MASTER

Optimized Schwarz methods for elliptic optimal control problems

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Abstract

In this thesis, we consider optimal control problems for second order elliptic partial differential equations with distributed control. We study and give a review of overlapping domain decomposition methods for scalar elliptic problems in the sense of the classical additive and optimized Schwarz methods. In this work, we will present construction of the methods as well as an analysis. We will discuss Fourier convergence analysis for these algorithms. Next, we will extend the known results from the scalar elliptic problems to optimal control systems. Finally, we will perform numerical experiments to illustrate the convergence behaviors for both additive Schwarz and optimized Schwarz methods for scalar elliptic problems as well as for optimal control problems.

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Chapter 1

Introduction

This work is concerned with construction, analysis and numerical study of a class of domain decomposition methods for the solutions of both scalar elliptic problems and elliptic optimal control problems with distributed control. In particular, we study and give a review of overlapping domain decomposition methods in the sense of the classical additive Schwarz and optimized Schwarz methods.

Domain decomposition is nowadays a very crucial active area of research in the numerical approximation of boundary value problems for PDEs. The key idea of domain decomposition techniques is to split the original computational domain, say Ω, into smaller components Ω_i, $i = 1, \cdots, M$, which may or may not overlap, called subdomains, and then solve the boundary value problem on each subdomain. Clearly, additional interface conditions have to be imposed on $\partial\Omega_i \cap \partial\Omega_j$ [5]. In this thesis, we are interested in domain decomposition that overlaps.

The additive Schwarz domain decomposition method with Dirichlet transmission conditions leads to convergence rates which are not uniform with respect to frequency: high frequency components converge rapidly whereas low frequency components converge only slowly [2]. Optimized Schwarz methods are a class of domain decomposition methods with greatly enhanced convergence properties. These iterative methods can be used with or without overlap. They converge necessary faster than classical Schwarz methods at the same cost per iteration [1].

The ultimate aim of modelling and simulating application problems is to achieve better understanding of real world systems eventually with the purpose of being able to influence these systems in a desired way . This purpose has motivated the formulation of a domain decomposition method for optimal control problems governed by an elliptic PDE. The control problem is formulated as the minimization of the cost functional where the state of the system is characterized by the governing equations and the action of the control. The conditions for such a minimum result in a set of coupled equations called the optimality system [3].

This thesis is organized as follows. In chapter 2, we introduce and discuss overlapping domain decomposition methods for scalar elliptic problems. We study the convergence analysis for additive Schwarz and the optimized Schwarz methods with the application of the Fourier transform. In chapter 3, we perform numerical experiments to illustrate the convergence behaviors of the additive Schwarz and optimized Schwarz methods. In chapter 4, we discuss elliptic optimal control problems, specifically with distributed control. We will investigate the Fourier convergence analysis for the domain decomposition methods that will be generated from the elliptic optimal problems. We will further consider numerical simulations to study the convergence behavior of the Schwarz algorithms that will be treated in this chapter.

Chapter 2

Additive Schwarz and optimized Schwarz methods

In this chapter, we consider an elliptic partial differential equation with Dirchlet boundary condition. Our solution methods will based on additive Schwarz domain decomposition. We give an overview for the convergence analysis of the additive Schwarz methods and optimized Schwarz methods using Fourier transform. Our review is based on a research paper by Gander [1].

2.1 Model Problem

Consider the second order linear scalar boundary value problem (2.1). Find u such that

$$\begin{aligned} -\Delta u + \eta u &= f \text{ in } \Omega = \mathbb{R}^2, \\ u &= g \text{ on } \Gamma = \partial\Omega, \end{aligned} \tag{2.1}$$

where $\eta \in \mathbb{R}, f, g$ are given.

2.2 The additive Schwarz algorithm

In this section, our main objective is to review and understand the mathematical framework of additive Schwarz domain decompositions methods and in particular Fourier convergence analysis of this method.

We decompose the domain Ω into two overlapping subdomains namely:

$$\Omega_1 = (-\infty, L) \times \mathbb{R} \text{ and } \Omega_2 = (0, \infty) \times \mathbb{R}$$

where L is the size of the overlap.

Let $u_1^0,\ u_2^0$, be the given initial data, then for $n \geq 0$, we fine two sequences $u_1^{n+1},\ u_2^{n+1}$ and solve the subdomain problems concurrently. Therefore the additive Schwarz algorithm for this decomposition is given by:

$$\begin{aligned} -\Delta u_1^{n+1} + \eta u_1^{n+1} &= f && \text{in } \Omega_1, \\ u_1^{n+1}(x,y) &= g && \text{on } \partial\Omega_1 \cap \partial\Omega, \\ u_1^{n+1}(L,y) &= u_2^n(L,y), && \text{on } \partial\Omega_1 \cap \Omega_2, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} -\Delta u_2^{n+1} + \eta u_2^{n+1} &= f && \text{in } \Omega_2, \\ u_2^{n+1}(x,y) &= g && \text{on } \partial\Omega_2 \cap \partial\Omega, \\ u_2^{n+1}(0,y) &= u_1^n(0,y), && \text{on } \partial\Omega_2 \cap \Omega_1. \end{aligned} \tag{2.3}$$

Similar formulations can be seen in [2] and also the book by Quarteroni and Valli [7]

2.3 Fourier Transform

Our convergence analysis will be based on Fourier transform, and we therefore state the formulas for this transform. We denote the Fourier transform of a function $f(x)$ by

$$\hat{f}(k) = \mathcal{F}(f) := \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} e^{-i\langle k,x\rangle}\ f(x)\,dx, \quad k \in \mathbb{R}^s$$

and the inverse transform of $\hat{f}(k)$ is given by

$$f(x) = \mathcal{F}^{-1}(\hat{f}) := \frac{1}{(2\pi)^{s/2}} \int_{\mathbb{R}^s} e^{i\langle k,x\rangle}\ \hat{f}(k)\,dk \quad x \in \mathbb{R}^s$$

See the book by Kreiss and Lorenz [8] and reference therein.

Under appropriate conditions on f, one can show that

$$\mathcal{F}(f') := iw\mathcal{F}(f).$$

See the book by Erwin Kreyszig [6] for the proof.

2.3.1 Convergence analysis of Schwarz algorithm

In other to investigate the convergence of the additive Schwarz algorithm, we re-write the initial boundary problems (2.2,2.3) in terms of errors. For that we use the error equation:

$$e_j^{n+1} = u - u_j^{n+1} \tag{2.4}$$

where u is the solution of equation (2.1) and u_j^{n+1}, for $i = 1, 2$ are the iterates. Then:

$$\begin{aligned} -\Delta e_1^{n+1} + \eta e_1^{n+1} &= 0 && \text{in } \Omega_1, \\ e_1^{n+1}(x,y) &= 0 && \text{on } \partial\Omega_1 \cap \partial\Omega, \\ e_1^{n+1}(L,y) &= e_2^n(L,y), && \text{on } \partial\Omega_1 \cap \Omega_2, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} -\Delta e_2^{n+1} + \eta e_2^{n+1} &= 0 && \text{in } \Omega_2, \\ e_2^{n+1}(x,y) &= 0 && \text{on } \partial\Omega_2 \cap \partial\Omega, \\ e_2^{n+1}(0,y) &= e_1^n(0,y), && \text{on } \partial\Omega_2 \cap \Omega_1. \end{aligned} \tag{2.6}$$

By taking a Fourier transform in the y direction of the additive Schwarz algorithm (2.5, 2.6), we obtain

$$\begin{aligned} (\eta + k^2 - \partial_{xx})\hat{e}_1^{n+1} &= 0, && x < L,\ k \in \mathbb{R}, \\ \hat{e}_1^{n+1}(-\infty, k) &= 0 && k \in \mathbb{R}, \\ \hat{e}_1^{n+1}(L,k) &= \hat{e}_2^n(L,k), && k \in \mathbb{R}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} (\eta + k^2 - \partial_{xx})\hat{e}_2^{n+1} &= 0 && x > 0,\ k \in \mathbb{R}, \\ \hat{e}_2^{n+1}(\infty, k) &= 0 && k \in \mathbb{R}, \\ \hat{e}_2^{n+1}(0,k) &= \hat{e}_1^n(0,k), && k \in \mathbb{R}. \end{aligned} \tag{2.8}$$

The general solution of these differential equations in Fourier domain takes the form

$$\hat{e}_j^{n+1}(x,k) = C_j(k)e^{\lambda_1(k)x} + D_j(k)e^{\lambda_2(k)x},\ j = 1, 2, \tag{2.9}$$

where $\lambda_j(k) = \pm\sqrt{k^2 + \eta},\ j = 1, 2$ with $\lambda_1(k) = \sqrt{k^2 + \eta}$ and $\lambda_2(k) = -\sqrt{k^2 + \eta}$.

By using the condition on the iterates at infinity we have

$$\hat{e}_1^{n+1}(x,k) = C_1(k)e^{\sqrt{k^2+\eta}\ x} \tag{2.10}$$

and

$$\hat{e}_2^{n+1}(x,k) = D_2(k)e^{-\sqrt{k^2+\eta}\ x}. \tag{2.11}$$

Hence by applying the transmission boundary conditions we obtain the following subdomain solutions

$$\hat{e}_1^{n+1}(x,k) = \hat{e}_2^{n}(L,k)e^{\sqrt{k^2+\eta}(x-L)} \tag{2.12}$$

and

$$\hat{e}_2^{n+1}(x,k) = \hat{e}_1^{n}(0,k)e^{-\sqrt{k^2+\eta}\ x}. \tag{2.13}$$

Next, we investigate the convergence factor of this algorithm as follows:

Re-writing equation (2.13) and evaluating the resulting equation at $x = L$ we have

$$\hat{e}_2^{n}(L,k) = \hat{e}_1^{n-1}(0,k)e^{-\sqrt{k^2+\eta}\ L}. \tag{2.14}$$

Thus inserting equation (2.14) into equation (2.12) yields

$$\hat{e}_1^{n+1}(x,k) = \hat{e}_1^{n-1}(0,k)e^{-\sqrt{k^2+\eta}\ L} \cdot e^{\sqrt{k^2+\eta}(x-L)}. \tag{2.15}$$

Hence evaluating this equation at $x = 0$, gives

$$\hat{e}_1^{n+1}(0,k) = \hat{e}_1^{n-1}(0,k)e^{-2\sqrt{k^2+\eta}\ L}. \tag{2.16}$$

Similarly, we find a relation for the iterates with subscripts one as:

From equation (2.12) we have,

$$\hat{e}_1^{n}(x,k) = \hat{e}_2^{n-1}(L,k)e^{-2\sqrt{k^2+\eta}(x-L)} \tag{2.17}$$

Evaluating equation (2.17) at $x = 0$ and then substituting the result into equation (2.13) we obtain

$$\hat{e}_2^{n+1}(x,k) = \hat{e}_2^{n-1}(L,k)e^{-\sqrt{k^2+\eta}\ L} \cdot e^{-\sqrt{k^2+\eta}\ x}. \tag{2.18}$$

Finally, evaluating (2.18) at $x = L$ gives a similar result as

$$\hat{e}_2^{n+1}(L,k) = \hat{e}_2^{n-1}(L,k)e^{-2\sqrt{k^2+\eta}\ L}. \tag{2.19}$$

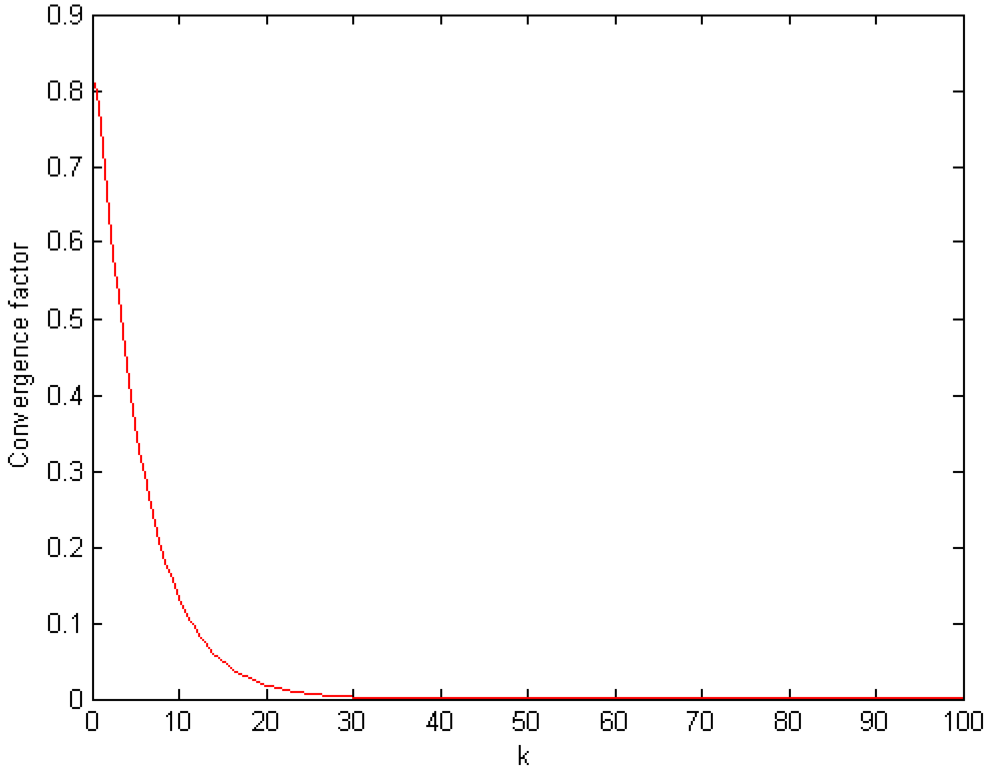

Figure 2.1: Convergence factor ρ_{cla} for the additive Schwarz method as a function of k with overlap L $= \frac{1}{10}$ and problem parameter $\eta = 1$.

Hence from these two relations (2.16, 2.19), the convergence factor $\rho_{add}(k, \eta, L)$ is given by

$$\rho_{add} = \rho_{cla}(k, \eta, L) = e^{-2\sqrt{k^2+\eta}\ L} < 1\ \forall k \in \mathbb{R} \tag{2.20}$$

The convergence factor depends on the parameter η and the size of the overlap L From Figure 2.1 above, it is clear that, the additive Schwarz algorithm performs well at high frequencies but converges slowly at lower frequencies. Hence there is the need for improvement in convergence. In the following section, we will introduce a new Schwarz algorithm which will take care of this drawback.

2.4 Optimized Schwarz algorithm

This algorithm considers a new transmission condition which replaces the Dirichlet transmission in the additive Schwarz algorithm of the previous section. The modified algorithm is then given by:

$$\begin{aligned} -\Delta u_1^{n+1} + \eta u_1^{n+1} &= f && \text{in } \Omega_1, \\ u_1^{n+1}(x,y) &= g && \text{on } \partial\Omega_1 \cap \partial\Omega, \\ (\partial x + S_1)u_1^{n+1}(L,y) &= (\partial x + S_1)u_2^n(L,y), && \text{on } \partial\Omega_1 \cap \Omega_2, \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} -\Delta u_2^{n+1} + \eta u_2^{n+1} &= f && \text{in } \Omega_2, \\ u_2^{n+1}(x,y) &= g && \text{on } \partial\Omega_2 \cap \partial\Omega, \\ (\partial x + S_2)u_2^{n+1}(0,y) &= (\partial x + S_2)u_1^n(0,y), && \text{on } \partial\Omega_2 \cap \Omega_1. \end{aligned} \tag{2.22}$$

where $S_j = 1, 2$ are linear operators along the interface in the y direction. These linear operators are chosen to optimize the performance of the new algorithm.

2.4.1 Convergence analysis of Optimized Schwarz algorithm

By converting the algorithm (2.21, 2.22) into boundary value problems for the error as the one shown above and taking Fourier transform in the direction of y yields

$$\begin{aligned} (\eta + k^2 - \partial_{xx})\hat{e}_1^{n+1} &= 0, && x < L, \ k \in \mathbb{R}, \\ \hat{e}_1^{n+1}(-\infty, k) &= 0 && k \in \mathbb{R}, \\ (\partial_x + \sigma(k)_1)\hat{e}_1^{n+1}(L,k) &= (\partial_x + \sigma(k)_1)\hat{e}_2^n(L,k), && k \in \mathbb{R}, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} (\eta + k^2 - \partial_{xx})\hat{e}_2^{n+1} &= 0 && x > 0, \ k \in \mathbb{R}, \\ \hat{e}_2^{n+1}(\infty, k) &= 0 && k \in \mathbb{R}, \\ (\partial_x + \sigma(k)_2)\hat{e}_2^{n+1}(0,k) &= (\partial_x + \sigma(k)_2)\hat{e}_1^n(0,k), && k \in \mathbb{R}. \end{aligned} \tag{2.24}$$

where $\sigma_j(k)$ is the symbol of the operator S_j, given by

$$\sigma_j(k)\hat{u}_j^{n+1} = (\widehat{S_j(u_j^{n+1})}).$$

The subdomain solutions are of the same form as the additive Schwarz algorithm discussed in subsection 2.3.1.

$$\hat{e}_j^{n+1}(x,k) = M_j(k)e^{\lambda_1(k)x} + Q_j(k)e^{\lambda_2(k)x}, \ j = 1, 2 \tag{2.25}$$

where $\lambda_j(k) = \pm\sqrt{k^2 + \eta}, \ j = 1, 2$ with $\lambda_1(k) = \sqrt{k^2 + \eta}$ and $\lambda_2(k) = -\sqrt{k^2 + \eta}$

Therefore, by applying the conditions on the iterates at infinity we get the following:

$$\hat{e}_1^{n+1}(x,k) = M_1(k)e^{\sqrt{k^2+\eta}\ x}, \tag{2.26}$$

and

$$\hat{e}_2^{n+1}(x,k) = Q_2(k)e^{-\sqrt{k^2+\eta}\ x}. \tag{2.27}$$

Evaluating the equations (2.26), (2.27) at $x = L$ and $x = 0$ respectively and solving for the constants $M_1(k), Q_2(k)$, we have

$$\hat{e}_1^{n+1}(x,k) = \hat{e}_1^{n+1}(L,k)e^{\sqrt{k^2+\eta}\ (x-L)} \tag{2.28}$$

and

$$\hat{e}_2^{n+1}(x,k) = \hat{e}_2^{n+1}(0,k)e^{-\sqrt{k^2+\eta}\ x} \tag{2.29}$$

Since

$$\frac{\partial \hat{e}_1^{n+1}}{\partial x} = \sqrt{k^2+\eta}\hat{e}_1^{n+1}, \qquad \frac{\partial \hat{e}_2^{n+1}}{\partial x} = -\sqrt{k^2+\eta}\hat{e}_2^{n+1}, \tag{2.30}$$

we can state that;

$$\frac{\partial \hat{e}_1^{n+1}}{\partial x}(L,k) = \sqrt{k^2+\eta}\hat{e}_1^{n+1}(L,y), \qquad \frac{\partial \hat{e}_2^{n+1}}{\partial x}(L,k) = -\sqrt{k^2+\eta}\hat{e}_2^{n+1}(L,k), \tag{2.31}$$

$$\frac{\partial \hat{e}_1^{n+1}}{\partial x}(0,k) = \sqrt{k^2+\eta}\hat{e}_1^{n+1}(0,k), \qquad \frac{\partial \hat{e}_2^{n+1}}{\partial x}(0,k) = -\sqrt{k^2+\eta}\hat{e}_2^{n+1}(0,k). \tag{2.32}$$

Substituting (2.31), (2.32) into the transmission boundary conditions of the algorithm (2.23), (2.24) together with (2.28), (2.28) gives the subdomain solutions in Fourier space as follows;

$$\hat{e}_1^{n+1}(x,k) = \frac{\sigma_1(k) - \sqrt{k^2+\eta}}{\sigma_1(k) + \sqrt{k^2+\eta}} e^{\sqrt{k^2+\eta}(x-L)}\hat{e}_2^{n}(L,k),$$

$$\hat{e}_2^{n+1}(x,k) = \frac{\sigma_2(k) + \sqrt{k^2+\eta}}{\sigma_2(k) - \sqrt{k^2+\eta}} e^{-\sqrt{k^2+\eta}\ x}\hat{e}_1^{n}(0,k).$$

The difference between these solutions and the additive Schwarz algorithm solutions is the fraction in front of the exponentials. By following a similar procedure as in the case of the additive Schwarz, we can state the new convergence factor as;

$$\rho_{opt} = \rho_{opt}(k, L, \eta, \sigma_1, \sigma_2) := \frac{\sigma_1(k) - \sqrt{k^2+\eta}}{\sigma_1(k) + \sqrt{k^2+\eta}} \cdot \frac{\sigma_2(k) + \sqrt{k^2+\eta}}{\sigma_2(k) - \sqrt{k^2+\eta}} e^{-\sqrt{k^2+\eta}L} \tag{2.33}$$

From this new convergence factor by setting:

$$\sigma_1(k) = \sqrt{k^2+\eta}, \qquad \sigma_(k) = -\sqrt{k^2+\eta}, \tag{2.34}$$

makes the factors vanish identically [1] and the algorithm converges in two iterations.

Since we will perform numerical implementation for this algorithm, in the following chapter, we need to back-transform the transmission conditions from the Fourier domain into the physical domain to obtain the differential operators S_1 and S_2. For this transformations, we will require the following relations

$$S_1(u_1^{n+1}) = \mathcal{F}_k^{-1}(\sigma_1 \hat{u}_1^{n+1}), \qquad S_2(u_2^{n+1}) = \mathcal{F}_k^{-1}(\sigma_2 \hat{u}_2^{n+1}), \tag{2.35}$$

Furthermore, since the symbols $\sigma_j(k)$ contains a radical sign, it makes the differential operator S_j non-local. Therefore, to rectify this non-locality property, we will in the following section approximate them by polynomials in ik.

2.4.2 Polynomial approximation for symbols of transmission conditions

By approximating the symbols of the transmission conditions we had in subsection 2.4.1 by polynomials in ik, we set

$$\sigma_1^{app}(k) = p_1 + q_1 k^2, \qquad \sigma_2^{app}(k) = -p_2 - q_2 k^2. \tag{2.36}$$

The parameters $p_j \geq 0,\ q_j \geq 0$ for $j = 1, 2$ are chosen to optimized the convergence performance of the optimized Schwarz algorithm.

We did not include first order terms in our approximations due to symmetric nature of the symbols. Now, by incorporating the approximate expressions into the convergence factor 2.33 gives

$$\rho_{opt} = \frac{\sqrt{k^2+\eta} - p_1 - q_1 k^2}{\sqrt{k^2+\eta} + p_1 + q_1 k^2} \cdot \frac{\sqrt{k^2+\eta} - p_2 - q_2 k^2}{\sqrt{k^2+\eta} + p_2 + q_2 k^2} e^{-2\sqrt{k^2+\eta}L}$$

2.4.3 Low-frequency approximation

This approximation technique will be applied in our numerical implementations for the scalar elliptic problems. As we have identified earlier in subsection 2.3.1, low frequencies lead to slow convergence in the additive Schwarz algorithm. But the optimized Schwarz algorithm has ability to withstand this deficiency to generate good convergence performance. This can be done by using Taylor series expansion for the symbols $\sigma_j(k)$ of the differential operator S_j.

$$\sigma_1(k) = \sqrt{\eta} + \frac{1}{2\sqrt{\eta}}k^2 + 0(k^4) \text{ and}$$

$$\sigma_2(k) = -\sqrt{\eta} - \frac{1}{2\sqrt{\eta}}k^2 + 0(k^4)$$

From these approximate expressions, we can state the convergence factors for the zeroth and second order optimized algorithm respectfully as follows:

$$\rho_{To} = \left(\frac{\sqrt{k^2+\eta} - \sqrt{\eta}}{\sqrt{k^2+\eta} + \sqrt{\eta}}\right)^2 e^{-2\sqrt{k^2+\eta}L}$$

$$\rho_{T2} = \left(\frac{\sqrt{k^2+\eta} - \sqrt{\eta} - \frac{1}{2\sqrt{\eta}}k^2}{\sqrt{k^2+\eta} + \sqrt{\eta} + \frac{1}{2\sqrt{\eta}}k^2}\right)^2 e^{-2\sqrt{k^2+\eta}L}$$

Remark 2.1. *We remark that, for the numerical experiments that will be carried out later, we will stick to zeroth order low-frequency approximation (Hence, we will neglect the second order differential term that will appear in the differential operator after the back-transformation of the symbols into the physical domain)*

In the following, we make plots for the convergence factors for the additive Schwarz algorithm and the zeroth order low-frequency approximation of the optimized Schwarz algorithm. In this graph, it will be observed that, the optimized Schwarz algorithm performs much better.

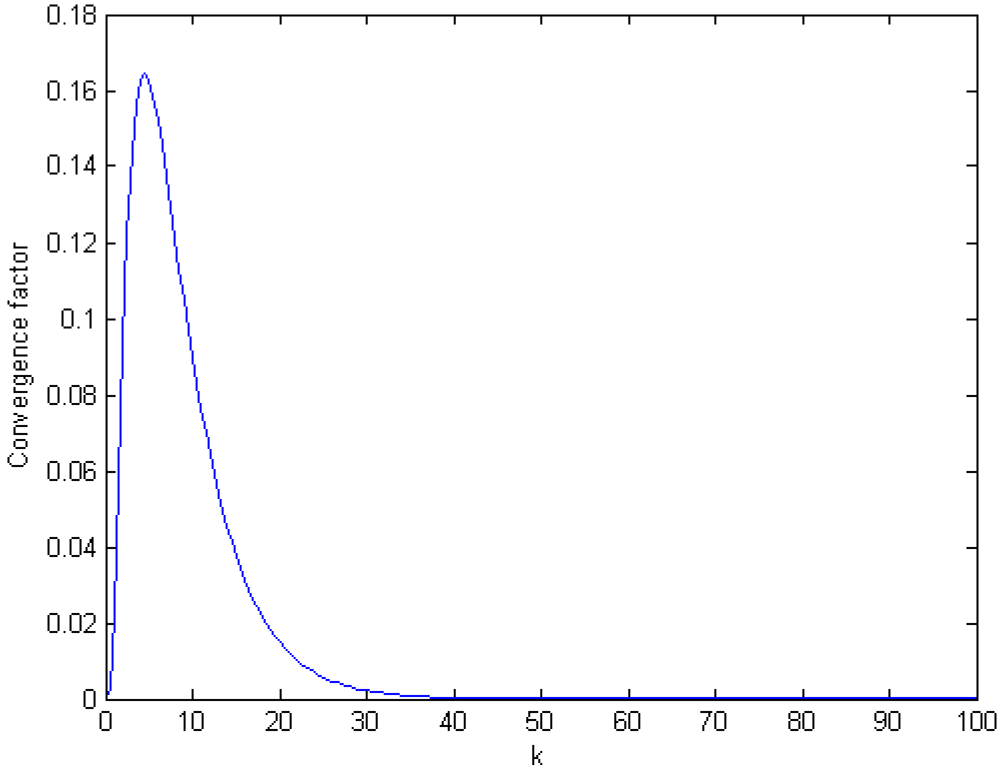

Figure 2.2: Convergence factor of the zeroth order optimized Schwarz algorithm as a function of k with overlap $L = \frac{1}{10}$ and problem parameter $\eta = 1$.

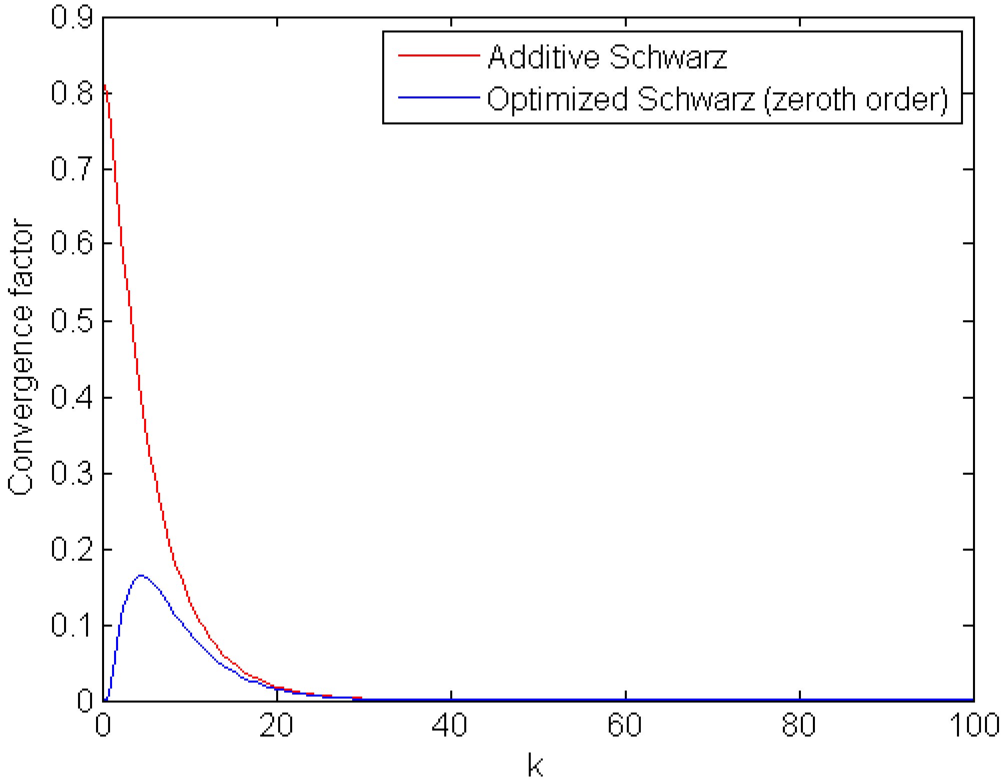

Figure 2.3: Convergence factors for both the additive Schwarz algorithm and the optimized Schwarz algorithm as a function of k with overlap $L = \frac{1}{10}$ and problem parameter $\eta = 1$.

Next, we consider transforming the differential operator into the physical domain which will be useful in the numerical implementation. Inserting equation (2.36) into equation (2.35), we obtain the transmission operators as

$$S_1 = (p_1 - q_1 \frac{\partial^2}{\partial y^2}) u_1^{n+1},$$

$$S_2 = (-p_2 + q_2 \frac{\partial^2}{\partial y^2}) u_2^{n+1}.$$

Hence substituting the obtained transmission expressions into equation (2.21,2.22) yields

$$
\begin{aligned}
-\Delta u_1^{n+1} + \eta u_1^{n+1} &= f && \text{in } \Omega_1, \\
u_1^{n+1}(x,y) &= g && \text{on } \partial\Omega_1 \cap \partial\Omega, \\
(p_1 + \partial_x - q_1 \partial_{yy}) u_1^{n+1}(L,y) &= (p_1 + \partial_x - q_1 \partial_{yy}) u_2^n(L,y), && \text{on } \partial\Omega_1 \cap \Omega_2,
\end{aligned}
\tag{2.37}
$$

and

$$
\begin{aligned}
-\Delta u_2^{n+1} + \eta u_2^{n+1} &= f && \text{in } \Omega_2, \\
u_2^{n+1}(x,y) &= g && \text{on } \partial\Omega_2 \cap \partial\Omega, \\
(-p_2 + \partial_x + q_2 \partial_{yy}) u_2^{n+1}(0,y) &= (-p_2 + \partial_x + q_2 \partial_{yy}) u_1^n(0,y), && \text{on } \partial\Omega_2 \cap \Omega_1.
\end{aligned}
\tag{2.38}
$$

Chapter 3

Discretization and Numerical Experiments

In this chapter, we consider numerical experiments for the two algorithms discussed in chapter 2. We use a finite difference discretization with the classical five-point discretization for the Laplacian on a uniform mesh with mesh parameter h. We solve the scalar elliptic problem on the rectangular domain $\Omega = (-1, 1) \times (0, 1)$,

$$\begin{aligned} -\Delta u + \eta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \Gamma = \partial\Omega. \end{aligned} \tag{3.1}$$

We decompose the rectangular domain Ω into two subdomains $\Omega_1 = (-1, 0.1) \times (0, 1)$, $\Omega_2 = (0, 1) \times (0, 1)$, where 0.1 is the size of the overlap (ie:$L = 0.1$). We perform numerical simulations for boundary value problems for the error, $f = 0$, $g = 0$, and use a random initial guess so that all the frequency components are present.

In section 3.1, we will present the discretization of the boundary problems in the additive Schwarz algorithm and the next section will consider discretization of the boundary problem in the optimized Schwarz algorithm.

Remark 3.1. *In this chapter, and also in chapter 4 we will make plots for the errors against the number of iterations. We will use the L_2 norm for this work. The plots will be made on a semilogy scale to show the linear convergence behavior of the algorithms.*

3.1 Additive Schwarz algorithm: Discretization

3.1.1 Discretization of subdomain Ω_1

We consider the subdomain

$$\begin{aligned}\Omega_1 &= (a_{x1}, b_{x1}) \times (a_{y1}, b_{y1}),\\ &= (-1, 0.1) \times (0, 1)\end{aligned}$$

with

$$h_{x1} = \frac{(b_{x1} - a_{x1})}{(N_{x1} + 1)},\ h_{y1} = \frac{(b_{y1} - a_{y1})}{(N_{y1} + 1)}$$

as the grid spacing in the x and y directions respectively, where $N_{x1},\ N_{y1}$ are the internal grid points in the respective coordination directions. Using the classical five-point discretization for the Laplacian on a uniform mesh for the elliptic problem we have:

$$-\frac{1}{h_{x1}^2}\left(u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}\right) - \frac{1}{h_{y1}^2}\left(u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}\right) + \eta u_{i,j}^{n+1} = f_{i,j}$$

The boundary conditions for this subdomain are as follows:

for $1 \leq i \leq N_{x1}$ and $1 \leq j \leq N_{y1}$, we have

$$\begin{aligned}u_{0,j}^{n+1} &= g_1(a_{x1}, y_j)\\ u_{N_{x1}+1,j}^{n+1} &= g_2(a_{x1}, y_j)\\ u_{i,0}^{n+1} &= g_3(x_i, a_{y1})\\ u_{i,N_{y1}+1}^{n+1} &= g_4(x_i, b_{y1})\end{aligned}$$

The boundary condition $u_{N_{x1}+1,j}^{n+1} = g_2(a_{x1}, y_j)$ is the transmission boundary condition in the additive Schwarz algorithm. In this subdomain, $f = 0,\ g_1 = 0,\ g_3 = 0$ and $g_4 = 0$.

From this, we can write the large linear system in the form:

$$A_1 u = b_1$$

By using the backslash command in matlab ($A_1 \setminus b_1$), we are able to compute the solution u.

3.1.2 Discretization of subdomain Ω_2

In this subdomain, let

$$\begin{aligned}\Omega_1 &= (a_{x2}, b_{x2}) \times (a_{y2}, b_{y2}),\\ &= (0,1) \times (0,1)\end{aligned}$$

with

$$h_{x2} = \frac{(b_{x2} - a_{x2})}{(N_{x2}+1)},\ \ h_{y2} = \frac{(b_{y2} - a_{y2})}{(N_{y2}+1)}.$$

Using the classical five-point discretization for the Laplacian on a uniform mesh for the elliptic problem we get:

$$-\frac{1}{h_{x2}^2}\left(u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}\right) - \frac{1}{h_{y2}^2}\left(u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}\right) + \eta u_{i,j}^{n+1} = f_{i,j}$$

The boundary conditions for this subdomain are given as:

for $1 \leq i \leq N_{x1}$ and $1 \leq j \leq N_{y1}$, we have

$$\begin{aligned} u_{0,j}^{n+1} &= g_1(a_{x2}, y_j)\\ u_{N_{x2}+1,j}^{n+1} &= g_2(a_{x2}, y_j)\\ u_{i,0}^{n+1} &= g_3(x_i, a_{y2})\\ u_{i,N_{y2}+1}^{n+1} &= g_4(x_i, b_{y2})\end{aligned}$$

where $f = 0,\ g_2 = 0,\ g_3 = 0$ and $g_4 = 0$.

The boundary condition g_1 is the transmission boundary condition in the additive Schwarz algorithm.

From this discretization, we write the large linear system in the form:

$$A_2 u = b_2$$

For the rest of the algorithms in the next section and also in chapter 4, backslash command in matlab was used to compute the numerical solutions.

3.1.3 Simulation results for additive Schwarz algorithm

In this section, we recall the boundary value problems for the errors we had in chapter 2. As stated above, we simulated the boundary problems for the error. From subsection 2.3.1, we considered the following equations:

$$\begin{aligned} -\Delta e_1^{n+1} + \eta e_1^{n+1} &= 0 && \text{in } \Omega_1, \\ e_1^{n+1}(x,y) &= 0 && \text{on } \partial\Omega_1 \cap \partial\Omega, \\ e_1^{n+1}(L,y) &= e_2^{n}(L,y), && \text{on } \partial\Omega_1 \cap \Omega_2, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} -\Delta e_2^{n+1} + \eta e_2^{n+1} &= 0 && \text{in } \Omega_2, \\ e_2^{n+1}(x,y) &= 0 && \text{on } \partial\Omega_2 \cap \partial\Omega, \\ e_2^{n+1}(0,y) &= e_1^{n}(0,y), && \text{on } \partial\Omega_2 \cap \Omega_1. \end{aligned} \tag{3.3}$$

.

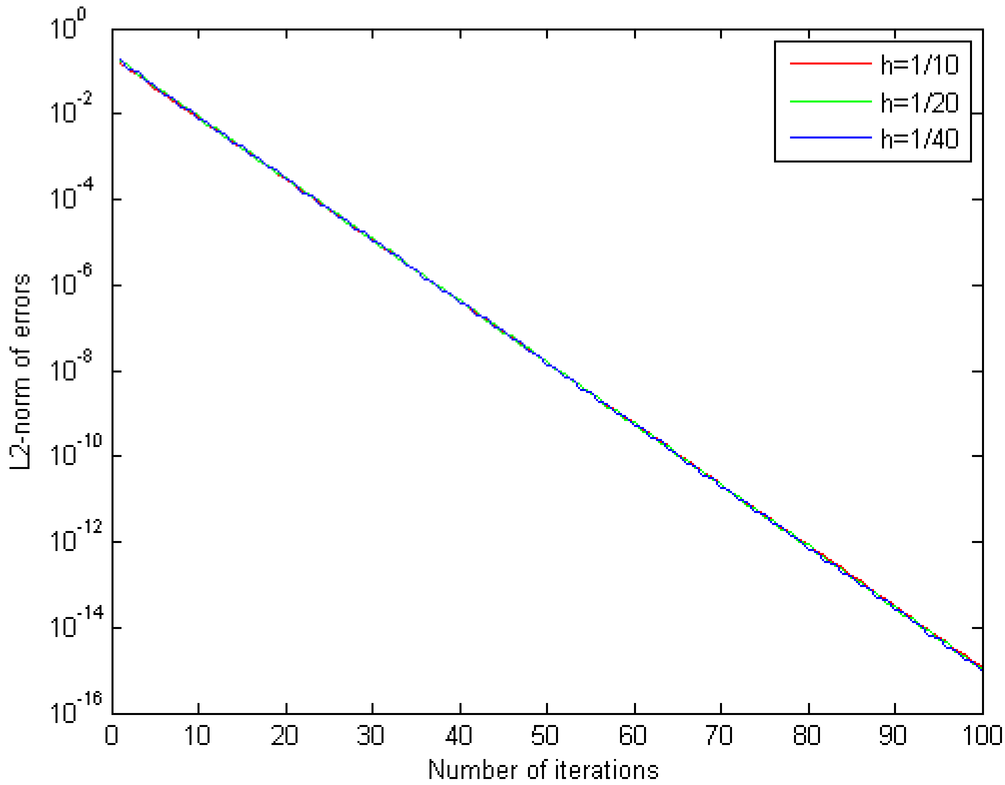

Figure 3.1: A plot of the errors for the additive Schwarz algorithm for three different mesh sizes with fixed overlap size.

From Figure 3.1 above, it shows that rate of the convergence is independent of the mesh size and this is a very significant result.

3.2 Optimized Schwarz algorithm: Discretization

3.2.1 Discretization of subdomain Ω_1

The dicretization in this method is the same as the one we obtained in the additive Schwarz method, the only difference in this situation is the replacement of the Dirichlet boundary condition by the interface boundary condition. We therefore recall the discretized equation as:

$$-\frac{1}{h_{x1}^2}\left(u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}\right) - \frac{1}{h_{y1}^2}\left(u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}\right) + \eta u_{i,j}^{n+1} = f_{i,j} \quad (3.4)$$

We now discretize the interface boundary using central differences:

$$\frac{1}{2h_{x1}}\left(u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}\right) + pu_{i,j}^{n+1} = g_2$$

Solving for $u_{i+1,j}^{n+1}$:

$$u_{i+1,j}^{n+1} = 2h_{x1}g_2 - 2h_{x1}pu_{i,j}^{n+1} + u_{i-1,j}^{n+1} \tag{3.5}$$

Substituting (3.5) into (3.5) yields:

$$-\frac{1}{h_{x1}^2}u_{i-1,j}^{n+1} + \frac{1}{h_{x1}^2}\left(2+h_{x1}p\right)u_{i,j}^{n+1} - \frac{1}{h_{y1}^2}\left(u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}\right) + \eta u_{i,j}^{n+1} = f_{i,j} + \frac{2}{h_{x1}}g_2$$

for $1 \leq i \leq N_{x1}$ and $1 \leq j \leq N_{y1}$

The remaining conditions boundary conditions for this subdomain are given by:

for $1 \leq i \leq N_{x1}$ and $1 \leq j \leq N_{y1}$, we have

$$\begin{aligned} u_{0,j}^{n+1} &= g_1(a_{x2}, y_j) \\ u_{i,0}^{n+1} &= g_3(x_i, a_{y2}) \\ u_{i,N_{y2}+1}^{n+1} &= g_4(x_i, b_{y2}) \end{aligned}$$

where $f = 0$, $g_1 = 0$, $g_3 = 0$ and $g_4 = 0$.

3.2.2 Discretization of subdomain Ω_2

The dicretization in this subdomain is also the same as the one we obtained in the additive Schwarz method, therefore we recall the discretized equation as:

$$-\frac{1}{h_{x2}^2}\left(u_{i-1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i+1,j}^{n+1}\right) - \frac{1}{h_{y2}^2}\left(u_{i,j-1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j+1}^{n+1}\right) + \eta u_{i,j}^{n+1} = f_{i,j} \tag{3.6}$$

We now discretize the interface boundary using central differences:

$$-\frac{1}{2h_{x2}}\left(u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}\right) + pu_{i,j}^{n+1} = g_1 \tag{3.7}$$

From (3.7) we have:

$$u_{i-1,j}^{n+1} = 2h_{x2}g_2 - 2h_{x2}pu_{i,j}^{n+1} + u_{i+1,j}^{n+1} \tag{3.8}$$

Hence equation (3.6) becomes:

$$-\frac{1}{h_{x1}^2}u_{i+1,j}^{n+1}+\frac{1}{h_{x1}^2}\Big(2+h_{x1}p\Big)u_{i,j}^{n+1}-\frac{1}{h_{y2}^2}\Big(u_{i,j-1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j+1}^{n+1}\Big)+\eta u_{i,j}^{n+1}=f_{i,j}+\frac{2}{h_{x2}}g_1$$

for $i = 0$ and $1 \leq j \leq N_{y1}$.

The other boundary conditions are defined as:

for $0 \leq i \leq N_{x1}$ and $1 \leq j \leq N_{y1}$, we have

$$\begin{aligned} u_{N_{x2}+1,j}^{n+1} &= g_2(a_{x2}, y_j) \\ u_{i,0}^{n+1} &= g_3(x_i, a_{y2}) \\ u_{i,N_{y2}+1}^{n+1} &= g_4(x_i, b_{y2}) \end{aligned}$$

where $f = 0,\ g_2 = 0,\ g_3 = 0$ and $g_4 = 0$.

3.2.3 Simulation Results for Optimized Schwarz algorithm

This section considers the simulated results for the Optimized Schwarz algorithm also discussed in chapter 2.

$$\begin{aligned} -\Delta e_1^{n+1} + \eta e_1^{n+1} &= 0 && \text{in } \Omega_1, \\ e_1^{n+1}(x,y) &= 0 && \text{on } \partial\Omega_1 \cap \partial\Omega, \\ (\partial_x + p)e_1^{n+1}(L,y) &= (\partial_x + p)e_2^{n}(L,y), && \text{on } \partial\Omega_1 \cap \Omega_2, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} -\Delta e_2^{n+1} + \eta e_2^{n+1} &= 0 && \text{in } \Omega_2, \\ e_2^{n+1}(x,y) &= 0 && \text{on } \partial\Omega_2 \cap \partial\Omega, \\ (\partial_x - p)e_2^{n+1}(0,y) &= (\partial_x - p)e_1^{n}(0,y), && \text{on } \partial\Omega_1 \cap \Omega_1. \end{aligned} \tag{3.10}$$

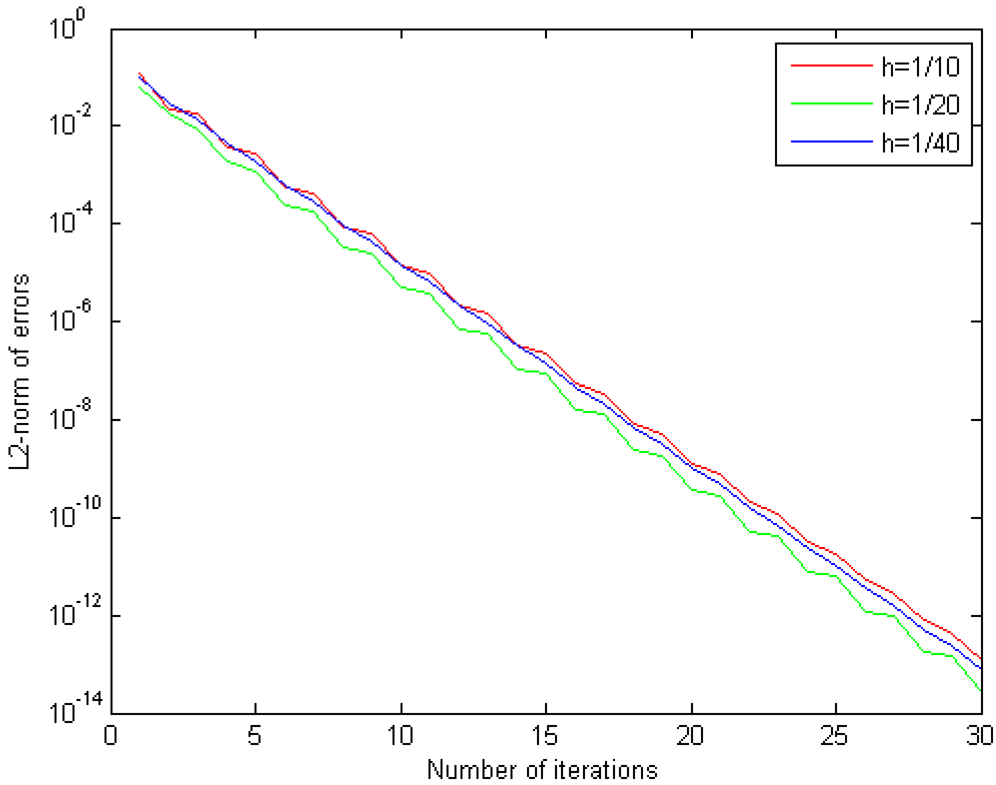

Figure 3.2: A plot of the errors for the optimized Schwarz algorithm for three different mesh sizes with fixed overlap size.

As we have stated before, in the algorithm, we used the zeroth order low frequency approximation. The convergence pattern in this case is also mesh independent. From Figures 3.1 and 3.2, it is very clear that, the improved algorithm converges faster then additive Schwarz algorithm

Chapter 4

Optimal Control Problems

In this chapter, we will extend the technique for scalar elliptic PDE problems discussed in chapter 2, to elliptic optimal control problems. We will discuss additive Schwarz methods and optimized Schwarz methods. We will extend the Fourier convergence analysis presented in chapter 2 to these optimal control problems. We will then give plots for the two convergence factors and then compare them accordingly.. Finally, in this chapter, we will performance numerical experiments to study convergence behaviour of the model problems.

4.1 Elliptic distributed optimal control problem

We consider an elliptic distributed control problem whereby the cost functional is of the tracking type. We want to minimize the functional

$$J(y,u) = \frac{1}{2}\int_{\Omega}(y-y_d)^2\,dx + \frac{\mu}{2}\int_{\Omega}u^2\,dx$$

subject to the state constraint,

$$\begin{aligned} -\Delta y + \eta y &= f + u, \quad \text{in } \Omega, \\ y &= g \text{ on } \Gamma, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, with boundary Γ, y_d is the desired state and $\mu > 0$ is the weight of the cost of the control.

4.1.1 Optimality Systems

Let p be the adjoint state , then we obtain the optimality systems which consists of the state equation (4.1), the adjoint state equation (4.2) and the control equation (4.3):
1. The adjoint state equation:

$$\begin{aligned} -\Delta y + \eta y &= f + u, \quad \text{in } \Omega \\ y &= g \text{ on } \Gamma, \end{aligned} \tag{4.1}$$

2. The control equation:

$$\begin{aligned} -\Delta p + \eta p &= -(y - y_d), \quad \text{in } \Omega \\ p &= 0 \text{ on } \Gamma, \end{aligned} \tag{4.2}$$

3. The state equation:

$$\mu u - p = 0 \text{ in } \Omega. \tag{4.3}$$

By eliminating the control u from the state equation (4.1) we obtain the optimality system,

$$\begin{cases} -\Delta y + \eta y = f + \frac{1}{\mu} p, \quad \text{in } \Omega, \\ y = g \text{ on } \Gamma, \\ -\Delta p + \eta p = -(y - y_d), \quad \text{in } \Omega \\ p = 0 \text{ on } \Gamma. \end{cases} \tag{4.4}$$

.

see [3], [4] for reference.

4.1.2 Additive Schwarz algorithm for coupled system

In this subsection, we formulate an additive Schwarz algorithm for the coupled systems obtained in subsection 4.1.1. We will consider Fourier convergence analysis for this algorithm to determine and study the convergence behavior.

We decompose the domain Ω into two overlapping subdomains namely:

$$\Omega_1 = (-\infty, L) \times \mathbb{R} \text{ and } \Omega_2 = (0, \infty) \times \mathbb{R}$$

where L is the size of the overlap.

Let $y_1^0,\ p_1^0,\ y_2^0,\ p_2^0$, be the given initial data, then for $n \geq 0$, we define four sequences $y_1^{n+1},\ p_1^{n+1},\ y_2^{n+1},\ p_2^{n+1}$ and solve the subdomain problems simultaneously. Therefore the additive Schwarz algorithm for the coupled systems is given by:

$$
\begin{cases}
-\Delta y_1^{n+1} + \eta y_1^{n+1} = f + \frac{1}{\mu} p_1^{n+1}, & \text{in } \Omega_1, \\
y_1^{n+1}(x_1, x_2) = g, & \text{on } \partial\Omega_1 \cap \partial\Omega \\
y_1^{n+1}(L, x_2) = y_2^{n}(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2 \\
-\Delta p_1^{n+1} + \eta p_1^{n+1} = -(y_1^{n+1} - y_d), & \text{in } \Omega_1, \\
p_1^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega \\
p_1^{n+1}(L, x_2) = p_2^{n}(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2
\end{cases}
\tag{4.5}
$$

and

$$
\begin{cases}
-\Delta y_2^{n+1} + \eta y_2^{n+1} = f + \frac{1}{\mu} p_2^{n+1}, & \text{in } \Omega_2, \\
y_2^{n+1}(x_1, x_2) = g, & \text{on } \partial\Omega_2 \cap \partial\Omega \\
y_2^{n+1}(0, x_2) = y_1^{n}(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1 \\
-\Delta p_2^{n+1} + \eta p_2^{n+1} = -(y_2^{n+1} - y_d), & \text{in } \Omega_2, \\
p_2^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega \\
p_2^{n+1}(0, x_2) = p_1^{n}(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1
\end{cases}
\tag{4.6}
$$

4.1.3 Fourier convergence analysis for additive Schwarz algorithm

This subsection considers the convergence analysis for the additive Schwarz algorithm formulated in subsection 4.1.2. We define the errors by

$$
e_j^{n+1} = y - y_j^{n+1}, \; j = 1, 2, \tag{4.7}
$$

$$
E_j^{n+1} = p - p_j^{n+1}, \tag{4.8}
$$

where y, p are solutions of the equation (4.4) and y_j^{n+1}, p_j^{n+1} represent the iterates. Using these relations gives the following boundary value problems for the errors:

$$
\begin{cases}
-\Delta e_1^{n+1} + \eta e_1^{n+1} = \frac{1}{\mu} E_1^{n+1}, & \text{in } \Omega_1, \\
e_1^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega \\
e_1^{n+1}(L, x_2) = e_2^n(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2 \\
-\Delta E_1^{n+1} + \eta E_1^{n+1} = -e_1^{n+1}, & \text{in } \Omega_1, \\
E_1^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega \\
E_1^{n+1}(L, x_2) = E_2^n(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2
\end{cases}
\tag{4.9}
$$

and

$$
\begin{cases}
-\Delta e_2^{n+1} + \eta e_2^{n+1} = \frac{1}{\mu} E_2^{n+1}, & \text{in } \Omega_2, \\
e_2^{n+1}(x_1, x_2) = 0, , & \text{on } \partial\Omega_2 \cap \partial\Omega \\
e_2^{n+1}(0, x_2) = e_1^n(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1 \\
-\Delta E_2^{n+1} + \eta E_2^{n+1} = -e_2^{n+1}, & \text{in } \Omega_2, \\
E_2^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega \\
E_2^{n+1}(0, x_2) = E_1^n(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1
\end{cases}
\tag{4.10}
$$

Taking Fourier transform in the x_2-direction, the error boundary value problems (4.9), (4.10) becomes:

$$\begin{cases} (\eta + k^2 - \partial_{x_1}^2)\hat{e}_1^{n+1} = \frac{1}{\mu}\hat{E}_1^{n+1}, & x < L,\ k \in \mathbb{R},\ \text{in } \Omega_1, \\ \hat{e}_1^{n+1}(-\infty, k) = 0, & k \in \mathbb{R} \\ \hat{e}_1^{n+1}(L, k) = \hat{e}_2^{n}(L, k), & k \in \mathbb{R} \\ (\eta + k^2 - \partial_{x_1}^2)\hat{E}_1^{n+1} = -\hat{e}_1^{n+1}, & x < L,\ k \in \mathbb{R},\ \text{in } \Omega_1 \\ \hat{E}_1^{n+1}(-\infty, k) = 0, & k \in \mathbb{R} \\ \hat{E}_1^{n+1}(L, k) = \hat{E}_2^{n}(L, k), & k \in \mathbb{R} \end{cases} \tag{4.11}$$

and

$$\begin{cases} (\eta + k^2 - \partial_{x_1}^2)\hat{e}_2^{n+1} = \frac{1}{\mu}\hat{E}_2^{n+1}, & x > 0,\ k \in \mathbb{R},\ \text{in } \Omega_2, \\ \hat{e}_2^{n+1}(\infty, k) = 0, & k \in \mathbb{R} \\ \hat{e}_2^{n+1}(0, k) = \hat{e}_1^{n}(0, k), & k \in \mathbb{R} \\ (\eta + k^2 - \partial_{x_1}^2)\hat{E}_2^{n+1} = -\hat{e}_2^{n+1}, & x > 0,\ k \in \mathbb{R},\ \text{in } \Omega_1 \\ \hat{E}_2^{n+1}(\infty, k) = 0, & k \in \mathbb{R} \\ \hat{E}_2^{n+1}(0, k) = \hat{E}_1^{n}(0, k), & k \in \mathbb{R} \end{cases} \tag{4.12}$$

In order to get the general solutions, we solve the coupled differential system in Ω_1 and Ω_2. First, we determine, the solutions in Ω_1.

$$\begin{aligned} \partial_{x_1}^2 \hat{e}_1^{n+1} &= (\eta + k^2)\hat{e}_1^{n+1} - \frac{1}{\mu}\hat{E}_2^{n+1} \qquad \text{in } \Omega_1, \\ \partial_{x_1}^2 \hat{E}_1^{n+1} &= (\eta + k^2)\hat{E}_1^{n+1} + \hat{e}_1^{n+1}, \end{aligned} \tag{4.13}$$

In matrix-vector notation we have,

$$\begin{bmatrix} \partial_{x_1}^2 \hat{e}_1^{n+1} \\ \partial_{x_1}^2 \hat{E}_1^{n+1} \end{bmatrix} = \begin{bmatrix} \eta + k^2 & -\frac{1}{\mu} \\ 1 & \eta + k^2 \end{bmatrix} \begin{bmatrix} \hat{e}_1^{n+1} \\ \hat{E}_1^{n+1} \end{bmatrix}. \tag{4.14}$$

This is a second order coupled differential equation and we need to diagonalize the coefficient matrix A given by

$$A = \begin{bmatrix} \eta + k^2 & -\frac{1}{\mu} \\ 1 & \eta + k^2 \end{bmatrix}$$

in order to decouple the systems.

The eigenvalues of matrix A are $\lambda_1 = (k^2 + \eta) + \frac{i}{\sqrt{\mu}}$, $\quad \lambda_2 = (k^2 + \eta) - \frac{i}{\sqrt{\mu}}$ and their corresponding eigenvectors are

$$\begin{bmatrix} \frac{i}{\sqrt{\mu}} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{i}{\sqrt{\mu}} \\ 1 \end{bmatrix}.$$

Hence $A = B\Lambda B^{-1}$ with

$$B = \begin{bmatrix} \frac{i}{\sqrt{\mu}} & -\frac{i}{\sqrt{\mu}} \\ 1 & 1 \end{bmatrix}, \; B^{-1} = \begin{bmatrix} \frac{\sqrt{\mu}}{2i} & \frac{1}{2} \\ -\frac{\sqrt{\mu}}{2i} & \frac{1}{2} \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} (k^2 + \eta) + \frac{i}{\sqrt{\mu}} & 0 \\ 0 & (k^2 + \eta) - \frac{i}{\sqrt{\mu}} \end{bmatrix}$$

Setting $z = Bg$, where $z = \begin{bmatrix} \hat{e}_1^{n+1} \\ \hat{E}_1^{n+1} \end{bmatrix}$ and $g = \begin{bmatrix} \hat{g}_1^{n+1} \\ \hat{g}_2^{n+1} \end{bmatrix}$, the coupled system (4.14) reduces to

$$\begin{bmatrix} \partial_{x_1}^2 \hat{g}_1^{n+1} \\ \partial_{x_1}^2 \hat{g}_2^{n+1} \end{bmatrix} = \begin{bmatrix} (k^2 + \eta) + \frac{i}{\sqrt{\mu}} & 0 \\ 0 & (k^2 + \eta) - \frac{i}{\sqrt{\mu}} \end{bmatrix} \begin{bmatrix} \hat{g}_1^{n+1} \\ \hat{g}_2^{n+1} \end{bmatrix} \tag{4.15}$$

Therefore equation (4.15)decouples:

$$\partial_{x_1}^2 \hat{g}_1^{n+1} = \left((k^2 + \eta) + \frac{i}{\sqrt{\mu}} \right) \hat{g}_1^{n+1} \tag{4.16}$$

$$\partial_{x_1}^2 \hat{g}_2^{n+1} = \left((k^2 + \eta) - \frac{i}{\sqrt{\mu}} \right) \hat{g}_2^{n+1}. \tag{4.17}$$

These differential equations have the following solutions:

$$\hat{g}_1^{n+1}(x_1,k) = C_1(k)e^{ax_1} + C_2(k)e^{-ax_1}, \tag{4.18}$$

$$\hat{g}_2^{n+1}(x_1,k) = C_3(k)e^{bx_1} + C_4(k)e^{-bx_1}, \tag{4.19}$$

where

$$a = \sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}+(k^2+\eta)}{2}} + i\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}-(k^2+\eta)}{2}},$$

$$b = \sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}+(k^2+\eta)}{2}} - i\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}-(k^2+\eta)}{2}},$$

$a^2 = (k^2+\eta) + \frac{i}{\sqrt{\mu}}$, $b^2 = (k^2+\eta) - \frac{i}{\sqrt{\mu}}$, and $C_1(k), C_2(k), C_3(k), C_4(k)$ are constants.

The general solution of the coupled differential system (4.14) now follows by using $z = Bg$, then we have

$$\begin{bmatrix} \hat{e}_1^{n+1} \\ \\ \hat{E}_2^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{i}{\sqrt{\mu}} & -\frac{i}{\sqrt{\mu}} \\ \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{g}_1^{n+1} \\ \\ \hat{g}_2^{n+1} \end{bmatrix} \tag{4.20}$$

$$\hat{e}_1^{n+1}(x_1,k) = \frac{i}{\sqrt{\mu}}\Big[C_1(k)e^{ax_1} + C_2(k)e^{-ax_1}\Big] - \frac{i}{\sqrt{\mu}}\Big[C_3(k)e^{bx_1} + C_4(k)e^{-bx_1}\Big], \tag{4.21}$$

$$\hat{E}_1^{n+1}(x_1,k) = C_1(k)e^{ax_1} + C_2(k)e^{-ax_1} + C_3(k)e^{bx_1} + C_4(k)e^{-bx_1}. \tag{4.22}$$

By means of the boundary condition at infinity ($\hat{e}_1^{n+1}(-\infty,k) = 0$), $\hat{e}_1^{n+1}(x_1,k)$ takes the form

$$\hat{e}_1^{n+1}(x_1,k) = \frac{i}{\sqrt{\mu}}\Big[C_1(k)e^{ax_1} - C_3(k)e^{bx_1}\Big] \tag{4.23}$$

By applying the boundary condition $\hat{e}_1^{n+1}(L,k) = \hat{e}_2^{n+1}(L,k)$ yields

$$\begin{aligned} \hat{e}_1^{n+1}(L,k) &= \frac{i}{\sqrt{\mu}}\Big[C_1(k)e^{aL} - C_3(k)e^{bL}\Big] \\ &= \hat{e}_2^{n}(L,k). \end{aligned} \tag{4.24}$$

Hence

$$C_1(k)e^{aL} - C_3(k)e^{bL} = \frac{\sqrt{\mu}}{i}\hat{e}_2^n(L,k). \tag{4.25}$$

Similarly, $\hat{E}_1^{n+1}(x_1,k)$ gives the result,

$$C_1(k)e^{aL} + C_3(k)e^{bL} = \hat{E}_2^n(L,k) \tag{4.26}$$

Solving equations (4.25, 4.26) gives

$$C_1(k) = \frac{1}{2}\left[\frac{\sqrt{\mu}}{i}\hat{e}_2^n(L,k) + \hat{E}_2^n(L,k)\right]e^{-aL}, \tag{4.27}$$

$$C_3(k) = \frac{1}{2}\left[\hat{E}_2^n(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^n(L,k)\right]e^{-bL}. \tag{4.28}$$

Hence the general solution in subdomain Ω_1 is then obtained as:

$$\hat{e}_1^{n+1}(x_1,k) = \frac{1}{2}\left(e^{a(x_1-L)} + e^{b(x_1-L)}\right)\hat{e}_2^n(L,k) + \frac{i}{2\sqrt{\mu}}\left(e^{a(x_1-L)} - e^{b(x_1-L)}\right)\hat{E}_2^n(L,k), \tag{4.29}$$

$$\hat{E}_1^{n+1}(x_1,k) = \frac{\sqrt{\mu}}{2i}\left(e^{a(x_1-L)} - e^{b(x_1-L)}\right)\hat{e}_2^n(L,k) + \frac{1}{2}\left(e^{a(x_1-L)} + e^{b(x_1-L)}\right)\hat{E}_2^n(L,k), \tag{4.30}$$

Since the coupled system in subdomain Ω_2 has a similar form as in Ω_1, we will neglect all the steps involving the determination of the general solution. Finally, we obtain

$$\hat{e}_2^{n+1}(x_1,k) = \frac{1}{2}\left(e^{-ax_1} + e^{-bx_1}\right)\hat{e}_1^n(0,k) + \frac{i}{2\sqrt{\mu}}\left(e^{-ax_1} - e^{-bx_1}\right)\hat{E}_1^n(0,k), \tag{4.31}$$

$$\hat{E}_2^{n+1}(x_1,k) = \frac{\sqrt{\mu}}{2i}\left(e^{-ax_1} - e^{-bx_1}\right)\hat{e}_1^n(0,k) + \frac{1}{2}\left(e^{-ax_1} + e^{-bx_1}\right)\hat{E}_1^n(0,k). \tag{4.32}$$

We then determine the convergence factor for this algorithm. By following similar steps as in subsection 2.3.1, we have the convergence relations as:

$$\begin{bmatrix} \hat{e}_1^{n+1}(0,k) \\ \\ \hat{E}_1^{n+1}(0,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e^{-2aL} + e^{-2bL)}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-2aL} - e^{-2bL)}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-2aL} - e^{-2bL)}\right) & \frac{1}{2}\left(e^{-2aL} + e^{-2bL)}\right) \end{bmatrix} \begin{bmatrix} \hat{e}_1^{n-1}(0,k) \\ \\ \hat{E}_1^{n-1}(0,k) \end{bmatrix} \tag{4.33}$$

and

$$\begin{bmatrix} \hat{e}_2^{n+1}(L,k) \\ \\ \hat{E}_2^{n+1}(L,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e^{-2aL} + e^{-2bL)}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-2aL} - e^{-2bL)}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-2aL} - e^{-2bL)}\right) & \frac{1}{2}\left(e^{-2aL} + e^{-2bL)}\right) \end{bmatrix} \begin{bmatrix} \hat{e}_2^{n-1}(L,k) \\ \\ \hat{E}_2^{n-1}(L,k) \end{bmatrix} \tag{4.34}$$

where the matrix:

$$\begin{bmatrix} \frac{1}{2}\left(e^{-2aL} + e^{-2bL)}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-2aL} - e^{-2bL)}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-2aL} - e^{-2bL)}\right) & \frac{1}{2}\left(e^{-2aL} + e^{-2bL)}\right) \end{bmatrix}$$

is the convergence matrix.

The eigenvalues for this matrix are $\lambda_1 = e^{-2aL}$ and $\lambda_2 = e^{-2bL}$

In determining the convergence factor from these eigenvalues, we will use the same long expressions we previously stated for a and b .

The absolute value of the eigenvalues can be written as:

$$|\lambda_1| = \left|\exp\left(-2L\left(\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}+(k^2+\eta)}{2}} + i\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}-(k^2+\eta)}{2}}\right)\right)\right|$$

$$= \exp\left(-2L\left(\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}+(k^2+\eta)}{2}}\right)\right)$$

$$|\lambda_2| = \left|\exp\left(-2L\left(\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}+(k^2+\eta)}{2}} - i\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}-(k^2+\eta)}{2}}\right)\right)\right|$$

$$= \exp\left(-2L\left(\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}+(k^2+\eta)}{2}}\right)\right)$$

Hence the convergence factor for this algorithm is given by:

$$\rho = \exp\left(-2L\left(\sqrt{\frac{\sqrt{(k^2+\eta)^2+\frac{1}{\mu}}+(k^2+\eta)}{2}}\right)\right) < 1, \ \forall k \in \mathbb{R}$$

This show that the algorithm converges which can be seen in the graph below. We can see that, for our estimated convergence factor or spectral radius of the convergence matrix, the additive Schwarz algorithm performs well at high frequencies but converges slowly at lower frequencies. Therefore in the next subsection, we will again make a modification to remedy this drawback.

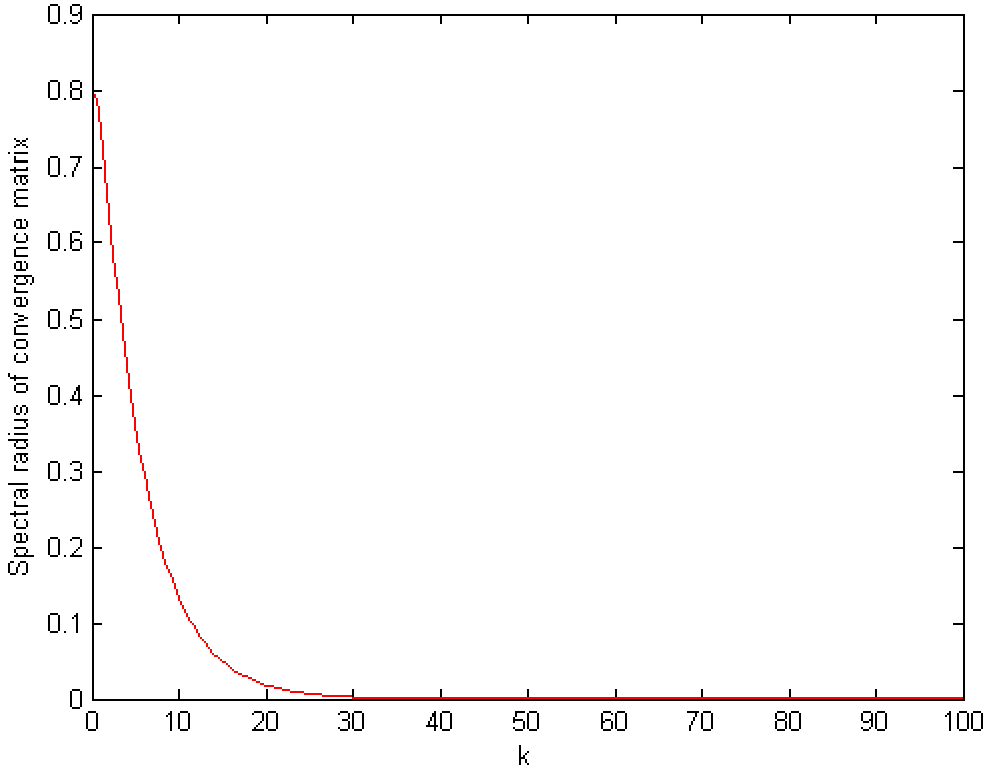

Figure 4.1: This plot depict the convergence behavior of the additive Schwarz algorithm

4.1.4 Optimized Schwarz algorithm for the coupled system

This subsection presents the optimized Schwarz algorithm for the coupled system described in 4.1.1 and the following subsection will consider the Fourier convergence analysis for this method. The formulation of this algorithm is the same as the additive Schwarz algorithm, only the Dirichlet transmission boundary conditions have been replaced by interface boundary conditions:

$$\begin{cases} -\Delta y_1^{n+1} + \eta y_1^{n+1} = f + \frac{1}{\mu} p_1^{n+1}, & \text{in } \Omega_1, \\ y_1^{n+1}(x_1, x_2) = g, & \text{on } \partial\Omega_1 \cap \partial\Omega \\ (\partial x_1 + S_1) y_1^{n+1}(L, x_2) = (\partial x_1 + S_1) y_2^n(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2 \\ -\Delta p_1^{n+1} + \eta p_1^{n+1} = -(y_1^{n+1} - y_d), & \text{in } \Omega_1, \\ p_1^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega \\ (\partial x_1 + S_1) p_1^{n+1}(L, x_2) = (\partial x_1 + S_1) p_2^n(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2 \end{cases} \tag{4.35}$$

and

$$\begin{cases} -\Delta y_2^{n+1} + \eta y_2^{n+1} = f + \frac{1}{\mu} p_2^{n+1}, & \text{in } \Omega_2, \\ y_2^{n+1}(x_1, x_2) = g, & \text{on } \partial\Omega_2 \cap \partial\Omega \\ (\partial x_1 + S_2) y_2^{n+1}(0, x_2) = (\partial x_1 + S_2) y_1^n(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1 \\ -\Delta p_2^{n+1} + \eta p_2^{n+1} = -(y_2^{n+1} - y_d), & \text{in } \Omega_2, \\ p_2^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega \\ (\partial x_1 + S_2) p_2^{n+1}(0, x_2) = (\partial x_1 + S_2) p_1^n(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1 \end{cases} \tag{4.36}$$

where $S_j = 1, 2$ are linear operators along the interface in the x_2 direction. These linear operators are chosen to optimize the performance of the new algorithm.

4.1.5 Fourier convergence analysis for Optimized Additive Schwarz

This subsection presents the convergence analysis for the optimized Schwarz algorithm formulated in subsection 4.1.2. We define the errors

$$e_j^{n+1} = y - y_j^{n+1}, \ j = 1, 2, \tag{4.37}$$
$$E_j^{n+1} = p - p_j^{n+1}, \tag{4.38}$$

where y, p are solutions of the equation (4.4) and y_j^{n+1}, p_j^{n+1} represent the iterates. Using these relations gives the following boundary value problems:

$$\begin{cases} -\Delta e_1^{n+1} + \eta e_1^{n+1} = \frac{1}{\mu} E_1^{n+1}, & \text{in } \Omega_1, \\ e_1^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega \\ (\partial x_1 + S_1) e_1^{n+1}(L, x_2) = (\partial x_1 + S_1) e_2^n(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2 \\ -\Delta E_1^{n+1} + \eta E_1^{n+1} = -e_1^{n+1}, & \text{in } \Omega_1, \\ E_1^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega \\ (\partial x_1 + S_1) E_1^{n+1}(L, x_2) = (\partial x_1 + S_1) E_2^n(L, x_2), & \text{on } \partial\Omega_1 \cap \Omega_2 \end{cases} \tag{4.39}$$

and

$$\begin{cases} -\Delta e_2^{n+1} + \eta e_2^{n+1} = \frac{1}{\mu} E_2^{n+1}, & \text{in } \Omega_2, \\ e_2^{n+1}(x_1, x_2) = 0,, & \text{on } \partial\Omega_2 \cap \partial\Omega \\ (\partial x_1 + S_2) e_2^{n+1}(0, x_2) = (\partial x_1 + S_2) e_1^n(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1 \\ -\Delta E_2^{n+1} + \eta E_2^{n+1} = -e_2^{n+1}, & \text{in } \Omega_2, \\ E_2^{n+1}(x_1, x_2) = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega \\ (\partial x_1 + S_2) E_2^{n+1}(0, x_2) = (\partial x_1 + S_2) E_1^n(0, x_2), & \text{on } \partial\Omega_2 \cap \Omega_1 \end{cases} \tag{4.40}$$

By taking the Fourier transform in the x_2 direction, boundary value problems (4.39), (4.40) gives:

$$\begin{cases} (\eta + k^2 - \partial_{x_1}^2)\hat{e}_1^{n+1} = \frac{1}{\mu}\hat{E}_1^{n+1}, & x < L,\ k \in \mathbb{R},\ \text{in } \Omega_1, \\ \hat{e}_1^{n+1}(-\infty, k) = 0, & k \in \mathbb{R} \\ (\partial_x + \sigma(k)_1)\hat{e}_1^{n+1}(L, k) = (\partial_x + \sigma(k)_1)\hat{e}_2^{n}(L, k), & k \in \mathbb{R} \\ (\eta + k^2 - \partial_{x_1}^2)\hat{E}_1^{n+1} = -\hat{e}_1^{n+1}, & x < L,\ k \in \mathbb{R},\ \text{in } \Omega_1 \\ \hat{E}_1^{n+1}(-\infty, k) = 0, & k \in \mathbb{R} \\ (\partial_x + \sigma(k)_1)\hat{E}_1^{n+1}(L, k) = (\partial_x + \sigma(k)_1)\hat{E}_2^{n}(L, k), & k \in \mathbb{R} \end{cases} \tag{4.41}$$

and

$$\begin{cases} (\eta + k^2 - \partial_{x_1}^2)\hat{e}_2^{n+1} = \frac{1}{\mu}\hat{E}_2^{n+1}, & x > 0,\ k \in \mathbb{R},\ \text{in } \Omega_2, \\ \hat{e}_2^{n+1}(\infty, k) = 0, & k \in \mathbb{R} \\ (\partial_x + \sigma(k)_2)\hat{e}_2^{n+1}(0, k) = (\partial_x + \sigma(k)_2)\hat{e}_1^{n}(0, k), & k \in \mathbb{R} \\ (\eta + k^2 - \partial_{x_1}^2)\hat{E}_2^{n+1} = -\hat{e}_2^{n+1}, & x > 0,\ k \in \mathbb{R},\ \text{in } \Omega_1 \\ \hat{E}_2^{n+1}(\infty, k) = 0, & k \in \mathbb{R} \\ (\partial_x + \sigma(k)_2)\hat{E}_2^{n+1}(0, k) = (\partial_x + \sigma(k)_2)\hat{E}_1^{n}(0, k), & k \in \mathbb{R} \end{cases} \tag{4.42}$$

The general solution of these differential equations take the same form as in the case of the additive Schwarz algorithm with the exception of the interface boundary conditions. Hence we will recall the expressions for $\hat{e}_1^{n+1}$ and $\hat{E}_2^{n+1}$ just after we applied the boundary conditions at infinity.

$$\begin{cases} \hat{e}_1^{n+1}(x_1, k) &= \frac{i}{\sqrt{\mu}}\Big[C_1(k)e^{ax_1} - C_3(k)e^{bx_1}\Big] \\ \hat{E}_1^{n+1}(x_1, k) &= C_1(k)e^{ax_1} + C_3(k)e^{bx_1} \end{cases} \tag{4.43}$$

Setting $x = L$ into the above equation yields:

$$\begin{cases} \hat{e}_1^{n+1}(L,k) & = \frac{i}{\sqrt{\mu}}\Big[C_1(k)e^{aL} - C_3(k)e^{bL}\Big] \\ \\ \hat{E}_1^{n+1}(L,k) & = C_1(k)e^{aL} + C_3(k)e^{bL} \end{cases} \tag{4.44}$$

Solving for the constants $C_1(k)$, $C_3(k)$ in terms of $\hat{e}_1^{n+1}(L,k)$ and $\hat{E}_1^{n+1}(L,k)$ we have:

$$\begin{cases} C_1(k) & = \frac{1}{2}\Big[\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(L,k) + \hat{E}_2^{n+1}(L,k)\Big]e^{-aL} \\ \\ C_3(k) & = \frac{1}{2}\Big[\hat{E}_2^{n+1}(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(L,k)\Big]e^{-bL} \end{cases} \tag{4.45}$$

Hence substituting the expressions in equation (4.45) into equation (4.44) we obtain

$$\begin{cases} \hat{e}_1^{n+1}(x_1,k) & = \frac{i}{2\sqrt{\mu}}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k) + \hat{E}_1^{n+1}(L,k)\Big)e^{a(x_1-L)} \\ \\ & \quad -\frac{i}{2\sqrt{\mu}}\Big(\hat{E}_1^{n+1}(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)\Big)e^{b(x_1-L)} \\ \\ \hat{E}_1^{n+1}(x_1,k) & = \frac{1}{2}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k) + \hat{E}_1^{n+1}(L,k)\Big)e^{a(x_1-L)} + \frac{1}{2}\Big(\hat{E}_1^{n+1}(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)\Big)e^{b(x_1-L)} \end{cases} \tag{4.46}$$

Upon further simplifications gives

$$\hat{e}_1^{n+1}(x_1,k) = \frac{1}{2}\Big(e^{a(x_1-L)} + e^{b(x_1-L)}\Big)\hat{e}_1^{n+1}(L,k) + \frac{i}{2\sqrt{\mu}}\Big(e^{a(x_1-L)} - e^{b(x_1-L)}\Big)\hat{E}_1^{n+1}(L,k) \tag{4.47}$$

$$\hat{E}_1^{n+1}(x_1,k) = \frac{\sqrt{\mu}}{2i}\Big(e^{a(x_1-L)} - e^{b(x_1-L)}\Big)\hat{e}_1^{n+1}(L,k) + \frac{1}{2}\Big(e^{a(x_1-L)} + e^{b(x_1-L)}\Big)\hat{E}_1^{n+1}(L,k) \tag{4.48}$$

Re-arranging and writing it in matrix-vector form gives

$$\begin{bmatrix} \hat{e}_1^{n+1}(x_1,k) \\ \\ \hat{E}_1^{n+1}(x_1,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Big(e^{a(x_1-L)} + e^{b(x_1-L))}\Big) & \frac{i}{2\sqrt{\mu}}\Big(e^{a(x_1-L)} - e^{b(x_1-L)}\Big) \\ \\ \frac{\sqrt{\mu}}{2i}\Big(e^{a(x_1-L)} - e^{b(x_1-L)}\Big) & \frac{1}{2}\Big(e^{a(x_1-L)} + e^{b(x_1-L)}\Big) \end{bmatrix} \begin{bmatrix} \hat{e}_1^{n+1}(L,k) \\ \\ \hat{E}_1^{n+1}(L,k) \end{bmatrix} \tag{4.49}$$

Since the interface boundary conditions involves first derivatives we proceed in the following way. Differentiating (4.46) with respect to the variable x_1 and evaluating the resulting relations at $x_1 = L$ we obtain:

$$\begin{cases} \partial_{x_1}\hat{e}_1^{n+1}(L,k) & = \frac{i}{2\sqrt{\mu}}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k) + \hat{E}_1^{n+1}(L,k)\Big)a - \frac{i}{2\sqrt{\mu}}\Big(\hat{E}_1^{n+1}(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)\Big)b \\ \\ \partial_{x_1}\hat{E}_1^{n+1}(L,k) & = \frac{1}{2}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k) + \hat{E}_1^{n+1}(L,k)\Big)a + \frac{1}{2}\Big(\hat{E}_1^{n+1}(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)\Big)b \end{cases} \tag{4.50}$$

Since we will need $\partial_{x_1}\hat{e}_1^n(0,k)$ and $\partial_{x_1}\hat{E}_1^n(0,k)$ for the interface boundary conditions in the subdomain Ω_2, we state similar results for $\partial_{x_1}\hat{e}_1^{n+1}(0,k)$ and $\partial_{x_1}\hat{E}_1^{n+1}(0,k)$ as:

$$\begin{cases} \partial_{x_1}\hat{e}_1^{n+1}(0,k) & = \frac{i}{2\sqrt{\mu}}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k) + \hat{E}_1^{n+1}(0,k)\Big)a - \frac{i}{2\sqrt{\mu}}\Big(\hat{E}_1^{n+1}(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(0,k)\Big)b \\ \\ \partial_{x_1}\hat{E}_1^{n+1}(0,k) & = \frac{1}{2}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(0,k) + \hat{E}_1^{n+1}(0,k)\Big)a + \frac{1}{2}\Big(\hat{E}_1^{n+1}(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(0,k)\Big)b \end{cases} \tag{4.51}$$

From our previous experience in dealing with the solutions in the subdomain Ω_2, we can easily state the following for $e_2^{n+1}(x_1,k)$ and $E_2^{n+1}(x_1,k)$.

$$\begin{cases} \hat{e}_2^{n+1}(x_1,k) & = \frac{i}{\sqrt{\mu}}\Big[M_2(k)e^{-ax_1} - M_4(k)e^{-bx_1}\Big] \\ \\ \hat{E}_2^{n+1}(x_1,k) & = M_2(k)e^{-ax_1} + M_4(k)e^{-bx_1} \end{cases} \tag{4.52}$$

By inserting $x = 0$ in equation (4.52) we have,

$$\begin{cases} \hat{e}_2^{n+1}(0,k) & = \frac{i}{\sqrt{\mu}}\Big(M_2(k) - M_4(k)\Big) \\ \\ \hat{E}_2^{n+1}(0,k) & = M_2(k) + M_4(k) \end{cases} \tag{4.53}$$

Therefore expressing the constants $M_2(k)$, $M_4(k)$ in terms of $\hat{e}_2^{n+1}(0,k)$ and $\hat{E}_2^{n+1}(0,k)$ we obtain

$$\begin{cases} M_2(k) & = \frac{1}{2}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k) + \hat{E}_2^{n+1}(0,k)\Big) \\ \\ M_4(k) & = \frac{1}{2}\Big(\hat{E}_2^{n+1}(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)\Big) \end{cases} \tag{4.54}$$

Hence by using these constants, equation (4.52) becomes:

$$\begin{cases} \hat{e}_2^{n+1}(x_1,k) & = \frac{i}{2\sqrt{\mu}}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k) + \hat{E}_2^{n+1}(0,k)\Big)e^{-ax_1} - \frac{i}{2\sqrt{\mu}}\Big(\hat{E}_2^{n+1}(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)\Big)e^{-bx_1} \\ \\ \hat{E}_2^{n+1}(x_1,k) & = \frac{1}{2}\Big(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k) + \hat{E}_1^{n+1}(0,k)\Big)e^{-ax_1} + \frac{1}{2}\Big(\hat{E}_2^{n+1}(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)\Big)e^{-bx_1} \end{cases} \tag{4.55}$$

Next, we re-arrange equation (4.55) that can easily be written in matrix-vector form.

$$\hat{e}_2^{n+1}(x_1,k) = \frac{1}{2}\Big(e^{-ax_1} + e^{-bx_1}\Big)\hat{e}_2^{n+1}(0,k) + \frac{i}{2\sqrt{\mu}}\Big(e^{-ax_1} - e^{-bx_1}\Big)\hat{E}_2^{n+1}(0,k) \tag{4.56}$$

$$\hat{E}_2^{n+1}(x_1,k) = \frac{\sqrt{\mu}}{2i}\Big(e^{-ax_1} - e^{-bx_1}\Big)\hat{e}_2^{n}(0,k) + \frac{1}{2}\Big(e^{-ax_1} + e^{-bx_1}\Big)\hat{E}_2^{n+1}(0,k) \tag{4.57}$$

Therefore in matrix-vector notation, we have:

$$\begin{bmatrix} \hat{e}_2^{n+1}(x_1,k) \\ \\ \hat{E}_2^{n+1}(x_1,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Big(e^{-ax_1} + e^{-bx_1}\Big) & \frac{i}{2\sqrt{\mu}}\Big(e^{-ax_1} - e^{-bx_1}\Big) \\ \\ \frac{\sqrt{\mu}}{2i}\Big(e^{-ax_1} - e^{-bx_1}\Big) & \frac{1}{2}\Big(e^{-ax_1} + e^{-bx_1}\Big) \end{bmatrix} \begin{bmatrix} \hat{e}_2^{n+1}(0,k) \\ \\ \hat{E}_2^{n+1}(0,k) \end{bmatrix} \tag{4.58}$$

Differentiating (4.55) with respect to the variable x_1 and evaluating the result at $x_1 = 0$ we obtain:

$$\begin{cases} \partial_{x_1}\hat{e}_2^{n+1}(0,k) & = -\frac{i}{2\sqrt{\mu}}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k) + \hat{E}_2^{n+1}(0,k)\right)a + \frac{i}{2\sqrt{\mu}}\left(\hat{E}_2^{n+1}(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)\right)b \\ \partial_{x_1}\hat{E}_2^{n+1}(0,k) & = -\frac{1}{2}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k) + \hat{E}_1^{n+1}(0,k)\right)a - \frac{1}{2}\left(\hat{E}_2^{n+1}(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)\right)b \end{cases} \tag{4.59}$$

Since we will need $\partial_{x_1}\hat{e}_2^n(L,k)$ and $\partial_{x_1}\hat{E}_2^n(L,k)$, for the interface boundary condition in the subdomain$-\Omega_1$, we state similar results for $\partial_{x_1}\hat{e}_2^{n+1}(L,k)$ and $\partial_{x_1}\hat{E}_2^{n+1}(L,k)$ as follows:

$$\begin{cases} \partial_{x_1}\hat{e}_2^{n+1}(L,k) & = -\frac{i}{2\sqrt{\mu}}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(L,k) + \hat{E}_2^{n+1}(L,k)\right)a + \frac{i}{2\sqrt{\mu}}\left(\hat{E}_2^{n+1}(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(L,k)\right)b \\ \partial_{x_1}\hat{E}_2^{n+1}(L,k) & = -\frac{1}{2}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(L,k) + \hat{E}_1^{n+1}(L,k)\right)a - \frac{1}{2}\left(\hat{E}_2^{n+1}(L,k) - \frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(L,k)\right)b \end{cases} \tag{4.60}$$

Now, after obtaining all the terms involving the interface boundary conditions in this algorithm, we can determine the convergence factor which will turn out to be more complicated than the one we obtain in the additive Schwarz algorithm.

We recall the interface boundary conditions from the equations (4.41), (4.42) respectively as:

$$\begin{cases} (\partial x_1 + \sigma_1(k))e_1^{n+1}(L,k) = (\partial x_1 + \sigma_1(k))e_2^n(L,k) \\ (\partial x_1 + \sigma_1(k))E_1^{n+1}(L,k) = (\partial x_1 + \sigma_1(k))E_2^n(L,k). \end{cases} \tag{4.61}$$

and

$$\begin{cases} (\partial x_1 + \sigma_2(k))e_2^{n+1}(0,k) & = (\partial x_1 + \sigma_2(k))e_1^n(0,k) \\ (\partial x_1 + \sigma_2(k))E_2^{n+1}(0,k) & = (\partial x_1 + \sigma_2(k))E_1^n(0,k), \end{cases} \tag{4.62}$$

First, we consider the interface boundary conditions (4.61). By insetting all the terms already determined we obtain:

$$(\partial x_1 + \sigma_1(k))e_1^{n+1}(L,k) = (\partial x_1 + \sigma_1(k))e_2^n(L,k),$$

$$\partial x_1 e_1^{n+1}(L,k) + \sigma_1(k)e_1^{n+1}(L,k) = \partial x_1 e_2^n(L,k) + \sigma_1(k)e_2^n(L,k)$$

$$\frac{i}{2\sqrt{\mu}}\left(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)+\hat{E}_1^{n+1}(L,k)\right)a-\frac{i}{2\sqrt{\mu}}\left(\hat{E}_1^{n+1}(L,k)-\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)\right)b+\sigma_1(k)e_1^{n+1}(L,k)$$

$$=-\frac{i}{2\sqrt{\mu}}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n}(L,k)+\hat{E}_2^{n}(L,k)\right)a+\frac{i}{2\sqrt{\mu}}\left(\hat{E}_2^{n}(L,k)-\frac{\sqrt{\mu}}{i}\hat{e}_2^{n}(L,k)\right)b+\sigma_1(k)e_2^{n}(L,x_2)$$

$$(2\sigma_1(k)+a+b)\hat{e}_1^{n+1}(L,k)+\frac{i}{\sqrt{\mu}}(a-b)\hat{E}_1^{n+1}(L,k) = (2\sigma_1(k)-a-b)\hat{e}_2^{n}(L,k)+\frac{i}{\sqrt{\mu}}(b-a)\hat{E}_2^{n}(L,k) \quad (4.63)$$

and

$$(\partial x_1 + \sigma_1(k))E_1^{n+1}(L,k) = (\partial x_1 + \sigma_1(k))E_2^n(L,k),$$

$$\partial x_1 E_1^{n+1}(L,xk) + \sigma_1(k)E_1^{n+1}(L,k) = \partial x_1 E_2^n(L,k) + \sigma_1(k)E_2^n(L,k)$$

$$\frac{1}{2}\left(\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)+\hat{E}_1^{n+1}(L,k)\right)a+\frac{1}{2}\left(\hat{E}_1^{n+1}(L,k)-\frac{\sqrt{\mu}}{i}\hat{e}_1^{n+1}(L,k)\right)b+\sigma_1(k)E_1^{n+1}(L,k)$$

$$=-\frac{1}{2}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n}(L,k)+\hat{E}_2^{n}(L,k)\right)a-\frac{1}{2}\left(\hat{E}_2^{n}(L,k)-\frac{\sqrt{\mu}}{i}\hat{e}_1^{n}(L,k)\right)b+\sigma_1(k)E_2^{n}(L,k)$$

$$(a-b)\hat{e}_1^{n+1}(L,k)+\frac{i}{\sqrt{\mu}}(2\sigma_1(k)+a+b)\hat{E}_1^{n+1}(L,k) = (b-a)\hat{e}_2^{n}(L,k)+\frac{i}{\sqrt{\mu}}(2\sigma_1(k)-a-b)\hat{E}_2^{n}(L,k) \quad (4.64)$$

$$
\begin{bmatrix} \hat{e}_1^{n+1}(L,k) \\ \\ \\ \hat{E}_1^{n+1}(L,k) \end{bmatrix} = \begin{bmatrix} (2\sigma_1(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)+a+b) \end{bmatrix}^{-1} \cdot \begin{bmatrix} (2\sigma_1(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)-a-b) \end{bmatrix} \begin{bmatrix} \hat{e}_2^{n}(L,k) \\ \\ \\ \hat{E}_2^{n}(L,k) \end{bmatrix} \tag{4.65}
$$

Inserting the equation (4.65) into equation (4.49) gives

$$
\begin{bmatrix} \hat{e}_1^{n+1}(x_1,k) \\ \\ \hat{E}_1^{n+1}(x_1,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e^{a(x_1-L)}+e^{b(x_1-L))}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{a(x_1-L)}-e^{b(x_1-L)}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{a(x_1-L)}-e^{b(x_1-L)}\right) & \frac{1}{2}\left(e^{a(x_1-L)}+e^{b(x_1-L)}\right) \end{bmatrix} \cdot \begin{bmatrix} (2\sigma_1(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)+a+b) \end{bmatrix}^{-1} \begin{bmatrix} (2\sigma_1(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)-a-b) \end{bmatrix} \begin{bmatrix} \hat{e}_2^{n}(L,k) \\ \\ \\ \hat{E}_2^{n}(L,k) \end{bmatrix} \tag{4.66}
$$

Next, we consider similar derivations for the interface boundary conditions (4.62)

$$(\partial x_1 + \sigma_2(k))e_2^{n+1}(0,k) = (\partial x_1 + \sigma_2(k))e_1^n(0,k),$$

$$\partial x_1 e_2^{n+1}(0,x_2) + \sigma_2(k)e_2^{n+1}(0,k) = \partial x_1 e_1^n(0,k) + \sigma_2(k)e_1^n(0,x_2)$$

$$-\frac{i}{2\sqrt{\mu}}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)+\hat{E}_2^{n+1}(0,x_2)\right)a+\frac{i}{2\sqrt{\mu}}\left(\hat{E}_2^{n+1}(0,k)-\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)\right)b+\sigma_2(k)e_2^{n+1}(0,x_2)$$

$$= \frac{i}{2\sqrt{\mu}}\left(\frac{\sqrt{\mu}}{i}\hat{e}_1^n(0,k) + \hat{E}_1^n(0,k)\right)a - \frac{1}{2\sqrt{\mu}}\left(\hat{E}_1^n(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^n(0,k)\right)b + \sigma_2(k)e_1^n(0,x_2)$$

$$(2\sigma_2(k) - a - b)\hat{e}_2^{n+1}(0,k) + \frac{i}{\sqrt{\mu}}(b-a)\hat{E}_2^{n+1}(0,k) = (2\sigma_1(k) + a + b)\hat{e}_1^n(0,k) + \frac{i}{\sqrt{\mu}}(a-b)\hat{E}_1^n(0,k) \tag{4.67}$$

and

$$(\partial x_1 + \sigma_2(k))E_2^{n+1}(0,k) = (\partial x_1 + \sigma_2(k))E_1^n(0,k),$$

$$\partial x_1 E_2^{n+1}(0,k) + \sigma_2(k)E_2^{n+1}(0,k) = \partial x_1 E_1^n(0,k) + \sigma_2(k)E_1^n(0,k)$$

$$-\frac{1}{2}\left(\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)+\hat{E}_2^{n+1}(0,x_2)\right)a-\frac{1}{2}\left(\hat{E}_2^{n+1}(0,k)-\frac{\sqrt{\mu}}{i}\hat{e}_2^{n+1}(0,k)\right)b+\sigma_2(k)E_2^{n+1}(0,x_2)$$

$$= \frac{1}{2}\left(\frac{\sqrt{\mu}}{i}\hat{e}_1^n(0,k) + \hat{E}_1^n(0,k)\right)a + \frac{1}{2}\left(\hat{E}_1^n(0,k) - \frac{\sqrt{\mu}}{i}\hat{e}_1^n(0,k)\right)b + \sigma_1(k)E_1^n(0,x_2)$$

$$(b-a)\hat{e}_2^{n+1}(0,k) + \frac{i}{\sqrt{\mu}}(2\sigma_2(k) - a - b)\hat{E}_2^{n+1}(0,k) = (a-b)\hat{e}_1^n(0,k) + \frac{i}{\sqrt{\mu}}(2\sigma_2(k) + a + b)\hat{E}_1^n(0,k) \tag{4.68}$$

$$\begin{bmatrix} \hat{e}_1^{n+1}(0,k) \\ \\ \hat{E}_1^{n+1}(0,k) \end{bmatrix} = \begin{bmatrix} (2\sigma_2(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)-a-b) \end{bmatrix}^{-1} \cdot \begin{bmatrix} (2\sigma_2(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)+a+b) \end{bmatrix} \begin{bmatrix} \hat{e}_1^{n}(0,k) \\ \\ \hat{E}_1^{n}(0,k) \end{bmatrix} \tag{4.69}$$

Inserting the matrix equation (4.69) into equation (4.58) we have;

$$\begin{bmatrix} \hat{e}_2^{n+1}(x_1,k) \\ \\ \hat{E}_2^{n+1}(x_1,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(e^{-ax_1}+e^{-bx_1}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-ax_1}-e^{-bx_1}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-ax_1}-e^{-bx_1}\right) & \frac{1}{2}\left(e^{-ax_1}+e^{-bx_1}\right) \end{bmatrix} \cdot \begin{bmatrix} (2\sigma_2(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)-a-b) \end{bmatrix}^{-1} \begin{bmatrix} (2\sigma_2(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)+a+b) \end{bmatrix} \begin{bmatrix} \hat{e}_1^{n}(0,k) \\ \\ \hat{E}_1^{n}(0,k) \end{bmatrix} \tag{4.70}$$

Now that we have obtained similar relations for $\begin{bmatrix} \hat{e}_1^{n+1}(x_1,k) \\ \\ \hat{E}_1^{n+1}(x_1,k) \end{bmatrix}$ and $\begin{bmatrix} \hat{e}_2^{n+1}(x_1,k) \\ \\ \hat{E}_2^{n+1}(x_1,k) \end{bmatrix}$, we can determine the convergence factor for the optimized Schwarz algorithm by using equations (4.66) and (4.70)

We evaluate equation (4.66) at $x_1 = 0$:

$$\begin{bmatrix} \hat{e}_1^{n+1}(0,k) \\ \\ \hat{E}_1^{n+1}(0,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Big(e^{-aL} + e^{-bL)}\Big) & \frac{i}{2\sqrt{\mu}}\Big(e^{-aL} - e^{-bL}\Big) \\ \\ \frac{\sqrt{\mu}}{2i}\Big(e^{-aL} - e^{-bL}\Big) & \frac{1}{2}\Big(e^{-aL} + e^{-bL}\Big) \end{bmatrix} . \begin{bmatrix} (2\sigma_1(k) + a + b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k) + a + b) \end{bmatrix}^{-1} \begin{bmatrix} (2\sigma_1(k) - a - b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k) - a - b) \end{bmatrix} \begin{bmatrix} \hat{e}_2^{n}(L,k) \\ \\ \hat{E}_2^{n}(L,k) \end{bmatrix} \tag{4.71}$$

Next, we re-write equation (4.70) and evaluate the result at $x_1 = L$

$$\begin{bmatrix} \hat{e}_2^{n}(L,k) \\ \\ \hat{E}_2^{n}(L,k) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Big(e^{-aL} + e^{-bL}\Big) & \frac{i}{2\sqrt{\mu}}\Big(e^{-aL} - e^{-bL}\Big) \\ \\ \frac{\sqrt{\mu}}{2i}\Big(e^{-aL} - e^{-bL}\Big) & \frac{1}{2}\Big(e^{-aL} + e^{-bL}\Big) \end{bmatrix} . \begin{bmatrix} (2\sigma_2(k) - a - b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k) - a - b) \end{bmatrix}^{-1} \begin{bmatrix} (2\sigma_2(k) + a + b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k) + a + b) \end{bmatrix} \begin{bmatrix} \hat{e}_1^{n-1}(0,k) \\ \\ \hat{E}_1^{n-1}(0,k) \end{bmatrix} \tag{4.72}$$

Therefore by inserting equation (4.71) into (4.72) we obtain

$$\begin{bmatrix} \hat{e}_1^{n+1}(0,k) \\ \\ \hat{E}_1^{n+1}(0,k) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2}\left(e^{-aL}+e^{-bL)}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-aL}-e^{-bL}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-aL}-e^{-bL}\right) & \frac{1}{2}\left(e^{-aL}+e^{-bL}\right) \end{bmatrix} \begin{bmatrix} (2\sigma_1(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)+a+b) \end{bmatrix}^{-1} \cdot$$

$$\begin{bmatrix} (2\sigma_1(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)-a-b) \end{bmatrix} \begin{bmatrix} \frac{1}{2}\left(e^{-aL}+e^{-bL}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-aL}-e^{-bL}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-aL}-e^{-bL}\right) & \frac{1}{2}\left(e^{-aL}+e^{-bL}\right) \end{bmatrix} \cdot$$

$$\begin{bmatrix} (2\sigma_2(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)-a-b) \end{bmatrix}^{-1} \begin{bmatrix} (2\sigma_2(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)+a+b) \end{bmatrix} \begin{bmatrix} \hat{e}_1^{n-1}(0,k) \\ \\ \hat{E}_1^{n-1}(0,k) \end{bmatrix}$$

Similarly we have;

$$\begin{bmatrix} \hat{e}_2^{n+1}(L,k) \\ \\ \hat{E}_2^{n+1}(L,k) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{2}\left(e^{-aL}+e^{-bL)}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-aL}-e^{-bL}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-aL}-e^{-bL}\right) & \frac{1}{2}\left(e^{-aL}+e^{-bL}\right) \end{bmatrix} \begin{bmatrix} (2\sigma_2(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)-a-b) \end{bmatrix}^{-1}.$$

$$\begin{bmatrix} (2\sigma_2(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_2(k)+a+b) \end{bmatrix} \begin{bmatrix} \frac{1}{2}\left(e^{-aL}+e^{-bL}\right) & \frac{i}{2\sqrt{\mu}}\left(e^{-aL}-e^{-bL}\right) \\ \\ \frac{\sqrt{\mu}}{2i}\left(e^{-aL}-e^{-bL}\right) & \frac{1}{2}\left(e^{-aL}+e^{-bL}\right) \end{bmatrix}.$$

$$\begin{bmatrix} (2\sigma_1(k)+a+b) & \frac{i}{\sqrt{\mu}}(a-b) \\ \\ (a-b) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)+a+b) \end{bmatrix}^{-1} \begin{bmatrix} (2\sigma_1(k)-a-b) & \frac{i}{\sqrt{\mu}}(b-a) \\ \\ (b-a) & \frac{i}{\sqrt{\mu}}(2\sigma_1(k)-a-b) \end{bmatrix} \begin{bmatrix} \hat{e}_2^{n-1}(L,k) \\ \\ \hat{E}_2^{n-1}(L,k) \end{bmatrix}$$

By using matlab, we computed the eigenvalues for the matrix that resulted from the multiplication of the six matrices involved in the above expressions as:

$$\exp(-\Re(bL))^2. \left| \frac{(b^2+\sigma_2 b-\sigma_1 b-\sigma_2\sigma_1)}{(-b^2+\sigma_2 b-\sigma_1 b+\sigma_2\sigma_1)} \right|$$

and

$$\exp(-\Re(aL))^2. \left| \frac{(a^2+\sigma_2 a-\sigma_1 a-\sigma_2\sigma_1)}{(a^2-\sigma_2 a+\sigma_1 a-\sigma_2\sigma_1)} \right|$$

From the six matrices involved in the last two matrix-vector relations, it can be seen that, all the operators σ_1, σ_2 are appearing in the leading diagonals in some of the matrices. We therefore make choices for these operators that will make the diagonals zeros. When we are able to realize these operators, we then consider low-frequency approximation.

$\sigma(k)$	Choice	Low-frequency approximation
σ_1	$\frac{1}{2}(a+b)$	$\sqrt{\dfrac{\sqrt{\eta^2+\frac{1}{\mu}}+\eta}{2}}$
σ_2	$-\frac{1}{2}(a+b)$	$-\sqrt{\dfrac{\sqrt{\eta^2+\frac{1}{\mu}}+\eta}{2}}$

Table 4.1:

These choices corresponds to a similar situation we had in the scalar elliptic problems. The approximation we considered was the zeroth order low-frequency approximations by setting $k = 0$ into the expressions σ_1 and σ_2. This was a very interesting result from the Fourier convergence analysis.

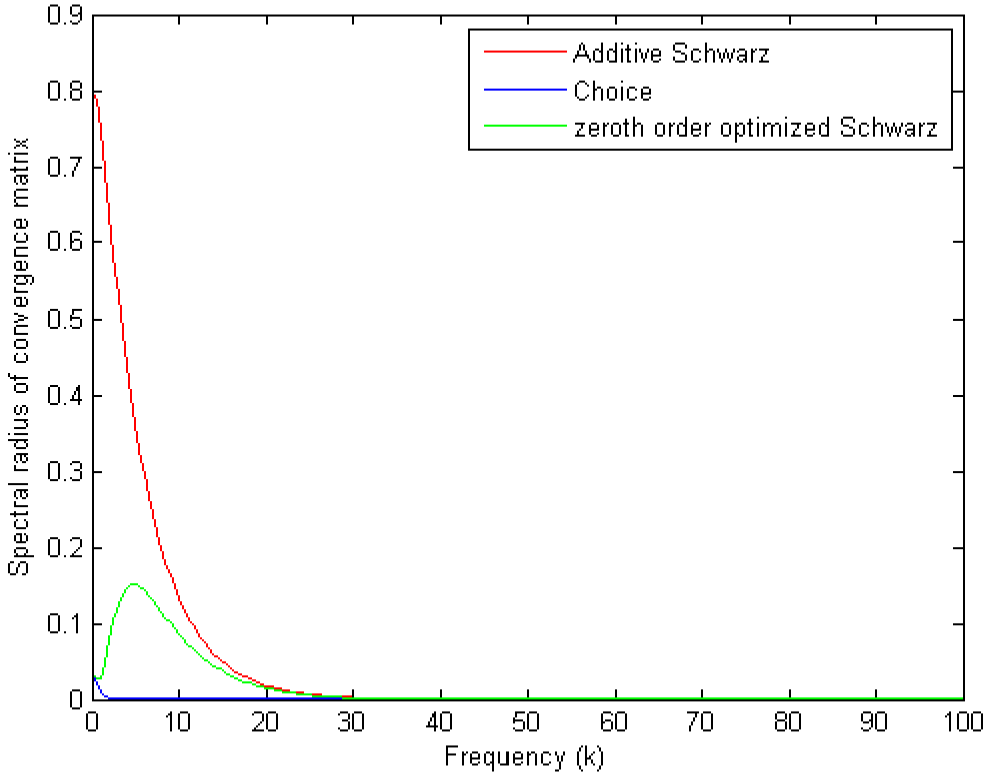

Figure 4.2: This graph show the convergence pattern for the optimized Schwarz algorithm.

From the graph, it can easily be seen that, the convergence behavior of the zeroth-order low-frequency approximation for the optimized Schwarz algorithm is better than the additive Schwarz algorithm.

4.2 Discretization and Numerical Experiments

In this section, we consider numerical experiments for the two algorithms discussed above. We use a finite difference discretization with the classical five-point discretization for the Laplacian on a uniform mesh with mesh parameter h. We solve the coupled optimality systems on the rectangular domain $\Omega = (-1, 1) \times (0, 1)$,

$$\begin{cases} -\Delta y + \eta y = f + \frac{1}{\mu} p, \quad \text{in } \Omega, \\ \\ y = g \text{ on } \Gamma, \\ \\ -\Delta p + \eta p = -(y - y_d), \quad \text{in } \Omega \\ p = 0 \text{ on } \Gamma. \end{cases} \tag{4.73}$$

We decompose the rectangular domain Ω into two subdomains $\Omega_1 = (-1, 0.1) \times (0, 1)$, $\Omega_2 = (0, 1) \times (0, 1)$, where 0.1 is the size of the overlap (ie:$L = 0.1$). We perform numerical simulations for the boundary value problems for the error, $f = 0,\ g = 0,\ y_d = 0$ and use a random initial guess so that all the frequency components are present.

In our numerical implementation, we adapted the same boundary conditions we had in chapter 3. The boundary value problems are similar, just that in this chapter the boundary problems are coupled. Therefore to avoid repetitions, we will only state the form of the linear systems.

4.2.1 Additive Schwarz and optimized Schwarz: Discretization

Since the boundary problems are coupled, the large linear systems to be solved result in a matrix form

1. **Discretization of subdomain Ω_1**

$$A_1 y_1 - \frac{1}{\mu} p_1 = b_1$$
$$A_1 p_1 + y_1 = b_2$$

$$\begin{bmatrix} A_1 & -\frac{1}{\mu} I \\ \\ I & A_1 \end{bmatrix} \begin{bmatrix} y_1 \\ \\ p_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ \\ b_2 \end{bmatrix}$$

2. **Discretization of subdomain Ω_2**

$$A_2 y_2 - \frac{1}{\mu} p_2 = b_1$$
$$A_2 p_2 + y_2 = b_2$$

$$\begin{bmatrix} A_2 & -\frac{1}{\mu}I \\ I & A_2 \end{bmatrix} \begin{bmatrix} y_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The matrices A_1 and A_2 are the same as the ones we obtained in chapter 3.

We will like to mention that, the discretization in the optimized Schwarz case is also of the same form as above, but now the interface boundary condition discretization we had in chapter 3 we will be employed.

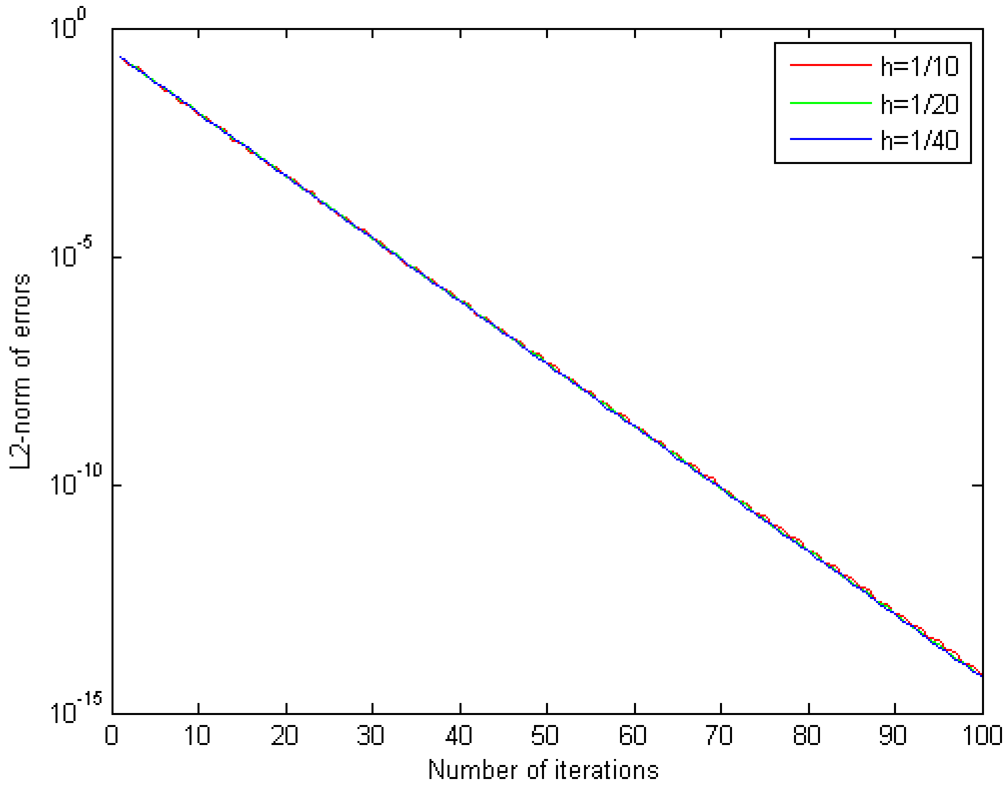

Figure 4.3: A plot of the errors for the additive Schwarz algorithm for three different mesh sizes with fixed overlap size.

We have again demonstrated that, the rate of convergence for the additive Schwarz algorithm is mesh independent.

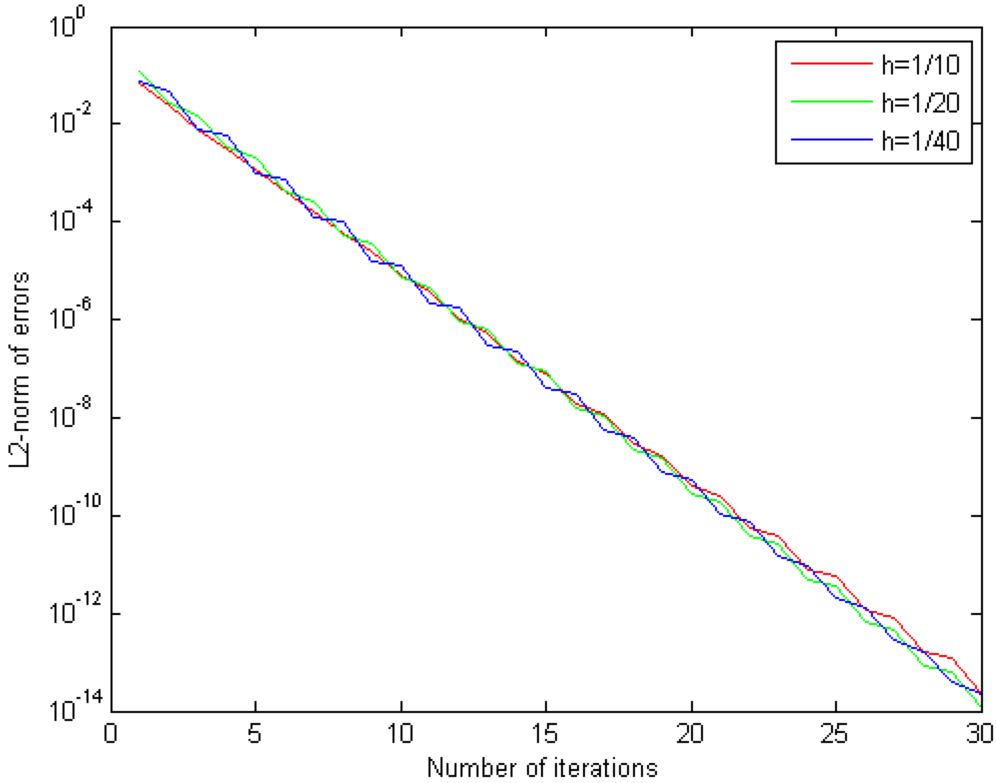

Figure 4.4: A plot of the errors for the optimized Schwarz algorithm for three different mesh sizes with fixed overlap size. The convergence pattern in this case is also mesh independent

From Figures 4.3 and 4.4 above, we can observed that, the optimized Schwarz algorithm converges faster than the additive Schwarz algorithm.

Chapter 5

Summary and Conclusion

This thesis provides a study of both additive Schwarz and optimized Schwarz overlapping domain decomposition methods applied to scalar elliptic problems and elliptic optimal control problems. We have presented a review of Fourier convergence analysis for the additive Schwarz methods and optimized Schwarz methods for the scalar elliptic boundary value problems.

We have made a tremendous extension of the Fourier convergence analysis in the scalar elliptic problems to elliptic optimal control problems. We have obtained a significant results in the optimal control problems. This new results will be very useful in the study of domain decomposition methods.

In conclusion, we have demonstrated that, Fourier convergence analysis for both algorithms is possible for the elliptic optimal control problems. Both our analytical and numerical simulations suggested that optimized Schwarz algorithm performs much better the additive Schwarz algorithm.

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