MASTER

Generalized Brauer algebras

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Generalized Brauer Algebras

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Chapter 1

Introduction

This thesis is the result of nearly a year of research under the supervision of dr. Arjeh Cohen. It began with the question of giving a neat description of the classical Brauer algebra in terms of the Coxeter group of type A_n, which is nothing but the symmetric group on $n+1$ symbols, and its accompanying root system. The purpose was to extend such a description to other simply laced, spherical Coxeter graphs. One of the reasons was to shed more light on the newly introduced BMW algebras of simply laced type. See for example [4] or [22] for the relation between the BMW algebra of type A and the classical Brauer algebra.

The BMW algebras of simply laced type are a generalization of an algebra introduced by Birman, Wenzl and Murakami. Their purpose was to give a description of the Kauffman link invariant in terms of a trace on an algebra. As such the original BMW algebras are strongly connected to knot theory in $\mathbb{R}^3$. The connection is outlined in the first chapter of this thesis. The BMW algebra of type D_n seems to have some connection to knot theory on a cylinder.

There is also a connection to the representation theory of Artin groups. In particular, the Lawrence-Krammer like representations of the Artin groups of simply laced type factor throught the BMW algebra of the same type. This was first shown by Zinno [29] for the original BMW algebra and later by Gijsbers [13], for the other types.

In the first chapter the construction of the original BMW algebra in terms of the Kauffman link invariant is outlined. It is fairly self contained, and begins with basic facts of knot theory. Proofs are usually omitted.

The second chapter deals with Coxeter groups. Some results are proved that are of importance for the description of the generalized Brauer algebra in the third chapter.

The third chapter is where the generalized Brauer algebras are defined. A representation of those algebras is constructed, and by means of the representation we arrive at an upper bound for the dimension of the generalized Brauer algebras. These upper bounds coincide with the upper bounds for the dimension of the BMW algebra, as in [13]. It is conjectured that the bounds are tight.

knot
link
isotopic

Chapter 2

Knot Theory

In this section we will not be concerned with exact technical details, but instead we try to give an overview of the application of the classical Brauer algebra in mathematical knot theory. The path from knots and links to the Brauer algebra is schematically depicted in Figure 2.1.

Links	$\longrightarrow$	Link Diagrams	$\longleftrightarrow$	Braids
		$\downarrow$		$\downarrow$
		Kauffman Polynomial	$\longleftrightarrow$	BMW Algebra
				$\downarrow$
				Brauer Algebra

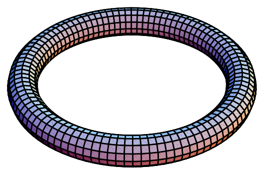
Figure 2.1: *The path from knots and links to the Brauer algebra.*

2.1 Links, knots and diagrams

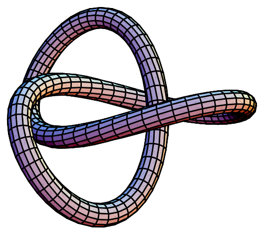
Knot theory is a branch of topology in which mathematics is applied to analyse pieces of twine whose ends are fused. It is of use in such diverse fields as quantum physics and molecular biology [28].

The primary objects of study in knot theory are knots and links. A *knot* is to be thought as a single piece of twine positioned in an ambient space of which the ends are fused together. A *link* is a finite collection of one or more knots lying in the same space. Continuous deformations of the ambient space can transform the link, but such deformations are considered non-essential. Two links that can be transformed into one another by a sequence of such deformations are called *isotopic*. Isotopy defines an equivalence relation on the collection of all links.

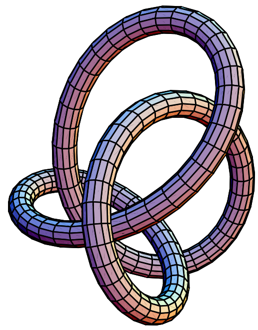
Isotopy of links and knots depends on the ambient space. Different choices of ambient space lead to different knot theories. A common choice is three-dimensional Euclidean space, denoted $\mathbb{E}^3$. Euclidean spaces of higher or lower dimension do not



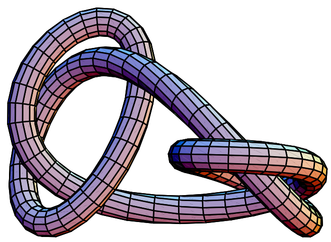
(a) Unknot



(b) Trefoil Knot



(c) Figure Eight Knot



(d) Granny Knot

Figure 2.2: *Knots in three-dimensional Euclidean space.*

lead to interesting knot theories. Let $m > 3$. Every link in $\mathbb{E}^m$ is equivalent to a collection of non-intertwined unknots. The proof of this fact consists of two parts. First, the link is repositioned by a deformation of $\mathbb{E}^m$ such that it is completely contained in a three dimensional subspace. One piece of strand can locally be lifted into the orthogonal complement of this three dimensional subspace, and as such be pulled through another piece of strand of the link. In this way the link can be untangled to form a collection of unknots.

If the ambient space is two-dimensional Euclidean space then every knot is equivalent to a circle. This follows from a theorem of Schoenflies on extensions of homeomorphisms from a Jordan curve to the circle. See for example [17].

In mathematical language, a knot is an embedding of the circle in the ambient space. Similarly, a link is an embedding of a disjoint union of circles in the ambient space. These embedding need to be either tame, smooth or piecewise linear. A tame embedding of a collection of closed curves is an embedding of those curves that extends to small tubes around them. In other words, links are characterised by the condition that they can be drawn as in Figure 2.2. Finally, an *oriented link* is a tame, smooth or piecewise linear embedding of a collection of oriented circles.

oriented link
unknot
trivial knot
knot invariant
invariant
knot
link diagram
transversal
over
under

A formal definition of isotopy would lead us too far afield, and is not very important for our purposes. Instead we refer the reader to [16] or [24]. It suffices to say that the isotopy class of a link L is denoted $[L]$.

The simplest link is called the *unknot*, which is also known as the *trivial knot*. It is the closed curve $\mathbb{S} \to \mathbb{E}^3$ parametrized by

$$\mathbf{x}(t) \mapsto (\cos t, \sin t, 0),$$

and is depicted in Figure 2.2(a). Given some complicated knot it is a natural question to ask whether it is isotopic to the unknot. In general this question cannot be easily solved by simply looking at a pictorial representation of the knot. Different methods have been proposed to solve this question and its generalization of determining the isotopy classes of knots. One such method is by defining a function on the set of knots that is constant on the isotopy classes. Such a function is called a *knot invariant* or simply an *invariant*.

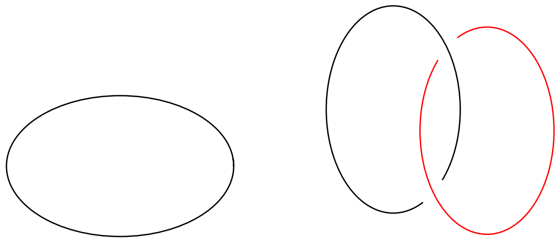
A knot invariant should be easy to calculate and distinguish as many non-isotopic knots as possible. Determining a complete invariant has been a difficult question which even now is not completely solved. One of the earliest examples of a knot invariant was defined by Alexander in [2]. It is now known as the Alexander-polynomial or Alexander-Conway polynomial. Other examples are the Jones polynomial [19], the HOMFLYPT polynomial [11], [25] and the Kauffman polynomial [21]. For a short historical overview of the subject see for example [4]. In subsequent sections we shall be concerned only with the latter three, since they give good examples of how to connect algebraic structures to knot theory. For reference purposes these polynomials and their notations are collected in Table 2.1. The last column lists the algebra connected with the invariant. It will be explained in Paragraph 2.2.

Notation	**Invariant**	**Algebra**
$V_L(t)$	Jones polynomial	Temperley-Lieb algebra
$P_L(l,m)$	HOMFLYPT polynomial	Hecke algebra
$K_L(l,m)$	Kauffman polynomial	BMW algebra

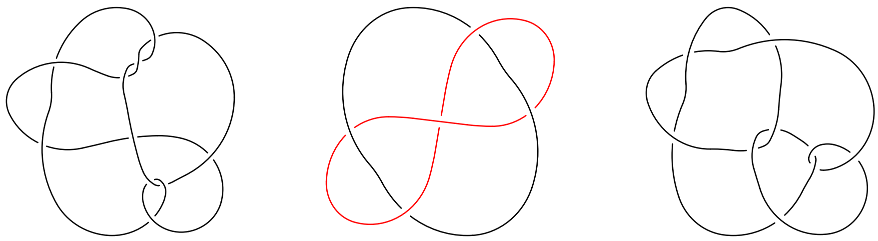
Table 2.1: *Knot invariants*

A common to way to represent a link or knot is by a planar diagram called a *knot*or *link diagram*. It is the projection of a knot or link on a suitably chosen plane. The projection has only a finite number of double points and these double points are *transversal*. A double point is assigned the label *over* or *under* depending on the position of the preimages of the crossing lines. Under- and over-crossing are depicted by erasing a small piece of the line near the double point. Examples of link diagrams are given in Figure 2.3.

One link can have many different link diagrams. For example both (a) and (d)



(a) Unknot (b) Intertwined unknots (c) Trefoil knot



(d) Millett unknot (e) Whitehead link (f) Another knot

Figure 2.3: *Link diagrams*

Reidemeister moves

equivalent

of Figure 2.3 are both representatives of the isotopy class of the unknot. Even for the same embedding of a link a different plane of projection results in a different link diagram. However, changing a link diagram according to the rules given in Figure 2.4 results in a link diagram of an equivalent knot. The rules are called the *Reidemeister moves* and are denoted R1, R2 and R3. In a sense it is surprising that these moves are sufficient; diagrams of isotopic knots can be changed into one another by applying a sequence of them. This is proved for example in [7]. It is called Reidemeister's Theorem and is given below. Let two link diagrams be called *equivalent* when there is a sequence of Reidemeister moves transforming one into the other. Then characterizing isotopy classes of links is the same as characterizing equivalence classes of link diagrams.

Theorem 2.1.1 (Reidemeister 1926) *Two links are isotopic if and only if their link diagrams can be transformed into one another by a sequence of Reidemeister moves* $R1$*,* $R2$ *and* $R3$*.*

The usual starting point from which to calculate link invariants are the oriented link diagrams. A link diagram is transformed into a weighted sum of other link diagrams by so called skein relations, until only diagrams of non-intertwined unknots remain. Specifying the value of the link invariant on the non-intertwined unknot diagrams then completely determines it. The skein relations depend on the specific invariant and are defined using the partial diagrams depicted in Figure 2.5. Let L denote a link (diagram). The interpretations of L_+, L_-, L_0 and L_∞ are as follows. Choose some crossing in L, and change it locally into one of the four partial diagrams of Figure 2.5. The resulting diagram is called L_-, L_+, L_0 or algebraic crossing number

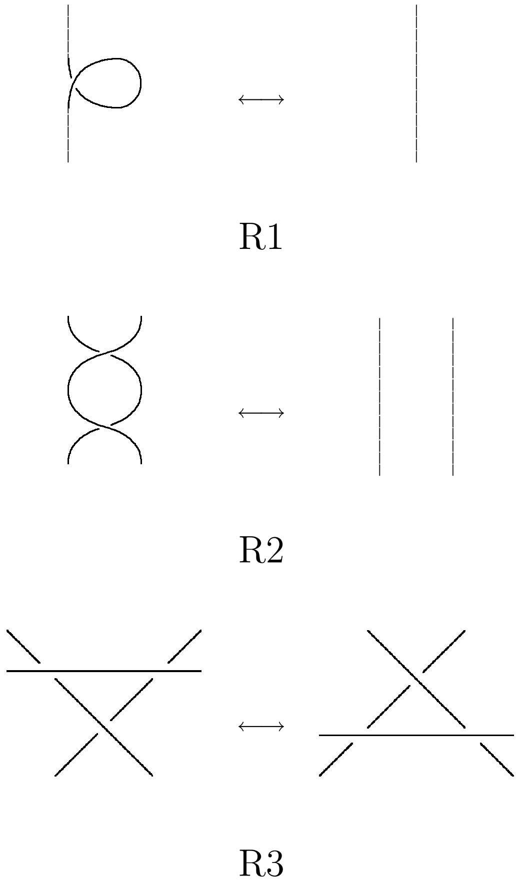


Figure 2.4: *The Reidemeister moves*

L_∞, according to the partial diagram chosen. The skein relation for the Jones and HOMFLYPT polynomials are given below. The symbol O denotes the unknot diagram. See [24, p. 13] for (2.1) and [4] for (2.2). Examples of calculations of the Jones polynomial using the skein relations below can be found for example in [24, page 9-12].

$$\begin{aligned} \frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) &= (\sqrt{t} + \frac{1}{\sqrt{t}})V_{L_0}(t), \\ V_O(t) &= 1. \qquad (2.1) \\ l^{-1}P_{L_+}(l,m) + lP_{L_-}(l,m) &= mP_{L_0}(l,m), \\ P_O(l,m) &= 1. \qquad (2.2) \end{aligned}$$

The skein relation for the Kauffman polynomial requires a short digression. We follow [4]. Let L be an oriented link represented as an oriented diagram D. The *algebraic crossing number* ε of D is defined as follows. Consider again the diagrams $-$ and $+$ as in Figure 2.5. Crossing that correspond to $-$ are assigned the value -1,

L-polynomial
braid
isotopy of braids

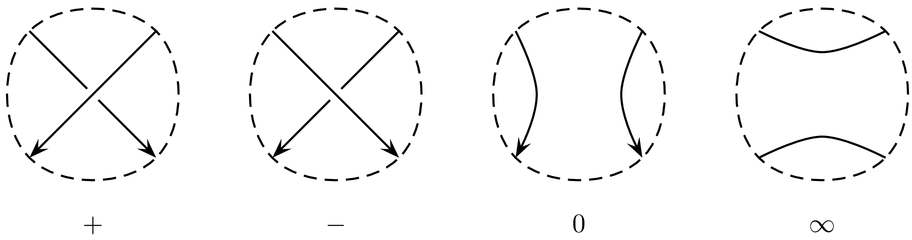


Figure 2.5: *Partial link diagrams*

while crossing that correspond to + are assigned the value 1. The algebraic crossing number of D is the sum of the values assigned to the crossings of D. Kauffman in [21] defines a precursor of the Kauffman polynomial $K_L(l, m)$ called the *L-polynomial*, here denoted as $\hat{K}_D(l, m)$. Its relation to the Kauffman polynomial is

$$\hat{K}_D(l, m) = l^{\varepsilon} K_L(l, m). \tag{2.3}$$

The skein relation of the L-polynomial is the following.

$$\begin{aligned} \hat{K}_{D_+}(l, m) + \hat{K}_{D_-}(l, m) &= m(\hat{K}_{D_0}(l, m) + \hat{K}_{D_\infty}(l, m)). \\ \hat{K}_O(l, m) &= 1 \end{aligned} \tag{2.4}$$

Moreover, if two diagrams can be transformed into one another by applying a sequence of the second and third Reidemeister moves, then their L-polynomials are equal. Applying the first Reidemeister move to a diagram multiplies the L-polynomial by a factor l or l^{-1}. These conditions completely determine $\hat{K}_D(l, m)$ on all oriented diagrams. By (2.3) the $K_L(l, m)$ is completely determined on all links. Without proof the following theorem is stated. Its proof can be found in [21].

Theorem 2.1.2 *The L-polynomial is an invariant on the collection of all link diagrams under sequences of R2 and R3 Reidemeister moves. The Kauffman polynomial is an invariant of links.*

2.2 Braids

A *braid* is a mathematical formalization of the everyday concept of a braid. Let $n \in \mathbb{N}$. Consider the union b of n curves that are embedded in $\mathbb{R}^2 \times [0, 1]$ and whose boundary ∂b consists of the points

$$\partial b = \{(1, 0), (2, 0), \ldots, (n, 0)\} \times \{0, 1\}.$$

Moreover, suppose that none of the curves have a critical point with respect to the third (vertical) coordinate. Then b is called a braid on n strands. Two braids on the same number of strands are called *isotopic* if they are related by an isotopy of $\mathbb{R}^2 \times [0, 1]$ that preserves the boundary and the vertical coordinate. Isotopy defines an equivalence relation on the set of braids, and as was the case with knots, two braids are considered the same if they are isotopic.

braid group

Let B_n denote the collection of isotopy classes of all braids on n strands. There is a composition defined on B_n. Let b and c braids on n strands, and let $[b]$ and $[c]$ denote their isotopy classes. Then the composition of $[b]$ and $[c]$, denoted by juxtaposition, is the isotopy class of the braid obtained by identifying the lower boundary points of b with the upper boundary points of c. Without proof it is stated that this indeed defines a composition that makes B_n into a group. It is called the *braid group* on n strands.

There is an algebraic characterization of the braid group B_n in terms of generators and relations. Then B_n is the group defined by the relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, && \text{if } |i-j| > 1, \\ \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j, && \text{if } |i-j| = 1. \end{aligned}$$

on the generators $\sigma_1, \ldots, \sigma_{n-1}$. These generators and their inverses are represented diagrammatically in Figure 2.6. In particular, it follows from this presentation of B_{n+1} that its subgroup generated by $\sigma_1, \ldots, \sigma_{n-1}$ is isomorphic to B_n. This subgroup consists of all isotopy classes of braids that have a representative for which $(1, 0, n)$ is connected to $(0, 0, n)$ by a straight line that is not intertwined with any of the other curves. Conversely, the map $\jmath_n$ that maps B_n into B_{n+1} by adding a straight line connecting $(1, 0, n)$ and $(0, 0, n)$ that is not intertwined with any of the other strands is a monomorphism. The subscript n is omitted from $\jmath_n$ when it does not lead to confusion.

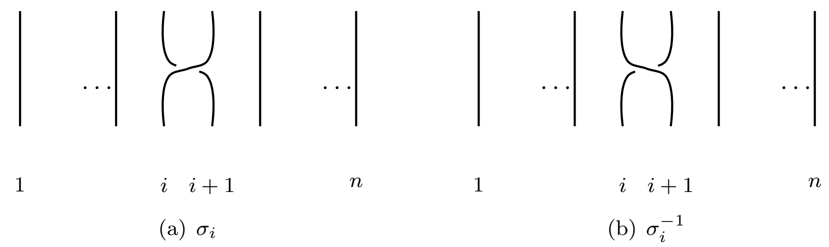


Figure 2.6: *Fundamental generators σ_i and their inverses for the braid group B_n*

Remark. The braid group on n strands is connected to the Coxeter diagram A_{n-1}, through the algebraic representation just given. After all, the generators are indexed by the vertices of A_{n-1}, while the relations depend on the edges of A_{n-1}. Generators σ_i and σ_j commute if and only if there is no edge in A_{n-1} between vertex i and j. Similarly, these generators do not commute when there is an edge between i and j. This observation allows for the construction by generators and relations of braid-like groups related to the other simply laced Coxeter diagrams. Such groups are commonly known as Artin groups, in honour of Emil Artin, who introduced the groups B_n. The Artin group of type A_{n-1} is isomorphic to B_n.

The collection of all isotopy classes of braids is denoted B_∞. It is the (disjoint) union of the individual braid groups, i.e., Markov moves

$$B_\infty = \coprod_{i=1}^{\infty} B_n. \tag{2.5}$$

There is an operation on B_∞ that maps an isotopy class of braids onto an isotopy class of links. Let b be a braid on n strands. The closure of b, denoted $\hat{b}$, is the link obtained from b by identifying $(1, 0, i)$ with $(0, 0, i)$, for each $i = 1, \ldots, n$. Taking into account the fact that closures of isotopic braids are isotopic links this indeed defines a function mapping elements of B_∞ to isotopy classes of links, by

$$[b] \mapsto [\hat{b}], \qquad [b] \in B_\infty. \tag{2.6}$$

A natural question to ask is whether this map is a surjection. It was proved by Alexander in 1923 [1] that indeed it is. Every link is isotopic to the closure of a braid. This is the first step in reducing the study of knots and links to the study of braids. The first of which is combinatorically and topologically oriented, while the second is more algebraic. The second step is characterizing the fibres of the closure map. It was taken by Markov around 1935 when he showed that certain equivalence classes, defined by so called *Markov moves*, are exactly the fibres of the closure map on B_∞. These moves are listed below.

M1 $\alpha\beta \leftrightarrow \beta\alpha$ for any $\alpha, \beta \in B_n$.

M2 $\imath_n(\beta)\sigma_n \leftrightarrow \beta \leftrightarrow \imath_n(\beta)\sigma_n^{-1}$ for any $\beta \in B_n$.

The final result is summarized in the next theorem, commonly known as Markov's Theorem. A short example follows to illustrate it.

Theorem 2.2.1 (Markov) *Let L be a link. There exists $\beta \in B_\infty$ such that L is isotopic to $\hat{\beta}$. Moreover, $\beta_1, \beta_2 \in B_\infty$ have isotopic closures if and only if they can be transformed into one another by a sequence of Markov moves.*

Example. Let L be the unknot and let α denote the isotopy class of the trivial braid on 1 strand. Clearly, $\hat{\alpha} = [L]$. According to Markov's Theorem it holds that the closure of $i_1(\alpha)\sigma_1$ is the isotopy class of the unknot. Now $i_1(\alpha)$ is simply the unity element of B_2. Thus $i_1(\alpha)\sigma_1 = \sigma_1$. The closure of σ_1 is seen to be in the isotopy class of L by applying the first Reidemeister move once.

The theorem gives another way to arrive at link invariants. A function that is invariant under the Markov moves lifts to an invariant of links by the closure operation. Vice versa, to every link invariant corresponds a function on B_∞ that is invariant under the Markov moves, as follows,

$$f(L) = f(\hat{\beta}) = g(\beta).$$

where f is an invariant of links, and g is the corresponding invariant on B_∞.

A method to construct functions on B_∞ invariant on Markov classes is by considering representations of the braid groups, as follows. Let (A_n) denote a sequence of algebras with unity indexed by the natural numbers, and assume that for each n there is a monomorphism $\imath_n : A_n \hookrightarrow A_{n+1}$. Denote the disjoint union of (A_n) by A_∞ and write $\imath$ for the function from A_∞ to itself defined by

$$\imath(a) = \imath_n(a), \qquad \text{when } a \in A_n.$$

For each $n \in \mathbb{N}$ let there be a homomorphism of groups $\kappa_n : B_n \to A_n^\times$. They determine a function κ on B_∞ to A_∞ by the expression.

$$\kappa(\beta) = \kappa_n(\beta), \qquad \text{when } \beta \in B_n.$$

Lifting of the braid to a braid group on more strands is required to factor through κ. The following relation is assumed to hold.

$$\imath \circ \kappa = \kappa \circ \jmath.$$

Denote the common basefield of (A_n) by $\mathbb{K}$. In the cases we consider $\mathbb{K}$ is a field of complex numbers. Suppose that each algebra admits a linear map $\phi_n : A_n \to \mathbb{K}$. Let the function determined on A_∞ by (ϕ_n) be denoted ϕ and suppose that it satisfies the following conditions.

M1' $\phi(ab) = \phi_n(ba)$ for all $n \in \mathbb{N}$ and $a, b \in A_n$.

M2' $\phi(\imath(a)\kappa_{n+1}(\sigma_n)) = \phi(a) = \phi(\imath(a)\kappa_{n+1}(\sigma_n^{-1}))$ for all $n \in \mathbb{N}$ and $a \in A_n$.

Then the composition $\phi \circ \kappa$ is constant on the Markov classes. By Theorem 2.2.1 it is a link invariant.

Of the three link invariants listed in Table 2.1 both the HOMFLYPT polynomial and the Jones polynomial were originally constructed by means of a representation on the braid group in a certain algebra. The last column lists the sequence of algebras, denoted (A_n) in our exposition.

Remark. Although the HOMFLYPT polynomial was indeed constructed by means of a representation of the braid group, in [11], it was also independently constructed around the same time using purely combinatorial methods in [25]. Because of this the HOMFLYPT polynomial is also known as the HOMFLY polynomial, but we follow Jones [20] in adding the PT to HOMFLY.

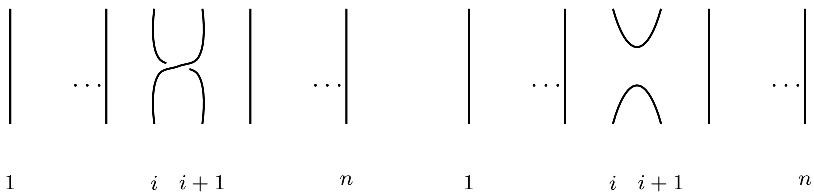
2.3 The BMW Algebra

The procedure outlined in the previous paragraph is applied to a specific sequence of algebras. These algebras were introduced independently by Birman and Wenzl [4] and Murakami [23] with the purpose of finding a representation of the braid group that would give rise to the Kauffman polynomial. They are henceforth known as Birman-Murakami-Wenzl algebras, or BMW algebras.

There are various definition of these algebras. The most convenient for our purposes is the one found in [13], since it is closest to the definition of the Brauer algebra in terms and generators and relations. Let l and m be complex numbers. Fix a natural number n. The BMW algebra of degree n is the algebra defined by the following relations on the generators $g_1, \ldots, g_{n-1}$ and $e_1, \ldots e_{n-1}$ over $\mathbb{C}$,

$$\begin{array}{rrcl} & g_i e_i & = & l^{-1} e_i \\ & m e_i & = & l(g_i^2 + m g_i - 1) \\ |i-j| > 1, & g_i g_j & = & g_j g_i \\ |i-j| = 1, & g_i g_j g_i & = & g_j g_i g_j \end{array}$$

It contains an identity element e. The algebra is denoted $\mathscr{C}_n(l,m)$. From the relations it follows that $\sigma_i \mapsto g_i$ defines a homomorphism of B_n into $\mathscr{C}_n(l,m)^\times$. This gives rise to a representation of the generators g_i as braid diagrams. The e_i are represented by diagrams as in Figure 2.7



(a) g_i (b) e_i

Figure 2.7: *Generators g_i and e_i for the BMW algebra $\mathscr{C}_n(l,m)$*

Multiplication of the generators by a fixed scalar leads to another set of generators for the algebra. The relations are appropriatly modified. If we set $G_i = \mathrm{i}\, g_i$ and $E_i = -e_i$ and consider $\mathscr{C}_n(l,m)$ as generated by the set consisting of G_i and E_i we arrive at the defining relations in [4], if l is replaced by $-\mathrm{i}\, l$. In particular, the following relation is of importance, since it expresses the skein relation (2.4) for the L-polynomial in the BMW algebra.

$$G_i + G_i^{-1} = m(1 + E_i). \tag{2.7}$$

Given a choice of ϕ, not all choices of κ lead to link invariants. If κ maps σ_i to G_i the resulting object is the L-polynomial, which is not an invariant of links. If instead κ maps σ_i to $l^{-1} G_i$ the resulting link invariant is the Kauffman polynomial.

It still is not clear whether functions $\phi_n : \mathscr{C}_n(l,m) \to \mathbb{C}$ exist for each natural number n, such that the compound function ϕ satisfies the conditions M1' and M2'. The following proposition affirms the existence of such functions, and gives a way to construct them. It is copied from [4]. Let z denote the complex number (whenever it exists),

$$z = \frac{m}{l + l^{-1} - m}.$$

and set $\imath$ to be the injection that maps a word in $\mathscr{C}_n(l,m)$ to the same word in $\mathscr{C}_{n+1}(l,m)$ for each natural number n.

Proposition 2.3.1 *Each $\mathscr{C}_n(l,m)$ supports a linear functional* $\mathrm{tr}_n : \mathscr{C}_n(l,m) \to \mathbb{C}$ *which is characterized by the following properties. Let $a, b \in \mathscr{C}_n(l,m)$.*

(i) $\mathrm{tr}_n(ab) = \mathrm{tr}_n(ba)$,

(ii) $\mathrm{tr}_{n+1}(\imath(a)(l^{-1}G_n)) = \mathrm{tr}_{n+1}(\imath(a)(lG_n^{-1})) = \mathrm{tr}_{n+1}(\imath(a)E_n) = z\,\mathrm{tr}_n(a)$.

Set $\kappa : \sigma_i \mapsto l^{-1}G_i$. The composition $\mathrm{tr} \circ \kappa$ is not expected to be a link invariant, since tr does not satisfy the the condition M2'. Set ϕ_n equal to $z^{-n+1}\,\mathrm{tr}_n$. Then the compound function ϕ does satisfy M2', and hence $\phi \circ \kappa$ is invariant on Markov classes. It corresponds to the Kauffman polynomial, as shown below.

The Kauffman polynomial was defined in terms of the L-polynomial and the algebraic crossing number ε. Let L be an oriented link that is the closure of an oriented braid b. Let β be the isotopy class of b and suppose that $\beta \in B_n$ for some natural number n. Write β as a product of generators,

$$\beta = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_k}$$

It is stated as a fact that the crossing number ε of $[L] = \hat{\beta}$ is $\varepsilon_1 + \dots + \varepsilon_r$. Set $\hat{\kappa} : \sigma_i \to G_i$. Then

$$\begin{aligned} \kappa(\beta) &= \kappa(\sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}) \\ &= (l^{-1}G_{i_1})^{\varepsilon_1} \dots (l^{-1}G_{i_k})^{\varepsilon_k} \\ &= l^{-\varepsilon} G_{i_1} \dots G_{i_k} \\ &= l^{-\varepsilon} \hat{\kappa}(\beta). \end{aligned}$$

By (2.3) it follows that $\phi(\kappa(\beta)) = K_L$ if and only if $\phi(\hat{\kappa}(\beta)) = \hat{K}_D$, where D is a link diagram of L.

Proposition 2.3.2 *It holds that $\phi(\hat{\kappa}(\beta)) = \hat{K}_D$. Thus, $\phi(\kappa(\beta)) = K_L$.*

Proof. It is sufficient to show that $\phi \circ \hat{\kappa}$ satisfies the defining properties of the L-polynomial. In particular, it needs to satisfy the relations expressed in (2.4).

- Let O be the planar circle diagram of the unknot. The trivial braid in B_1 closes to the circle and gets mapped to e by $\hat{\kappa}$. Hence, $\phi(\mathsf{e}) = \phi_1(\mathsf{e}) = z^0\,\mathrm{tr}_1(\mathsf{e}) = 1$. This shows that the first relation is satisfied.

- Choose a crossing of D and create the diagrams D_-, D_+, D_0 and D_∞ according to Figure 2.5. Choose $\beta \in B_n$ such that its closure is D_0. Then there is an $i = 1, \dots, n-1$ such that $\beta\sigma_i^{-1}$ closes to D_- and $\beta\sigma_i$ closes to D_+. By (2.7)

$$\begin{aligned} \phi(\hat{\kappa}(\beta\sigma_i)) + \phi(\hat{\kappa}(\beta\sigma_i^{-1})) &= \phi(\hat{\kappa}(\beta)(G_i + G_i^{-1})) \\ &= \phi(\hat{\kappa}(\beta)(m(1 + E_i))) \\ &= m(\phi(\hat{\kappa}(\beta)) + \phi(\hat{\kappa}(\beta)E_i)). \end{aligned}$$

The last term stands for D_∞ as can be deduced from the diagram of E_i.

This establishes the proposition. □

Coxeter matrix
Coxeter group
Coxeter generators
irreducible
Coxeter graph

Chapter 3

Coxeter groups

In the first section relevant definitions are given and some well known facts on Coxeter groups are stated. The main reference is *Reflection groups and Coxeter groups* by James E. Humphreys [18]. In the second section admissible sets are defined, and some results on their structure are derived. The third section is concerned with distinguishing a canonical set of coset representatives of a reflection subgroup. The results in that section were obtained by Matthew Dyer in *Subgroups of Coxeter systems* [10]. Finally, the last section combines the results such that they become applicable in the last chapter. In it a characterization of the normalizer of an admissible set is given.

3.1 Preliminaries

Definitions, basic results and examples

Let S be a finite set. A *Coxeter matrix* is a symmetric function $M : S^2 \to \mathbb{N}_\infty$ that satisfies $M(s,s) = 1$ and $M(s,t) \neq 1$, for all s and t in S. The rank of M is defined as the size of S. Let S^* denote the free group with generators S. Let $S^*(M)$ be defined as the minimal normal subgroup of S^* generated by all words of the form

$$(st)^{M(s,t)}, \qquad s,t \in S. \tag{3.1}$$

It is a fact that the quotient of S^* by $S^*(M)$ is a group. This group is denoted $W(M)$ and called the *Coxeter group* of type M. The set S is known as the set of *Coxeter generators* of $W(M)$ and the pair consisting of $W(M)$ and S is called the Coxeter system of type M. Note that the Coxeter generators are involutions, since $M(s,s) = 1$. The rank of $W(M)$ is defined as the rank of M. A Coxeter group is said to be *irreducible* if it cannot be decomposed as a direct product of Coxeter groups, besides the trivial decomposition.

With each Coxeter matrix M there is associated a unique undirected labeled graph called the *Coxeter graph* of M. It is defined by taking S to be the set of nodes, and the collection of all pairs (s,t) satisfying $M(s,t) > 2$ to be the set of edges. The fact that this graph is indeed undirected follows from the symmetry of

M. In the representation of the Coxeter graph as a diagram the label 3 is often omitted. Conversely, unlabeled edges are understood to have label 3.

simply laced
spherical
Coxeter group
Coxeter system

Usually there is no distinction made between a Coxeter matrix and its corresponding graph. We will for example speak of the vertices and edges of M. In this context, the relation of being end vertices of the same edge is denoted by $\sim$, i.e. the fact that i and j are vertices of M that are connected by an edge in M would be expressed as $i \sim j$.

A Coxeter graph M is said to be *simply laced*, if and only if all of its edges are unlabeled (have label 3). Moreover, it is said to be *spherical* if the Coxeter group of type M is finite. It is said to be irreducible if it is connected. A Coxeter group is irreducible if and only if its Coxeter graph is irreducible [18, Section 6.1]. By the classification of finite, irreducible Coxeter groups it follows that the only irreducible types M that are simply laced and spherical are A_m, for $m \in \mathbb{N}$, D_m, for $m \geq 4$, E_6, E_7 and E_8. The vertices of these graphs are labeled as in Figure 3.1. Details on the classification are found in Chapter 2 of [18].

Remark. The term *spherical* for a Coxeter graph of a finite Coxeter group comes from the fact that a Coxeter group is finite if and only if it isomorphic to a group generated by the reflections in the bounding hyperplanes of a spherical simplex. See page 122 of [9]. A hint of the proof is given in the part on the geometric representation.

Example. Let $M = A_2$. Then M is simply laced and spherical. Indeed $W(M)$ is finite. Let s_1 and s_2 be the elements indexed by the vertices of M. Then $s_1 \sim s_2$. Thus, $W(M)$ is the group generated by the involution s_1 and s_2 that satisfy the relation $(s_1 s_2)^3 = 1$. This relation is equivalent to

$$s_1 s_2 s_1 = s_2 s_1 s_2.$$

Hence $W(M)$ consists of the elements 1, s_1, s_2, $s_1 s_2$, $s_2 s_1$ and $s_1 s_2 s_1$.

A *Coxeter group* is a group W that is isomorphic to the Coxeter group of a certain type M. Note that the type need not be determined by W. A *Coxeter system* is a Coxeter group W paired with an isomorphism $W(M) \to W$. It is usually represented as a pair $(W, \phi(S))$, where ϕ is the aforementioned isomorphism. The rank is the size of S and the type is M. These definitions coincide with the previous ones in the case that $W = W(M)$ and $\phi : w \mapsto w$.

Examples.

- The symmetric group on three symbols, denoted as Σ_3 is a Coxeter group. It suffices to determine the image of $(1,2)$ and $(2,3)$ under the isomorphism, since they form a set of generators for Σ_3. Set

$$(1,2) \mapsto s_1 \text{ and } (2,3) \mapsto s_2.$$

Indeed, $s_1^2 = s_2^2 = 1$ and $(s_1 s_2)^3 = 1$. These relations suffice. Hence, $(\Sigma_3, \{(1,2),(2,3)\})$ is a Coxeter system of type A_2. In general, Σ_m, paired

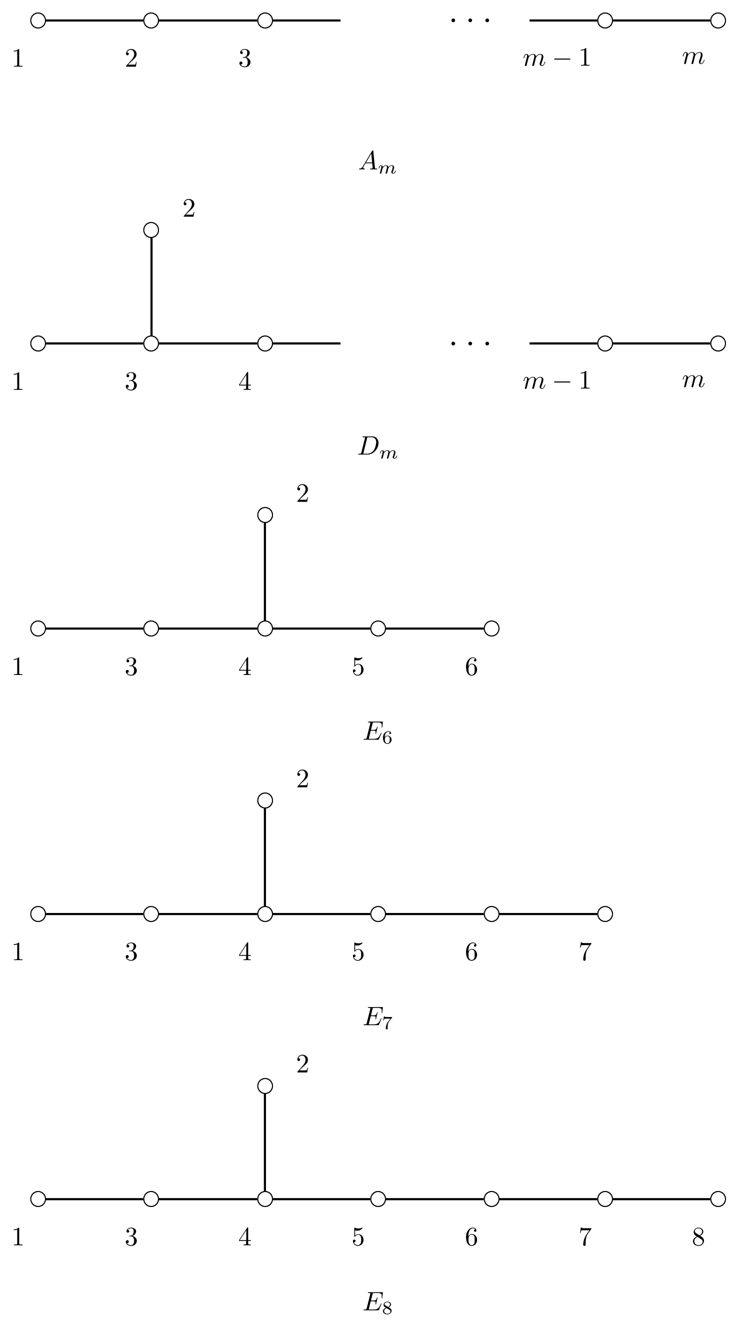


Figure 3.1: *Irreducible Coxeter graphs of simply laced, spherical type*

with the set of all transpositions interchanging i and $i+1$, is a Coxeter system of type A_{m-1}.

conjugation
conjugate
reflection
reflection subgroup
length function

- Consider the group consisting of all orthogonal transformations fixing a regular m-sided polygon centered at the origin. It is called the dihedral group of order $2m$ and is often denoted as $\mathcal{D}_m$ (notation $\mathcal{D}_{2m}$ is sometimes seen as well). It is generated by two reflections in lines that meet at an angle of π/m. Write r and t for these reflections. Obviously, $r^2 = t^2 = 1$. Since rt is a rotation about an angle of size $2\pi/m$ the order of rt is m. Again, $(\mathcal{D}_m, \{r,t\})$ is a Coxeter system. The graph consisting of two vertices connected by an edge labeled m is its Coxeter graph. This graph is usually called $I_2(m)$. It is not simply laced, unless $m = 3$ or $m = 2$. Note that $I_2(3) = A_2$ and $I_2(2) = A_1^2$.

There are groups that are not Coxeter groups. The smallest example is the cyclic group of order 3, since the group of order 2 is a Coxeter group (it is of type A_1) and the fact that it is generated by a single element of order 3. The braid group introduced in the introduction is also not a Coxeter group.

From this point onwards let (W,S) denote a Coxeter system of type M. Let w and v be elements of W. The *conjugation* of w by v, denoted $w * v$ is defined by

$$w * v = v^{-1}wv. \tag{3.2}$$

It is known that this defines a right action of W on itself. Elements that are in the same orbit under this action are called *conjugate*.

A *reflection* of (W,S) is an element of W that is conjugate to an element of S. Note that reflections are always involutions, since the elements of S are. The set of all reflections is denoted $\mathcal{R}(W,S)$. A *reflection subgroup* is a subgroup V of W that is generated by the reflections it contains, i.e.

$$V = \langle V \cap \mathcal{R}(W,S) \rangle. \tag{3.3}$$

Being a reflection subgroup depends not only on W but also on S, and hence it is only appropriate to speak of reflection subgroups of Coxeter systems. Not all subgroups of W are reflection subgroups of (W,S), as the following example shows. Any reflection subgroup is also a Coxeter group, as is proved in the next section.

Example. Consider the Coxeter group $W(A_3)$. Denote the elements of S by s_1, s_2 and s_3, corresponding to the common labeling of the vertices of A_3. Then the group generated by s_1s_3 is not a reflection subgroup of $W(A_3)$. After all, it does not contain any reflections.

The *length function* on (W,S), usually denoted ℓ_S (or simply ℓ, when reference to S is superfluous), assigns to each group element the length of the shortest word over S representing it, i.e.

$$\ell_S(w) = \min\{n \in \mathbb{N} : w \in S^n\}, \qquad w \in W. \tag{3.4}$$

With the convention $\min \emptyset = \infty$, it is well defined as a function from W to $\mathbb{N}_\infty$. A word $s_1 \dots s_k$ over S is called reduced if $\ell_S(s_1 \dots s_k) = k$. Note that there may be different reduced words representing the same element of W.

root system
perpendicular
Weyl group
geometric representation
positive root

The geometric representation of a finite Coxeter group

In this section some facts on root systems are recollected, in particular the fact that the Weyl groups of crystallographic root systems are finite Coxeter groups. For a more thorough treatment we refer the reader to [18] or [12].

Fix a Coxeter system (W, S) of type M. Associate with each element of $s \in S$ a symbol α_s and define $\Delta(W, S)$, or simply Δ, to be the set consisting of all α_s, where s runs over S. Let E denote the vector space freely generated by Δ over $\mathbb{R}$. Define a symmetric bilinear form B on E by the following expression.

$$B(\alpha_s, \alpha_{s'}) = -\cos \frac{\pi}{m(s, s')}, \qquad s, s' \in S.$$

The bilinear form is positive definite (and hence an inner product) if and only if the Coxeter group W is finite [18, Section 6.4]. For the rest of the paragraph assume that this is the case. Instead of $B(v, v') = 0$ we shall also write $v \perp v'$ and say that v is *perpendicular* to v'. The subspace consisting of all elements of E perpendicular to v is denoted $v^\perp$.

Let $v \in E$. The reflection corresponding to v is the involution that maps v to $-v$ and fixes $v^\perp$. Denote it by r_v. It is given by the expression

$$x \cdot r_v = x - \frac{2B(x, v)}{B(v, v)} v, \qquad x \in E.$$

Denote by $\mathcal{W}(\Delta)$ the group generated by the reflections with corresponding vectors in Δ. The root system of (W, S) is the closure of Δ under $\mathcal{W}(\Delta)$. It is usually denoted Φ, and we shall do so for the rest of this paragraph. Its elements are called roots. Without proof it is stated that $\mathcal{W}(\Delta) = \mathcal{W}(\Phi)$. The group $\mathcal{W}(\Phi)$ is known as the *Weyl group* of Φ.

The importance of this construction is that it defines a faithful representation σ of W on E, by

$$\sigma(s) = r_{\alpha_s}, \qquad s \in S.$$

For proof of this result see [18, Section 5.3]. From now on we shall simply equate W with $\sigma(W)$. By the *geometric representation* we mean σ.

There is an interpretation of the length function on (W, S) that shall be explored in Section 3.3. For now it suffices to state the following result. A root is called *positive* if it is contained $\mathbb{R}^+\Delta$. It is called negative if its additive inverse is positive. It is a fact that every root is either positive or negative. Denote the positive roots of Φ by Φ^+. Then it holds that

$$\Phi = \Phi^+ \cup -\Phi^+.$$

Moreover, since $0 \notin \Phi$ obviously $\Phi^+ \cap -\Phi^+ = \emptyset$. Now let ℓ denote the length function of (W, S). It holds that

scaled inner product

$$\ell(sw) > \ell(w) \text{ if and only if } \alpha_s \cdot w > 0.$$

See for example [18, Section 5.4].

If (W, S) is simply laced all roots in Φ have the same length. Moreover, then $W = \mathcal{W}(\Phi)$ acts transitively on Φ. This is an important property of the root system of a simply laced Coxeter group. Since in this thesis the only Coxeter groups under consideration are those of simply laced type we make the following convention. Let the *scaled inner product* (β, β') of $\beta, \beta' \in \Phi$ be defined by

$$(\beta, \beta') = \frac{2B(\beta, \beta')}{B(\beta, \beta')}.$$

The only possible values of scaled inner product are -2, 1, 0, 1 and 2, since B comes from a Coxeter group of simply laced type. Moreover, $(\beta, \beta') = 2$ if and only if $\beta = \beta'$.

Under the geometric representation every reflection $r \in \mathcal{R}(W, S)$ has a unique corresponding positive root. If $r = s \cdot w$, for some $s \in S$ and $w \in W$ the this root is is either $\alpha_s \cdot w$ or $-\alpha_s \cdot w$, depending on which is positive. These are the only two roots corresponding to r. By α_r is meant the positive root corresponding to r. If $X \subset \mathcal{R}(W, S)$ then Φ_X is the collection of all roots corresponding to reflections in X. Note that $\Phi_X = -\Phi_X$ and that the size of Φ_X is twice the size of X.

Remark. The centered dot denotes both conjugation in W as well as the action of $\mathcal{W}(\Phi)$ on Φ. This may seem confusing, but in a sense, these actions coincide. Let r be a reflection in (W, S) with corresponding root β, .i.e., $r = r_\beta$. Then for each $w \in W$ it holds that $r_\beta \cdot w = r_{\beta \cdot w}$. See for example Section 1.2 of [18].

Examples. We give the root systems obtained from the Coxeter systems of type A_m and D_m (cf. page 41 of [18]). Denote the standard basis vectors of $\mathbb{R}^n$ by ε_i.

- $\mathbf{A_m}$. Let V be the hyperplane of $\mathbb{R}^{m+1}$ consisting of vectors whose coordinated add up to 0. Write α_i for α_{s_i} and set

  $$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \qquad i = 1, \dots m.$$

  Then $\alpha_i \in V$ and the angle between α_i and α_{i+1} is indeed $\pi/3$. The entire root system Φ consists of the vectors $\varepsilon_i - \varepsilon_j$. A root is negative if and only if $i > j$. Note that the reflection in $\varepsilon_i - \varepsilon_j$ corresponds to the transposition (i, j) under the geometric representation.

- $\mathbf{D_m}$. Set $\alpha_1 = \varepsilon_1 + \varepsilon_2$ and

  $$\alpha_i = \varepsilon_{i-1} - \varepsilon_i,$$

  when $i = 2, \dots, m$. The elements of Φ are of the form $\pm\varepsilon_i \pm \varepsilon_j$, where $i \neq j$. The positive roots are $\varepsilon_i + \varepsilon_j$ and $\varepsilon_i - \varepsilon_j$, for $i < j$.

There are two lemmas on simply laced root systems and the geometric representation that wel shall use later on. They are stated and proved next.

Lemma 3.1.1 *Let M be simply laced and let β_1, β_2 and β_3 form a pairwise orthogonal set of roots in Φ. Suppose there exists another root that orthogonal to neither β_1, β_2 nor β_3. Fix some $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ with ζ_i either -1 or 1, for $i = 1, 2, 3$. Then there is a unique root γ in Φ that satisfies*

$$\begin{aligned}(\gamma, \beta_1) &= \zeta_1,\\ (\gamma, \beta_2) &= \zeta_2,\\ (\gamma, \beta_3) &= \zeta_3.\end{aligned}$$

Proof. Let γ' be a root connecting β_1, β_2 and β_3. Let I be the set of indices i satisfying $(\gamma, \beta_i) \neq \zeta_i$. Let w be the product of reflections with corresponding roots β_i, where i runs over I. Since β_1, β_2 and β_3 are assumed to pairwise orthogonal w is an involution. Let j be either 1, 2 or 3. Then

$$\begin{aligned}(\gamma, \beta_j) &= (\gamma' \cdot w, \beta_j)\\ &= (\gamma', \beta_j \cdot w).\end{aligned}$$

If j was not in I, then w fixes β_j and the inner product is ζ_i by definition of γ'. If however j was in I, then $\beta_j \cdot w = -\beta_j$. Again, by definition of γ' the inner product is ζ_i.

We proceed to show unicity. Suppose $\delta \in \Phi$ is another root having inner product ζ_i with each of the three roots β_1, β_2 and β_3. Since $\gamma - \zeta_1\beta_1 - \zeta_2\beta_2 - \zeta_3\beta_3$ is a root in a root system of simply laced type it holds that

$$\begin{aligned}(\gamma - \zeta_1\beta_1 - \zeta_2\beta_2 - \zeta_3\beta_3, \delta) &= (\gamma, \delta) - \zeta_1(\beta_1, \delta) - \zeta_2(\beta_2, \delta) - \zeta_3(\beta_3, \delta)\\ &= (\gamma, \delta) - 3\\ &= -5, -4, -3, -2, -1.\end{aligned}$$

The only possible value for (γ, δ) is 2. Hence $\gamma = \delta$. □

Lemma 3.1.2 *Let (W, S) be an irreducible Coxeter system, and let $\beta_1, \beta_2 \in \Phi$ such that $\beta_1 \neq -\beta_2$. Then there is a root $\gamma \in \Phi$ such that $\gamma \not\perp \beta_1$ and $\gamma \not\perp \beta_2$.*

3.2 Admissibility

This section is concerned with the notion of admissibility, as introduced in the third chapter of [13]. There the main purpose is to give a condition for the orbit poset to have a top element. We will not be very concerned with the poset structure, however. Instead the focus is on characterizing the smallest admissible set that contains a given set, the so called admissible closure. Admissible closures have a rather important role in Chapter 4 on generalized Brauer algebras. In particular, it will be shown that essentialy the only orbits one needs to consider in the generalized Brauer algebra are those that are admissible.

In this section W is a Coxeter group and $S \subseteq W$ is a set of fundamental generators for W such that (W, S) is a Coxeter system of type M. For many of the propositions in the second part it is required that M is simply laced and spherical. Remember that $\mathcal{R}(W, S)$ is the set of reflections of (W, S), that is, $\mathcal{R}(W, S)$ consists of the elements of W that are conjugate to a fundamental generator.

commuting set
admissible set

Definitions and basic results

A subset of W is said to be *commuting* if its elements commute pairwise. For our purposes it will be enough to consider commuting sets of reflections. By means of the geometric representation these correspond bijectively to orthogonal sets of positive roots, and their negatives. This fact will be used in many of the proofs in this section. The reason we consider sets of reflections instead of roots is that the sign of a root is not important for our purposes.

By $\mathcal{X}(W, S)$ is meant the collection of all commuting sets of reflections of (W, S), i.e.,

$$\mathcal{X}(W, S) = \{X \subseteq \mathcal{R}(W, S) \mid X \text{ is commuting}\}. \tag{3.5}$$

The conjugation action of W on itself extends to $\mathcal{X}(W, S)$, since the conjugate of a reflection is again a reflection and the conjugates of commuting reflections again commute. The orbit of $X \in \mathcal{X}(W, S)$ under the conjugation action by some subgroup $V \leq W$ is denoted $\Omega_V(X)$. Usually $V = W$; then, instead of $\Omega_W(X)$ we write $\Omega(X)$ if there is no danger of confusion.

The collection $\mathcal{X}(W, S)$ is a poset under the inclusion order. Its maximal elements are the sets X for which there are no reflections of (W, S) that commute with each element of X. It shall later be seen that the maximal elements are exactly those sets that coincide with their centralizer.

In the article [8] another partial order is defined on $\mathcal{X}(W, S)$. It will here be denoted $\preceq$. Its definition is repeated here, but can be found in the proof of Proposition 3.3.1. of the aforementioned reference. Let X and Y be elements of $\mathcal{X}(W, S)$. Then $X \preceq Y$ if either $X = Y$ or

$$\min\{\ell(r) \mid r \in X \backslash Y\} < \min\{\ell(r) \mid r \in Y \backslash X\}. \tag{3.6}$$

It will not be proved here that this indeed defines a partial order on (W, S). Instead we refer to chapter 3 of [13].

An important class of sets are those satisfying the property of admissibility that was introduced in [13]. The definition of this property is stated below. However, it is different from that in the earlier mentioned source, where it appears as a property equivalent to admissibility (cf. Definition 3.2.2 and Proposition 3.2.4(ii)). Also, in [13] admissibility is a property of an orbit of sets, and not of a single set. It will later be shown that the definitions coincide, i.e. that an orbit is admissible if and only if one of its elements is.

Definition 3.2.1 A set X in $\mathcal{X}(W, S)$ is called *admissible* when for all pairs t_1, t_2 of commuting reflections of (W, S) it holds that if conjugation by $t_1 t_2$ moves an element of X to another element of X then t_1t_2 is in the normalizer of X. admissible orbit

A subset of $\mathcal{X}(W,S)$ is said to be *admissible* if each of its elements is. admissible closure

The collection of all admissible sets in $\mathcal{X}(W,S)$ is denoted $\mathcal{A}(W,S)$. Note that this notation is similar to that for the Artin system corresponding to (W,S). This should not lead to confusion, since we are not concerned with Artin systems here. It may not be a priori clear whether $\mathcal{X}(W,S) = \mathcal{A}(W,S)$. The following example shows that this is not the case in general. derived set

Example. Let $M = D_4$ and set $X = \{s_1, s_2, s_4\}$. It is shown that X is not admissible. Set $t_1 = s_3s_1s_3$ and $t_2 = s_3s_2s_3$. Then r and t commute and conjugation by their product interchanges s_1 and s_2. Then $s_4 \cdot t_1t_2 = s_4 \cdot s_3s_1s_2s_3$. This is the reflection corresponding to the positive root $\alpha_1+\alpha_2+2\alpha_3+\alpha_4$. In particular it is not equal to s_4 and that shows that t_1t_2 is not in the normalizer of X, and hence that X is not admissible.

Let $\Omega(X)$ denote the orbit of some set $X \in \mathcal{X}(W,S)$ under the conjugation action by W. The poset $\Omega(X)$ under the order relation $\leq$ defined by (3.6) is known to have a top element if it is admissible. This was proved in Gijsbers' thesis [13], and the proof shall not be repeated here. The following proposition is mainly for ease of reference later on.

Proposition 3.2.1 *Let M be simply laced and spherical. If Ω is an admissible orbit in $\mathcal{X}(W,S)$ under conjugation by W then it has a top element with respect to the $\leq$ order.*

Proof. This is Corollary 3.3.6. of [13]. It will not be proved here. □

The next lemma shows that for each $X \in \mathcal{X}(W,S)$ there is a unique admissible set that contains X and is minimal with respect to the inclusion order. It is called the *admissible closure* of X and denoted X^c. The operator $X \mapsto X^c$ is increasing by definition. Define the *derived set* of X, denoted X', by

$$X' = X^c \backslash X. \tag{3.7}$$

Note that a set is admissible if and only if its derived set is empty.

Lemma 3.2.2 *The collection $\mathcal{A}(W,S)$ is closed under intersection of sets. Moreover, it contains every maximal element, with respect to the inclusion order, of $\mathcal{X}(W,S)$.*

Proof. Let X and Y be admissible sets in $\mathcal{X}(W,S)$. Consider their intersection $X\cap Y$. Suppose that there exist commuting reflection t_1 and t_2 of (W,S) such that there is some element of $X\cap Y$ that gets mapped to another element of $X\cap Y$ under conjugation by t_1t_2. Since X and Y are admissible it holds that t_1t_2 is an element of the intersections of the normalizer of X and the normalizer of Y. Let $r \in X\cap Y$. Then $r\cdot t_1t_2 \in X\cap Y$. It follows that t_1t_2 is in the normalizer of $X\cap Y$, and hence that $X\cap Y$ is admissible.

The last statement follows from part (iv) of Proposition 3.2.4. of [13] and inspection of Table 2 of that same source. □

connected reflections
connecting reflection
connected roots
companion reflection

Example. Let $M = E_6$ and put $X = \{s_2, s_3, s_5\}$. Then X is not admissible. This follows from the same argument as used in the previous example or from Table 2 of [13]. Note that the set $Y = \{s_2, s_3, s_5, s_4 \cdot s_5 s_2 s_3 s_4\}$ is admissible. In the inclusion order Y covers X, i.e., there is no element Z of $\mathcal{X}(W,S)$ satisfying $X \subset Z \subset Y$. Hence, Y is the admissible closure of X.

Characterization of the admissible closure

This paragraph is devoted to giving an intrinsic characterization of the admissible closure of a set $X \in \mathcal{X}(W,S)$. This is done by means of a companion function, that adds to each triple of reflections connected by some other reflection a fourth reflection (cf. Proposition 3.2.4. of [13]). Most of the work is done in the proof of Lemma 3.2.4, while the main result is stated in Theorem 3.2.6.

Some of the proofs in this section, mainly that of lemmas 3.2.3 and 3.2.4 make heavy use of the geometric representation of the Coxeter group W. A root system Φ is fixed such that the Weyl group of Φ is isomorphic to W. Where manipulation of roots is necessary there is no distinction made between the Weyl group of Φ and the Coxeter group W. For example, for a reflection r of (W,S) we shall speak of roots corresponding to r. Note that in lemmas 3.2.3 and 3.2.4 introduce the requirement that M is simply laced. For the conclusion of the latter lemma to hold it is even required that M is spherical, and hence that Φ (and W) is finite. Both requirements are also premisses for Proposition 3.2.5 and Theorem 3.2.6

Consider the pairwise commuting triple of reflections r_1, r_2 and r_3. They are said to be *connected* if there exists another reflection that commutes with neither. This reflection is then said to *connect* the triple. In terms of the root system Φ it means that there exists a root γ in Φ such that the roots corresponding to r_1, r_2 and r_3 are non-perpendicular to γ. Similarly, a triple of orthogonal roots is said to be *connected* if there is a root that is non-perpendicular to every root in the triple.

A reflection r of (W,S) is said to be a *companion reflection* of r_1, r_2 and r_3 if there exists a reflection t connecting r_1, r_2 and r_3 such that

$$r = r_i * (t r_j r_k t) \tag{3.8}$$

for some choice of different indices i, j and k. If such a companion reflection exists it is unique, as the following lemma shows.

Lemma 3.2.3 *Suppose that M is simply laced. Let r_1, r_2 and r_3 be a commuting triple of reflections of (W,S). It has a unique companion reflection if and only if it is connected.*

Proof. Suppose that r_1, r_2 and r_3 are connected by a reflection t. Let β_1, β_2 and β_3 be roots corresponding to respectively r_1, r_2 and r_3 in the geometric representation.

Likewise, let γ be the root corresponding to t. For each $i = 1, 2, 3$ consider the reflection $r_i' = r_i \cdot t$. It has corresponding root $\beta_i - (\beta_i, \gamma)\gamma$. Denote this root β_i'. Let $j = 1, 2, 3$. Then

$$\begin{aligned} (\beta_i', \beta_j)\beta_i' &= -(\beta_i, \gamma)(\beta_j, \gamma)\beta_i' \\ &= -(\beta_i, \gamma)(\beta_j, \gamma)\beta_i + (\beta_i, \gamma)^2(\beta_i, \gamma)\gamma \\ &= -(\beta_i, \gamma)(\beta_j, \gamma)\beta_i + (\beta_j, \gamma)\gamma \\ &= (\beta_j, \gamma)(\gamma - (\beta_i, \gamma)\beta_i). \end{aligned}$$

Denote the reflections corresponding to β_i' by r_i'. Since t is an involution it holds that $r_1 * tr_2r_3t = r_i * r_2'r_3'$. The root corresponding to the result is $\beta_1 \cdot r_2'r_3'$. Thus,

$$\begin{aligned} \beta_1 \cdot r_2'r_3' &= \beta_1 - (\beta_1, \beta_2')\beta_2' - (\beta_1, \beta_3')\beta_3' \\ &= \beta_1 - (\beta_1, \gamma)(2\gamma - (\beta_2, \gamma)\beta_2 - (\beta_3, \gamma)\beta_3)) \\ &= -(\beta_1, \gamma)(2\gamma - (\beta_1, \gamma)\beta_1 - (\beta_2, \gamma)\beta_2 - (\beta_3, \gamma)\beta_3). \end{aligned}$$

This expression is invariant under permutation of the indices 1, 2 and 3, up to a minus sign. It follows that the reflections corresponding to $\beta_1 \cdot r_2'r_3'$, $\beta_2 \cdot r_1'r_3'$ and $\beta_3 \cdot r_1'r_2'$ are the same. This establishes that the companion reflection does not depend on the particular order of r_1, r_2 and r_3.

Let t' be another reflection satisfying the conditions of the lemma. It will be demonstrated that $r_1 * tr_2r_3t = r_1 \cdot t'r_2r_3t'$. Let γ' be the root corresponding to t'. It was shown in the previous paragraph that roots corresponding to $r_1 * tr_2r_3t$ and $r_1 * t'r_2r_3t'$ are $2\gamma - (\beta_1, \gamma)\beta_3 - (\beta_2, \gamma)\beta_3 - (\beta_3, \gamma)\beta_3$ and $2\gamma' - (\beta_1, \gamma')\beta_1 - (\beta_2, \gamma')\beta_2 - (\beta_3, \gamma')\beta_3$, respectively. A short calculation shows that the inner product of these roots is equal to

$$\begin{aligned} & 4(\gamma, \gamma') - 2((\beta_1, \gamma)(\beta_1, \gamma') - (\beta_2, \gamma)(\beta_2, \gamma') - (\beta_3, \gamma)(\beta_3, \gamma')) \\ =\ & 2(2(\gamma, \gamma') - (\beta_1, \gamma)(\beta_1, \gamma') - (\beta_2, \gamma)(\beta_2, \gamma') - (\beta_3, \gamma)(\beta_3, \gamma')) \\ =\ & -2, 0, 2. \end{aligned}$$

However, the inner product is not equal to 0, since $(\beta_1, \gamma)(\beta_1, \gamma') + (\beta_2, \gamma)(\beta_2, \gamma') + (\beta_3, \gamma)(\beta_3, \gamma')$ is always odd. This follows from the fact that M is simply laced and the conditions on both γ and γ' (all (β_i, γ) and (β_i, γ') are odd). The inner product must equal 2 or -2 and it follows that the roots described above differ only up to a minus sign. Hence, their corresponding reflections are equal. This proves the choice of connecting reflection is inconsequential. This established the *if* direction.

The *only if* direction is a direct consequence of the definition of companion reflection. □

The collection of all connected, commuting triples of reflections of (W, S) is denoted $\mathcal{C}$. Define a function that assigns to each triple of reflections in $\mathcal{C}$ its unique companion reflection. Such a function exists by Lemma 3.2.3. Denote it by cp, i.e.,

$\mathrm{cp} : \mathcal{C} \to \mathcal{R}(W, S)$, and define it as follows.

$$\mathrm{cp}(r_1, r_2, r_3) = r_1 * (tr_2r_3t) \tag{3.9}$$

where r_1, r_2 and r_3 form a pairwise computing triple connected by t. By the Lemma 3.2.3 it follows that cp is symmetric in its argument. It does not depend on the arrangement of the triple. Let $X \in \mathcal{X}(W, S)$. The function cp is extended to $\mathcal{X}(W, S)$ by applying it to all connected triples of reflections contained in X and joining the results.

Example. Let $M = E_7$ and set $X = \{s_2, s_3, s_7\}$. Let t be the reflection corresponding to the root $\alpha_4 + \alpha_5 + \alpha_6$. Then t connects s_2, s_3 and s_7. This can be read from the diagram. The companion reflection r is given by the following expression.

$$r = \mathrm{cp}(s_2, s_3, s_7) = s_2 * ts_3s_7t.$$

The root corresponding to r is the image of the root corresponding to s_2 under ts_3s_7t, i.e.,

$$\alpha_2 * ts_3s_7t = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7.$$

Since there is only one connected triple contained in X it follows that $\mathrm{cp}(X) = \{r\}$. Note that $X \cup \mathrm{cp}(X)$ is admissible. It will be shown that this is not a coincidence.

Some properties on the companion function are collected in the next lemma. They will be required in the proof of the Proposition 3.2.5 later in this paragraph. The proofs are rather technical.

Lemma 3.2.4 *Suppose that M is simply laced and spherical. Let r_1, r_2, r_3, r_4 and r_5 form a fivesome of commuting reflections of (W, S). Assume that r_1, r_2 and r_3 form a connected triple. Then*

(i) $\mathrm{cp}(r_1, r_2, r_3)$ *commutes with r_i, for $i = 1, \ldots, 5$.*

(ii) $\mathrm{cp}(r_1, r_2, r_3)$, *$r_2$ and r_3 always form a connected triple, and* $\mathrm{cp}(\mathrm{cp}(r_1, r_2, r_3), r_2, r_3) = r_1$.

(iii) If $\mathrm{cp}(r_1, r_2, r_3)$, *$r_4$ and r_5 form a connected triple then*

$$\mathrm{cp}(\mathrm{cp}(r_1, r_2, r_3), r_4, r_5) = r_i,$$

for some $i = 1, 2, 3$.

(iv) If $\mathrm{cp}(r_1, r_2, r_3)$, *$r_3$ and r_4 form a connected triple then*

$$\mathrm{cp}(\mathrm{cp}(r_1, r_2, r_3), r_3, r_4) = \mathrm{cp}(r_1, r_2, r_4).$$

Proof. The first and second statements have short proofs, while the proof of the third and the fourth statement are somewhat more complicated. In the rest of this proof $\beta_i \in \Phi$ denotes a root that correspond to the reflection r_i.

(i) Set $r = \text{cp}(r_1, r_2, r_3)$. First we show that r commutes with r_4. Let t be a reflection connecting r_1, r_2 and r_3. Then t does not commute with r. Since W is finite, and t connects r_1, r_2, r_3 and r it follows that t commutes with r_4, or that $r = r_4$. In the first case,

$$\begin{aligned} r_4 r &= r_4(r_1 * t r_2 r_3 t) \\ &= (r_1 * t r_2 r_3 t) r_4 \\ &= r r_4. \end{aligned}$$

In the second case ($r = r_4$) the reflections also commute.

The case for r_5 is similar.

Choose some $i = 1, 2, 3$. Set $j, k = 1, 2, 3$ to be different indices not equal to i. It will be shown that r_i commutes with r. By Lemma 3.2.3 it follows that $r = r_j \cdot t r_i r_k t$. Moreover, $r_i = r_k \cdot t r_i r_k t$. Since r_j and r_k commute, their conjugates under $t r_i r_k t$ also commute. This proves the statement.

(iii) Let t be a reflection of (W, S) connecting $\text{cp}(r_1, r_2, r_3)$, r_4 and r_5. Then it is also connected to $\text{cp}(\text{cp}(r_1, r_2, r_3), r_4, r_5)$. It will be shown that t does not commute with at least one reflection in r_1, r_2 and r_3. Let γ be a root corresponding to t, and let δ be the root that has inner product -1 with each β_i for $i = 1, 2, 3$. Then $2\delta + \beta_1 + \beta_2 + \beta_3$ is a root corresponding to $\text{cp}(r_1, r_2, r_3)$. Consider,

$$\begin{aligned} -1, 1 &= (\gamma, 2\delta + \beta_1 + \beta_2 + \beta_3) \\ &= 2(\gamma, \delta) + (\gamma, \beta_1) + (\gamma, \beta_2) + (\gamma, \beta_3), \end{aligned}$$

in which the first identity follows from the definition of t. Thus, there is an index $i = 1, 2, 3$ such that $(\beta_i, \delta) \neq 0$. Moreover, for this i the reflection r_i commutes with r_4, r_5 and $\text{cp}(r_1, r_2, r_3)$. Since W is finite it must follow that $\text{cp}(\text{cp}(r_1, r_2, r_3), r_4, r_5) = r_i)$.

(iv) Let γ' be the root that has inner product -1 with each β_i for $i = 1, 2, 3$. Then $2\gamma' + \beta_1 + \beta_2 + \beta_3$ is a root corresponding to $\text{cp}(r_1, r_2, r_3)$. Fix a root γ that has inner product -1 with this root, and inner product -1 with β_4 and β_5. Set

$$\delta = 2\gamma + 2\gamma' + \beta_1 + \beta_2 + 2\beta_3 + \beta_4.$$

Then δ is a root corresponding to $\text{cp}(\text{cp}(\beta_1, \beta_2, \beta_3), \beta_3, \beta_4)$. Consider $1/2(\delta - \beta_2 - \beta_1 - \beta_4)$. It equals $\gamma + \gamma' + \beta_3$. Assume for a moment that it is a root. Then,

$$\begin{aligned} (\beta_1, \gamma + \gamma' + \beta_3) &= -1, \\ (\beta_2, \gamma + \gamma' + \beta_3) &= -1, \\ (\beta_4, \gamma + \gamma' + \beta_3) &= -1. \end{aligned}$$

These statements follow again by finiteness of Φ and the choice of γ', γ. Hence, δ would be a root corresponding to $\mathrm{cp}(r_1, r_2, r_4)$. It is sufficient to show that $\gamma + \gamma' + \beta_3$ is a root.

By definition of γ and γ' it holds that$(\gamma, 2\gamma' + \beta_1 + \beta_2 + \beta_3) = -1$. Thus,

$$\begin{aligned} -1 &= (\gamma, 2\gamma' + \beta_1 + \beta_2 + \beta_3) \\ &= 2(\gamma, \gamma') + (\gamma, \beta_3) \\ &= 2(\gamma, \gamma') - 1 \end{aligned}$$

It follows that $(\gamma, \gamma') = 0$. Hence, $\gamma' \cdot r_3 r_\gamma = \gamma' + \beta_3 + \gamma \in \Phi$. This proves the statement.

□

The next proposition leads up to a characterization of the admissible closure. It gives an upper bound for the admissible closure. Remember that the admissible closure of a commuting set was defined as the smallest admissible set containing that commuting set.

Proposition 3.2.5 *Suppose that M is simply laced and spherical. Let $X \in \mathcal{X}(W, S)$. Then $X \cup \mathrm{cp}(X) \in \mathcal{A}(W, S)$.*

Proof. The proof is divided into two parts. Define X_n as follows,

$$X_n = \bigcup_{i=0}^{n} \mathrm{cp}^i(X)$$

In the first part it is shown that X_∞ is admissible. Secondly, it is demonstrated that cp is an idempotent, i.e., that $\mathrm{cp}^2 = \mathrm{cp}$.

Note that X_∞ is a commuting set, by part (i) of Lemma 3.2.4.

For the first part we make use of characterization (iii) of admissibility, as given in Proposition 3.2.4. of [13]. Assume that there is some reflection t that moves only three reflections in X_∞ by conjugation. Call these reflection r_1, r_2 and r_3. Let j be the minimal index such that X_j contains r_1, r_2 and r_3. Set $r = \mathrm{cp}(r_1, r_2, r_3)$. Hence $r \in X_{j+1}$. Moreover, r is not equal to r_1, r_2 or r_3. Now t does not commute with r, and hence there are four reflection in X_∞ not commuting with t. This contradicts the choice of t, and hence such a t cannot exist. This shows that X_∞ is admissible.

Consider reflections r_1, r_2, r_3, r_4, r_5, r_6 and r_7 in X. Whenever we write down the companion reflection of some triple it is assumed to exist. We shall not state explicitly that the arguments of cp form a connected triple. Consider, $\mathrm{cp}(\mathrm{cp}(r_1, r_2, r_3), r_4, r_5)$. There are three possible cases.

- Suppose r_1, r_2, r_3, r_4 and r_5 are all different. Then $\mathrm{cp}(\mathrm{cp}(r_1, r_2, r_3), r_4, r_5) = r_i$, for some $i = 1, 2, 3$. This follows from part (ii) of Lemma 3.2.4.

- Suppose that one of r_4, r_5 equals one of r_1, r_2, r_3. Without loss of generality it is assumed that $r_4 = r_3$, since cp is symmetric in its arguments. Thus,

$\text{cp}(\text{cp}(r_1, r_2, r_3), r_4, r_5) = \text{cp}(r_1, r_2, r_5)$, as follows from part (ii) of Lemma 3.2.4.

- Suppose that both r_4 and r_5 equal one of r_1, r_2, r_3. Without loss of generality $r_4 = r_2$ and $r_5 = r_3$. Then $\text{cp}(\text{cp}(r_1, r_2, r_3), r_2, r_3) = r_1$. This is part (ii) of Lemma 3.2.4.

Now consider $\text{cp}(\text{cp}(r_1, r_2, r_3), \text{cp}(r_4, r_5, r_6), r_7)$. The following cases are distinguished.

- Suppose r_1, r_2, r_3, $\text{cp}(r_4, r_5, r_6)$ and r_7 are all different. Then by part (ii) of Lemma 3.2.4 it follows that $\text{cp}(\text{cp}(r_1, r_2, r_3), \text{cp}(r_4, r_5, r_6), r_7) = r_i$ for some $i = 1, 2, 3$.

- Suppose that one $\text{cp}(r_4, r_5, r_6)$, r_7 equals one of r_3. Then by part (ii) of Lemma 3.2.4 it follows that $\text{cp}(\text{cp}(r_1, r_2, r_3), \text{cp}(r_4, r_5, r_6), r_7)$ equals $\text{cp}(r_1, r_2, r_7)$ or $\text{cp}(r_1, r_2, \text{cp}(r_4, r_5, r_6))$. The latter case was already taken care of and is equal to some r_i or some $\text{cp}(r_i, r_j, r_k)$.

- Suppose that both $\text{cp}(r_4, r_5, r_6)$ and r_7 equal some r_1, r_2 or r_3. Then $\text{cp}(\text{cp}(r_1, r_2, r_3),\ \text{cp}(r_4, r_5, r_6), r_7) = r_i$, for some $i = 1, 2, 3$, by part (ii) of the lemma.

The case in which cp has three companion reflections as arguments can be reduced to the case in which it has only two, similarly to what was done above. That case is already dealt with. This shows that applying cp to X twice does not result in reflections that were not already contained in $X \cup \text{cp}(X)$. Hence, $\text{cp}^2 = \text{cp}$, and the proposition is proved. □

Theorem 3.2.6 *Suppose that M is simply laced and spherical. Let $X \in \mathcal{X}(W, S)$. Then $X \cup \text{cp}(X) = X^c$.*

Proof. Let $X \in \mathcal{X}(W, S)$. Assume that X is not admissible. In light of Proposition 3.2.5 it is sufficient to show that there does not exist an admissible set $Y \in \mathcal{X}(W, S)$ such that X is contained in Y, and that Y is strictly contained in $X \cup \text{cp}(X)$.

Suppose such an Y exists. Then there is some reflection r of (W, S) such that $r \in X \cup \text{cp}(X)$ and $r \in Y$. Then $r \notin X$, and hence $r \in \text{cp}(X)$. There exists some connected triple of reflections $r_1, r_2, r_3 \in X \subseteq Y$ such that $r = \text{cp}(r_1, r_2, r_3)$. Consider a reflection t of (W, S) that connects r_1, r_2 and r_3. Then $r_2 * t$ and $r_3 \cdot t$ are commuting reflections, and their product moves r_2 and r_3 by conjugation. However $r_1 * (r_2 * t)(r_3 * t) = r_1 * t r_2 r_3 t = r$, and $r \notin Y$. Thus $(r_2 * t)(r_3 * t)$ is not an element of the normalizer of Y. This shows that Y is not admissible, which contradicts the choice of Y. Hence, there is no such Y. This proves the theorem. □

The theorem gives an indication of how to construct the admissible closure of a set $X \in \mathcal{X}(W, S)$. It involves finding all connected triples of X, reflections connecting them, calculating the companion reflections by means of (3.9) and adding the results to X. However, it is not a priori obvious how to find a reflection connecting a triple quickly.

The next proposition is the last of this paragraph. The proof requires the previous theorem. It shows that the orbit of a commuting set resembles that of its admissible closure.

Proposition 3.2.7 *Suppose that M is simply laced and spherical. Let $X \in \mathcal{X}(W,S)$. Then,*

(i) For each $w \in W$ it holds that $X^c * w = (X * w)^c$.

(ii) $C(X) = C(X^c)$.

Proof. Both proofs make heavy use of Theorem 3.2.6.

(i) Let $r \in X^c * w$. Then by Theorem 3.2.6 it holds that $r \in X * w \cup \mathrm{cp}(X) * w$. Suppose that $r \in X * w$. Since admissible closure is increasing it holds that $r \in (X * w)^c$. If $r \in \mathrm{cp}(X) * w$, there is some connected triple of reflection r_1, r_2, $r_3 \in X$ such that $r = \mathrm{cp}(r_1, r_2, r_3) \cdot w$. Hence, it is sufficient to show that

$$\mathrm{cp}(r_1, r_2, r_3) * w = \mathrm{cp}(r_1 * w, r_2 * w, r_3 * w).$$

Let t be some reflection of (W,S) connecting r_1, r_2 and r_3. By (3.9),

$$\begin{aligned}
\mathrm{cp}(r_1 * w, r_2 * w, r_3 * w) &= r_1 * w(t(r_2 * w)(r_3 * w)t \\
&= r_1 * w(t((r_2 r_3) * w)t) \\
&= w^{-1} r_1 (t * w^{-1}) r_2 r_3 (t * w^{-1}) w \\
&= \mathrm{cp}(r_1, r_2, r_3) * w.
\end{aligned}$$

The last statement is a consequence of the fact that $t * w^{-1}$ is a reflection of (W,S) connecting r_1, r_2 and r_3. This proves that $X^c * w \subseteq (X * w)^c$.

Now let $r \in (X * w)^c$. Then $r \in X * w$ or $r \in \mathrm{cp}(X * w)$. In the former case $r \in X^c * w$. In the latter case there exist reflections connected $r_1, r_2, r_3 \in X$ such that $r = \mathrm{cp}(r_1 * w, r_2 * w, r_3 * w)$. Thus $r = \mathrm{cp}(r_1, r_2, r_3) * w$, and hence $r \in \mathrm{cp}(X) * w$. This proves that $r \in X^c * w$ and hence that $(X * w)^c \subseteq X^c * w$.

(ii) It is sufficient to show that $C(X) \subseteq C(\mathrm{cp}(X))$. Since $C(X)$ is a subgroup of W generated by reflections it suffices to show that every reflection in $C(X)$ is contained in $C(\mathrm{cp}(X))$. Let $q \in C(X)$ be a reflection and $r \in \mathrm{cp}(X)$. Then there are reflection $r_1, r_2, r_3 \in C(X)$ such that $r = \mathrm{cp}(r_1, r_2, r_3)$. Consider Lemma 3.2.4(i) and put $r_4 = q$. Then r_1, r_2, r_3, r_4 satisfy the conditions of the lemma. Thus q commutes with $r = \mathrm{cp}(r_1, r_2, r_3)$. Hence $q \in C(\mathrm{cp}(X))$. This proves the statement.

□

Corollary 3.2.8 *Assume that M is simply laced and spherical. Let Ω be an orbit under the conjugation action of W on $\mathcal{X}(W,S)$. Then Ω is admissible if and only if one of its elements is.*

3.3 Reflection subgroups

multiplicity
reflection cocyle

The results set out in this paragraph were obtained by Matthew Dyer in an article published in 1990 [10]. They generalize some facts on parabolic subgroups to arbitrary reflection subgroups of Coxeter systems. Most propositions here correspond to a proposition of Dyer in the aforementioned article. If so, the bold faced text in parentheses immediately following a proposition in this section refers to the corresponding proposition of Dyer. Theorem 3.3.5 is the most important of this section for purposes of the thesis. By means of it we arrive at a decomposition of the normalizer of a certain subset of the reflections.

Following [10] we introduce the free vector space $\mathcal{M}(W, S)$ over the field $\mathbb{F}_2$ with a basis indexed by the reflections of (W, S). Its dimension is $|\mathcal{R}(W, S)|$. The conjugation action defined earlier by (3.2) extends linearly to this vector space. The extension is also denoted by a horizontally centered dot. For any $m \in \mathcal{M}(W, S)$ and $r \in \mathcal{R}(W, S)$ the coefficient $m_r \in \mathbb{F}_2$ is defined by

$$m = \sum_{r \in R} m_r r.$$

By the *multiplicity* of r in m is meant m_r. On $\mathcal{M}(W, S)$ a norm $\| \cdot \|$ is defined by

$$\|m\| = \| \sum_{r \in R} m_r r\| = \sum_{r \in R} \|m_r\|, \qquad m \in M.$$

A *reflection cocyle* on (W, S) is a function μ from W to $\mathcal{M}(W, S)$ that satisfies the following conditions.

(i) For all $w, v \in W$ it holds that $\mu(wv) = \mu(w) + \mu(v) * w^{-1}$.

(ii) For all $s \in S$ it holds that $\mu(s) = s$.

Such a function exists and (i) and (ii) define it uniquely. It is henceforth denoted μ_S or simply μ. The following lemma shows the relation between the length function and the reflection cocyle on (W, S).

Lemma 3.3.1 (Lemma 2.3) *Let W be a Coxeter group and S a set of Coxeter generators for W. Then the following statements hold, for each $w \in W$.*

(i) The length $\ell_S(w)$ of w equals $\|\mu(w)\|$.

(ii) For every reflection $r \in \mathcal{R}(W, S)$ either $\ell_S(rw) < \ell_S(w)$ or $\ell_S(rw) > \ell_S(w)$. Moreover, the first option is equivalent to $\mu(w)_r = 1$, while the second is equivalent to $\mu(w)_r = 0$.

Proof. This proof is basically a specialization of the proof of Lemma 2.3 of [10].

(i) Let $w \in W$ and let $w = s_1 \ldots s_m$ be a reduced expression for w. Then

$$\begin{aligned} \mu(w) &= s_1 + \mu(s_2 \ldots s_m) * s_1 \\ &= s_1 + s_2 * s_1 + \mu(s_3 \ldots s_m) * s_2 s_1 \\ &= s_1 + s_2 * s_1 + s_3 * s_2 s_1 + \ldots s_m * s_{m-1} \ldots s_1. \end{aligned} \tag{3.10}$$

For each $i = 1, \dots, m$ define r_i to be the reflection $s_i * s_{i-1} \dots s_1$. Assume that there exist different indices i and j such that $r_i = r_j$. Suppose without loss of generality that $i < j$. Then

$$s_i \dots s_{j-1} = s_{i+1} \dots s_j.$$

By this equality w also admits the expression

$$\begin{aligned} w &= s_1 \dots s_{i-1} s_{i+1} \dots s_j s_j \dots s_m \\ &= s_1 \dots s_{i-1} s_{i+1} \dots s_{j-1} s_{j+1} \dots s_m, \end{aligned}$$

contradicting the fact that $s_1 \dots s_m$ is a reduced expression for w. Hence, all of the reflections r_i, $i = 1, \dots, m$ are different. Now

$$\|\mu(w)\| = \sum_{r \in R} \|\mu(w)_r\| = \sum_{r \in R} \|\mu(w)_r\| = m = \ell_S(w),$$

since the only reflections that have multiplicity 1 in w are the r_i, for $i = 1, \dots, m$.

(ii) Let $r \in \mathcal{R}(W, S)$ be any reflection. We will show that $\mu(r)_r = 1$ and hence

$$\mu(rw)_r = \mu(w)_r + 1, \tag{3.11}$$

for any $w \in W$. Suppose $r = s * w$, for certain $s \in S$ and $w \in W$. Let $w = s_1 \dots s_m$ be a reduced expression for w. Then

$$\begin{aligned} \mu(r) &= \mu(w^{-1} s w) = \mu(w^{-1}) + \mu(sw) * w \\ &= \mu(w^{-1}) + \mu(s) * w + \mu(w) * sw \end{aligned}$$

By (3.10) and the expression for r it follows that

$$\mu(r) = r + \sum_{i=1}^{m} s_i * s_{i+1} \dots s_m + \Big(\sum_{i=1}^{m} s_i * s_{i-1} \dots s_1 \Big) * sw.$$

Let r_i denote $s_i * s_{i+1} \dots s_m$ and consider the following,

$$\begin{aligned} r_i * r &= (s_i * s_{i+1} \dots s_m) * r \\ &= (s_i * s_{i+1} \dots s_m) * u^{-1} s w \\ &= (s_i * s_{i+1} \dots s_m w^{-1}) * sw \\ &= (s_i * s_i \dots s_1) * sw \\ &= (s_i * s_{i+1} \dots s_1) * sw. \end{aligned}$$

Hence,

$$\mu(r) = r + \sum_{i=1}^{m} (r_i + r_i * r), \tag{3.12}$$

and this proves that $\mu(r)_r = 1$, since the summands beyond the first do not contain r.

Let $r \in \mathcal{R}(W,S)$ be any reflection and let $w \in W$ satisfy $\mu(w)_r = 1$. Fix a reduced expression $w = s_1 \dots s_m$ for w. By (3.10) the reflection r is of the form $r = s_i * s_{i-1} \dots s_1$ for certain $i = 1, \dots, m$. Consider the composition of r and w;

$$\begin{aligned} rw &= (s_i * s_{i-1} \dots s_1) s_1 \dots s_m \\ &= (s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1) s_1 \dots s_m \\ &= s_1 \dots s_{i-1} s_{i+1} \dots s_m. \end{aligned}$$

It follows that $\ell_S(rw) < \ell_S(w)$. Now assume that $\mu(w)_r = 1$. Then by (3.11) the multiplicity of r in rw is 0. Hence

$$\ell_S(w) = \ell_S(r(rw)) > \ell_S(rw),$$

by the previous argument.

□

The next lemma is a list of statements about the length function that will be of relevance in this section. Lemma 3.3.1 has a subtle part in its proof, by means of a reflection cocycle.

Lemma 3.3.2 (Lemma 2.7) *Let (W,S) be a Coxeter system with length function ℓ_S.*

(i) Let $w \in W$. Suppose that $w = t_1 \dots t_k$ for certain $k \in \mathbb{N}$ and $t_1, \dots, t_k \in S$. Then $k \equiv \ell_S(w) \pmod 2$.

(ii) Let $r \in \mathcal{R}(W,S)$ and $\ell_S(r) = 2k+1$. Then there exist $w \in W$ and $s \in S$ such that $t = s * w$ and $\ell_S(w) = k$.

Proof.

(i) By Proposition 5.1 of [18] there is a unique epimorphism $\epsilon : W \to \{-1, 1\}$. By Proposition 5.2 it is given by $\epsilon(w) = (-1)^{\ell_S(w)}$. Thus,

$$1 = \epsilon(1) = \epsilon(s_1 \dots s_m t_k \dots t_1) = (-1)^{m+k}.$$

It follows that $m + k$ is even, and hence $m \equiv k \pmod 2$.

(ii) There exist $s_1 \dots s_{2k+1} \in S$ such that $r = s_1 \dots s_{2k+1}$. Set $w \in W$ equal to $w = s_1 \dots s_k$. Then $\ell_S(rws_{k+1}) = k < \ell_S(ws_{k+1})$. By part (i) of Lemma 3.3.1 it follows that $\mu(ws_{k+1})_r = 1$. The definition of the reflection cocycle forces $r = s_i * s_{i-1} \dots s_1$ for certain $i \in \mathbb{N}$. Moreover, $i = k+1$ since the length of r is $2k+1$.

$\square$

Let X be some subset of W and fix an element $m \in \mathcal{M}(W,S)$. The projection of m to X is defined as follows.

$$m \cap X = \sum_{r \in X \cap R} m_r r, \tag{3.13}$$

where $R = \mathcal{R}(W,S)$. This should not cause confusion with the normal use of the intersection sign. The function $\Gamma : \mathcal{P}(W) \to \mathcal{P}(R)$ assigns to each subset of W a set of reflections that are minimal in a certain sense. Its definition is

$$\Gamma(X) = \{r \in R \mid \mu(r) \cap X = r\}. \tag{3.14}$$

Dyer denotes this function by χ and has a slightly different definition. The importance of Γ is that it maps a reflection subgroup to a set of Coxeter generators for this reflection subgroup. The next lemma is an important step in the proof.

Lemma 3.3.3 (Lemma 3.2) *Let (W,S) be a Coxeter system and let H be some subgroup of W. Then the following statements hold.*

(i) If $s \in S$ then

$$\Gamma(H * s) = \begin{cases} \Gamma(H) * s & \text{if } s \notin H, \\ \Gamma(H) & \text{if } s \in H. \end{cases}$$

(ii) If $r \in H \cap R$ there exist $m \in \mathbb{N}$ and $r_0, r_1, \ldots r_m \in \Gamma(H)$ such that $r = r_0 * r_1 r_2 \ldots r_m$.

(iii) For every $w \in W$ and $u \in H$ it holds that

$$\mu(uw) \cap H = \mu(u) \cap H + (\mu(w) \cap H) * u^{-1}.$$

Proof. *(i)* Suppose $s \in H$. Then $H * s = H$, since H is a group. Thus $\Gamma(H * s) = \Gamma(H)$.

Suppose $s \notin H$. Let $r \in R$. Consider,

$$\begin{aligned} \mu(r * s) &= \mu(s^{-1} r s) = \mu(srs) = s + \mu(rs) * s \\ &= s + \mu(r) * s + s * rs = s * s + \mu(r) * s + (s * r) * s \\ &= \Big(s + \mu(r) + (s * r)\Big) * s. \end{aligned}$$

Thus,

$$\mu(r * s) \cap H * s = (\mu(r) \cap G) * s, \tag{3.15}$$

since $s \notin H$ (or rather $\mu(s) \cap H = 0$). Take $r \in \Gamma(H) * s$. There exists $r_0 \in \Gamma(H)$ such that $r = r_0 * s$. By (3.15) it holds that $\mu(r_0 * s) \cap H * s = (\mu(r_0) \cap H) * s = r_0 * s$. Thus, $r \in \Gamma(H * s)$. Take $r \in \Gamma(H * s)$. Then in particular $r \in H * s$. There exists a reflection r_0 in G such that $r = r_0 * s$. Again by (3.15) it holds that $s + \mu(r_0) + s * r_0 \cap H = r_0$. Since $s \notin H$ this gives $\mu(r_0) \cap H = r_0$. Thus, $r_0 \in \Gamma(H)$ and $r = r_0 * s \in \Gamma(H) * s$.

(ii) This is proved by induction on the length of r. Assume $\ell_S(r) = 1$. Then $r \in S \cap H$ and hence $r \in \Gamma(H)$. Take $r_0 = r$ and $m = 0$.

Let $r \in H \cap R$ and assume (ii) proved for all $k < \ell_S(r)$. Suppose that $\ell_S(r) > 1$. From Lemma 3.3.2 it follows that there exists $s \in S$ such that $\ell_S(r * s) < \ell_S(r)$. Set $r' = r * s$. Then $r' \in H * s \cap R$. By induction there exist $r_0, r_1, \ldots, r_m \in \Gamma(H * s)$ such that $r' = r_0 * r_1 \ldots r_m$. Consider the cases $s \in H * s$ and $s \notin H * s$.

First assume the former. Then $s \in H$. Hence $H * s = H$ and thus $s \in \Gamma(H)$ (since $s \in S$). Then also $r' \in H$ and $r_0, r_1, \ldots, r_m \in H$. Since H is a group $r = r' * s^{-1} = r' * s = r_0 * r_1 \ldots r_m s \in H$. This rounds up the first case.

Assume that $s \notin H * s$. For each $i = 1, \ldots, m$ set $t_i = r_i * s^{-1} = r_i * s$. Then $t_i \in H$ for each i. Moreover, $t_i \in \Gamma(H)$ since $r_i \in \Gamma(H * s) = \Gamma(H) * s$ by part (i) of this lemma. Consider,

$$\begin{aligned} t_0 * t_1 \ldots t_m &= (r_0 * s) * (r_1 *)(r_2 * s) \ldots (r_m * s) \\ &= r_0 * ((s r_1 \ldots r_m) * s) = r_0 * (r_1 \ldots r_m s) \\ &= (r_0 * (r_1 \ldots r_m)) * s = r' * s = r. \end{aligned}$$

It follows that $r = t_0 * (t_1 \ldots t_m)$ is the required expression for r.

(iii) Take u and w as stated in the lemma. Then,

$$\begin{aligned} \mu(uw) \cap H &= (\mu(u) + \mu(w) * u^{-1}) \cap H \\ &= (\mu(u) \cap H) + ((\mu(w) * u^{-1}) \cap H) \\ &= (\mu(u) \cap H) + (\mu(w) * u^{-1}) \cap (H * w^{-1}) \\ &= (\mu(u) \cap H) + ((\mu(w) \cap H) * w^{-1} \end{aligned}$$

□

The main theorem is a corollary to the previous lemma. It states that every reflection in a Coxeter subsystem of the form $(H, \Gamma(H))$ is also a reflection in the supersystem and that reflection cocycles behave nicely.

Also note that the proof of the part (ii) is constructive in the sense that it gives an algorithm to actually compute the representation of a reflection in H in terms of the elements of $\Gamma(H)$.

Theorem 3.3.4 (Theorem 3.3) *Let H be a reflection subgroup of (W, S). Then $(H, \Gamma(H))$ is a Coxeter system. Moreover, the following statements hold.*

(i) $H \cap \mathcal{R}(W, S) = \mathcal{R}(H, \Gamma(H))$.

(ii) The reflection cocycle of $(H, \Gamma(H))$ is $u \mapsto \mu(u) \cap H$.

Proof. *(i)* By part (ii) of Lemma 3.3.3 and the premise that H is a reflection subgroup it follows that $H = \langle \Gamma(H) \rangle$. For part *(ii)* it is sufficient to check whether $u \mapsto \mu(u) \cap H$ satisfies the conditions for the reflection cocycle. It satisfies the first condition by part (iii) of Lemma 3.3.3 and the second condition by the definition of Γ. □

To ensure that the components of the decomposition of the normalizer have empty intersection the following definition is required. Let (W, S) be a Coxeter system and $H \leq G$ subgroups of W. It is convenient to assume that H is a reflection subgroup, altough not strictly necessary. We will give an example illustrating this later on. The symbol $\mathcal{L}(G, H)$ is the set defined by,

$$\mathcal{L}(G, H) = \{w \in G : \ell(rw) > \ell(w) \text{ for all } r \in H \cap \mathcal{R}(W, S)\}. \tag{3.16}$$

It is an easy consequence of Lemma 3.3.1) that $\mathcal{L}(G, H)$ is exactly the set of $w \in G$ such that each reflection $r \in H \cap \mathcal{R}(W, S)$ has multiplicity 0 in $\mu(w)$. As will be shown later $\mathcal{L}(G, H)$ is a set of coset representatives for H in G if H is a reflection subgroup. Moreover, in that case the elements of $\mathcal{L}(G, H)$ are minimal with respect to the length function of the containing Coxeter system. This is proved in Theorem 3.3.5.

Example. To illustrate the point that $\mathcal{L}(G, H)$ may be trivial when H is not a reflection subgroup we give the following example. Take $M = A_3$ and denote the fundamental reflections of (W, S) by s_i, where i ranges over the vertices of M. Let H be group generated by $s_2s_1s_3s_2$ and put $G = W$. Then H is not a reflection subgroup of (W, S), since it does not contain any reflections of (W, S) at all. This means that the condition for being in $\mathcal{L}(G, H)$ is trivially satisfied for all $w \in W$ and hence $\mathcal{L}(G, H) = G = W$. If however, H is a (non-trivial) reflection subgroup there is always at least one reflection in H. This reflection is not contained in $\mathcal{L}(G, H)$. Then $\mathcal{L}(G, H) \neq G$.

Theorem 3.3.5 *Let (W, S) be a Coxeter system and $H \leq G \leq W$ such that H is a reflection subgroup of (W, S). Then $\mathcal{L}(G, H)$ is a set of coset representatives of H in G of minimal length. Moreover, if H is normal in G the following hold.*

(1) $\mathcal{L}(G, H)$ is a group.

(2) G decomposes as $H \rtimes \mathcal{L}(G, H)$.

(3) $\mathcal{L}(G, H)$ permutes $\Gamma(H)$.

(4) Let $w \in W$. Elements of Gw that have minimal length are contained in the same right $\mathcal{L}(G, H)$ coset.

Proof. Fix $w \in W$. Let $v \in Hw$ be an element of minimal length. For each reflection $r \in H$ the product rv lies in Hw. From Lemma 3.3.1 it follows that $\ell(rv) \neq \ell(v)$. Hence, $\ell(rv) > \ell(v)$ and thus $v \in \mathcal{L}(G, H)$. By definition of v there is $h \in H$ such that $v = hw$. Put $u = h^{-1}$. Then $w = uv$.

Suppose that there is another pair $u' \in H$ and $v' \in \mathcal{L}(G, H)$ satisfying $w = u'v'$ and $u' \neq u$. In particular $u'v' = uv$, or $u^{-1}u'v' = v$. Since H is a reflection subgroup it is a Coxeter group, by Theorem 3.3.4. Let $T = \Gamma(H)$ denote the canonical set of generators for H. Let $r \in \mathcal{R}(H, T)$ be a reflection. It is also an element of $\mathcal{R}(W, S) \cap H$ by Theorem 3.3.4. Consider,

$$\mu(v)_r = \mu(u^{-1}u'v')_r = \mu(u^{-1}u')_r + \mu(v')_{r*u'^{-1}u} = \mu(u^{-1}u')_r.$$

The last equality is a consequence of $v' \in \mathcal{L}(G,H)$ and the fact that $r * u'^{-1}u$ is a reflection in H. Since $v \in \mathcal{L}(G,H)$ it must hold that $\mu(u^{-1}u')_r = 0$ for all $r \in \mathcal{R}(H,T)$. By Lemma 3.3.2 the length of $\mu(u^{-1}u')$ equals zero. It follows that $u = u'$.

Suppose that H is normal in G.

(1) Let $u, v \in \mathcal{L}(G,H)$ and $r \in \mathcal{R}(W,S) \cap H$. Then,

$$\mu(uv)_r = \mu(u)_r + (\mu(v) * u^{-1})_r = \mu(v)_{r*u} = 0,$$

by normality of H in G. This establishes $uv \in \mathcal{L}(G,H)$. Since $0 = \mu(1) = \mu(u^{-1}u) = \mu(u^{-1}) + \mu(u) * u$ it follows that $\mu(u^{-1})_r = \mu(u)_{r*u^{-1}} = 0$. Thus, $u^{-1} \in \mathcal{L}(G,H)$.

(2) The fact that H is normal in G is direct consequence of the premisses. However, we still need to show that $H \cap \mathcal{L}(G,H)$ is the trivial group. Let $u \in H \cap \mathcal{L}(G,H)$. The multiplicity of r in $\mu(u)$ is 0 for all $r \in \mathcal{R}(H,T)$. Since H is a reflection group $u = e$.

(3) Let $t \in \Gamma(H)$ and $w \in \mathcal{L}(G,H)$. Then,

$$\begin{aligned}
\mu(t * w) \cap H &= \mu(w^{-1}tw) \cap H \\
&= (\mu(w^{-1}) + \mu(tw) * w) \cap H \\
&= (\mu(w^{-1}) + \mu(t) * w + \mu(w) * t^{-1}w) \cap H \\
&= \mu(w^{-1}) \cap H + \mu(t) * w \cap H + \mu(w) * t^{-1}w \cap H.
\end{aligned}$$

By the normality of H in G it follows that $t*w \in H$. It needs to be shown that the length of $t*w$ with respect to $\Gamma(H)$ is 1. Since $\mathcal{L}(G,H)$ is a group, $w^{-1} \in \mathcal{L}(G,H)$ and hence $\mu(w^{-1}) \cap H = 0$. By the same argument $\mu(w) * t^{-1}w \cap H$ is equal to 0. Let r be a reflection of $(H, \Gamma(H)$. The multiplicity of r in t is non-zero if and only if $r = t$, by the fact that $t \in \Gamma(H)$. Thus, the multiplicity of r in $\mu(t) * w$ is non-zero if and only if $r = t * w$. The length of $t * w$ with respect to $\Gamma(H)$ is 1, and consequentially, $t * w \in \Gamma(H)$.

(4) Let $v \in Gw$ be of minimal length. For each $u \in H$ the following inequalities hold.

$$\ell(v) \leq \ell(uv) \text{ and } \ell(v) \leq \ell(v(u * w)). \tag{3.17}$$

Let $v' \in Hw$ be another element of minimal length. Set $u = v'w^{-1}$ and let $r \in \mathcal{R}(H,T)$. Consider,

$$\begin{aligned}
\mu(v'v^{-1})_r &= \mu(v')_r + (\mu(v^{-1}) * v'^{-1})_r \\
&= \mu(v^{-1})_{r*v'} \\
&= \mu(v^{-1})_{r*(uw)} \\
&= \mu(v^{-1})_{(r*u)*w} \\
&= 0
\end{aligned}$$

The last equality is a consequence of the fact that the length of $((r*u)*w)v^{-1}$ equals the length of $(v(r*u)*w)$. The length of the latter expression is larger than the length of v, since $(r*u) \in H$ and (3.17). Thus, the equality follows by Lemma 3.3.2. This proves that $v'v^{-1} \in \mathcal{L}(G, H)$.

□

3.4 The action on sets of commuting reflections

The objects under investigation in this section are groups related to sets of commuting reflections. It will be most convenient to define them in terms of the geometric representation. Let E denote n-dimensional Euclidean space and fix a root system $\Phi \subseteq E$. Let Δ denote a set of fundamental roots of Φ. The inner product restricted to Δ is encoded by a Coxeter graph M as is described in Section 3.1. Since Δ is a basis of E this determines the inner product on E. Let W denote the Weyl group of Φ and let S be the subset of W uniquely determined by the condition that $\Phi_S^+ = \Delta$. Then (W, S) is a Coxeter system of type M.

Subgroups of the normalizer

Fix a set $X \in \mathcal{X}(W, S)$. In this section there are three groups under examination. They depend on X, or more specific, on the set roots Φ_X. The largest is the normalizer of X, defined as the subgroup of W consisting of all elements that permute Φ_X. Equivalently, it is the normalizer of X in W. Therefore, we denote it $N(X)$. It contains the group of elements that fix Φ_X element-wise. This group is denoted $C_+(X)$. The choice of notation is motivated by the fact that $C_+(X)$ resembles the centralizer of X in W. To be able to state the definition of the third group the following result is required.

Lemma 3.4.1 *The group $C_+(X)$ is a reflection subgroup of (W, S) that is normal in $N(X)$. Moreover,*

$$C_+(X) = \mathcal{W}(\Phi_X^\perp \cap \Phi)$$

Proof. For the proof that $C_+(X)$ is a reflection subgroup of (W, S) we refer to [6].

Let $w \in N(X)$. Since $C_+(X)$ is a reflection subgroup of (W, S) by definition it is sufficient to show that for each generating reflection $r \in C_+(X)$ and $r \notin X$ it holds that $r * w \in C_+(X)$. Note that $r \cdot w$ is also a reflection. Suppose that it is an element of X. Then there is a reflection $t \in X$ such that $t = r * w$. Since $w \in N(X)$ this would imply that $r = t * w^{-1} \in X$, contradicting the assumption that $r \notin X$. This establishes that $C_+(X)$ normal is in $N(X)$.

We start with the $\supseteq$ part of the statement. Let $\beta \in \Phi$ satisfying $\beta \perp \Phi_X$. Since M is spherical this implies in particular that $\beta \notin \Phi_X$. Let $\gamma \in \Phi_X^+$. Then,

$$\gamma \cdot r_\beta = \gamma - (\beta, \gamma)\gamma = \gamma.$$

This shows that $r_\beta \in C_+(X)$.

We show the $\subseteq$ part of the statement. Let r be a generating reflection in $C_+(X)$. In particular $r \notin X$. There is $\beta \in \Phi$ such that $r = r_\beta$. Moreover, $\beta \notin \Phi_X$ since $r \notin X$. Let $\gamma \in \Phi_X$. Then $\gamma \cdot r_\beta = \gamma$. Consider,

$$\begin{aligned} \gamma &= \gamma \cdot r_\beta \\ &= \gamma - (\gamma, \beta)\beta \end{aligned}$$

Thus, $\beta \perp \gamma$. This shows that $\beta \perp \Phi_X$, and hence that $r_\beta \in \mathcal{W}(\Phi_X^\perp \cap \Phi)$. □

By Theorem 3.3.5 and the preceding lemma the set $\mathcal{L}(N(X), C_+(X))$ is a group. Since it is used often in this section and the current notation is somewhat cumbersome we write $L(X)$ instead. The decomposition stated and proved in Theorem 3.3.5 specializes to a decomposition of $N(X)$ in the following way.

$$N(X) = C_+(X) \rtimes L(X). \tag{3.18}$$

There is a possibility of confusion here, since we are considering two actions of W simultaneously. The first is the conjugation action, in which the set acted on is W itself. The second is the action on Φ determined by the geometric representation. These actions are very similar, due to the fact that the conjugate by w of a reflection corresponding to a root β is a reflection with corresponding root βw, i.e.,

$$r_\beta * w = w^{-1} r_\beta w = r_{(\beta w)}.$$

An important difference is that r_β fixes the group element r_β by conjugation, while it maps the root β to its negative. In other words, element-wise stabilizers of the actions differ. To add to confusion, it is sometimes easier to work with $C(X)$, while at other times it easier to work with $C_+(X)$. We require a way to get from one to the other.

Let $N_+(X)$ denote the subgroup of $N(X)$ that maps Φ_X^+ to Φ_X^+. Let $C_+(X)$ be the intersection of $C(X)$ with $N_+(X)$. It coincides with the earlier definition of $C_+(X)$. Similarly, $L_+(X)$ denotes the intersection of $L(X)$ and $N_+(X)$. The next lemma is an direct consequence of the exhange condition.

Lemma 3.4.2 *The following equations hold.*

$$N(X) = \langle X \rangle \rtimes N_+(X), \quad C(X) = \langle X \rangle \times C_+(X), \quad L(X) = \langle X \rangle \rtimes L_+(X).$$

Moreover, $\mathcal{L}(N(X), C(X)) = L_+(X)$.

In general $L_+(X)$ is not normal in $N(X)$. In particular it does not commute with the elements of $C(X)$, and hence it does not leave the elements $C(X)$ fixed under conjugation. Since its intersection with $C(X)$ is empty it must also act non-trivially on X. In fact, as the following lemma shows, $L_+(X)$ is exactly the group of all permutations of X that are also contained in W. This is not very surprising since $L_+(X)$ was already shown to be a collection of coset representatives of $C(X)$ in $N(X)$ (cf. Theorem 3.3.5). Similar results hold for $L(X)$ if instead of the conjugating action the action of W on Φ_X is considered. The next lemma collects the results, and is essential for the proof of the main theorem of this section.

Lemma 3.4.3 *The group $L_+(X)$ is permutation group on X. The group $L(X)$ is a permutation group on Φ_X.*

Proof. We will prove the statement on $L_+(X)$. The statement on $L(X)$ follows by Lemma 3.4.2.

The proof is based on the fact that $L_+(X)$ is a set of coset representatives of $C(X)$ in $N(X)$ (cf. Theorem 3.3.5). A map $\theta : L_+(X) \to \Sigma(X)$ is defined, and it is proved that θ is a homomorphism of groups. Finally it is shown that its kernel is trivial.

Enumerate the elements of X as follows

$$X = \{r_1, \ldots, r_k\}.$$

Consider some element $w \in L(X)$. Since $w \in N(X)$ it follows that for each $i = 1, \ldots, k$ there is some j such that $r_i * w = r_j$. Define $\theta(w)$ to be the permutation that maps r_i to r_j.

It is shown that θ is a homomorphism of groups. Let u and v be elements of $L(X)$. Let $i = 1, \ldots, k$. Then

$$\begin{aligned} r_i \cdot \theta(uv) &= r_i * uv \\ &= (r_i * u) * v \\ &= r_i \cdot \theta(u) * v \\ &= r_i \cdot \theta(u)) \cdot \theta(v) \end{aligned}$$

By this it follows that $\theta(uv) = \theta(u)\theta(v)$. Thus θ is a homomorphism of groups.

Let $w \in L_+(X)$ be such that $\theta(w) = \mathrm{id}$, where id is the identity permutation of $\Sigma(X)$. Then $r_i * w = r_i$, and hence w fixes X elementwise. Thus $w \in C(X)$. By part (ii) of Theorem 3.3.5 it follows that $w = 1$, in which 1 is the identity in W. □

In Section 4.3 a representation of W will be considered that depends on the orbit of X. It is induced from a representation of $N(X)$ under which only the $C_+(X)$ part acts non-trivially. The kernel contains $L(X)$, and thus the normal closure of $L(X)$ in $N(X)$. Denote it by $H(X)$ and write $W(X)$ for the quotient of $N(X)$ by $H(X)$. The latter is non-trivial whenever $L(X)$ is not normal in $N(X)$.

This nearly concludes the first part of this section. The last result shows that the groups introduced behave more or less as expected under the conjugation action of W on $\mathcal{X}(W, S)$.

Lemma 3.4.4 *Let $w \in W$. The following statements hold. For (3), (4) and (5) it is required that $\ell(rw) > \ell(w)$ for every reflection $r \in C_+(X)$.*

(1) $C(X * w) = C(X) * w$,

(2) $N(X * w) = N(X) * w$,

(3) $L(X * w) = L(X) * w$, residual group

(4) $H(X * w) = H(X) * w$,

(5) $W(X * w) = W(X) * w$.

Proof. All proofs are by straightforward calculation.

(1) Let $v \in C(X)$ and $r \in X$. Then $r * w \in X * w$. Moreover, all elements of $X * w$ are of this form. Then

$$\begin{aligned}(r * w) * (v * w) &= (w^{-1}rw) * (w^{-1}vw) \\ &= (r * v) * w \\ &= r * w.\end{aligned}$$

This proves that $v * w \in C(X * w)$. Thus, $C(X) * w \subseteq C(X * w)$. The other inclusion follows by $C(X * w) * w^{-1} \subseteq C(X * ww^{-1}) = C(X)$.

(2) Let $v \in N(X)$ and $r \in X$. Then,

$$(r * w) * (v * w) = (r * v) * w,$$

in which $r * v \in X$ by choice of v. Thus $N(X) * w \subseteq N(X * w)$. The other inclusion follows by the same argument used in (1).

(3) Let $v \in L(X)$ and r be a reflection in $C_+(X)$. Then $v * w \in N(X * w)$ by (2). It remains to show that the multiplicity of r in $v * w$ is zero.

$$\begin{aligned}\mu(v * w)_{r*w} &= \mu(w^{-1})_{r*w} + (\mu(vw) * w^{-1})_{r*w} \\ &= \mu(w^{-1})_{r*w} + (\mu(v) * w)_{r*w} + (\mu(w) * v^{-1}w)_{r*w} \\ &= \mu(w^{-1})_{r*w} + \mu(v)_r + \mu(w)_{r*v} \\ &= 0.\end{aligned}$$

Vanishing of the first and third term are consequences of the fact that $\mu(w)_r = 0$ for every reflection $r \in C_+(X)$.This is a consequence of the condition on w and Lemma 3.3.1. Thus, $L(X) * w \subseteq L(X * w)$. The other inclusion follows by a similar argument as that used in (2).

(4) This is an immediate consequence of (2) and (3)

(5) This is an immediate consequence of (2) and (4).

□

The residual group

The group $W(X)$ is called the *residual group* with respect to X after its role in the representation. By the first isomorphism theorem [3, p.11] it is isomorphic to a quotient of $C_+(X)$.

$$W(X) = N(X)/H(X) \cong C_+(X)/(H(X) \cap C_+(X)). \tag{3.19}$$

In fact $W(X)$ is a Coxeter group of type derived from the type of $C_+(X)$. Lemma 3.4.1 shows that the type of $C_+(X)$ is the same as the type of the root system $\Phi_X^\perp \cap \Phi$. There is an inductive procedure to determine it, which requires a short digression.

Denote the type of $C_+(X)$ paired with $\Gamma(C_+(X))$ by $M_0(X)$. Put $k = |X|$ and enumerate the elements of Φ_X^+ as follows,

$$\Phi_X^+ = \{\beta_1, \ldots, \beta_k\}.$$

Let, for all $j = 1, \ldots, k$, Ψ_j be the system of all roots in Φ perpendicular to $\beta_1, \ldots, \beta_j$. Then Ψ_j is determined from Ψ_{j-1} and β_j in the following manner.

$$\Phi = \Psi_0 \supset \Psi_1 \supset \ldots \supset \Psi_{k-1} \supset \Psi_k = \Phi_X^\perp \cap \Phi \text{ and } \Psi_{j-1} \cap \beta_j^\perp = \Psi_j.$$

The next lemma shows how to get the type of Ψ_j from Ψ_{j-1} and β_j.

Lemma 3.4.5 *Let Ψ be an irreducible root system of type M and $\beta \in \Psi$. The type of $\Psi \cap \beta^\perp$ is M restricted to the vertices i of M such that α_i is orthogonal to the highest root of Ψ.*

Proof. Let α_0 denote the highest root of Ψ. Let M' denote the Coxeter graph restricted to the vertices i of M such that $\alpha_i \perp \alpha_0$. The type of $\Psi \cap \alpha_0$ is M' [18, p.44]. Since Ψ is irreducible, there is some $w \in \mathcal{W}(\Psi)$ such that $\beta w = \alpha_0$. Since w acts as an orthogonal transformation on Ψ the sets $\Psi \cap \beta^\perp$ and $(\Psi \cap \beta^\perp)w$ are root systems of the same type. But the latter root system is equal to $\Psi \cap (\beta w)^\perp$. Since $\beta w = \alpha_0$ its type is M'. □

The vertices of M perpendicular to the highest root are exactly the vertices that are not connected to the extra vertex in the extended Coxeter graph of M. The extended graphs can be found on page 34 of [18].

Example. Let $M = D_5$ and put $X = \{s_1, s_2, s_4\}$. The group $C_+(X)$ is isomorphic to the Weyl group of all roots perpendicular to α_1, α_2 and α_4. The type of this group can be determined inductively using Lemma 3.4.5. Set

$$\begin{aligned} \Psi_1 &= \Phi \cap \alpha_1^\perp \\ \Psi_2 &= \Psi_1 \cap \alpha_2^\perp \\ \Psi_3 &= \Psi_2 \cap \alpha_3^\perp. \end{aligned}$$

The type of Ψ_1 is determined by examing the extended diagram of D_5. It is the diagram of D_5 to which an extra vertex is added, connected only to the vertex labeled 4 in D_5. Deleting the extra vertex and the vertex labeled 4 yields the type of Ψ_1. It is A_3A_1. Now α_2 is in the A_1 part of Ψ_1. Hence the type of Ψ_2 is A_3. For the final step, construct the extended diagram of A_3. It is the circle on 4 vertices, obtained from A_3 by adding an extra vertex, and connecting it to the nodes labeled 1 and 3 in A_3. By deleting the extra node and its neighbours one obtains A_1. This is the type of Ψ_3.

By Lemma 3.4.1 the type $M_0(X)$ is A_1.

M	$\|X\|$	$M_0(X)$	$M_0'(X)$	$L_+(X)$
A_n	t	A_{n-2t}	A_{n-2t}	Σ_t
D_n	t	$A_1^t D_{n-2t}$	$A_1 D_{n-2t}$	Σ_t
D_n	$2t$, $2t \neq n$	D_{n-2t}	A_{n-2t-1}	$W(B_t)$
D_n	n	$\emptyset$	$\emptyset$	$W(D_n)$
E_6	1	A_5	A_5	Σ_1
E_6	2	A_3	A_2	Σ_2
E_6	4	$\emptyset$	$\emptyset$	Σ_4
E_7	1	D_6	D_6	Σ_1
E_7	2	$A_1 D_4$	$A_1 D_4$	Σ_2
E_7	3	D_4	A_2	Σ_3
E_7	4	A_1^3	A_1	Σ_4
E_7	7	$\emptyset$	$\emptyset$	$L_3(2)$
E_8	1	E_7	E_7	Σ_1
E_8	2	D_6	A_5	Σ_2
E_8	4	D_4	A_2	Σ_3
E_8	8	$\emptyset$	$\emptyset$	$2^3 L_3(2)$

Table 3.1: *Types of the Coxeter groups $C_+(X)$ and $W(X)$ with respect to their canonical generators. The last column lists the structure of $L_+(X)$, for X admissible. This table is derived from Table 2 of [13]. The entries in the column labeled $M_0(X)$ correspond to the entries in the column labeled X of [13]. Likewise, $M_0'(X)$ is called Z. The entries in the last column are computed using the structure theorem for the normalizer (3.18). The type of the normalizer is obtained from [13].*

Now assume that X is admissible. Denote the orbit of X under W by Ω. By Proposition 3.2.1 there is a unique maximal element of Ω, say $\top$. Let M_Ω denote the Coxeter graph M restricted to the vertices i such that $s_i \in C_+(\top)$. These types are again listed in Table 3.1.

Proposition 3.4.6 *Let $\Omega \subseteq \mathcal{X}(W,S)$ be an admissible orbit under the conjugation action by W. Then $W(\top)$ is a Coxeter group and $s_i H(\top)$ form a set of Coxeter generators, in which i ranges over the vertices of M_Ω.*

Proof.

`** no proof **`

□

For each $w \in W$ let $\tau_X(w)$ denote an element of minimal length in the right coset of $N(X)$ in W. Then $\tau_X(w)$ is simply a smallest element mapping X to $X * w$ under the action of conjugation. In the geometric representation w maps Φ_X to Φ_{X*w}. Writing w as the product of $w\tau_X^{-1}(w)$ and $\tau_X(w)$ is to be interpreted as decomposing the map w into a part that permutes both Φ_X and $\Phi_X^\perp \cap \Phi$, and a part that moves Φ_X to Φ_{X*w} along the shortest possible path. Of course, $\tau_X(w)$ is not properly defined, since there may be multiple elements of minimal length in a given right coset of $N(X)$. It will be shown that any choice of $\tau_X(w)$ leads to the same result.

The element $w\tau_X(w)^{-1} \in N(X)$ by choice of $\tau_X(w)$. Take any other element w_0 of minimal length in $N(X)w$. By part (iii) of Theorem 3.3.5 the product $\tau_X(w)w_0^{-1} \in L_0(X)$. Hence, $w\tau_X(w)^{-1}L_0(X) = ww_0^{-1}L_0(X)$. In particular, $w\tau_X(w)^{-1}H(X)$ is indepent of the choice of $\tau_X(w)$. Define η_X to be the map $W \to W_X$ by,

$$\eta_X(w) = w\tau_X(w)^{-1}H(X). \tag{3.20}$$

It is well defined, and satisfies the properties collected in the next lemma.

Lemma 3.4.7 *Let (W,S) be a Coxeter system and $X \in \mathcal{X}(W,S)$. The following hold.*

(1) Let $v, w \in W$. Then

$$\eta_X(vw) = \eta_X(v)(\eta_{X*v}(w) * \tau_X(v)^{-1}).$$

(2) Let w_0 be an element of minimal length in a right $N(X)$ coset. Then $\eta_X(w_0) = H(X)$.

Proof.

(1) We need to show that

$$\eta_X(v)(\eta_{X*v}(w) * \tau_X(v)^{-1}) = vw\tau_X(vw)^{-1}H(X). \tag{3.21}$$

The first part of the proof is by straightforward calculation. We compute

$$\begin{aligned}
\eta_X(v)(\eta_{X*v}(w) * \tau_X(v)^{-1}) &= v\tau_X(v)^{-1}H(X) \\
&\quad (\tau_X(v)w\tau_{X*v}(w)^{-1}H(X*v)\tau_X(v)^{-1}) \\
&= H(X)vw\tau_{X*v}(w)^{-1}H(X*v)\tau_X(v)^{-1} \\
&= H(X)vw\tau_{X*v}(w)^{-1}\tau_X(v)^{-1}H(X) \\
&= vw(\tau_X(v)\tau_{X*v}(w))^{-1}H(X).
\end{aligned}$$

The second equation is a consequence of $v\tau_X(v)^{-1} \in N(X)$ and $H(X)$ is normal in $N(X)$. The third equation follows from Lemma 3.4.4. Finally, the fourth equation follows from $vw(\tau_X(v)\tau_{X*v}(w))^{-1} \in N(X)$ and $H(X)$ normal in $N(X)$. Hence, it is sufficient to show that

$$H(X)\tau_X(v)\tau_{X*v}(w) = H(X)\tau_X(vw).$$

We will in fact prove the stronger statement that $\tau_X(vw)(\tau_X(v)\tau_{X*v}(w))^{-1} \in L(X)$. Denote for the purposes of this proof $\tau_X(vw)$ by τ, $\tau_X(v)$ by v_0 and

$\tau_{X*v}(w)$ by w_0. We expand, using the defining condition (i) of the reflection cocycle (cf. Section 3.3).

$$\mu(\tau w_0^{-1} v_0^{-1}) \quad = \quad \mu(\tau) + \mu(w_0^{-1}) * \tau^{-1} + \mu(v_0^{-1}) * w_0 \tau^{-1}.$$

Let r be a reflection in $C_+(X)$. Then the multiplicity of r in τ is 0. Since w_0 is an element of minimal length in $N(X * v)w_0$, w_0^{-1} is an element of minimal length in $N(X * v_0 w_0)w_0^{-1}$. Moreover, $r * \tau \in C_+(X * \tau) = C_+(X * v_0 w_0)$. Thus the multiplicity of $r * \tau$ in w_0 is 0. This shows that the second term in the sum vanishes. The third term can be shown to vanish is a similar manner, as follows.

Again r is a reflection in $C_+(X)$. Then $r * \tau w_0^{-1}$ is a reflection in $C_+(X * v_0)$. Since v_0^{-1} is of minimal length the multiplicity of any reflection in $C_+(X * v_0)$ in v_0^{-1} vanishes.

Hence, $\tau(v_0 w_0)^{-1} \in L(X) \leq H(X)$ and thus, $H(X)\tau = H(X)v_0 w_0$. This proves (3.21) and hence (i).

(2) $\tau_X(w_0)$ is an element of minimal length in $N(X)w_0$. By the last part of Theorem 3.3.5 it holds that $w_0 \tau_X(w_0)^{-1} \in L(X)$. Thus $\eta(w_0) = w_0 \tau_X(w_0)^{-1} H(X) = H(X)$.

□

Examples. Two examples follow. The first is simpler since in that example $L(X)$ is normal in $N(X)$.

- Let (W, S) be a Coxeter system of type A_5 with corresponding root system Φ. The situation is depicted in Figure 3.2. Let r be the reflection corresponding to $\alpha_1 + \alpha_2 + \alpha_3$ Set X to be the set consisting of r and s_2. Then $X \in \mathcal{X}(W, S)$. A short calculation gives,

$$\begin{aligned} N(X) &= \langle r, s_2, s_5, s_1 s_3 \rangle, \\ C_+(X) &= \langle s_5 \rangle, \\ L(X) &= \langle r, s_2, s_1 s_3 \rangle. \end{aligned}$$

From these representations it follows that $L(X)$ is normal in $N(X)$, and hence that $N(X)$ is the direct product of $C_+(X)$ and $L(X)$. This is the general situation if $M = A_m$. In particular it holds that $W(X) = C_+(X)$ and $H(X) = L(X)$.

Put $w = s_5 s_1 s_2$. We will calculate $\eta_X(w)$. The first step is to determine an element of shortest length in $N(X)w$. Let the image of X under conjugation by w be denoted Y. Then

$$Y = \{r, s_2\} * w = \{s_3, s_1\},$$

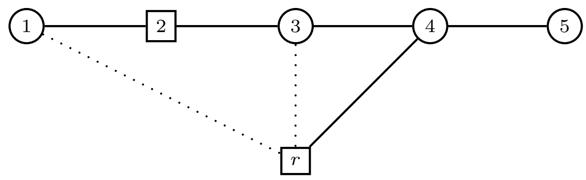


Figure 3.2: *The Coxeter graph A_5 to which a vertex representing r is added. Elements of X are represented by square boxes. A dotted line between vertices means the (positive) roots corresponding to those vertices have inner product 1, instead of -1.*

The reflections in Y all have length 3, while the longest reflection in X has length 5. Hence, there can be no element of W length smaller than 2 mapping X to Y by conjugation. However, $w_0 = s_3 s_2$ is an element of $N(X)w$, since $X \cdot w_0 = Y$. It is a minimal element by the previous argument. Thus,

$$\begin{aligned} \eta_X(s_5 s_1 s_2) &= s_5 s_1 s_2 s_2 s_3 H(X) \\ &= s_5 s_1 s_3 L(X) \\ &= s_5 L(X) \\ &= s_5 H(X). \end{aligned}$$

- Let (W, S) be a Coxeter system of type E_6 and set $X = \{s_2, s_5\}$. The situation is represented in Figure 3.3. By the inductive procedure outlined the type of $C_+(X)$ is A_3. From the diagram it follows that s_1 and s_3 commute with X, and hence $s_1, s_3 \in C_+(X)$. Moreover, s_1 and s_3 are canonical generators for $C_+(X)$. The root $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$ is perpendicular to α_2 and α_5 and hence the reflection r that corresponds to it commutes with s_2 and s_5. Thus, $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \in C_+(X)$. Moreover, the root is perpendicular to α_3 and has inner product -1 with α_1. Thus $\ell(s_1 r) > \ell(r)$ and $\ell(s_3 r) > \ell(r)$. Hence, $r \in \Gamma(C_+(X))$. The subgraph on 1, 3 and r is the Coxeter graph of the system $C_+(X)$ paired with $\Gamma(C_+(X))$.

Set $w = (s_2 s_5) * s_4$. Then $w \in N(X)$. Obviously, $\ell(s_1 w) > \ell(w)$ and $\ell(s_3 w) > \ell(w)$. Consider,

$$(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5) \cdot w = \alpha_3 > 0.$$

Thus, $\ell(rw) > \ell(w)$. It follows that $w \in L(X)$. Moreover, $r * w = s_3$. Thus,

$$\begin{aligned} r^{-1} s_3 &= r^{-1}(r * w) \\ &= r^{-1}(w^{-1} r w) \\ &= (w^{-1} * r) w. \end{aligned}$$

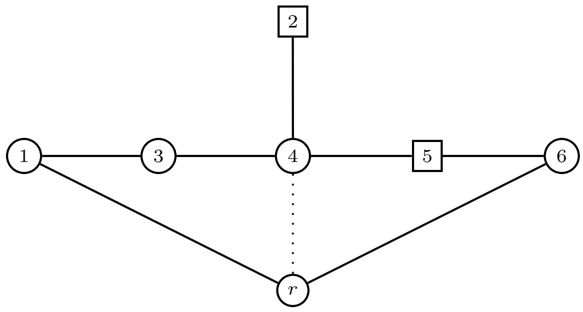


Figure 3.3: *The Coxeter graph E_6 to which a vertex representing r is added. Elements of X are represented by square boxes. A dotted line between vertices means the (positive) roots corresponding to those vertices have inner product* 1.

Now $w^{-1}*r \in H(X)$ by the normality of $H(X)$. Then X is admissible, because it contains only two elements. Since $H(X)$ is a group $(w^{-1} * r)w \in H(X)$. It follows that $s_3H(X) = rH(X)$. By Proposition 3.4.6 and Table 3.1 the type of $W(X)$ is A_2. Hence, $s_1H(X)$ must be distinct from $s_3H(X)$. We can conclude that

$$W(X) = \langle s_1H(X), s_3H(X)\rangle.$$

Moreover, by Lemma 3.4.3 it follows that $L_+(X)$ is the group generated by w, since it is isomorphic to a subgroup of $\Sigma(X) \cong \Sigma_2$. Hence $L(X)$ is the group generated by w and X.

Put $v = s_6s_1s_3s_4s_2s_4$. Then v maps α_2 to $-\alpha_4$ and α_5 to $\alpha_2+\alpha_4+\alpha_5+\alpha_6$. An element of minimal length mapping X to $X * v$ by conjugation is $v_0 = s_6s_4s_2$. This follows from an argument similar to that in the previous example. Hence, vv_0^{-1} is an element of $N(X)$.

$$\begin{aligned}
\eta_X(v) &= vv_0^{-1}H(X) \\
&= s_6s_1s_3s_4s_2s_4s_2s_4s_6H(X) \\
&= s_1s_3H(X) \\
&= (s_1H(X))(s_3H(X))
\end{aligned}$$

Generators of the normal closure

In the rest of this paragraph the results obtained earlier on admissible closures (cf. Section 3.2) are combined with those of the previous part of this section. The focus is on the normalizer of a the admissible closure of a set $X \in \mathcal{X}(W, S)$. A set of generators is exhibited for $H(X^c)$.

Lemma 3.4.8 *Let the reflections r_1 and r_2 be connected by a reflection t. There is a reflection t_0 such that for each $r \in C(r_1, r_2)$ it holds that $\ell(rt_0) > \ell(t_0)$ and there exists an element $u \in C(r_1, r_2)$ such that $t = t_0 * u$.*

Proof. Let Φ be a root system of type M and assume without loss of generality that $W = \mathcal{W}(\Phi)$. Then $C_+(X) = \mathcal{W}(\Phi_X^\perp \cap \Phi)$ by Lemma 3.4.1.

Let $\gamma > 0$ be the root corresponding to t. Define γ_1 to be a maximal, positive root satisfying $r_{\gamma_1} \in C(r_1, r_2)$ such that $\gamma - \gamma_1 \in \Phi^+$, if such a root exists. Assume that it doesn't. Then t satisfies the conditions for t_0 as the following argument shows.

Assume that $t \neq t_0$. Then there exists a positive root β satifying $r_\beta \in C(r_1, r_2)$ such that $\beta \cdot t < 0$. Consider,

$$\beta \cdot t = \beta - (\beta, \gamma)\gamma < 0$$

Thus, $(\beta, \gamma) > 0$. It follows that $\gamma - \beta > 0$ and that β is a root satisfying the conditions for γ_1. In particular, there is such a root of maximal height.

Thus, suppose γ_1 exists. Set $\gamma_1' = \gamma - \gamma_1$. It is positive and connects β_1 and β_2. Define γ_2 to be a maximal, positive root satisfying $r_{\gamma_2} \in C(r_1, r_2)$ such that $\gamma_1' - \gamma_2 \in \Phi^+$. If such a root does not exist, then putting t_0 equal to the reflection corresponding to γ_1' and u equal to the reflection corresponding to γ_1 fulfills the condition of the lemma, by the argument of the previous paragraph.

The procedure of choosing γ_1, γ_2 can be continued inductively as follows. Assume that we have a sequence of roots $\gamma_1, \dots, \gamma_k \in \Phi^+$ and a sequence of roots $\gamma_1', \dots, \gamma_k' \in \Phi^+$ such that

$$\begin{aligned} \gamma &= \gamma_1' + \gamma_1 \\ \gamma_1' &= \gamma_2' + \gamma_2 \\ \gamma_2' &= \gamma_3' + \gamma_3 \\ &\vdots \\ \gamma_{k-1}' &= \gamma_k' + \gamma_k. \end{aligned}$$

and γ_i satisfies $r_{\gamma_i} \in C(r_1, r_2)$. Moreover, assume that the sequence ends at k, i.e., there is no root γ_{k+1} satisfying $r_{\gamma_{k+1}} \in C(r_1, r_2)$ such that $\gamma_k' - \gamma_{k+1} \in \Phi^+$. Set t_0 equal to the reflection corresponding to γ_k'. Then t_0 satisfies the condition of the lemma, by the previous argument. It remains to construct $u \in C(r_1, r_2)$.

Write t_i for the reflection corresponding to γ_i. Then $t_i \in C(r_1, r_2)$ by choice of γ_i. Set $u = t_k \dots t_1$. Then,

$$\begin{aligned} \gamma &= \gamma_1' \cdot t_1 \\ \gamma_1' &= \gamma_2' \cdot t_2 \\ &\vdots \\ \gamma_{k-1}' &= \gamma_k' \cdot t_k. \end{aligned}$$

Set $u = t_k t_{k-1} \dots t_2 t_1$. Thus, $\gamma = \gamma'_k \cdot u$. This implies that $t = t_0 * u$. Since $t_i \in C(r_1, r_2)$ and the fact that $C(r_1, r_2)$ is a group it follows that $u \in C(r_1, r_2)$. This establishes the lemma. □

Lemma 3.4.9 *Let $X \in \mathcal{X}(W, S)$ and let $r_1, r_2 \in X$. Fix a reflection t that connects r_1 and r_2. Then $(r_1 r_2) * t$ is an element of $H(X^c)$. Moreover, let t_0 denote the reflection from the previous lemma. Then $(r_1 r_2) * t_0 \in L_+(X^c)$.*

Proof. It is shown that $(r_1 r_2) * t$ is an element of $N(X^c)$. Since the admissibility operator is increasing both $r_1, r_2 * X^c$. Moreover, X^c is admissible by definition. Consider,

$$\begin{aligned} r_1 * (r_1 r_2 * t) &= t r_2 r_1 t r_1 t r_1 r_2 t \\ &= t r_2 t r_2 t \\ &= r_2. \end{aligned}$$

Since $(r_1 r_2) * t$ is an involution, $r_2 * (r_1 r_2 * t) = r_1$. Note that $(r_1 r_2) * t$ is the product of $r_1 * t$ and $r_2 * t$. These reflections commute,

$$\begin{aligned} (r_1 * t)(r_2 * t) &= (r_1 r_2) * t \\ &= (r_2 r_1) * t \\ &= (r_2 * t)(r_1 * t). \end{aligned}$$

By the definition of admissibility (Definition 3.2.1) it follows that $(r_1 r_2) * t \in N(X^c)$.

By Lemma 3.4.8 there is reflection t_0 such that for each $r \in C(r_1, r_2)$ the length of $r t_0$ is larger than the length of t_0, and an element $u \in C(r_1, r_2)$ such that $t = t_0 * u$. Consider,

$$\begin{aligned} r_1 r_2 * t &= u^{-1} t_0 u r_1 r_2 u^{-1} t_0 u \\ &= u^{-1} t_0 r_1 r_2 t_0 u. \end{aligned}$$

This shows that it is sufficient to prove that $r_1 r_2 * t_0 \in L_+(X^c)$.

Let r be a reflection in $C(X^c)$. It needs to be shown that (cf. Lemma 3.4.2),

$$\ell(r t_0 r_1 r_2 t_0) > \ell(t_0 r_1 r_2 t_0). \tag{3.22}$$

By the results in Section 3.3 this is equivalent to $\beta \cdot t_0 r_1 r_2 t_0 < 0$, where β is the unique positive root corresponding to r (cf. Section 3.1). If $\beta \cdot t_0 = \beta$ then the statement (3.22) holds. Hence, we only need to consider reflections r that have a corresponding root that is not fixed by t_0.

For this purpose consider the group generated by t_0 and all reflections $r \in C(r_1, r_2)$ that do not commute with t_0. It contains r_1 and r_2 by choice of t. It is a reflection subgroup of W by definition, and by Theorem 3.3.4 it is a Coxeter group. Denote it by G, and let $\Gamma(G)$ denote its set of fundamental generators (cf. (3.14) in Section 3.3). We will show that $t_0, r_1, r_2 \in \Gamma(G)$. Let $r' \in C(r_1, r_2)$. Then,

$\ell_S(r't_0) > \ell_S(t_0$ by choice of t_0. Also, $\ell_S(r'r_1) > \ell_S(r_1)$ and ...

Moreover $(H, \Gamma(H))$ is a Coxeter system of type D_n or of type A_3. The case A_3 is trivial, since in that case there are no reflections in $C_+(r_1, r_2)$ that do not commute with t_0.

Let $t \in \Gamma(H')$. Then $t = f(r_1, r_2, t_0)$ for some function f.

`** Proof is unfinished **`

□

The final result now follows.

Theorem 3.4.10 *Let $X \in \mathcal{X}(W, S)$. Then $H(X^c)$ is generated by the union of X^c and the set containing exactly all $(r_1 r_2) * t$, where $r_1, r_2 \in X$ and $t \in \mathcal{R}(W, S)$ such that t commutes with neither r_1 nor r_2.*

Proof. The structure of the action by conjugation of $L_+(X^c)$ on X^c can be deduced from Table 2 of Chapter 3 of [13]. It is reprinted in the last column of Table 3.1, of this thesis. From the structure it follows in particular that, for each type and each orbit, $L_+(X^c)$ is generated by elements having cycle structure (2), or cycle structure (2)(2), as a permutation group on X^c.

Moreover, $L(X^c) = \langle X^c \rangle \rtimes L_+(X^c)$ by Lemma 3.4.2. Since $\langle X^c \rangle$ is also normal in $N(X^c)$ the group $H(X^c)$ is generated by X^c and the normal closure of $L_+(X^c)$ in $N(X^c)$. Hence, it is sufficient to show that $L_+(X^c)$ is generated by those $(r_1 r_2) * t$ that are elements of $L_+(X^c)$.

By Lemma 3.4.3 the group $L_+(X^c)$ acts on X^c by conjugation as a permutation group. Enumerate the element of X^c as follows,

$$X^c = \{r_1, \ldots, r_k\}.$$

Suppose, without loss of generality, that there is some a $w \in L_+(X^c)$ that interchanges r_2 and r_1 and fixes all others. Define Y to be $X^c \backslash \{r_1, r_2\}$. Then $w \in C(Y)$. Moreover, for each $r \in Y$ it holds that $\ell(rw) > \ell(w)$, since $w \in L_+(X^c)$. Thus, $w \in C_+(Y)$, by the fact that $\langle Y \rangle$ is normal in $C(Y)$ (cf. Lemma 3.4.3). Suppose without loss of generality that there exists a root system Φ of type M such that $W = \mathcal{W}(\Phi)$. By Lemma 3.4.1, part (iii), it holds that $C_+(Y) = \mathcal{W}(\Phi_Y^\perp \cap \Phi)$. Let β_i denote a root corresponding to r_i, for $i = 1, 2$. Since X^c is a commuting set, $\beta_1, \beta_2 \in \Phi_Y^\perp \cap \Phi$. Moreover, β_1 and β_2 are in the same irreducible component of $\Phi_Y^\perp \cap \Phi$, since $w \in \mathcal{W}(\Phi_Y^\perp \cap \Phi)$ and $\beta_1 \cdot w = \beta_2$. Hence, there exists a root $\gamma \in \Phi_Y^\perp \cap \Phi$ connecting β_1 and β_2. Denote the reflection corresponding to γ by t.

By Lemma 3.4.8 there is some reflection $t_0 \in C(Y)$ such that for each $r \in C(X)$ it holds that $\ell(rt_0) > \ell(t_0)$, that connects r_1 and r_2. By Lemma 3.4.9 it follows that $t' r_1 r_2 t' \in L_+(X^c)$. Since $w \in L_+(X^c)$ and the fact that $L_+(X^c)$ is a group (Theorem 3.3.5) it holds that $t_0 r_1 r_2 t_0 w^{-1} \in L_+(X^c)$. But also $t_0 r_1 r_2 t_0 w^{-1} \in C(X^c)$. Since the intersection of $L_+(X^c)$ and $C(X^c)$ is trivial, it must hold that $t_0 r_1 r_2 t_0 = w$.

Suppose, without loss of generality, that there is a $w \in L_+(X^c)$ that acts by conjugation on X^c as $(r_1, r_2)(r_3, r_4)$, and that there is no element in $L_+(X^c)$ acting as either (r_1, r_2) or (r_3, r_4). If there were, then this case reduces to the previous one. Let Y be $X \backslash \{r_1, r_2, r_3, r_4\}$. By Lemma 3.4.1 it holds that $C_+(Y) = \mathcal{W}(\oplus_{\mathcal{Y}}^{\perp} \cap \oplus$. Hence there is some reflection$t \in \mathcal{W}(\Phi_Y^{\perp} \cap \Phi)$ connecting r_1 and r_2, by the argument of the previous paragraph. Moreover, it needs to be connected to either r_3 or r_4, since if not then there would be some element in $L(X^c)$ interchanging r_1 and r_2, and fixing r_3 and r_4. Thus, r_1, r_2 and r_3 form a connected triple of reflections. Moreover, $r_4\,\mathrm{cp}(r_1, r_2, r_3)$, by unicity of the companion reflection. Thus tr_1r_2t acts as $r_1, r_2)(r_3, r_4)$ on X^c. Since $tr_1r_2tw^{-1} \in L_+(X^c) \cap C(X)$ it follows that $w = tr_1r_2t$.

This establishes the proposition. □

A final example follows, to illustrate most of the concepts defined in this section.

Example. Let $M = E_7$ and set $X = \{s_2, s_3, s_5, s_7\}$. The configuration is shown in Figure 3.4. The set X is strictly contained in its admissible closure X^c, which is a set containing exactly seven reflections. From Lemma 3.4.3 it follows that $L_+(X^c)$ is isomorphic to some subgroup of Σ_7. Table 2 in Chapter 3 of [13] shows that $L_+(X^c)$ is isomorphic to $L_3(2)$, the automorphism group of the Fano plane. Set $t_1, t_2, t_3 \in \mathcal{R}(W, S)$ as follows (to avoid lengthy subscripts we write $r(\beta)$ instead of r_β).

$$\begin{aligned} t_1 &= r(\alpha_4), \\ t_2 &= r(\alpha_4 + \alpha_5 + \alpha_6), \\ t_3 &= r(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6). \end{aligned}$$

Then t_1 is connected to s_2, s_3 and s_5, while t_2 is connected to s_2, s_3 and s_7. Finally t_3 is connected to s_3, s_5 and s_7. Note that there is no reflection connecting s_2, s_5 and s_7. The reason is that the root system perpendicular to α_2 and α_5 is disconnected, and α_7 is an element of the component of type A_1.

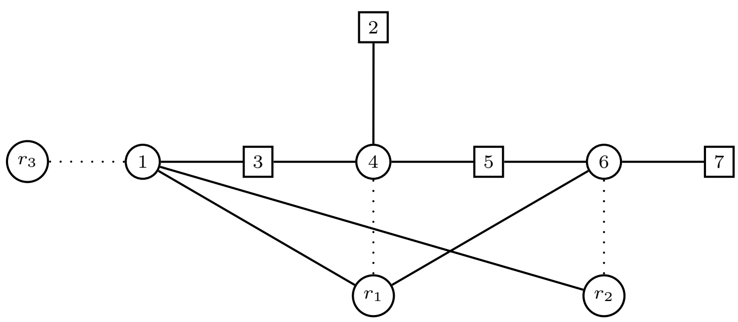


Figure 3.4: *The Coxeter graph E_7 to which a vertices representing r_1, r_2 and r_3 are added. Elements of X are represented by square boxes. A dotted line between vertices means the (positive) roots corresponding to those vertices have inner product 1.*

Hence, by Proposition 3.2.3 three triples have a companion reflection. Denote the companion reflections r_1, r_2 and r_3, respectively. Then

$$
\begin{aligned}
r_1 &= s_2 * ((s_3 s_5) * t_1) \\
&= r(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5), \\
r_2 &= s_2 * ((s_3 s_7) * t_2) \\
&= r(\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7), \\
r_3 &= s_3 * ((s_5 s_7) * t_3) \\
&= r(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7).
\end{aligned}
$$

The r_i are all different, and hence $X^c = X \cup \{r_1, r_2, r_3\}$, since the size of X^c is seven. Now consider the groups,

$$
\begin{aligned}
L_1 &= \langle (s_2 s_3) * t_1, (s_2 s_5) * t_1, (s_3 s_5) * t_1 \rangle, \\
L_2 &= \langle (s_2 s_3) * t_2, (s_2 s_7) * t_2, (s_3 s_7) * t_2 \rangle, \\
L_3 &= \langle (s_3 s_5) * t_3, (s_3 s_7) * t_3, (s_5 s_7) * t_3 \rangle.
\end{aligned}
$$

They are all isomorphic to the Klein 4-group. Consider them to be subgroups of the permutation group on X^c, under the conjugation homomorphism described in Lemma 3.4.3. As such they move exactly four reflections. Let the vertices of the Fano plane be labeled by elements of X^c, as in Figure 3.5. Each group L_i fixes three reflections. These reflections correspond to lines in the Fano plane. The fixed line of L_1 is $\{r_3, s_7, r_2\}$, the fixed line of L_2 is $\{r_3, s_5, r_1\}$ and the fixed line of L_3 is $\{r_1, r_2, s_2\}$. Conjugation of the groups L_i by elements in $\langle L_1 \cup L_2 \cup L_3 \rangle$ results in other groups isomorphic to the Klein 4-group. This way we arrive at a total of seven groups, including L_1, L_2 and L_3. The fixed lines of these groups are exactly the lines of the Fano plane depicted in the figure.

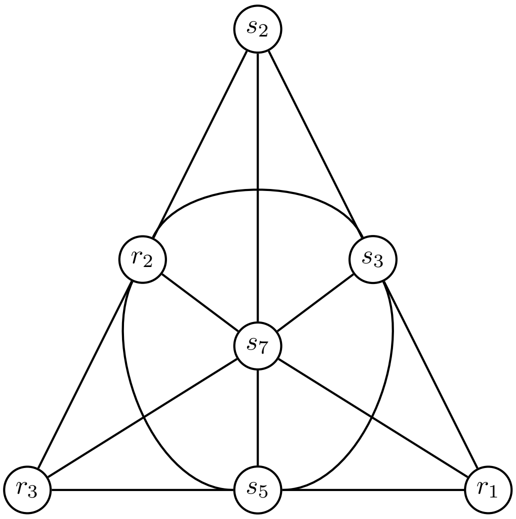


Figure 3.5: *Vertices of the Fano plane labeled by elements of* X^c

Chapter 4

Generalized Brauer algebras

This chapter is concerned with introducing the generalized Brauer algebras, for each type M as described above. The algebra of type A_n shall be shown to correspond to the classical centralizer algebra of Brauer as introduced in [5]. We shall not be concerned with this algebra in its role as the centralizer of some group representation on a tensor product. Instead we shall regard and introduce purely as algebra linearly generated by a set of diagrams.

The goal here is to apply the result of the previous chapter to the class of generalized Brauer algebras to construct a representation and finally exhibit a basis for the generalized Brauer algebra. The means by which to proceed are the simply laced Coxeter groups. In particular this basis is shown to correspond to a collection of triples (X, Y, w), where $X \in \mathcal{X}(W, S)$, $Y \in \Omega(X)$ and u is an element of W_{X^c}. In the case of the classical Brauer algebra this correspondence is known, and fairly easy to exhibit. This will be done in Section 4.1. The rest of the chapter is devoted to extending the results to the generalized Brauer algebras.

4.1 The classical Brauer algebra

For each natural number n let $[n]$ denote the set of all natural numbers smaller than or equal to n. The size of this set is n, since it does not include zero. Fix some natural number m, and consider the set of partitions of $[2m]$ whose elements all have size two. In [22] such a partition is called an m-connector. An m-connector is made into a diagram by arranging m nodes labeled $1, \ldots, m$ in a line and arranging m nodes labeled $m+1, \ldots, 2m$ in a line directly below the previous one. Two nodes are connected if they are in the same element of the partition. The procedure is illustrated in Figure 4.1.Without proof it is stated that it is a bijection.

Let $\mathbb{K}$ be a field and δ some parameter. Consider the vector space of all formal $\mathbb{K}(\delta)$ linear combinations of Brauer diagrams of degree m. It is be made into an associative algebra by defining a multiplication on the set of diagrams and extending it linearly. This product of diagrams is given by identifying the lower vertices of the first diagrams with the upper vertices of the second diagram. Circles are replaced

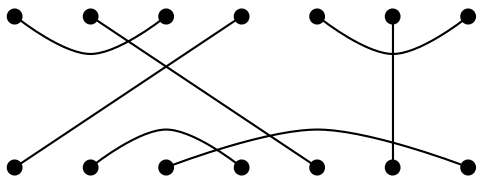


Figure 4.1: *A Brauer diagram of degree* 7 *corresponding to the* 7*-connector* $\{\{1,3\},\{2,12\},\{4,8\},\{5,7\},\{6,13\},\{9,11\},\{10,14\}\}$

fundamental transpositions
top strands
bottom strands
free points
free vertices

by δ. The structure of this algebra was examined in [14]. Results on the semisimplicity were obtained in [27] after a conjecture by Hanlon and Wales [14]. When we wish to refer to the Brauer algebra as an algebra of diagrams we denote it by $\mathrm{Br}_{\mathbb{K}}(m,\delta)$.

Brauer diagrams as triples

Let (W,S) be the Coxeter system of type A_{m-1}, represented as the symmetric group on m symbols generated by the *fundamental transpositions* $(i,i+1)$, where $i=1,\ldots,n-1$. Let b be a Brauer diagram. As announced in the introduction of this chapter there is a correspondence between triples of the from (X,Y,w), where $X\in\mathcal{X}(W,S)$, $Y\in\Omega(X)$ and $w\in W_{X^c}$, with Brauer diagrams.An example illustrates to which triple (X,Y,w) the diagram b corresponds. But first the group W_{X^c} is examined, and it is shown that W_{X^c} equals $C_+(X)$.

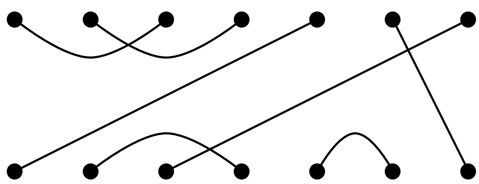


Figure 4.2: *The Brauer diagram b*

Put $m=7$ and let b be the diagram shown in Figure 4.2. There two strands connecting upper vertices (so called *top strands*) and two strands connecting lower vertices (so called *bottom strands*). The vertices in the top line not connected to a top strand are called *free points* or *free vertices*. For the diagram under consideration

$$\begin{aligned} T &= \{\{1,3\},\{2,4\}\},\\ B &= \{\{2,4\},\{5,6\}\}. \end{aligned}$$

Moreover, the free points are those labeled 5, 6 or 7. Set X equal to those transpositions interchanging the end vertices of the top strands, and do so similarly for Y with respect to the bottom strands, i.e.,

$$\begin{aligned} X &= \{(1,3),(2,4)\}, \\ Y &= \{(2,4),(5,6)\}. \end{aligned}$$

The elements of X commute mutually, as do the elements of Y. Moreover, both sets consist of reflections of the Coxeter system (W, S), since

$$(1,3) = (2,3)(1,3)(2,3), (2,4) = (2,3)(3,4)(2,3)$$

and $(5,6)$ is already an element of S. Hence, both X and Y are elements of $\mathcal{X}(W,S)$.

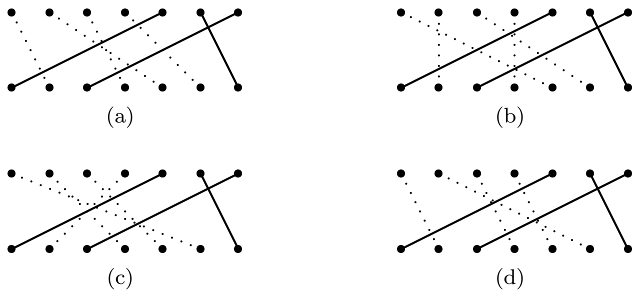


Figure 4.3: *Some choices of Brauer (permutation) diagrams for w. Solid lines represent strands connected to free points.*

Now, let w be the element of W whose restriction to the free points behaves as b, i.e.,

$$5 * w = 1, 6 * w = 7 \text{ and } 7 * w = 3.$$

Furthermore, it maps T to B (or equivalently, maps X to Y by conjugation). There are 8 possible choices for w, and some of them are depicted in Figure 4.3. In particular $Y \in \Omega(X)$. Put w equal to (d) of Figure 4.3, i.e.,

$$w = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 4 & 5 & 1 & 7 & 3. \end{matrix}$$

Then b can be decomposed as follows, in which E_X is the diagram that has both top and bottom strands corresponding to X, and for which the permutation on the free points is trivial (see Figure 4.4).

Now let w_0 be an element of minimal length mapping T to B, or rather mapping X to Y by conjugation. The length function is of course that of (W, S). Then $w \in N(X)w_0$. We shall proceed to determine $\pi_X(w)$, as defined in () and show that it is a natural choice for the third element of the triple, previously called u.

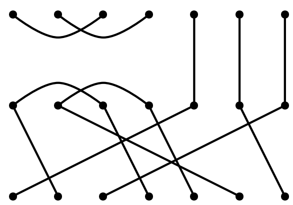


Figure 4.4: *The Brauer diagram b written as E_X times w*

Without proof it is stated that w_0 is unique and that it is given by

$$w_0 = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 4 & 6 & 1 & 3 & 7. \end{matrix} .$$

and hence ww_0^{-1},

$$ww_0^{-1} = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 3 & 2 & 5 & 7 & 6 \end{matrix} . \tag{4.1}$$

Remark. A method to find w_0 is to first draw lines connecting the ends of a top vertex to the ends of a bottom vertex, for each top vertex, such that there are as little crossings as possible. Now do the same thing for the vertices not yet connected, again making as little crossings as possible.

By the theorem on the decomposition of the normalizer (Theorem 3.3.5 and in particular (3.18)) there are $u \in C_+(X)$ and $v \in \langle X \rangle \rtimes L(X)$ such that $ww_0^{-1} = uv$. Moreover, such u and v are unique. The group $C_+(X)$ consists of all the transpositions commuting with all elements of X, divided out by elements of X themselves. In this case,

$$C_+(X) = \langle (5, 6), (6, 7) \rangle.$$

Hence, they are exactly the elements that act trivially on 1, 2, 3 and 4 (we will see that this does not hold for the other types). From (4.1) it follows that $u = (5)(6, 7)$, and that $v = (1)(3)(2, 4)$. Indeed, $ww_0^{-1} = uv$ and $v \in L(X)$. The triple corresponding to b is (X, Y, u) and the final decomposition of b is depicted in Figure 4.5.

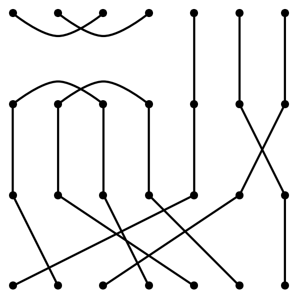


Figure 4.5: *The Brauer diagram b written as $E_X uw_0$*

Vice versa, given a triple (X, Y, u) it is possible to construct a Brauer diagram. The procedure is the inverse of the procedure displayed. Let w_0 be an element of minimal length mapping X to Y. Such an element exists by the condition that $Y \in \Omega(X)$. Then the Brauer diagram corresponding to (X, Y, u) is $E_X u w_0$.

Brauer monoid

Remark. Note that it was required that $u \in W(X^c)$, and not simply $u \in C_+(X)$. However, since $M = A_m$, W_{X^c} is $C_+(X)$. This is because every set in $\mathcal{X}(W, S)$ is admissible, and because $L(X)$ is already normal in $N(X)$.

4.2 Definition

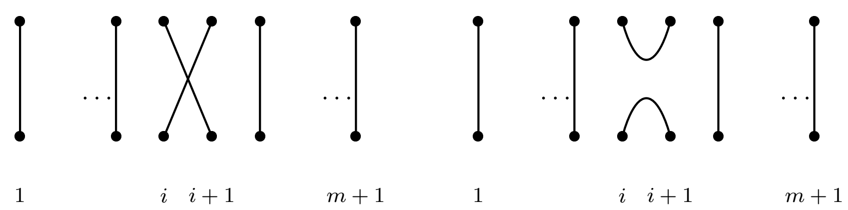
Let M be a simply laced, spherical Coxeter graph. Fix a parameter δ. Let S and E be sets of symbols indexed by the vertices of M, i.e.,

$$\begin{aligned} S &= \{s_i \mid i \text{ is a vertex of } M\}, \\ E &= \{e_i \mid i \text{ is a vertex of } M\}. \end{aligned}$$

The *Brauer monoid* of type M is the monoid defined by the relations in Table 4.1 in the free monoid generated by $E \cup S \cup \{\delta\}$. In the case that $M = A_m$ this monoid is thought of as the set of Brauer diagrams of degree $m + 1$ multiplied by a power of δ. The specific correspondence is given by

$$\begin{aligned} e_i &\mapsto E_i \\ s_i &\mapsto S_i, \end{aligned}$$

where E_i and S_i are depicted in Figure 4.6. That this correspondence is indeed an isomorphism was already proved by Brauer in 1937 [5]. The proof will not be repeated here. The distinction between the (classical) Brauer diagram monoid of degree $m+1$ and the Brauer monoid of type A_m is conceptual in the sense that the first is thought of as an monoid generated by diagrams, while the second is thought of as an abstract free monoid in which certain relations hold.



(a) S_i (b) E_i

Figure 4.6: *Fundamental generators for the classical Brauer algebra*

In particular does the existence of the diagram monoid mean that there is indeed a structure satisfying the relations in the case that $M = A_m$, namely the monoid generated by the Brauer diagrams and δ. This is not all that clear in the other cases. In fact there is no guarantee that there exist models for the Brauer monoid that are not trivial. However, in Section 4.3 such a model is constructed.

Brauer algebra
fundamental reflections
fundamental elements

Br1	δs_i	$=$	$s_i\delta$
Br2	δe_i	$=$	$e_i\delta$
W1	s_i^2	$=$	1
Br3	$e_i s_i$	$=$	e_i
Br4	$s_i e_i$	$=$	e_i
Br5	e_i^2	$=$	δe_i
	for $i \not\sim j$		
W2	$s_i s_j$	$=$	$s_j s_i$
Br6	$e_i s_j$	$=$	$s_j e_i$
Br7	$e_i e_j$	$=$	$e_j e_i$
	for $i \sim j$		
W3	$s_i s_j s_i$	$=$	$s_j s_i s_j$
Br8	$s_j e_i s_j$	$=$	$s_i e_j s_i$
Br9	$s_j s_i e_j$	$=$	$e_i e_j$

Table 4.1: *Defining relations for the Brauer monoid of type M.*

Let $\mathbb{K}$ be a field, and denote the field of fractions of δ over $\mathbb{K}$ by $\mathbb{K}(\delta)$. The *Brauer algebra* of type M over $\mathbb{K}(\delta)$ is the algebra linearly generated over $\mathbb{K}(\delta)$ by the elements of the Brauer monoid of type M. It is denoted $\mathrm{Br}_{\mathbb{K}}(M, \delta)$, but usually the shorter form $\mathrm{Br}(M)$ is used, unless confusion is likely to occur. Note that the base field of $\mathrm{Br}(M)$ is $\mathbb{K}(\delta)$ (as opposed to $\mathbb{K}$).

The next theorem binds this chapter to the previous one. It is an extension of the fact that the permutation diagrams are Brauer diagrams.

Theorem 4.2.1 *Let M be simply laced and let W denote the submonoid of the Brauer monoid of type M generated by S. Then (W, S) is a Coxeter system of type M. Moreover, the subalgebra of* $\mathrm{Br}(M)$ *generated by W is isomorphic to the group algebra of W over $\mathbb{K}(\delta)$.*

From this point onward, W always denotes the submonoid of the Brauer monoid generated by S and (W, S) is always thought as Coxeter system of simply laced type. In particular, we speak of the set of reflections of (W, S) and of the geometric representation of W. The collection $\mathcal{X}(W, S)$ has the same meaning as it had in Section 3.2. In a sense $\mathrm{Br}(M)$ is thought of as the group algebra of the Coxeter group of type M to which a set of (nearly) idempotent elements is added.

Analogously to the terminology of the Coxeter groups the elements of S are called the *fundamental reflections*. The elements of $E \cup S$ are called the *fundamental elements*.

Example. Let $M = A_2$. Then $\mathrm{Br}(M)$ is the algebra generated by s_1, s_2 and e_1, e_2 subject to the relations of Table 4.1. The monoid $W = \langle s_1, s_2 \rangle$ is the Coxeter group of type A_2, which is isomorphic to the symmetric group on 3 symbols. Under the isomorphism $s_i \mapsto S_i$, $e_i \mapsto E_i$ the elements of W correspond to permutation diagrams, i.e., those Brauer diagrams in which upper vertices are only connected to lower vertices.

Remark. At this point a natural question would be if there also exist Brauer algebras whose type is a Coxeter graph with edges that have a label larger than 3 (non simply-laced type). The answer to this question is not clear. The relations Br8 and Br9 have no known equivalent for non-simply laced types. However, work has been done to construct a Temperley-Lieb algebra of non simply-laced type type B_m (cf. [26]) and a BMW algebra of type B_m (cf. [15]).

4.3 A representation

A definition of the generalized Brauer algebra in terms of generators and relations was given in the previous section. However, it is not a priori clear that it defines a non-trivial algebra. In this section it will be shown that there indeed exists an algebra that is isomorphic to a quotient of the generalized Brauer algebra. It is a subalgebra of the endomorphism algebra of a monoid whose generators are parametrized by the sets in an admissible orbit. In fact, the map from the abstractly defined Brauer algebra into this subset is an representation.

Let $\Omega \subseteq \mathcal{X}(W, S)$ be an admissible orbit under W. Denote its top element $\top$. Such an element exists by Proposition 3.2.1. Associate with each $X \in \Omega$ a symbol ξ_X. Let M_Ω denote the free module over $\mathbb{K}[W(\top)]$ generated by ξ_X, with $X \in \Omega$. Note that it is also a vector space over $\mathbb{K}$.

Remark. In the case that $M = A_m$ the symbol ξ_X is thought of as the diagram $w_X E_X$, in which E_X is the diagram introduced in the example of Section 4.1 and w_X is a minimal element of W that maps X to $\top$.

The maps τ_X and η_X are at the basis of the representation. They were defined in the second part of Section 3.4 (see (3.20). Note that τ_X maps an element $w \in W$ to an element of minimal length in $N(X)w$ and that η_X maps w to $w\tau_X(w)^{-1}H(X)$.

Define a map $\rho_\Omega : W \to \mathrm{End}(M_\Omega)$ by the following expression.

$$\xi_X \rho_\Omega(w) = \bar{\eta}_X(w)\xi_{X*w}. \tag{4.2}$$

It is a group homomorphism, as will be shown shortly. By Lemma 3.4.7 it follows that

$$\begin{aligned} \eta_\top(w_X^{-1}w) &= \eta_\top(w_X^{-1})(\eta_X(w) * \tau_\top(w_X^{-1})^{-1}) \\ &= \eta_X(w) * w_X. \end{aligned}$$

This allows us to prove that $\rho_\Omega : W \to \mathrm{End}(M_\Omega)$ is a homomorphism of groups and hence a representation of W on M_Ω. For ease of notation we will henceforth write $\tilde{\eta}_X(w)$ instead of $\eta_X(w) * w_X$, i.e.,

$$\tilde{\eta}_X : w \mapsto \eta_X(w) * w_X. \tag{4.3}$$

Proposition 4.3.1 *Let $\Omega \subseteq \mathcal{X}(W,S)$ be an admissible orbit under the conjugation action of W. Then ρ_Ω is a representation of W.*

Proof. We need to show that ρ_Ω is a homomorphism of groups into $\mathrm{End}(M_\Omega)$. Let $X \in \Omega$ and $v, w \in W$. Then,

$$\begin{aligned}
\xi_X \rho_\Omega(vw) &= \eta_X(vw) * w_X \xi_{X*vw} \\
&= (\eta_X(v) * w_X)(\eta_{X*v}(w) * \tau_X(v)^{-1} w_X)\xi_{X*vw} \\
&= \eta_X(v) * w_X)(\eta_{X*v}(w) * w_{X*v})\xi_{X*vw} \\
&= \eta_X(v) * w_X \xi_{X*v}\rho_\Omega(w) \\
&= \xi_X \rho_\Omega(v)\rho_\Omega(w).
\end{aligned}$$

□

Next ρ_Ω is extended to $\mathrm{Br}(M)$. Its extension will also be denoted ρ_Ω. This is done by first specifying the images of the Brauer monoid under the extension and then requiring that ρ_Ω is a linear map. Specification on the Brauer monoid is by giving the images of δ and of the fundamental elements e_i and requiring that ρ_Ω factors through the product. We will end up with a representation satisfying the following conditions. Let a and b be elements of the Brauer monoid.

$$\begin{aligned}
\rho_\Omega(ab) &= \rho_\Omega(a)\rho_\Omega(b) \\
\rho_\Omega(a+b) &= \rho_\Omega(a) + \rho_\Omega(b).
\end{aligned}$$

First, set $\rho_\Omega(\delta) = D$, in which D is an (invertable) element of the center of $\mathrm{End}(M_\Omega)$, e.g. $D = \mathrm{id}$. Define ρ_Ω on e_i as follows.

$$\xi_X \rho_\Omega(e_i) = \begin{cases} \xi_X D & \text{if } s_i \in X \\ 0 & \text{if } s_i \in C_+(X) \\ \tilde{\eta}_X(s_i r)\xi_{X*s_i r} & \text{where } r \in X \text{ and } r \sim s_i. \end{cases} \tag{4.4}$$

Here $r \sim s_i$ denotes that r and s_i do not commute. At first glance it may not be obvious that the map is well defined if s_i is neither in $C_+(X)$ nor in X, since it involves a choice of r. For the explanation we need to recall the last part of Section 3.4, in which a set of generators for $H(X^c)$ was exhibited. These generators are of the form tr_1r_2t, where $r_1, r_2 \in X$ and t is a reflection connecting them.

Since X is assumed to be admissible, $X^c = X$, and $H(X) \trianglelefteq N(X)$. Thus $X * s_ir_1r_2s_i = X$. Rewriting the expression yields, $X * s_ir_1 = X * s_ir_2$. This shows that the index of ξ does not depend on the specific choice of r. The coefficient $\eta_X(s_ir)$ is left.

It needs to be shown that $\eta_X(s_ir_1) = \eta_X(s_ir_2)$. Note that $\tau_X(s_ir_1) = \tau_X(s_ir_2)$, since $X * s_ir_1 = X * s_ir_2$. Thus,

$$\begin{aligned}
\eta_X(s_ir_1) &= s_ir_1\tau^{-1}H(X), \\
\eta_X(s_ir_2) &= s_ir_2\tau^{-1}H(X),
\end{aligned}$$

in which $\tau = \tau_X(s_i r_1)$. Hence,

$$s_i r_1 \tau^{-1}(s_i r_2 \tau^{-1})^{-1} \quad = \quad s_i r_1 r_2 s_i,$$

which is an element of $H(X)$. This proves that $\eta_X(s_i r_1) = \eta_X(s_i r_2)$ and hence that the map ρ_Ω is well defined on $\mathrm{Br}(M)$.

Theorem 4.3.2 *The function ρ_Ω is a a representation of the Brauer algebra* $\mathrm{Br}(M)$.

Proof. To show that ρ_Ω is a representation of $\mathrm{Br}(M)$ we need to show that it respects the relations in Figure 4.1.

Firstly, it is trivial that the relations Br1 and Br2 are respected. Also, that W1, W2 and W3 are respected was shown in Lemma 4.3.1. In the proofs we leave out ρ_Ω. Instead of $\xi_X \rho_\Omega(b)$ we simply write $\xi_X b$.

- **Br3.** The cases $s_i \in X$ or $s_i \in C_+(X)$ are trivial. Assume $s_i \notin C_+(X) \cup X$. Note that s_i is an element of minimal length in $N(X)s_i$. Thus, $\eta_X(s_i) = 1$, by Lemma 3.4.7, part (ii). Take $t \in X$ such that $t \sim s_i$.

$$\begin{aligned}
\xi_X e_i s_i &= \eta_X(s_i t) * w_X \xi_{X * s_i t} s_i \\
(1) &= \eta_X(s_i t s_i) * w_X \xi_{X * s_i t s_i} \\
(\mathrm{W}'3) &= \eta_X(t s_i t) * w_X \xi_{X * t s_i t} \\
(2) &= \eta_X(s_i t) * w_X \xi_{X * s_i t} \\
&= \xi_X e_i.
\end{aligned}$$

- **Br4.** The cases $s_i \in X$ or $s_i \in C_+(X)$ are trivial. Assume $s_i \notin C_+(X) \cup X$. Take $t \in X$ such that $t \sim s_i$.

$$\begin{aligned}
\xi_X s_i e_i &= \eta_X(s_i) * w_X \xi_{X * s_i} e_i \\
&= (\eta_X(s_i) * w_X)(\eta_{X * s_i}(s_i t s_i t) * w_{X * s_i}) \xi_{X * s_i s_i t s_i t} \\
(1) &= \eta_X(s_i s_i s_i t s_i) * w_X \xi_{X * s_i s_i (s_i t s_i)} \\
(\mathrm{W}'1) &= \eta_X(s_i t s_i) * w_X \xi_{X * s_i t s_i} \\
(\mathrm{W}'3) &= \eta_X(t s_i t) * w_X \xi_{X * t s_i t} \\
(2) &= \eta_X(s_i t) * w_X \xi_{X * s_i t} \\
&= \eta_X e_i.
\end{aligned}$$

- **Br5.** From now on let $\tilde{\eta}_X(w)$ denote $\tilde{\eta}_X(w)$. Again the case $s_i \in C_+(X) \cup X$ is trivial. Assume that it is not the case. Take $t \in X$ such that $t \sim s_i$. Then,

$$\begin{aligned}
\xi_X e_i^2 &= \tilde{\eta}_X(s_i t) \xi_{X * s_i t} e_i \\
&= (\tilde{\eta}_X(s_i t) \xi_{X * s_i t} D \\
&= (\xi_X e_i) D \\
&= (\xi_X D) e_i \\
&= \xi_X \delta e_i.
\end{aligned}$$

The third equation is because D is contained in the center of $\mathrm{End}(M_\Omega)$ and hence commutes with $\rho_\Omega(e_i)$, here written as e_i. The second equation is $s_i \in X * s_i t$, namely $s_i = t * s_i t$ by W$'$3 and W$'$1.

- **Br6.** If $s_i \in C_+(X)$, then $s_i \in C_+(X * s_j)$. Then both sides map to zero. If $s_i \in X$, then $s_i \in X * s_j$. Thus both sides map to $\xi_X s_j D$. Assume that $s_i \notin X \cup C_+(X)$. Then

$$\begin{aligned} \xi_X e_i s_j &= \tilde{\eta}_X(s_i t)\xi_{X*s_i t} s_j \\ (1) \quad &= \tilde{\eta}_X(s_i t s_j)\xi_{X*s_i t s_j}. \end{aligned}$$

  Note that $s_i t s_j = s_i s_j^2 t s_j = s_j s_i (s_j t s_j)$. This shows that $\xi_X e_i s_j = \xi_X s_j e_i$.

- **Br7.** Assume that $s_i \in C_+(X)$. Then $s_i \in C_+(X * s_j)$ and both the left hand and right hand side get mapped to 0 by ρ_Ω. Assume that $s_i \in X$. Then $s_i \in X * s_j$. Hence $\xi_X s_i s_j = (\xi_X s_j)D = \xi_X s_j s_i$. The case remaining is s_i *not* $\in C_+(X) \cup X$. By symmetry of the previous argument we can assume that $s_j \notin C_+(X) \cup X$. Hence $s_j \notin C_+(X * s_i) \cup X * s_i$. Let $t, t' \in X$ such that $s_i \sim t$ and $s_j \sim t'$.

  Assume that $t \not\sim s_j$ and $t' \not\sim s_i$. Then

$$\begin{aligned} \xi_X e_i e_j &= \tilde{\eta}_X(s_i t s_j t')\xi_{X*s_i t s_j t'} \\ &= \tilde{\eta}_X(s_j t' s_i t)\xi_{X*s_j t' s_i t} \\ &= \xi_X e_j e_i. \end{aligned}$$

  Assume without loss of generality that $t \sim s_j$. Then $t * s_i \sim s_j$. Thus,

$$\begin{aligned} \xi_X e_i e_j &= \tilde{\eta}_X(s_i t)\xi_{X*s_i t} e_j \\ &= \tilde{\eta}_X(s_i t s_j (t * s_i))\xi_{X*s_i t s_j (t*s_i)}. \end{aligned}$$

  Consider,

$$\begin{aligned} s_i t s_j (t * s_i) &= s_i t s_j s_i t s_i \\ &= s_i s_j s_j t s_j s_i t s_i \\ &= s_j s_i t s_j t t s_i t \\ &= s_j s_i t s_j s_i t \\ &= s_j s_i t s_i s_j t \\ &= s_j t s_i t s_j t \\ &= s_j t s_i (t * s_j). \end{aligned}$$

  Combining this with the expression obtained for $\xi_X e_i e_j$ we get,

$$\xi_X e_i e_j = \xi_X e_j e_i.$$

- **Br8.** Note that $e_t = s_j e_i s_j = s_i e_j s_i$, in which $t = s_i s_j s_i = s_j s_i s_j$. Hence, it is sufficient to show that the ρ_Ω image of e_t under the representation does not depend on the specific way of writing e_t.

Suppose $t \in C_+(X)$. Write $t = s_i * w$. Then $s_i \in C_+(X * w^{-1})$ by Lemma 3.4.4. Hence,

$$\begin{aligned} \xi_X e_t &= \xi_X w^{-1} e_i w \\ &= \tilde{\eta}_X (w^{-1} \xi_{X*w^{-1}} e_i w \\ &= 0. \end{aligned}$$

Suppose $t \in X$ and $t = s_i * w$. Then $t \in X * w^{-1}$. Thus,

$$\begin{aligned} \xi_X e_t &= \xi_X w^{-1} e_i * w \\ &= \tilde{\eta}_X (w^{-1}) \xi_{X*w^{-1}} e_i w \\ &= \tilde{\eta}_X (w^{-1}) \xi_{X*w^{-1}} w \\ &= \tilde{\eta}_X (w^{-1} w) \xi_{X*w^{-1} w} \\ &= \xi_X \end{aligned}$$

Suppose $t \notin C_+(X) \cup X$. Let $r \in X$ such that $t \sim r$. Write $t = s_i * w$ and $r = r' * w$ for some $r' \in X * w^{-1}$.

$$\begin{aligned} \xi_X e_t &= \xi_X w^{-1} e_i w \\ &= \tilde{\eta}_X (w^{-1}) \xi_{X*w^{-1}} e_i w \\ &= \tilde{\eta}_X (w^{-1} s_i r' w) \xi_{X*w^{-1} s_i r' w} \\ &= \tilde{\eta}_X (tr) \xi_{X*tr}. \end{aligned}$$

This proves that the image of e_t does not depend on the specific choice of w and i. Thus, ρ_Ω preserves the relation Br8.

- **Br9.** Instead of Br9 we prove that another relation is preserves and that this relation, together with Br8 implies Br9. It is the relation

$$e_i s_j s_i = e_i e_j,$$

which is the mirror image of Br9. Assume it holds and consider,

$$\begin{aligned} s_j s_i e_j &= s_j s_i e_j s_i s_i \\ (\text{Br8}) &= s_j s_j e_i s_j s_i \\ (\text{W}'1) &= e_i s_j s_i \\ &= e_i e_j. \end{aligned}$$

It is sufficient to show that $e_i s_j s_i = e_i e_j$ is preserved by ρ_Ω.

Assume that $s_i \in C_+(X)$. Then both sides are mapped to 0. Assume that

$s_i \in X$. Then,

$$\begin{aligned}
\xi_X e_i s_j s_i &= \xi_X s_j s_i D \\
&= \tilde{\eta}_X(s_j s_i)\xi_{X * s_j s_i} D \\
&= \xi_X e_j D \\
&= \xi_X D e_j \\
&= \xi_X e_i e_j.
\end{aligned}$$

Assume that $s_i \notin C_+(X) \cup X$. Let $r \in X$ such that $r \sim s_i$. Then $s_i \in X * s_i r$. Under ρ_Ω , the right hand side equals,

$$\begin{aligned}
\xi_X e_i e_j &= \tilde{\eta}_X(s_i r)\xi_{X * s_i r} e_j \\
&= \tilde{\eta}_X(s_i r s_j s_i)\xi_{X * s_i r s_j s_i}
\end{aligned}$$

The left hand side equals,

$$\begin{aligned}
\xi_X e_i s_j s_i &= \tilde{\eta}_X(s_i r)\xi_{X * s_i r} s_j s_i \\
&= \tilde{\eta}_X(s_i r s_j s_i)\xi_{X * s_i r s_j s_i}.
\end{aligned}$$

Both sides are equal. Thus the relation is preserved. Since Br8 was already shown to be preserved it follows that Br9 is preserved.

These calculations show that ρ_Ω is indeed a representation. $\square$

Example. Let Ω be the orbit of $X_1 = \{s_1, s_2\}$. The representation of $\mathrm{Br}(D_4)$ is calculated for the fundamental elements. From Table 1 of [13] it follows that $|\Omega| = 6$. Let $r(\beta)$ denote r_β. The elements of Ω are,

$$\begin{aligned}
X_1 &= \{s_1, s_2\}, \\
X_2 &= \{r(\alpha_2 + \alpha_3), r(\alpha_1 + \alpha_3)\}, \\
X_3 &= \{r(\alpha_3), r(\alpha_1 + \alpha_2 + \alpha_3)\}, \\
X_4 &= \{r(\alpha_3 + \alpha_4), r(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\}, \\
X_5 &= \{r(\alpha_4), r(\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4)\}, \\
X_6 &= \{r(\alpha_2 + \alpha_3 + \alpha_4), r(\alpha_1 + \alpha_3 + \alpha_4)\}.
\end{aligned}$$

Set $\xi_i = \xi_{X_i}$. The highest element of this orbit is X_6, and hence the type of $W(\top)$ is A_1. It is generated by $s_3 H(\top)$. Put $s = s_3 H(\top)$. Consider the element s_1. The images of X_i under s_1 are,

$$\begin{array}{rcl} X_1 * s_1 & = & X_1, \\ X_2 * s_1 & = & X_3, \\ X_3 * s_1 & = & X_2, \\ X_4 * s_1 & = & X_6, \\ X_5 * s_1 & = & X_5, \\ X_6 * s_1 & = & X_4. \end{array}$$

Note that if s_1 moves a set X_i it is a shortest element in the coset $N(X_i)s_1$. Hence, s_1 is represented by the matrix,

$$\rho_\Omega(s_1) = \begin{pmatrix} \mathrm{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \mathrm{s} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Now consider the image of e_1. Then

$$\begin{array}{rcl} \xi_1 & \mapsto & \xi_1 Z \\ \xi_2 & \mapsto & s\xi_1 \\ \xi_3 & \mapsto & \xi_1 \\ \xi_4 & \mapsto & s\xi_1 \\ \xi_5 & \mapsto & 0 \\ \xi_6 & \mapsto & s\xi_1. \end{array}$$

and thus,

$$\rho_\Omega(e_1) = \begin{pmatrix} \mathrm{D} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{s} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.4 Parametrization of the idempotents

This section serves to introduce a non-diagrammatic equivalent of the diagram E_X that was used in the example of the previous section, to illustrate the fact that Brauer diagrams correspond to triples. This is the first step to generalize the example to the other simply laced, spherical types. Remember that the diagram

E_X is the Brauer diagram whose top strands and bottom strands correspond to the transpositions (or reflections) in X. The strands on the free points go straightly downwards. The next lemma handles the case in which there is only one top strand.

Lemma 4.4.1 *Let i and j be vertices of M. For each u and v in W it holds that $e_i * u = e_j * v$ if and only if $s_i * u = s_j * v$.*

Proof. We start with the *if* direction. Note that $s_i * uv^{-1} = s_j$. Denote the smallest element mapping s_i to s_j under conjugation by w_0. Properties of this element are collected in Lemma 2.4.1. of [13]. In particular $e_i * w_0 = e_j$. Moreover, $uv^{-1} \in N(s_i)w_0$. Write $uv^{-1} = ww_0$, where $w \in N(s_i)$. Note that $N(s_i) = C(s_i)$, and that elements of $C(s_i)$ commute with e_i (cf. Lemma 2.4.9. of [13]). Thus,

$$\begin{aligned} e_i * uv^{-1} &= e_i * ww_0 \\ &= e_i * w_0 \\ &= e_j. \end{aligned}$$

The second identity is a consequence of Lemma 2.4.1. of [13]. This proves the *if* direction.

We continue with the *only if* direction. Suppose that $e_i * u = e_j * v$. Let w_0 be the element of smallest length mapping s_i to s_j under conjugation. Then, $e_j = e_i * w$ by (iv) of Lemma 2.4.1. of [13]. Thus,

$$e_i * (uv^{-1}w_0^{-1}) = e_i. \tag{4.5}$$

This means that $uv^{-1}w_0^{-1} \in N(s_i)$. Indeed, write $w' = uv^{-1}w_0^{-1}$ and $r = s_i * w'$. It shall be shown that $r = s_i$ and hence that $w' \in N(s_i)$.

Let $\Omega \subseteq \mathcal{X}(W,S)$ be the admissible orbit containing $\{r\}$ and $\{s_i\}$. By (4.5) it holds that $\rho_\Omega(e_i) = \rho_\Omega(e_i * w')$. In particular, they map ξ_r to the same element. Consider,

$$\begin{aligned} \xi_r \rho_\Omega(e_i * w') &= \tilde{\eta}_r(w'^{-1})\xi_{s_i}\rho_\Omega(e_i w') \\ &= \tilde{\eta}_r(w'^{-1})\xi_{s_i}\rho_\Omega(w')D \\ &= \tilde{\eta}_r(w'^{-1}w')\xi_r D \\ &= \xi_r D \end{aligned}$$

Thus, $\xi_r \rho_\Omega(e_i) = \xi_r D$. From the definition of the representation (4.4) it follows that $r \sim s_i$ or $r = s_i$. Assume the first. Then $\xi_r = \xi_{r*s_i r}$. In other words, $r = r * s_i r$. Calculation shows that $r * s_i = r$. Since, $r \sim s_i$ it holds that $s_i = r$.

This proves the *only if* direction. □

Choose a reflection $t \in \mathcal{R}(W,S)$ and a vertex i of M, such that there exists $w \in W$ such that $t = s_i * w$. Define the element e_t of $\mathrm{Br}(M)$ as follows.

$$e_t = e_i * w. \tag{4.6}$$

The previous lemma implies that the definition does not depend on the exact choice of w and i. It shows that e_t is well defined and that $t \mapsto e_t$ is a bijection of the set of reflections of (W, S) into the set of such elements e_t. Note that $e_i = e_{s_i}$.

The next step is to extend the definition of the previous paragraph to arbitrary commuting sets of reflections. The manner in which to proceed is straightforward. Let $X \in \mathcal{X}(W, S)$. Define the element e_X of $\mathrm{Br}(M)$ as follows.

$$e_X = \prod_{t \in X} e_t \tag{4.7}$$

This definition is unambiguous by virtue of (4.6) and the fact that e_t and e_r commute whenever r and t commute (cf. [13, Lemma 2.4.9.]). It is the element e_X that should be thought of as the equivalent of E_X. Indeed, under the correspondence $s_i \mapsto S_i$, $e_i \mapsto E_i$ it gets mapped to E_X, when $M = A_{m-1}$. The next lemma shows that e_X behaves reasonably well under conjugation by W, for any simply laced, spherical M.

Lemma 4.4.2 *Let X and Y be elements of $\mathcal{X}(W, S)$. For each u and v in W it holds that $e_X * u = e_Y * v$ if $X * u = Y * v$.*

Proof. Suppose $w \in W$ is such that $X * w = Y$. The sizes of X and Y are equal since they are in the same orbit under conjugation by W. Set $k = |X|$. Enumerate the elements of X and Y as $x_1, \ldots, x_k$, respectively $y_1, \ldots, y_k$, satisfying $x_i * w = y_i$. Then,

$$e_X * w = \prod_{i=1}^{k} (e_{x_i} * w) = \prod_{i=1}^{k} e_{y_i} = e_Y,$$

by Lemma 4.4.1. The lemma follows by setting $w = uv^{-1}$. Then $e_X * uv^{-1} = e_Y$, and hence $e_X * u = e_Y * v$. □

It is in general not true that the converse of the above lemma holds, as the following short example shows. This is an important difference between $\mathrm{Br}(A_m)$ and the Brauer algebras of other types. In $\mathrm{Br}(A_m)$ the converse does hold. In other types some sort of a converse to the previous lemma holds, involving admissible sets. Proving it requires results from the next paragraph, and is hence postponed until Section 4.6.

Example. Let $M = D_4$. Put $t = s_3 s_1 s_2 s_3 s_2 s_1 s_3$ and set $X = \{s_1, s_2, s_4\}$ and $Y = \{s_1, s_2, t\}$. Furthermore, denote $s_3 s_1 s_2 s_3$ by w. Then it holds that

$$\begin{aligned} e_X &= e_X w \\ &= e_1 e_2 e_4 w \\ &= e_1 e_2 w e_t \\ &= e_1 e_2 e_t \\ &= e_Y. \end{aligned}$$

However, $X * w \neq Y * w$.

Some relations in $\mathrm{Br}(M)$ involving the elements e_r are collected in the following lemma. Basically they extend the relations already described for fundamental elements.

Lemma 4.4.3 *Let r and t be non-commuting reflections of (W, S). The following relations hold in $\mathrm{Br}(M)$.*

(i) $e_r r = r e_r = e_r$,

(ii) $e_r t e_r = e_r$,

(iii) $r t e_r = e_t r t = e_t e_r$,

(iv) $e_r e_t e_r = e_r$

If r and t do commute, then $e_r t = t e_r$.

Proof. Only (i) and (ii) are proved. For proof of the other relations we refer the reader to Proposition 2.3.3. and Lemma 2.4.9. of [13].

(i) Since M is connected, there is some vertex i of M and some $w \in w$ such that $s_i * w = r$. By definition of e_r it holds that $e_r = e_i * w$. Moreover, $r = s_i * w$. Thus,

$$\begin{aligned} e_r r &= (e_i * w)(s_i * w) \\ &= (e_i s_i) * w \\ &= e_i * w \\ &= e_r. \end{aligned}$$

Now $r e_r = (e_r r)^{\leftarrow} = e_r^{\leftarrow} = e_r$.

(ii) Let i and j be connected vertices of M and $w \in W$ such that $s_i * w = r$. Denote $s_j * w$ by t'. Then

$$\begin{aligned} e_r t' e_r &= (e_i s_j e_i) * w \\ &= e_i * w \\ &= e_r \end{aligned}$$

.

There are two possible cases, either t commutes with t' or it doesn't. Assume the first case. It shall be reduced to the second. Consider,

$$\begin{aligned} e_r t e_r &= e_r r t r e_r \\ &= e_r (t * r) e_r. \end{aligned}$$

The reflection $t * r$ does not commute with t', since r does not commute with t' and t does.

Let γ, γ' and β be roots corresponding to t, t' and r. Set $\delta = \gamma \cdot t'$. Then,

$$\begin{aligned}(\delta, \beta) &= (\gamma - (\gamma, \gamma')\gamma', \beta) \\ &= (\gamma, \beta) - (\gamma, \gamma')(\gamma', \beta) \\ &= -2, 0, 2,\end{aligned}$$

since all inner products in the expression are non-zero. It follows that $\delta = \pm\beta$ or $\delta \perp \beta$. In either case $e_r * r_\delta = e_r$. Thus,

$$\begin{aligned}e_r t e_r &= (e_r * r_\delta) t (e_r * r_\delta) \\ &= (e_r (r_\delta * t) e_r) * r_\delta \\ &= (e_r t' e_r) * r_\delta \\ &= e_r * r_\delta \\ &= e_r,\end{aligned}$$

in which the equations are consequences of the calculations above. This proves the statement.

□

In the next lemma the image of e_X under the representation ρ is considered. It states that the elements e_X have some of the properties of projections.

Lemma 4.4.4 *Let $X \in \mathcal{X}(W, S)$ be an admissible set and fix a set $Y \in \mathcal{X}(W, S)$. Note that Y is not necessarily admissible. Then*

$$\xi_X e_Y = w \xi_Z \delta^k \text{ or } \xi_X e_Y = 0,$$

in which $k \in \mathbb{N}$, $k < |Y|$ and $Z \in \mathcal{X}(W, S)$ such that $Y \subseteq Z$. Moreover, $k = |Y|$ if and only if $Y \subseteq X$.

Proof. The proof is by induction on the size of Y.

Suppose that $|Y| = 1$. There exists a reflection r such that $Y = \{r\}$ and hence $e_Y = e_r$. There are three cases to consider.

(1) If $r \in X$ then $\xi_X e_Y = \xi_X e_r = \xi_X \delta$. In this case $k = 1$ and $Z = X$. Thus $Y \subseteq Z$ and $k = |Y|$.

(2) If $r \in C_+(X)$ then $\xi_X e_Y = \xi_X e_r = 0$. The proposition holds in this case as well.

(3) If $r \notin C(X)$, i.e., $r \notin C_+(X)$ and $r \notin X$. There is $t \in X$ such that $r \sim t$. Then

$$\begin{aligned}\xi_X e_Y &= \xi_X e_r \\ &= \tilde{\eta}_X(rt) \xi_{X * rt}.\end{aligned}$$

Obviously, $\tilde{\eta}_X(rt) \in W(\top)$. Moreover, $t * rt = r$, and hence $r \in X * rt$. Here $k = 0$.

Let $|Y| > 1$ and suppose the proposition is proved for all elements of $\mathcal{X}(W,S)$ of cardinality smaller than Y. Let $r \in Y$ and set $Y_0 = Y \backslash \{r\}$. We compute

$$\xi_X e_Y = \xi_X e_{Y_0} e_r \tag{4.8}$$

By the induction hypotheses there are two cases to consider, either $\xi_X e_{Y_0} = 0$ or $\xi_X e_{Y_0} = w \xi_Z \delta^k$ with $Y_0 \subseteq Z$, $w \in W(\top)$ and $k \leq |Y_0|$. In the first case $\xi_X e_Y = 0$, by (4.8).

Consider the second case. Then

$$\xi_X e_Y = w \xi_Z \delta^k e_r = w \xi_Z e_r \delta^k, \tag{4.9}$$

in which $Z = Y_0 \cup Z_0$ for some $Z_0 \in \mathcal{X}(W,S)$. Since Y is a set of commuting reflections and $Y_0 \subseteq Y$ it holds that r commutes with the elements of Y_0. Moreover, every element of Z_0 commutes with the elements of Y_0 since Z is a set of commuting reflections. Again, there are three cases to consider,

(1) $r \in Z$. Then $\xi_X e_Y = w \xi_Z \delta^{k+1}$. Since, by the induction hypothesis $k \leq |Y_0| = |Y| - 1$ it holds that $k + 1 \leq |Y|$. Moreover, $Y = Y_0 \cup \{r\} \subseteq Z$. This proves the proposition in this case.

(2) $r \in C_+(Z)$. Then $\xi_Z e_r = 0$ and hence $\xi_X e_Y = 0$ by (4.9).

(3) $r \notin C(Z)$. This is equivalent to $r \notin Z$ and $r \notin C_+(Z)$. Let $t \in Z$ such that $t \sim r$.

$$\begin{aligned} \xi_X e_Y &= w \xi_Z e_r \delta^k \\ &= w \tilde{\eta}_Z(rt) \xi_{Z * rt} \delta^k \end{aligned}$$

Obviously, $w\tilde{\eta}_Z(rt) \in W(\top)$. Moreover, $Z * rt = Z_0 * rt \cup Y_0 * rt$. As proved earlier $Y_0 * rt = Y_0$. Hence $Z * rt = Z_0 * rt \cup Y_0$. Again, $r \in Z_0 * rt$, since $t * rt = r$. Thus, $Y = Y_0 \cup \{r\} \subseteq Z * rt$. Furthermore, $k \leq |Y_0| < |Y|$. This proves the statement.

Suppose that $k = |Y|$. Enumerate the elements of Y as follows

$$Y = \{r_1, \ldots, r_l\}.$$

Define $Z_i \in \mathcal{X}(W,S)$, $w_i \in W(\top)$ and $k_i \in \mathbb{Z}$ by the following expression.

$$\xi_X e_{\{r_1,\ldots,r_i\}} = w_i \xi_{Z_i} \delta^{k_i}.$$

Then $k_i \leq i$ and $0 \leq k_{i+1} - k_i \leq 1$. Since $k_l = l$ it must hold that $k_i = i$ and thus $k_{i+1} = k_i + 1$. The only case in the induction in which this can happen is case 1. Thus, $r_{i+1} \in Z_i$ for each i and $Z_i = Z_{i+1}$. In particular, $Z_i = Z_0 = X$ and $Y \subseteq X$.

This proves the statement. □

4.5 Annihilated sets

annihilated set
centralizer

Let $X \in \mathcal{X}(W,S)$. The *annihilated set* of e_X, denoted $A(e_X)$, is defined by

$$A(e_X) = \{w \in W \mid e_X w = e_X\}. \tag{4.10}$$

and the *centralizer* of e_X, denoted $C(e_X)$, is defined by

$$C(e_X) = \{w \in W \mid e_X w = w e_X\}. \tag{4.11}$$

The aim of this paragraph is to reveal the structure of the annihilated set and the centralizer of e_X, for all $X \in \mathcal{X}(W,S)$. The main result is that both sets have an easy description in terms of the decomposition of the normalizer of the admissible closure of X (cf. Theorem 4.5.4). Note that it already follows from the previous paragraph that the group generated by X is annihilated by e_X and that the normalizer of X in W is contained in the centralizer of e_X, i.e.

$$\langle X\rangle \subseteq A(e_X) \text{ and } N(X) \subseteq C(e_X).$$

In the next lemma it is proved by straightforward calculation that both $A(e_X)$ and $C(e_X)$ are groups.

Lemma 4.5.1 *Both $A(e_X)$ and $C(e_X)$ are subgroups of W. Moreover, $A(e_X)$ is a normal subgroup of $C(e_X)$.*

Proof. The statements follow by simple calculations. Let $u, v \in C(e_X)$. Then $e_X uv = u e_X v = uv e_X$ and hence $uv \in C(e_X)$. Consider, $(e_X v^{-1})^{\leftarrow} = v e_X = e_X v = (v^{-1} e_X)^{\leftarrow}$. It follows that $v^{-1} \in C(e_X)$.

Let $u, v \in A(e_K)$. Then $e_K uv = e_K v = e_K$. Hence, $uv \in A(e_K)$. Consider, $e_K = e_K v v^{-1} = e_K v^{-1}$. Hence, $v^{-1} \in A(e_K)$. It follows that $A(e_K)$ is a group. To show that it is a subgroup of $C(e_K)$ consider $e_K v = e_K = (e_K)^{\leftarrow} = (e_K v^{-1})^{\leftarrow} = v e_K$. Hence, $v \in C(e_K)$.

Let $u, v \in A(e_X) \cap C(e_X)$. Then $e_X uv = e_X v = e_X$ and $uv e_X = u e_X v = u e_X = e_X u = e_X$. Hence, $uv \in A(e_X) \cap C(e_X)$. Consider, $(e_X v^{-1})^{\leftarrow} = v e_X = e_X$ and $(v^{-1} e_X)^{\leftarrow} = e_X v = e_X$. Hence $v^{-1} \in C(e_X) \cap A(e_X)$.Let $u \in C(e_X) \cap A(e_X)$ and $w \in C(e_X)$. Then $e_X w^{-1} u w = w^{-1} e_X u w = w^{-1} e_X w = e_X$ and $w u w^{-1} e_X = w u e_X w^{-1} = w e_X w^{-1} = e_X$. Hence, $u * w = C(e_X) \cap A(e_X)$ and the last statement follows. ☐

The rest of this section relates the objects defined in Section 3.4 to the generalized Brauer algebras.

Lemma 4.5.2 *Let t be a reflection of (W,S). For all r_1 and r_2 in X it holds that $(r_1 r_2) * t \in A(e_X)$.*

Proof. This proof is by simple calculation, using Lemma 4.4.3. Consider the expression $e_{r_1}e_{r_2}tr_1r_2t$,

$$\begin{aligned} e_{r_1}e_{r_2}tr_1r_2t &= r_{r_2}e_{r_1}tr_1r_2t \\ &= e_{r_2}e_{r_1}e_tr_2t \\ &= e_{r_2}e_{r_1}e_te_{r_2} \\ &= e_{r_1}e_{r_2}e_te_{r_2} \\ &= e_{r_1}e_{r_2}. \end{aligned}$$

This proves the lemma. □

The next lemma is independent of the previous one. The statement it expresses is surprising from the classical point of view. This is because every set in $\mathcal{X}(W,S)$ is admissible, if (W,S) is of type A_n. Note that the derived set X' of X was defined as $X' = X^c \backslash X$ (cf. (3.7)).

Lemma 4.5.3 *Let $X \in \mathcal{X}(W,S)$. Then it holds that*

$$e_{X^c} = \delta^{|X'|}e_X.$$

Proof. Let $r \in X'$. Then there exist a connected triple of reflections r_1, r_2 and r_3 in X such that $r = \mathrm{cp}(r_1, r_2, r_3)$, by Theorem 3.2.6. Let t be a reflection of (W,S) connecting r_1, r_2 and r_3. Then $r = r_1 * tr_2r_3t$. Morover, by Lemma 4.5.2 the element tr_1r_2t is contained in $A(e_X)$. Thus,

$$\begin{aligned} \delta e_Xtr_2r_3t &= e_Xe_{r_1}tr_2r_3t \\ &= e_Xtr_2r_3te_r \\ &= e_Xe_r. \end{aligned}$$

This procedure can be repeated for every $r \in X'$, and the lemma follows. □

Recall that $H(X)$ is defined to be the normal closure of $L(X)$ in $N(X)$. See Section 3.4

Theorem 4.5.4 *Let $X \in \mathcal{X}(W,S)$. Then $A(e_X) = H(X^c)$ and $C(e_X) = N(X^c)$.*

Proof. That $H(X^c) \subseteq A(e_X)$ is a direct consequence of Lemma 3.4.9 and Lemma 4.5.2. By Lemma 4.5.1 also $H(X^c) \subseteq C(e_X)$. By Lemma 2.4.9. of [13] and the fact that $C_+(X) \supseteq C_+(X^c)$ the statement $N(X^c) \subseteq C(e_X)$ holds.

The other inclusions follow from the representation constructed in the Section 4.3. Indeed, let k be the cardinality of the derived set X' of X, i.e. $k = |X^c \backslash X|$. By Lemma 4.5.3 it holds that $\delta^{-k}e^c_X = e_X$. Let $w \in W$ and consider e_Xw. Let

$\Omega \subseteq \mathcal{X}(W,S)$ denote the orbit of X^c. Then,

$$\begin{aligned}
\xi_{X^c}\rho_\Omega(\delta^{-k}e_X^c w) &= \xi_{X^c}\rho_\Omega(e_X^c w)Z^{-k} \\
&= \xi_{X^c}\rho_\Omega(w)Z^{|X^c|-k} \\
&= \xi_{X^c}\rho_\Omega(w)Z^{|X|} \\
&= \eta_{X^c}(w) * w_X \xi_{X^c * w} Z^{|X|}.
\end{aligned}$$

Assume that $e_X = e_X w$. Then $\eta_{X^c}(w) = H(X^c)$ and $X^c * w = X^c$. By the second identity $w \in N(X^c)$ and by the first $w \in H(X^c)$. □

4.6 A linear spanning set

In the beginning of this chapter (cf. Section 4.1) we gave an example of how to write an element of the Brauer monoid of type A_m as a triple consisting of a set of reflections X, a set of reflections Y in the same orbit as X, and a group element w. In this section the procedure is generalized to arbitrary simply laced, spherical types. The main ingredients are the annihilated set of e_X, the centralizer of e_X and the map η_X.

Fix a set $X \in \mathcal{X}(W,S)$. For each $w \in W$ there is some element u of $\eta_X(w)$ such that

$$w = u\tau_{X^c}(w),$$

where $\tau_{X^c}(w)$ is an element of minimal length in the coset $N(X^c)\tau_X(w)$. Note that η_{X^c} is defined by (3.20). The next lemma describes the regular action of $\mathrm{Br}(M)$ on the elements of the form e_X.

Lemma 4.6.1 *Let $X \in \mathcal{X}(W,S)$, $w \in W$.*

(i) Let $u \in \eta_{X^c}(w)$. Then $e_X w = u e_X \tau_{X^c}(w)$.

(ii) The product $e_X e_t$ can be expanded according to the following distinction of cases.

$$e_X e_t = \begin{cases} \delta e_X & \text{if } t \in X \\ e_{X\cup\{t\}} & \text{if } t \in C_+(X) \\ e_X rt & \text{where } r \in X, \text{ if } t \notin X \cup C_+(X). \end{cases}$$

Proof.

(i) There is $h \in H(X^c)$ such that $u = w\tau_{X^c}(w)^{-1}h$, by definition of u. Thus,

$$\begin{aligned}
e_X w &= e_X w \tau_{X^c}(w)^{-1}\tau_{X^c}(w) \\
&= w\tau_{X^c}(w)^{-1} e_X \tau_{X^c}(w) \\
&= w\tau_{X^c}(w)^{-1} h e_X \tau_{X^c}(w) \\
&= u e_X \tau_{X^c}(w).
\end{aligned}$$

The third equality follows from $H(X^c) = A(e_X)$, as expressed in Theorem 4.5.4.

(ii) Suppose that $r \in X$. Then $e_X e_r = e_{X\setminus\{r\}} e_r e_r = \delta e_{X\setminus\{r\}} e_r = \delta e_X$.

Suppose that $r \in C_+(X)$. Then $r \notin X$, and hence $X \cup \{r\} \in \mathcal{X}(W,S)$. Thus $e_X e_r = e_{X\cup\{r\}}$.

Suppose that $r \notin X \cup C_+(X)$. Then there is some $t \in X$ such that t and r do not commute. By the $\mathrm{Br}(M)$ generalized variant of (6) in Chapter 2 of [13] it follows that $e_X e_r = e_{X\setminus t} e_t e_r = e_{X\setminus t} e_t rt = e_X rt$. It still needs to be shown that this is independent of t.

Let $t' \in X$ be another reflection that does not commute with r. Then $e_X rt'tr = e_X$, by Lemma 4.5.2. Thus, $e_X rt' = e_X rt$.

□

Proposition 4.6.2 *Every element b of the Brauer monomial of type M has a representation of the form*

$$b = \delta^k e_X v, \qquad X \in \mathcal{A}(W,S),\ v \in W \text{ and } k \in \mathbb{Z}.$$

This representation is unique up to choice of v in $H(X)v$.

Proof. Let $b \in \mathrm{Br}(M)$ be a monomial of the Brauer monomial M. There exists $n \in \mathbb{N}$ such that there are $v_1, \dots, v_{n+1} \in W$ and $X_1, \dots, X_n \in \mathcal{X}(W,S)$ such that

$$b = \delta^d v_1 e_{X_1} v_2 e_{X_2} \dots e_{X_n} v_{n+1},$$

for some $d \in \mathbb{Z}$. Note that $w_i = 1$ may hold. Assume without loss of generality that $d = 0$.

The rest of the proof is by induction on n. First, assume that $n = 0$. Then $b = v_1$, and the statement holds in this case. Now take $n > 0$, and assume that the statement is proved for all $n' < n$. Choose $Y \in \mathcal{X}(W,S)$, $l \in \mathbb{Z}$ and $u \in W$ such that

$$\delta^l e_Y u = v_1 e_{X_1} v_2 e_{X_2} \dots e_{X_{n-1}} v_n.$$

Then,

$$\begin{aligned} b &= \delta^l e_Y u e_{X_n} v_{n+1} \\ &= \delta^l e_Y e_{X_n * u^{-1}} u v_{n+1} \end{aligned}$$

By Lemma 4.6.1 there is a set $Y' \in \mathcal{X}(W,S)$, an element $u' \in W$ and an integer l' such that $e_Y e_{X_n} = \delta^{l'} e_{Y'} u'$. Thus,

$$b = \delta^{l+l'} e_{Y'} u' u v_{n+1}.$$

By Lemma 4.5.3 there is an integer l_0 such that

$$b = \delta^{l+l'-l_0} e_{Y'^c} u' u v_{n+1}.$$

This proves the first part of the proposition.

Suppose that $\delta^l e_Y w$ is another such representation. Consider the image on ξ_X. Then

$$\begin{aligned}\xi_X \delta^k e_X v &= (\xi_X e_X v) D^k \\ &= (\xi_X v) D^{k+|X|} \\ &= \bar{\eta}_X(v) \xi_{X*v} D^{k+|X|}\end{aligned}$$

This equals,

$$\begin{aligned}\xi_X \delta^l e_Y w &= \xi_X e_Y w D^l \\ &= \bar{u} \xi_{Z*v} D^{l'}.\end{aligned}$$

Moreover, $Y \subseteq Z$ by Lemma 4.4.4 It follows that $X * v \subseteq Y * w$ and by symmetry $X * v = Y * w$. Thus $Y = X * vw^{-1}$. Thus,

$$\begin{aligned}\delta^k e_X v &= \delta^l (e_X * vw^{-1}) w \\ \delta^k e_X v &= \delta^l w v^{-1} e_X v \\ \delta^k e_X &= \delta^l w v^{-1} e_X\end{aligned}$$

Hence, $k = l$ and $vw^{-1} \in H(X)$. This proves the proposition. $\square$

Let p be the map whose domain is the Brauer monomial of type M. It maps each element $b = \delta^k e_X w$ to the quadruple $(X, X * w, \eta_X(w), k)$. It is proved in the following theorem that it is an injection.

Theorem 4.6.3 *The map p is an injection. Its image is the union of the sets $\{X\} \times \Omega(X) \times W_X \times \mathbb{Z}_{|\delta|}$, where X ranges over $\mathcal{A}(W, S)$.*

Proof. Let $b \in \mathrm{Br}(M)$ be a Brauer monomial. By Proposition 4.6.2 there exist $X \in \mathcal{A}(W, S)$, $v \in W$ and $k \in \mathbb{Z}$ such that

$$b = \delta^k e_X v.$$

Let $a \in \mathrm{Br}(M)$ be another Brauer monomial and suppose that $p(a) = p(b)$. Then by the last statement of Proposition 4.6.2 there is $h \in H(X)$ such that

$$a = \delta^k e_X w, \text{ and } w = hv.$$

Thus $a = \delta^k e_X hv = \delta^k e_X v = b$, by Theorem 4.5.4.

That the image of p is contained in the specified set is a consequence of its definition. Now let $X \in \mathcal{A}(W, S)$, $Y \in \Omega(X)$, $uH(X) \in W_X$ and $k \in \mathbb{Z}_|\delta|$. We construct a monomial b such that $p(b) = (X, Y, uH(X), k)$. Let w_0 be an element of minimal length mapping X to Y by conjugation. Set

$$b = \delta^k e_X u w_0.$$

Set $p(b) = (p_1, p_2, p_3, p_4)$. Then $p_1 = X$ and since $X * uw_0 = Y$, $p_2 = Y$. Also $p_4 = k$. Only p_3 remains. Consider,

$$\begin{aligned} \eta_X(uw_0) &= \eta_X(u)\eta_X(w_0) \\ &= \eta_X(u) \\ &= uH(X), \end{aligned}$$

by Lemma 3.4.7, since $u \in N(X)$ and w_0 is an element of minimal length in an $N(X)$ coset. □

The theorem allows us to calculate the size of the Brauer monoid of type M. It is an upper bound for the dimension of the Brauer algebra over $\mathbb{K}$. An upper bound for the dimension of the Brauer algebra over $\mathbb{K}(\delta)$ can also be derived from the theorem.

$$[\mathrm{Br}(M) : \mathbb{K}(\delta)] \leq \sum_{\Omega} |\Omega|^2 |W(\top_\Omega)| \tag{4.12}$$

In this formula Ω ranges over the collection of all admissible orbits under the action of W on $\mathcal{X}(W,S)$, and $\top_\Omega$ is the top element of Ω. The type of this group can be found in Table 3.1 of this thesis. The size of the orbits is known as well, and hence it is possible to calculate the actual upper bound for the dimensions of the Brauer algebras of different types. The results are shown in Table 4.2.

M	**Upper bound**
A_n	$(n+1)!!$
D_n	$(2^n+1)n!! - (2^{n-1}+1)n!$
E_6	$1,440,585$
E_7	$439,670,025$
E_8	$53,328,069,225$

Table 4.2: *Upper bounds for the dimension of* $\mathrm{Br}(M)$. *Note that* $k!!$ *denotes the product of the first* k *odd integers* $1, \ldots (2k-1)$.

On page 65 of [13] the same numbers are shown. There it was conjectured that these numbers are an upper bound for the dimension of the BMW algebras. Our results here show that indeed the dimension of the Brauer algebra is bounded by these numbers.

Appendix A

Notation

(W, S)	a Coxeter system.
$u * v$	the conjugate of u by v, i.e., $u * v = v^{-1}uv$, for $u, v \in W$.
$\beta \cdot w$	the image of $\beta \in \Phi$ under $w \in W$.
$\mathcal{X}(W, S)$	the collection of sets of commuting reflections of the Coxeter system (W, S). Its elements are usually denoted X, Y or Z.
$\mathcal{A}(W, S)$	the collection of admissible sets in $\mathcal{X}(W, S)$.
X^c	the admissible closure of X.
X'	the derived set of X, i.e. $X' = X^c \backslash X$.
$\mathrm{cp}(r_1, r_2, r_3)$	the companion reflection of the connected triple r_1, r_2, r_3.
$\top$	the top element of an admissible orbit of $\mathcal{X}(W, S)$.
μ	the reflection cocycle of a Coxeter system.
$\mu(w)_r$	the multiplicity of the reflection r in w.
$\Gamma(H)$	the canonical set of generators of the reflection subgroup H.
$N_W(X)$, $N(X)$	normalizer of X in W.
$C_W(X)$, $C(X)$	centralizer of X in W.
$L(X)$	the group of all $w \in N(X)$ such that for all reflections $r \in C_+(X)$ it holds that $\mu(w)_r = 0$.
$W(X)$	the residual group with respect to X.
$[m]$	the set $\{1, \ldots, m\}$.
B_n	the braid group on n strands.
$\mathscr{C}_n(l, m)$	the BMW algebra of degree n.
ℓ	the length function of (W, S).
$\mathrm{Br}(M, \delta)$	the generalized Brauer algebra of type M.

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