.3588</td>
<td>$N_1 = 6.1211$</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N_2 = 3.8780$</td>
<td></td>
<td></td>
<td>$N_2 = 3.8789$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.7604</td>
<td>$N_1 = 6.3439$</td>
<td>20</td>
<td>0.7604</td>
<td>$N_1 = 6.3511$</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N_2 = 3.1344$</td>
<td></td>
<td></td>
<td>$N_2 = 3.1342$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.3015</td>
<td>-</td>
<td>441</td>
<td>0.3015</td>
<td>-</td>
<td>256</td>
</tr>
<tr>
<td>7</td>
<td>0.6351</td>
<td>-</td>
<td>572</td>
<td>0.6351</td>
<td>-</td>
<td>200</td>
</tr>
<tr>
<td>8</td>
<td>0.2315</td>
<td>-</td>
<td>640</td>
<td>0.2315</td>
<td>-</td>
<td>720</td>
</tr>
<tr>
<td>9</td>
<td>0.2296</td>
<td>$N_1 = 3.9877$</td>
<td>3819</td>
<td>0.2296</td>
<td>$N_1 = 3.9880$</td>
<td>1044</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N_5 = 2.9602$</td>
<td></td>
<td></td>
<td>$N_5 = 2.9600$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$N_{10} = 2.5000$</td>
<td></td>
<td></td>
<td>$N_{10} = 2.5000$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.2927</td>
<td>-</td>
<td>1976</td>
<td>0.2927</td>
<td>-</td>
<td>1332</td>
</tr>
</tbody>
</table>

Table 4.1: Result of Glassey and Hong & GADDX

were carried out using the parameters given in Glassey and Hong [10]. The parameters are presented in Appendix B. The purpose of these ten cases is to investigate the computational efficiency of the algorithm in homogeneous lines (have equal processing rate). We took the results of Glassey and Hong’s continuous model and from Glassey and Hong’s paper [10]. These results of the comparison are presented in Table 4.1. To compare the computational efficiency of our method versus that of Glassey and Hong, the number of evaluations of two-stage lines are examined. The symbol $\#$ call in the tables represents the number of evaluations of two-stage lines. From Table 4.1, we can conclude that our continuous model converges more rapidly than the Glassey and Hong model in the case of homogeneous lines. It is noted that in these experiments the failure rate functions are constant, since all machines have the same processing rates. The next ten experiments, case 11-20, were carried out with the parameters presented in Appendix B to see how well our method performs for cases where machines have unequal production rates. To make a comparison fair, since the Glassey and Hong continuous model assumes that the failure rates are constant for all
### Table 4.2: Result of GADDX & Glassey and Hong & Simulation

<table>
<thead>
<tr>
<th>Case</th>
<th>GADDX Throughput</th>
<th>Error</th>
<th>$#$ call</th>
<th>Glassey and Hong Throughput</th>
<th>Error</th>
<th>$#$ call</th>
<th>Simulation Throughput</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.8332</td>
<td>0.24%</td>
<td>12</td>
<td>0.8356</td>
<td>0.53%</td>
<td>28</td>
<td>0.8312</td>
</tr>
<tr>
<td>12</td>
<td>0.8564</td>
<td>-0.57%</td>
<td>8</td>
<td>0.8588</td>
<td>-0.29%</td>
<td>46</td>
<td>0.8613</td>
</tr>
<tr>
<td>13</td>
<td>0.7250</td>
<td>0.15%</td>
<td>10</td>
<td>0.7280</td>
<td>0.57%</td>
<td>46</td>
<td>0.7239</td>
</tr>
<tr>
<td>14</td>
<td>0.8170</td>
<td>0.12%</td>
<td>12</td>
<td>0.8170</td>
<td>0.12%</td>
<td>50</td>
<td>0.8160</td>
</tr>
<tr>
<td>15</td>
<td>0.8738</td>
<td>0.09%</td>
<td>20</td>
<td>0.8754</td>
<td>0.27%</td>
<td>28</td>
<td>0.8730</td>
</tr>
<tr>
<td>16</td>
<td>0.8258</td>
<td>-0.55%</td>
<td>32</td>
<td>0.8352</td>
<td>0.58%</td>
<td>402</td>
<td>0.8304</td>
</tr>
<tr>
<td>17</td>
<td>0.7992</td>
<td>0.38%</td>
<td>24</td>
<td>0.8056</td>
<td>1.18%</td>
<td>495</td>
<td>0.7962</td>
</tr>
<tr>
<td>18</td>
<td>0.7472</td>
<td>0.62%</td>
<td>36</td>
<td>0.7476</td>
<td>0.67%</td>
<td>102</td>
<td>0.7426</td>
</tr>
<tr>
<td>19</td>
<td>0.8312</td>
<td>1.29%</td>
<td>48</td>
<td>0.8256</td>
<td>0.61%</td>
<td>116</td>
<td>0.8206</td>
</tr>
<tr>
<td>20</td>
<td>0.8313</td>
<td>0.65%</td>
<td>48</td>
<td>0.8323</td>
<td>0.77%</td>
<td>112</td>
<td>0.8259</td>
</tr>
</tbody>
</table>

### Table 4.3: Results of 3-machine model

<table>
<thead>
<tr>
<th>Case</th>
<th>GADDX constant</th>
<th>Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TH</td>
<td>Error</td>
</tr>
<tr>
<td>21</td>
<td>0.8624</td>
<td>0.09%</td>
</tr>
<tr>
<td>22</td>
<td>0.8525</td>
<td>-0.29%</td>
</tr>
<tr>
<td>23</td>
<td>0.7446</td>
<td>0.42%</td>
</tr>
<tr>
<td>24</td>
<td>0.7454</td>
<td>0.96%</td>
</tr>
<tr>
<td>25</td>
<td>0.8622</td>
<td>-0.13%</td>
</tr>
<tr>
<td>26</td>
<td>0.8572</td>
<td>0.36%</td>
</tr>
<tr>
<td>27</td>
<td>0.8332</td>
<td>-0.39%</td>
</tr>
<tr>
<td>28</td>
<td>0.8198</td>
<td>-0.07%</td>
</tr>
<tr>
<td>29</td>
<td>0.8962</td>
<td>0.82%</td>
</tr>
<tr>
<td>30</td>
<td>0.8807</td>
<td>0.17%</td>
</tr>
<tr>
<td>31</td>
<td>0.9011</td>
<td>-0.09%</td>
</tr>
<tr>
<td>32</td>
<td>0.9032</td>
<td>-0.12%</td>
</tr>
</tbody>
</table>
machines, we also use constant failure rates for the failure rate function of all machines in our model. Table 4.2 shows that our continuous model not only converges more rapidly than the Glassey and Hong continuous model but also is more accurate for most of the cases.

The next twelve cases of the three-stage line were also examined to compare the analytical solution of lines with constant failure rates to the simulation results. The performance measures such as throughput and buffer contents are presented. In the first six cases, all stages of the lines have equal processing rate while in the next six cases, stages of the line have unequal processing rates. All the parameters of these cases are given in Appendix B. As shown in Table 4.3, the analytical approximation performs very well both in cases of equal processing rates and in cases of unequal processing rates.

The numerical results presented above are only some of the numerical results from our experiments. For example, we also successfully reproduced the numerical results presented in Burman [2]. One of the most important of the results is shown in Table 4.4. This experiment shows the importance of failure rate functions. Case 33 represents a three-machine system of two reliable machines feeding an unreliable machine that processes material at twice the rate of the first two machines when it is operational (see the parameters in Appendix B). It is noted that in Burman’s model, the failure rates are proportional to the speeds. For this case, we can reduce our model to Burman’s model by using failure rate functions which are proportional to the speeds.

<table>
<thead>
<tr>
<th>Case 33</th>
<th>Model</th>
<th>Throughput</th>
<th>$N_1$</th>
<th>$N_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>GADDX (proportional)</td>
<td>0.800</td>
<td>10.000</td>
<td>4.000</td>
</tr>
<tr>
<td></td>
<td>GADDX (constant)</td>
<td>0.750</td>
<td>10.000</td>
<td>5.275</td>
</tr>
<tr>
<td></td>
<td>Simulation</td>
<td>0.799</td>
<td>9.996</td>
<td>3.998</td>
</tr>
</tbody>
</table>

Table 4.4: The importance of the failure rate function
Chapter 5

Performance Analysis at Heineken

After developing an analytical model in the previous chapter, in this chapter we imple-
ment this analytical model to approximate and predict the performance measures of some
production lines at Heineken. The production line under our consideration is production
line 15A. For other production lines such as 8A, 8B, 16A and 16B, we present only the
important results. The purpose of this chapter is not only to analyze the production lines
but also to optimize them by finding the optimal speed configurations or the optimal buffer
allocations. For this, we use an optimization method called the Cross-Entropy method.

5.1 Introduction

Before we analyze Heineken’s production lines in detail, we first give an overview of the
method which is used to optimize their production lines. This method is called the Cross-
Entropy method and was developed by R.Y. Rubinstein [14]. The Cross-Entropy method
involves the following iterative phases:

1. Generation of a sample of random data according to a specified random mechanism.

2. Update the parameters of the random mechanism, typically parameters of probability
density functions (pdf), on the basis of the data, to produce a "better" sample in the
next iteration.

Below we present the algorithm of the Cross-Entropy method. As an example we use a
truncated normal distribution for generating the random sample. The $n$-dimensional normal
distribution with independent components, mean vector $\mu = (\mu_1, \ldots, \mu_n)$ and variance
vector $\sigma^2 = (\sigma^2_1, \ldots, \sigma^2_n)$, is denoted by $N(\mu, \sigma^2)$.

Algorithm

1. initialize: Choose $\hat{\mu}_0$ and $\hat{\sigma}_0^2$. Set $t := 0$. 
2. **draw:** Increase $t$ by 1. Generate a random sample $\mathbf{X}_1, \ldots, \mathbf{X}_N$ from the $N(\hat{\mu}_{t-1}, \hat{\sigma}^2_{t-1})$ distribution.

**select:** Let $I$ be the indices of the $N$ best performing (=elite) samples.

**update:** for all $j = 1, \ldots, n$ let

\[
\bar{\mu}_{tj} := \frac{\sum_{i \in I} X_{ij}}{N_{\text{elite}}} \quad (5.1.1)
\]

and

\[
\bar{\sigma}_{tj}^2 := \frac{\sum_{i \in I} (X_{ij} - \bar{\mu}_{tj})^2}{N_{\text{elite}}} \quad (5.1.2)
\]

**smooth:**

\[
\hat{\mu}_t := \alpha \bar{\mu}_t + (1 - \alpha) \hat{\mu}_{t-1} \quad (5.1.3)
\]

\[
\hat{\sigma}_t := \alpha \bar{\sigma}_t^2 + (1 - \alpha) \hat{\sigma}_{t-1} \quad (5.1.4)
\]

3. **repeat step 2 until:** $\max_j(\sigma_{tj}) < \epsilon$

The term $\alpha$ is called the smoothing parameter, with $0.7 < \alpha \leq 1$. The reasons of using smoothing are to smooth out the values of smoothed parameters and to reduce the probability that some component of the smoothed parameters will be zero or one at the first few iterations which lead the algorithm converge to a wrong solution. By using the algorithm, we can find the optimal speed configurations or optimal buffer allocations for Heineken production lines. The detailed explanation of this method is given in [14].

### 5.2 Production Line 15A

In this section we analyze production line 15A. As an illustration that in the reality the up-and-down times behaviors are not exponentially distributed. However, as the first step of analyzing the production lines, we assume that these rates are exponentially distributed. Later on, we will see that under these assumptions the analytical model still perform well to approximate the throughput of the production lines. For the future model, the more general model based on this model should be developed so that the model will be more realistic.

Now, we would like to analyze the production line 15A which consists of 10 machines. Each machine has a failure rate (mean up time), a repair rate (mean down time), and a deterministic speed. There is also a buffer between each pair of machines. The parameters of this line are given in Appendix B. Furthermore, we also use the results from Chapter 2 for the failure rate functions of the machines. In production line 15A, the Inpakker/Crate-packer is the only machine for which we use a failure rate function. The failure rates of the
Depalletizer, Depacker, Bottle-washer, Filler, Labeler/ETIMA, Palletizer do not depend on the speed. We use a constant failure rate for these machines. Unfortunately, Heineken has no data available of the other three machines. According to their experience, the failure rates of the Crate-logo and Pasteurizer do not depend on the speed, whereas the failure rate of the EBI does. Hence, we assume that that Crate-logo and Pasteurizer have constant failure rates while the EBI failure rate is proportional to its speed. Using these parameters as an input to our analytical model as described in Chapter 4, we calculate the OPI value for this production line. We also calculate the OPI value under the assumption that all machines have constant failure rates as well as the assumption that all machines have failure rates which are proportional to their speeds. Next to our analytical approximation we calculate the OPI value by using simulation. The results are presented in Table 5.1. This table shows that both the analytical model and the simulation model approximate the actual OPI value well. In addition to the calculation of OPI value, we also calculate the other performance measures such as average buffer contents, blocking probabilities and starvation probabilities by using an analytical model. These results are presented in Table 5.2. One of the interesting thing from this table is that both blocking probabilities and starvation probabilities of the Bottle-washer, EBI, Filler and ETIMA are quite small. The reason of this is that the Filler is the bottleneck of the system. Moreover, the EBI has many short stops. Later we will see that these 4 machines play an important role to improve the OPI value.

**Sensitivity Analysis**

We perform a sensitivity analysis to investigate the behavior of the production lines. We start with a sensitivity analysis on the buffer size. In this analysis, we see what happens to the OPI value if we vary the buffer capacities. We do this by varying the capacity of one buffer at a time while keeping the other capacities constant. Figure 5.1 shows the sensitivity results for all buffers. It can be seen that the OPI value is the most sensitive to the capacity of Buffers 4, and 5. This means that the current OPI value (71.629), can be increased significantly if we increase one of these buffers. Some buffers such as buffer 1, 2 and 9, can be decreased without a big influence on the current OPI. We implemented

<table>
<thead>
<tr>
<th>Approximation Methods</th>
<th>OPI</th>
<th>95% Confidence Interval</th>
<th>Error to Actual OPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>GADDX (Line15A)</td>
<td>71.629</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>GADDX (constant)</td>
<td>71.628</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>GADDX (proportional)</td>
<td>71.740</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Simulation (exponential)</td>
<td>71.560</td>
<td>71.550 - 71.580</td>
<td>-0.22%</td>
</tr>
<tr>
<td>Actual OPI</td>
<td>71.581</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1: OPI Comparison Line 15A
<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Percentage Buffer Content</th>
<th>Blocking Probability</th>
<th>Starvation Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>94.46%</td>
<td>0.3055</td>
<td>0</td>
</tr>
<tr>
<td>Crate logo</td>
<td>90.19%</td>
<td>0.2274</td>
<td>0.0168</td>
</tr>
<tr>
<td>Depacker</td>
<td>88.07%</td>
<td>0.1658</td>
<td>0.0103</td>
</tr>
<tr>
<td>Bottle washer</td>
<td>62.53%</td>
<td>0.0716</td>
<td>0.0006</td>
</tr>
<tr>
<td>EBI</td>
<td>45.29%</td>
<td>0.0545</td>
<td>0.0569</td>
</tr>
<tr>
<td>Filler</td>
<td>6.97%</td>
<td>0.0059</td>
<td>0.0894</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>31.94%</td>
<td>0.0073</td>
<td>0.1527</td>
</tr>
<tr>
<td>ETIMA</td>
<td>4.60%</td>
<td>0.0005</td>
<td>0.0597</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>1.59%</td>
<td>0.0001</td>
<td>0.1904</td>
</tr>
<tr>
<td>Palletizer</td>
<td>n/a</td>
<td>0</td>
<td>0.2427</td>
</tr>
</tbody>
</table>

Table 5.2: Other Performance Measures of Line 15A
the cross-entropy method to find the minimum total buffer capacity to achieve the desired OPI. For example, we want to find the minimum buffer capacities to achieve at least the current OPI (71.629). The result is presented in Table 5.3. The second sensitivity analysis we can perform is failure rate analysis. In this case, we try to vary the failure rate of each machine while keeping the other failure rates constant. The result are plotted in Figure 5.2. As calculated using GADDX, the current OPI value is 71.629. This value can increase significantly if we can improve the reliability of the Bottle-washer, EBI and Filler. The improvement will be around 2 – 3% if we can improve the reliability of one of these machines. Furthermore, Heineken has to pay attention to the failure rate of the Pasteurizer and Bottle-washer. If these values even increase only a little bit, the OPI value can drop sharply.

In addition to the failure rate analysis, we also analyze the sensitivity of the repair rate to support our failure rate analysis. The first results are plotted in Figures 5.3. We see that the increment of the repair rates of these machines cannot increase the current OPI. In Figure 5.4 we see that the current OPI can be increased if we are able to improve the repair rates of the Bottle-washer, EBI and Filler. The improvement of the Bottle-washer has the largest impact on the OPI value. Increasing the repair rate of the Filler and EBI has also a significant impact on the OPI. These results support the failure rate analysis results. Thus, we can conclude that both improving the reliability and improving the repair rate of the Bottle-washer, Filler and EBI gives a significant increase of the current OPI value.

The last sensitivity analysis is the machine speed analysis. The results of the machine speed analysis are easier to implement since the operator of Heineken can easily change the speed configurations of the machines. For this analysis, we try to vary each machine speed, while

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Current Average Speed (bottle/hour)</th>
<th>Optimal Average Speed (bottle/hour)</th>
<th>Current Buffer Size (in bottle)</th>
<th>Optimal Buffer Size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>48349</td>
<td>48160</td>
<td>1809</td>
<td>92</td>
</tr>
<tr>
<td>Crate logo</td>
<td>43284</td>
<td>39671</td>
<td>1464</td>
<td>690</td>
</tr>
<tr>
<td>Depacker</td>
<td>43284</td>
<td>39659</td>
<td>6895</td>
<td>1901</td>
</tr>
<tr>
<td>Bottle washer</td>
<td>40389</td>
<td>44971</td>
<td>3851</td>
<td>3825</td>
</tr>
<tr>
<td>EBI</td>
<td>37406</td>
<td>39365</td>
<td>270</td>
<td>2129</td>
</tr>
<tr>
<td>Filler</td>
<td>37406</td>
<td>37493</td>
<td>1088</td>
<td>1508</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>38154</td>
<td>39062</td>
<td>7013</td>
<td>1872</td>
</tr>
<tr>
<td>ETIMA</td>
<td>37094</td>
<td>41743</td>
<td>4595</td>
<td>897</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>40988</td>
<td>39738</td>
<td>8736</td>
<td>30</td>
</tr>
<tr>
<td>Palletizer</td>
<td>43037</td>
<td>42567</td>
<td>n/a</td>
<td>n/a</td>
</tr>
<tr>
<td>OPI value</td>
<td>71.629</td>
<td>73.660</td>
<td>71.629</td>
<td>71.654</td>
</tr>
</tbody>
</table>

Table 5.3: Optimization of Line 15A
Figure 5.2: Failure Rate Analysis Line 15A

Figure 5.3: Repair Rate Analysis(1) Line 15A
Figure 5.4: Repair Rate Analysis (2) Line 15A

Figure 5.5: Speed Sensitivity Analysis Line 15A
Table 5.4: The constraints of machine speeds

<table>
<thead>
<tr>
<th>Name</th>
<th>Speed Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>[0,54000]</td>
</tr>
<tr>
<td>Cratelogo</td>
<td>[0,53000]</td>
</tr>
<tr>
<td>Depacker</td>
<td>[0,53000]</td>
</tr>
<tr>
<td>Bottle washer</td>
<td>[0,45000]</td>
</tr>
<tr>
<td>EBI</td>
<td>[0,39400]</td>
</tr>
<tr>
<td>Filler</td>
<td>[0,37500]</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>[0,40170]</td>
</tr>
<tr>
<td>ETIMA</td>
<td>[0,43600]</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>[0,45700]</td>
</tr>
<tr>
<td>Palletizer</td>
<td>[0,47700]</td>
</tr>
</tbody>
</table>

keeping the other machine speeds constant. Figure 5.5 shows the results of this analysis. From this figure, it can be seen that the speed of the Bottle-washer speed has the highest influence on the OPI, followed by the EBI and Filler. For example, if we can increase the average speed of the Bottle washer from 40389 up to 44000, the current OPI value will increase around 1%.

The cross-entropy method as introduced before can help us to find the optimal speed configurations which maximize the OPI value. Heineken is interested in finding the optimal speed configurations which maximize the OPI value under the following constraints:

1. the average buffer content of all buffers before the filler is more than 10%, the average content of the buffer exactly before the filler is more than 50%;
2. the average buffer content of all buffers behind the filler is less than 60%;
3. the fraction of time that the filler is running at the current speed is more than 75%
4. the average speeds do not exceed the maximum speeds

It is noted that the maximum speed of the filler is 39400, however from the historical data of speed usage, it seems that this maximum speed is only used for about 50% of the time. Because of that, we set the average speed at 37500 (close to the current average speed). The result of the optimization is presented in Table 5.3. We can see that by using these optimal speed configurations, the OPI value increases to 73.660. It can also be seen that this speed configurations consist of the increase of the speeds of the Bottle-washer and EBI, as recommended before from the machine speed analysis.
Table 5.5: OPI Comparison Line 8A & 8B

<table>
<thead>
<tr>
<th>Approximation Methods</th>
<th>Line 8A</th>
<th>Line 8B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>OPI</td>
<td>Error</td>
</tr>
<tr>
<td>GADDX (Line8A/8B)</td>
<td>66.601</td>
<td>1.28%</td>
</tr>
<tr>
<td>GADDX (constant)</td>
<td>66.601</td>
<td>1.28%</td>
</tr>
<tr>
<td>GADDX (proportional)</td>
<td>66.639</td>
<td>1.34%</td>
</tr>
<tr>
<td>Simulation (exponential)</td>
<td>66.370</td>
<td>0.93%</td>
</tr>
<tr>
<td>Actual OPI</td>
<td>65.756</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.6: OPI Comparison Line 16A & 16B

<table>
<thead>
<tr>
<th>Approximation Methods</th>
<th>Line 16A</th>
<th>Line 16B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>OPI</td>
<td>Error</td>
</tr>
<tr>
<td>GADDX (Line16A/16B)</td>
<td>64.242</td>
<td>-2.27%</td>
</tr>
<tr>
<td>GADDX (constant)</td>
<td>64.241</td>
<td>-2.27%</td>
</tr>
<tr>
<td>GADDX (proportional)</td>
<td>64.263</td>
<td>-2.23%</td>
</tr>
<tr>
<td>Simulation (exponential)</td>
<td>63.260</td>
<td>-3.76%</td>
</tr>
<tr>
<td>Actual OPI</td>
<td>65.731</td>
<td>-</td>
</tr>
</tbody>
</table>

5.3 Other Production Lines

In the previous section, we analyzed production line 15A in detail. In this section, we only present the overview of the performance analysis results of the other production lines. We analyze production lines 8A, 8B, 16A and 16B. These production lines consist of 6 machines which are the Depalletizer, Filler, Pasterizer, Labeler, Carton-packer and Palletizer. To analyze these production lines, we assume that the failure rate functions of the Depalletizer, Filler and Palletizer are proportional to their speeds, while the failure rate of the Pasteurizer is constant. For the Labeler and Carton-packer, we use the results from Chapter 2.

Using the parameters as presented in Appendix B, we calculate the OPI values. Table 5.5 and Table 5.6 show the OPI values which are calculated by using GADDX, GADDX (constant), GADDX (proportional) and simulation. It can be seen that both the analytical model and the simulation model are able to predict the OPI values of Heineken’s production lines quite well. From the failure rate analysis and the repair rate analysis, we find that the improvement of the reliability of the Labeler/ETIMA significantly increases the OPI values. From the machine speed analysis, it is suggested to increase the speed of the Labeler/ETIMA to achieve a higher OPI value. Furthermore, we also calculated the optimal speed configurations as well as the optimal buffer allocations for these production lines by using the Cross-Entropy method. The results are presented in Appendix C.
Chapter 6

Conclusion

In this thesis, we use an analytical model based on a decomposition method to predict, approximate and analyze the production lines of Heineken. This model is able to analyze non-homogeneous flow lines with finite buffers, unreliable machines, deterministic processing times, operation dependent failures, exponentially distributed up-and-down times behavior. Since the failure rates of some machines at Heineken depend on their speeds, we also take this into account into the model. Furthermore, we use the GADDX algorithm which is very fast and accurate to approximate some performance measures such as throughput, average buffer content, blocking probability and starvation probability. A friendly software based on this analytical model was built to help the users to analyze the production lines. This software is able to calculate the performance measures, to do a buffer analysis, failure and repair rate analysis, machine speed analysis and to find the optimal speed configurations of the production lines as well as the optimal buffer allocation. At the end, we use this software to analyze and optimize some production lines at Heineken.

6.1 Suggestions

Below, some suggestions are made as a conclusion of the study at Heineken.

1. To increase the throughput of production line 15A, Heineken can do the following things:
   
   • increase the buffer capacity of the buffer behind the Bottle washer or EBI
   • improve the reliability of the Bottle-washer, EBI and Filler
   • reduce the down-time of the Bottle-washer, Filler and EBI
   • use the optimal speed configurations
2. The throughput of production line 8A, 8B, 16A and 16B can be increased by improving the reliability of the Labeler or by increasing the speed of the Labeler.

3. Use the software that we built to analyze the production lines using the up-to-date data.

6.2 Future Work

In this thesis we made some assumptions to build the model such as failure rates and repair rates are exponentially distributed. For the future improvement, a more advanced model can be developed by relaxing these assumptions to be more general. If our model assumes that the yield of the production is always 100%, in the future it is also possible to take into account a scrapped product/loss quality in the model. For example, we can include the probability that the machines produce a defect product. Furthermore, we assumed that the machine can only have one type of failure, in the next step we can also consider multiple types of failures. Finally, the continuous model can be extended by including assembly/disassembly components such that the model can handle a large number of important cases in industry.
Appendix A

Conditional Probability

In this Appendix, we will prove two propositions which are used in Chapter 4. These two propositions are related to conditional probability.

Proposition A.0.1. Let $U$, $V$ and $W$ be events. Let $V$ and $W$ are disjoint. Then,

$$\text{prob}[U|V \text{ or } W] = \text{prob}[U|V]\text{prob}[V|V \text{ or } W] + \text{prob}[U|W]\text{prob}[W|V \text{ or } W]$$  \hspace{1cm} (A.0.1)

Proof. Since $V$ and $W$ are disjoint, then $(U \text{ or } V)$ and $(U \text{ or } W)$ are disjoint. As a consequence

$$\text{prob}[U|V \text{ or } W] = \frac{\text{prob}(U \text{ and } V)}{\text{prob}(V \text{ or } W)} + \frac{\text{prob}(U \text{ and } W)}{\text{prob}(V \text{ or } W)}$$  \hspace{1cm} (A.0.2)

To analyze the first and second terms, note that

$$\text{prob}(U \text{ and } V) = \text{prob}[U|V]\text{prob}(V)$$  \hspace{1cm} (A.0.3)

$$\text{prob}(U \text{ and } W) = \text{prob}[U|W]\text{prob}(W)$$  \hspace{1cm} (A.0.4)

Then

$$\text{prob}[U|V \text{ or } W] = \frac{\text{prob}[U|V]\text{prob}(V)}{\text{prob}(V \text{ or } W)} + \frac{\text{prob}[U|W]\text{prob}(W)}{\text{prob}(V \text{ or } W)}$$  \hspace{1cm} (A.0.5)
It is also noted that
\[
\text{prob}[V | V \text{ or } W] = \frac{\text{prob}[V \text{ and } (V \text{ or } W)]}{\text{prob}(V \text{ or } W)} = \frac{\text{prob}(V)}{\text{prob}(V \text{ or } W)} \quad (A.0.6)
\]
In a similar way,
\[
\text{prob}[W | V \text{ or } W] = \frac{\text{prob}(W)}{\text{prob}(V \text{ or } W)} \quad (A.0.7)
\]
The proof is complete by combining equations (A.0.5), (A.0.6) and (A.0.7).

**Proposition A.0.2.** Let \( A, B, C \) and \( D \) be disjoint events. Then,
\[
\text{prob}[B \text{ or } C | A \text{ or } B \text{ or } C \text{ or } D] = \frac{\text{prob}(B | A \text{ or } B) \text{prob}(A \text{ or } B)}{\text{prob}(A \text{ or } B \text{ or } C \text{ or } D)} + \frac{\text{prob}(C | A \text{ or } D) \text{prob}(C \text{ or } D)}{\text{prob}(A \text{ or } B \text{ or } C \text{ or } D)} \quad (A.0.8)
\]

**Proof.** Since \( B \) and \( C \) are disjoint, the left term of (A.0.8) can be written as
\[
\text{prob}[B \text{ or } C | A \text{ or } B \text{ or } C \text{ or } D] = \text{prob}[B | A \text{ or } B \text{ or } C \text{ or } D] + \text{prob}[C | A \text{ or } B \text{ or } C \text{ or } D] \quad (A.0.9)
\]
The first term of (A.0.9) is given by
\[
\text{prob}[B | A \text{ or } B \text{ or } C \text{ or } D] = \frac{\text{prob}[B \text{ and } (A \text{ or } B \text{ or } C \text{ or } D)]}{\text{prob}(A \text{ or } B \text{ or } C \text{ or } D)} = \frac{\text{prob}(B)}{\text{prob}(A \text{ or } B \text{ or } C \text{ or } D)} \quad (A.0.10)
\]
It is also noted that from equation (A.0.7)
\[
\text{prob}[B | A \text{ or } B] = \frac{\text{prob}(B)}{\text{prob}(A \text{ or } B)} \quad (A.0.11)
\]
Substituting this into the previous equation results in
\[
\text{prob}[B | A \text{ or } B \text{ or } C \text{ or } D] = \frac{\text{prob}(B | A \text{ or } B) \text{prob}(A \text{ or } B)}{\text{prob}(A \text{ or } B \text{ or } C \text{ or } D)} \quad (A.0.12)
\]
The second term can be obtained in a similar manner. Combining first term and second term completes the proof. □
Appendix B

Experiment Parameters

In this Appendix we present the parameters which are used for numerical experiments in Chapter 4 and Chapter 5.

<table>
<thead>
<tr>
<th>Case</th>
<th>Parameters: ((p_i, r_i, \mu_i), N_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (3 machines)</td>
<td>((0.05, 0.1, 1), i = 1, 2, 3, N_1 = N_2 = 5)</td>
</tr>
<tr>
<td>2 (3 machines)</td>
<td>((0.002, 0.05, 1), i = 1, 2, 3, N_1 = N_2 = 5)</td>
</tr>
<tr>
<td>3 (3 machines)</td>
<td>((0.1, 0.1, 1), i = 1, 2, 3, N_1 = N_2 = 5)</td>
</tr>
<tr>
<td>4 (3 machines)</td>
<td>((0.1, 0.1, 1), i = 1, 2, 3, N_1 = N_2 = 10)</td>
</tr>
<tr>
<td>5 (3 machines)</td>
<td>((0.01, 0.07, 1), (0.013, 0.1, 1), (0.007, 0.05, 1), N_1 = N_2 = 10)</td>
</tr>
<tr>
<td>6 (10 machines)</td>
<td>((0.1, 0.1, 1), i = 1, \ldots, 10, N_i = 10, i = 1, \ldots, 9)</td>
</tr>
<tr>
<td>7 (12 machines)</td>
<td>((0.01, 0.07, 1), i = 1, 4, 7, 10, (0.013, 0.1, 1), i = 2, 5, 8, 11)</td>
</tr>
<tr>
<td>8 (17 machines)</td>
<td>((0.007, 0.05, 1), i = 3, 6, 9, 12, N_i = 10, i = 1, \ldots, 11)</td>
</tr>
<tr>
<td>9 (20 machines)</td>
<td>((0.1, 0.1, 1), i = 1, \ldots, 17, N_i = 5, i = 1, \ldots, 16)</td>
</tr>
<tr>
<td>10 (20 machines)</td>
<td>((0.1, 0.1, 1), i = 1, \ldots, 20, N_i = 5, i = 1, \ldots, 19)</td>
</tr>
</tbody>
</table>

Table B.1: Parameter values for comparison with Glassey and Hong (1993)
<table>
<thead>
<tr>
<th>Case</th>
<th>Parameters: ((p_i, r_i, \mu_i), N_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11 (3</td>
<td>(0.01, 0.09, 1), (0.02, 0.09, 1.1), (0.03, 0.09, 1.2), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>12 (3</td>
<td>(0.02, 0.1, 1.1), (0.01, 0.08, 1), (0.01, 0.1, 1), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>13 (3</td>
<td>(0.05, 0.1, 1.5), (0.02, 0.08, 1), (0.03, 0.07, 1.1), (N_1 = 30, N_2 = 70)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>14 (3</td>
<td>(0.01, 0.08, 1), (0.02, 0.2, 1), (0.05, 0.5, 0.9), (N_1 = 70, N_2 = 30)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>15 (3</td>
<td>(0.005, 0.05, 1), (0.005, 0.05, 1.1), (0.005, 0.05, 1), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>16 (4</td>
<td>(0.01, 0.08, 1), (i = 1, 4), (0.01, 0.07, 1), (i = 2, 3), (N_1 = 50, i = 1, \ldots, 3)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>17 (4</td>
<td>(0.01, 0.09, 1), (0.01, 0.07, 1), (0.01, 0.1, 0.9), (0.01, 0.08, 1),</td>
</tr>
<tr>
<td>machines)</td>
<td>(N_1 = N_2 = N_3 = 50)</td>
</tr>
<tr>
<td>18 (4</td>
<td>(0.02, 0.08, 1.1), (0.01, 0.07, 1), (0.01, 0.09, 1), (0.01, 0.05, 0.9),</td>
</tr>
<tr>
<td>machines)</td>
<td>(N_1 = 40, N_2 = 50, N_3 = 60)</td>
</tr>
<tr>
<td>19 (5</td>
<td>(0.01, 0.09, 1), (i = 1, 3, 5), (0.02, 0.09, 1.1), (i = 2, 4,)</td>
</tr>
<tr>
<td>machines)</td>
<td>(N_i = 50, i = 1, \ldots, 4)</td>
</tr>
<tr>
<td>20 (5</td>
<td>(0.01, 0.09, 1), (0.005, 0.009, 0.9), (0.012, 0.09, 1.1), (0.015, 0.09, 1.2),</td>
</tr>
<tr>
<td>machines)</td>
<td>(0.01, 0.09, 1), (N_1 = 30, N_2 = 40, N_3 = 50, N_4 = 60)</td>
</tr>
</tbody>
</table>

Table B.2: Parameter values for comparison with simulation

<table>
<thead>
<tr>
<th>Case</th>
<th>Parameters: ((p_i, r_i, \mu_i), N_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21 (3</td>
<td>(0.01, 0.09, 1), (i = 1, 2, 3), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>22 (3</td>
<td>(0.01, 0.09, 1), (i = 1, 2, 3, N_1 = 20, N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>23 (3</td>
<td>(0.02, 0.09, 1), (0.01, 0.09, 1), (0.03, 0.09, 1), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>24 (3</td>
<td>(0.02, 0.09, 1), (0.01, 0.09, 1), (0.03, 0.09, 1), (N_1 = 30, N_2 = 70)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>25 (3</td>
<td>(0.01, 0.09, 1), (0.01, 0.1, 1), (0.01, 0.08, 1), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>26 (3</td>
<td>(0.01, 0.09, 1), (0.01, 0.1, 1.1), (0.01, 0.08, 1), (N_1 = 70, N_2 = 30)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>27 (3</td>
<td>(0.01, 0.09, 1), (0.02, 0.09, 1.1), (0.03, 0.09, 1.2), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>28 (3</td>
<td>(0.01, 0.09, 1), (0.02, 0.09, 1.1), (0.03, 0.09, 1.2), (N_1 = 70, N_2 = 30)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>29 (3</td>
<td>(0.01, 0.09, 1), (0.01, 0.09, 1.2), (0.01, 0.09, 1.1), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>30 (3</td>
<td>(0.01, 0.09, 1), (0.01, 0.09, 1.2), (0.01, 0.09, 1.1), (N_1 = 10, N_2 = 80)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>31 (3</td>
<td>(0.01, 0.08, 1.2), (0.01, 0.09, 1.1), (0.01, 0.1, 1), (N_1 = N_2 = 50)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
<tr>
<td>32 (3</td>
<td>(0.01, 0.08, 1.2), (0.01, 0.09, 1.1), (0.01, 0.1, 1), (N_1 = 30, N_2 = 70)</td>
</tr>
<tr>
<td>machines)</td>
<td></td>
</tr>
</tbody>
</table>

Table B.3: Parameter values of 3-machine model

<table>
<thead>
<tr>
<th>(i)</th>
<th>(r_i)</th>
<th>(p_i)</th>
<th>(\mu_i)</th>
<th>(N_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000</td>
<td>0.000</td>
<td>1.000</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>1.000</td>
<td>0.000</td>
<td>1.000</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>0.100</td>
<td>0.100</td>
<td>2.000</td>
<td>n/a</td>
</tr>
</tbody>
</table>

Table B.4: Parameter values Case 33
<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Failure rate (per hour)</th>
<th>Repair rate (per hour)</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>0.1995</td>
<td>10.4758</td>
<td>41072</td>
<td>11364</td>
</tr>
<tr>
<td>Filler</td>
<td>2.8599</td>
<td>29.3875</td>
<td>39116</td>
<td>5510</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>0.3302</td>
<td>10.5884</td>
<td>39898</td>
<td>6405</td>
</tr>
<tr>
<td>ETIMA</td>
<td>6.9412</td>
<td>31.8541</td>
<td>38571</td>
<td>5866</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>10.3141</td>
<td>39.1237</td>
<td>49531</td>
<td>7930</td>
</tr>
<tr>
<td>Palletizer</td>
<td>0.4083</td>
<td>22.3431</td>
<td>52007</td>
<td>-</td>
</tr>
</tbody>
</table>

Table B.5: Input Production Line 8A

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Failure rate (per hour)</th>
<th>Repair rate (per hour)</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>0.6702</td>
<td>17.7516</td>
<td>40248</td>
<td>7637</td>
</tr>
<tr>
<td>Filler</td>
<td>2.5889</td>
<td>27.4351</td>
<td>38331</td>
<td>4212</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>0.3170</td>
<td>10.9333</td>
<td>39098</td>
<td>5043</td>
</tr>
<tr>
<td>ETIMA</td>
<td>3.8988*</td>
<td>30.0412</td>
<td>36843</td>
<td>5977</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>5.2876</td>
<td>35.3768</td>
<td>41858</td>
<td>5712</td>
</tr>
<tr>
<td>Palletizer</td>
<td>0.3286</td>
<td>18.4479</td>
<td>43951</td>
<td>-</td>
</tr>
</tbody>
</table>

*1 This value becomes 3.5434 if we assume the failure rate is constant or proportional.

Table B.6: Input Production Line 8B

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Failure rate (per hour)</th>
<th>Repair rate (per hour)</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>0.8103</td>
<td>16.1677</td>
<td>48349</td>
<td>1809</td>
</tr>
<tr>
<td>Cratelogo</td>
<td>1.6250</td>
<td>57.3408</td>
<td>43284</td>
<td>1464</td>
</tr>
<tr>
<td>Depacker</td>
<td>7.3333</td>
<td>62.3070</td>
<td>43284</td>
<td>6895</td>
</tr>
<tr>
<td>Bottle washer</td>
<td>2.5532</td>
<td>14.6104</td>
<td>40389</td>
<td>3851</td>
</tr>
<tr>
<td>EBI</td>
<td>1.4270</td>
<td>30.7223</td>
<td>37406</td>
<td>270</td>
</tr>
<tr>
<td>Filler</td>
<td>1.9570</td>
<td>29.9690</td>
<td>37406</td>
<td>1088</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>0.1383</td>
<td>15.9100</td>
<td>38154</td>
<td>7013</td>
</tr>
<tr>
<td>ETIMA</td>
<td>4.5838</td>
<td>46.9876</td>
<td>37094</td>
<td>4595</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>1.4177*</td>
<td>37.6381</td>
<td>40988</td>
<td>8736</td>
</tr>
<tr>
<td>Palletizer</td>
<td>0.6120</td>
<td>23.4593</td>
<td>43037</td>
<td>-</td>
</tr>
</tbody>
</table>

*1 This value becomes 1.7765 if we assume the failure rate is constant or proportional.

Table B.7: Input Production Line 15A
<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Failure rate (per hour)</th>
<th>Repair rate (per hour)</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>0.6929</td>
<td>15.1524</td>
<td>42457</td>
<td>8540</td>
</tr>
<tr>
<td>Filler</td>
<td>1.9432</td>
<td>23.0954</td>
<td>40435</td>
<td>4722</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>1.0801</td>
<td>17.1538</td>
<td>43306</td>
<td>8041</td>
</tr>
<tr>
<td>ETIMA</td>
<td>5.0351</td>
<td>37.1917</td>
<td>40629</td>
<td>7558</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>9.4956</td>
<td>45.1286</td>
<td>53922</td>
<td>6039</td>
</tr>
<tr>
<td>Palletizer</td>
<td>0.2937</td>
<td>24.8429</td>
<td>56618</td>
<td>-</td>
</tr>
</tbody>
</table>

Table B.8: Input Production Line 16A

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Failure rate (per hour)</th>
<th>Repair rate (per hour)</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depalletizer</td>
<td>1.2288</td>
<td>28.9817</td>
<td>41373</td>
<td>9211</td>
</tr>
<tr>
<td>Filler</td>
<td>2.4842</td>
<td>24.0920</td>
<td>39403</td>
<td>5099</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>0.9335</td>
<td>13.4260</td>
<td>40191</td>
<td>8471</td>
</tr>
<tr>
<td>ETIMA</td>
<td>7.4975</td>
<td>37.1482</td>
<td>41245</td>
<td>10495</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>11.8658*</td>
<td>38.8106</td>
<td>52011</td>
<td>7161</td>
</tr>
<tr>
<td>Palletizer</td>
<td>0.6957</td>
<td>20.5631</td>
<td>54611</td>
<td>-</td>
</tr>
</tbody>
</table>

*This value becomes 11.2277 if we assume the failure rate is constant or proportional.

Table B.9: Input Production Line 16B
Appendix C

Other Results

Below, we present the other results of performance analysis in Heineken. As mentioned before we applied the cross-entropy method to find the optimal speed configurations and optimal buffer sizes. The tables below show the optimal speed configurations, and buffer sizes as well as the OPI values of production line 8A, 8B, 16A and 16B.

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer Size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Current</td>
<td>Optimal</td>
</tr>
<tr>
<td>Depalletizer</td>
<td>41072</td>
<td>38835</td>
</tr>
<tr>
<td>Filler</td>
<td>39116</td>
<td>39179</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>39898</td>
<td>40699</td>
</tr>
<tr>
<td>ETIMA</td>
<td>38571</td>
<td>41998</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>49531</td>
<td>54908</td>
</tr>
<tr>
<td>Palletizer</td>
<td>52007</td>
<td>47878</td>
</tr>
<tr>
<td>OPI value</td>
<td>66.601</td>
<td>72.235</td>
</tr>
</tbody>
</table>

Table C.1: Optimization of Line 8A
<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer Size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Current</td>
<td>Optimal</td>
</tr>
<tr>
<td>Depalletizer</td>
<td>40248</td>
<td>41933</td>
</tr>
<tr>
<td>Filler</td>
<td>38331</td>
<td>38396</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>39098</td>
<td>40771</td>
</tr>
<tr>
<td>ETIMA</td>
<td>36843</td>
<td>41828</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>41858</td>
<td>47952</td>
</tr>
<tr>
<td>Palletizer</td>
<td>43951</td>
<td>41531</td>
</tr>
<tr>
<td>OPI value</td>
<td>73.431</td>
<td>79.035</td>
</tr>
</tbody>
</table>

Table C.2: Optimization of Line 8B

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer Size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Current</td>
<td>Optimal</td>
</tr>
<tr>
<td>Depalletizer</td>
<td>42457</td>
<td>46819</td>
</tr>
<tr>
<td>Filler</td>
<td>40435</td>
<td>40497</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>43306</td>
<td>45900</td>
</tr>
<tr>
<td>ETIMA</td>
<td>40629</td>
<td>45852</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>53922</td>
<td>54473</td>
</tr>
<tr>
<td>Palletizer</td>
<td>56618</td>
<td>57708</td>
</tr>
<tr>
<td>OPI value</td>
<td>64.242</td>
<td>66.775</td>
</tr>
</tbody>
</table>

Table C.3: Optimization of Line 16A

<table>
<thead>
<tr>
<th>Machine Name</th>
<th>Average Speed (bottle/hour)</th>
<th>Buffer Size (in bottle)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Current</td>
<td>Optimal</td>
</tr>
<tr>
<td>Depalletizer</td>
<td>41373</td>
<td>44727</td>
</tr>
<tr>
<td>Filler</td>
<td>39403</td>
<td>39489</td>
</tr>
<tr>
<td>Pasterizer</td>
<td>40191</td>
<td>45402</td>
</tr>
<tr>
<td>ETIMA</td>
<td>41245</td>
<td>46093</td>
</tr>
<tr>
<td>Crate-packer</td>
<td>52011</td>
<td>46545</td>
</tr>
<tr>
<td>Palletizer</td>
<td>54611</td>
<td>50057</td>
</tr>
<tr>
<td>OPI value</td>
<td>58.573</td>
<td>61.229</td>
</tr>
</tbody>
</table>

Table C.4: Optimization of Line 16B
Appendix D

Software Guide

In this Appendix, we explain how to use the software to analyze a flow line. The software consists of 2 main features. The first feature which is related to performance analysis and optimization will be explained in section D.1. Section D.2 explains the sensitivity analysis feature.

D.1 Performance Analysis

Performance Analysis and Optimization feature is able to calculate some performance measures such as throughput, average buffer content, the percentage of the average buffer content, blocking probabilities and starvation probabilities. The standard performance measurement, OPI NONA, used by Heineken is also included. Moreover this feature can also help Heineken to optimize their production lines. The optimization consists of finding an optimal speed configuration and determining the minimum buffer allocation to obtain the desired throughput. The details of this feature (see the red circles in the picture) are the following. See Figure D.1.

A. This box contains the input data. The users should load the data from Excel files (*.xlsx). The first column of Excel file should be the machine names. The second column is the failure rate of the machines (per hour). The third column is the repair rate of the machines (per hour). The fourth column is the average machine speeds (bottle/hour). The last column is the buffer capacities. As an example see Figure D.1. After the users load the data, this data will be displayed in the Table on this box. The users are also allowed to change this data. The software calculates automatically the isolated production rate of each machine. From this calculation we can see which
machines have the lowest efficiency if they are isolated. Furthermore, the users have to input the parameters which are used to calculate the OPI value e.g. Average Planned Down Time (APDT), Average Change Over Time (ACOT), Filler Nominal speed. The last input is failure rate functions. The users have to choose which failure rate function should be used for the analytical model. If we assume that the failure rates of all machines are proportional to speeds, then we should choose option 'proportional', if we assume that it is constant, we choose option 'constant'. Other options can be used if we want to analyze some production lines based on our finding about the relation between speed and failure rates.

B. This box aims to calculate the performance evaluation using the analytical model. The users simply click the button 'Calculate' to get the results. The table will display the results of the calculation of performance measures such as throughput, OPI value, average buffer contents, blocking probabilities, and starvation probabilities. It is also possible to save these results into Excel file.

C. If the users would like to optimize the current production lines, there are two possible optimizations. The first one is the optimal speed configurations and the second one is the optimal buffer allocation.

D. In this box, the users are allowed to change the optimization parameters e.g. standard
deviation of the samples. It is noted that if the standard deviation is small, then the optimization method will converge slower.

E. This box displays the input file name, status of the algorithm, and the number of iterations to find the optimal speeds/buffer capacities (if we use optimization feature).

F. Once we choose the optimization problem (box C), this box will be enabled. As an example, if we choose 'buffer allocation' problem in box C, then we should input the desired throughput. It is noted that the desired throughput cannot be greater than the minimum of isolated production rates in box A. Another example, if we choose 'speed configuration' in box C, then we can input the speed constraints (minimum and maximum) and also the percentage average buffer content constraints (minimum and maximum). Another constraint is the minimum fraction of time that the Filler is running at normal speed. The value is between 0 until 1.
G. Once we already input the data and the constraints, we can press the button ‘Optimize’ to get the optimal speeds/buffer capacities. This box will display the optimal speed/buffer capacities as well as the 95% confidence interval of these optimal values. Moreover, the throughput, OPI value and the fraction of time the Filler running at normal speed which are calculated using these optimal values are also given.

H. Here, we have 3 buttons. The first one is ‘Sensitivity analysis’ which is aimed to do a sensitivity analysis such as buffer analysis, failure rate analysis, repair rate analysis and speed analysis (see D.2). The second button is ‘Reset All’ which is used to clear all the input data and the results. The last button is ‘Exit’.

D.2 Sensitivity Analysis

The second feature of this software is sensitivity analysis. This feature will appear if the users press the button ‘Sensitivity Analysis’ in the performance analysis and optimization feature before. The following explains the red circles in the Figure D.2.

I. This box contains the list of the machines which we would like to analyze.

J. In this box the users can choose the type of analysis. For example, Buffer Analysis where the users can see which buffer capacity is the most sensitive to the chosen performance measure in the box K. The possible sensitivity analysis is the buffer analysis, the failure rate analysis, the repair rate analysis and the speed analysis.
K. The users should choose one of the performance measures which they would like to investigate. For example, if we choose speed analysis in box J and choose the option K in this box, it means that we investigate the effect of changing speed to the OPI value.

L. This box provides the parameters such as ACOT, APDT and filler nominal speed to calculate OPI. Moreover, the users are also allowed to choose the failure rate function as explained in the performance analysis and optimization feature (see box A).

M. In this box, the users can specify the interval of analysis; the lower and upper bounds of the interval as well as the step size.

N. This box shows the status of the file and the input data.

Others. The other parts are buttons 'Analyze', 'Load data', 'Save Output', 'Reset all', and 'Close'. It is noted that the input data should be in Excel file as described in the performance analysis and optimization feature. Finally, the axes on the right side of Figure D.2 will display the graphic after the users load the data and choose the parameters and type of the analysis.
Appendix E

Scatter Plots

Below are the scatter plots of speed (bottle/hour) versus breakdown rate (per hour).
Scatter Plots of speed (bottle/hour) versus breakdown duration (in second)
Appendix F

Speed Usage

This Appendix presents the speed usage of all 5 production lines. The data was taken from July 2008 until December 2008. The speeds displayed below are only the speeds which are used often.

<table>
<thead>
<tr>
<th>Line 1SA</th>
<th>Depalletizer</th>
<th>Palletizer</th>
<th>Usage</th>
<th>Usage</th>
<th>Usage</th>
<th>Usage</th>
<th>ETIMA</th>
<th>Usage</th>
<th>Usage</th>
<th>Usage</th>
<th>Usage</th>
<th>Usage</th>
</tr>
</thead>
<tbody>
<tr>
<td>51800</td>
<td>4.08%</td>
<td>52900</td>
<td>4.17%</td>
<td>45400</td>
<td>39.45%</td>
<td>39400</td>
<td>54.31%</td>
<td>39000</td>
<td>15.53%</td>
<td>45700</td>
<td>30.63%</td>
<td>43200</td>
</tr>
<tr>
<td>52700</td>
<td>3.95%</td>
<td>48700</td>
<td>4.15%</td>
<td>41100</td>
<td>28.58%</td>
<td>15200</td>
<td>8.14%</td>
<td>40700</td>
<td>13.15%</td>
<td>40900</td>
<td>11.87%</td>
<td>43500</td>
</tr>
<tr>
<td>46500</td>
<td>4.03%</td>
<td>58100</td>
<td>4.11%</td>
<td>19700</td>
<td>4.73%</td>
<td>18100</td>
<td>8.05%</td>
<td>100</td>
<td>10.56%</td>
<td>42000</td>
<td>9.80%</td>
<td>42000</td>
</tr>
<tr>
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<td>53000</td>
<td>3.87%</td>
<td>58800</td>
<td>4.20%</td>
<td>58600</td>
<td>3.13%</td>
<td>40600</td>
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<td>32100</td>
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<td>36400</td>
<td>4.57%</td>
<td>43600</td>
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<td>2.98%</td>
<td>27800</td>
<td>2.79%</td>
<td>38600</td>
<td>2.68%</td>
<td>23800</td>
<td>4.83%</td>
<td>37900</td>
<td>3.58%</td>
<td>44500</td>
</tr>
<tr>
<td>52900</td>
<td>3.19%</td>
<td>32300</td>
<td>2.88%</td>
<td>40300</td>
<td>2.62%</td>
<td>36700</td>
<td>2.83%</td>
<td>43700</td>
<td>4.29%</td>
<td>39500</td>
<td>3.66%</td>
<td>42600</td>
</tr>
<tr>
<td>54600</td>
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<td>53300</td>
<td>2.52%</td>
<td>41200</td>
<td>2.65%</td>
<td>30300</td>
<td>1.16%</td>
<td>43800</td>
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<td>36200</td>
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<td>39700</td>
<td>1.17%</td>
<td>40200</td>
<td>2.95%</td>
<td>37000</td>
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<td>45500</td>
</tr>
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<td>39400</td>
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<td>50400</td>
<td>0.93%</td>
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<td>39300</td>
<td>0.78%</td>
<td>37500</td>
<td>1.85%</td>
<td>33800</td>
<td>1.71%</td>
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<td>44400</td>
<td>1.56%</td>
<td>39500</td>
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<td>100</td>
<td>0.75%</td>
<td>40800</td>
<td>1.76%</td>
<td>35400</td>
<td>1.48%</td>
<td>46500</td>
</tr>
<tr>
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<td>1.52%</td>
<td>30100</td>
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<td>610</td>
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<td>31100</td>
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<td>1.30%</td>
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<td>0.57%</td>
<td>29700</td>
<td>0.68%</td>
<td>40100</td>
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<td>34600</td>
<td>1.06%</td>
<td>45400</td>
</tr>
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<td>1.26%</td>
<td>15200</td>
<td>0.14%</td>
<td>40500</td>
<td>0.87%</td>
<td>44100</td>
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