MASTER

On variants of the Grone-Merris conjecture

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Abstract

I studied the Laplacian spectrum of graphs, simplicial complexes and some conjectures related to them.

Chapter 1 discusses definitions which are used throughout the text, some linear algebra and some Laplacian spectrum related results. Definitions which are specific to a chapter are given in the chapter itself.

Chapter 2 discusses the proof of the Grone-Merris conjecture. Here I use the version of the proof given by Andries.

Chapter 3 is joint work with Jochem Berndsen. We prove the Brouwer conjecture for split graphs, cographs and regular graphs. We also show that the Brouwer conjecture is stronger than the Grone-Merris conjecture for some t if and only if the graph is nonsplit.

Chapter 4 concerns the the Laplace matrix of simplicial complexes and their spectrum. I have done current literature study and tried to connect the various existing definitions. I also calculate the spectrum of some classes of hypergraphs and proved the Duval-Reiner conjecture for some classes of hypergraphs. We give constructions of some new hypergraphs from existing hypergraphs and the effect of such constructions on the Laplacian spectrum of hypergraphs. This chapter is motivated by [8] and [14, Section 3.2, 3.3]. I give a more combinatorial way of looking at the same things.

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Contents

Chapter 1

Basics

1.1 Basic Definitions

A *graph* $\Gamma = (V\Gamma, E\Gamma)$ consists of a collection of vertices denoted by $V\Gamma$ and a collection of edges $E\Gamma$, where each edge is identified with a pair of vertices in $V\Gamma$. If the two vertices of an edge are the same then it is called a *loop*. If $|V\Gamma|$ and $|E\Gamma|$ are finite then Γ is a *finite graph* otherwise Γ is an *infinite graph*. In this work, we will consider finite graphs unless stated otherwise.

A graph $\Delta = (V\Delta, E\Delta)$ is a *subgraph* of a graph Γ if $V\Delta \subseteq V\Gamma$ and $E\Delta \subseteq E\Gamma$. For each subset $U \subseteq V\Gamma$, we define $[U] := \{uv \in E\Gamma | u \in U, v \in U\}$, and we let $\Gamma[U]$ denote the subgraph of Γ *induced* by U, that means $V\Gamma[U] = U, E\Gamma[U]) = [U]$.

A graph is called *complete* if all distinct vertices are adjacent to each other.

If the edges of a graph Γ are identified with *ordered* pairs of vertices, then Γ is *directed* . Hence we give an orientation for each edge e in $E\Gamma$, i.e. we define that edge e is from vertex v_1 to vertex v_2. In this case v_1 is the tail of e and v_2 is the head of e. A graph is *undirected* if there is no distinction between end vertices for an edge e.

We define the *degree* or *valency* of a vertex v in an undirected graph Γ to be the number of edges incident with v. A loop at vertex v contributes 2 to the degree of v. The degree of a vertex x is denoted by $\deg(x)$. In case of a directed graph we speak of *in-degree* and *out-degree*. The in-degree of vertex v is the number of edges with head v and the out-degree of vertex v, is the number of edges with tail v.

A graph Γ has *multiple edges* if $E\Gamma$ is a multiset. A graph Γ is *simple* if it has no multiple edges and no loops. We will in general work with simple graphs unless specified otherwise.

Let Γ be an undirected simple graph. The *adjacency matrix* of Γ is the 0-1 matrix A indexed by the vertex set $V\Gamma$ of Γ, where $A_{xy} = 1$ when there is a edge from x to y in Γ and $A_{xy} = 0$ otherwise. If the graph is not simple (but without loops), then A_{xy} equals number of edges from x to y.

Let Γ be an undirected simple graph. The *incidence matrix* of Γ is the 0-1 matrix M, with rows indexed by $V\Gamma$ and columns indexed by $E\Gamma$, where $M_{xe} = 1$ when vertex x is an endpoint of edge e and 0 otherwise.

Let Γ be a directed simple graph. The *directed incidence matrix* of Γ is the 0-± 1 matrix N, with rows indexed by $V\Gamma$ and columns indexed by $E\Gamma$, where $N_{xe} = -1, 1, 0$ when x is the head of the edge e, the tail of e, or not on e, respectively.

Let Γ be an undirected simple graph. The *Laplace matrix* or *Laplacian* of Γ is the matrix L indexed by the vertex set $V\Gamma$, with zero row sums, where $L_{xy} = -A_{xy}$ for $x \neq y$. If D is the diagonal matrix, indexed by the vertex set $V\Gamma$ such that D_{xx} is the degree or valency of x, then $L = D - A$. The matrix $|L| = D + A$ is called the *signless Laplace matrix* of Γ.

The (ordinary) *spectrum* of a graph Γ is by definition the spectrum of the adjacency matrix A, that is, its set of eigenvalues together with their multiplicities. The *Laplacian spectrum* of an undirected simple graph is the spectrum of the Laplacian L.

The *complement* of a graph Γ is a graph $\overline{\Gamma}$ on the same vertices such that two (different) vertices of $\overline{\Gamma}$ are adjacent if and only if they are not adjacent in Γ.

Let Γ be a graph with n vertices and m edges. If $e = uv \in E\Gamma$ and $w \notin V\Gamma$ then e is said to be *subdivided* when it is replaced by uw and wv and w is added to $V\Gamma$. The new graph has $n + 1$ vertices and $m + 1$ edges. If we subdivide all edges of Γ then we denote the resulting graph by $S(\Gamma)$. This graph has $n + m$ vertices and $2m$ edges.

The *line graph* of a graph Γ is the graph $\mathcal{L}(\Gamma)$ with vertices, the edges of Γ , and two edges of Γ are adjacent in $\mathcal{L}(\Gamma)$, if they share a vertex in Γ.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs on disjoint sets of n and m vertices, respectively. Then the *disjoint union* of Γ_1, Γ_2 is the graph $\Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

A *path* of length r from vertex x to vertex y in a graph is a sequence of vertices $(x, v_1, \ldots, v_{r-1}, y)$ such that consecutive vertices are adjacent and vertices v_i for $1 \leq i \leq r - 1$ are distinct and different from x and y. A path is a *cycle* if its starting and end vertex are same. A (nonempty) graph Γ is *connected* if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and another end in Y; otherwise the graph is *disconnected.* The *distance* $d(x, y)$ in Γ of two vertices x, y is the length of a shortest x-y path; if no such path exists, we set $d(x, y) = \infty$. The *diameter* of a graph Γ is the greatest distance between any two vertices. A *tree* is a connected graph with no cycles.

A subset of vertices S is called a *stable set* or *independent set* if all vertices in S are pairwise nonadjacent. A subset of vertices C is called a *clique* if the vertices in C are pairwise adjacent. So the induced subgraph on C is complete.

Let Γ be a graph with n vertices. Let's label the vertices of Γ with $\{1, 2, \ldots, n\}$ such that if $i \leq j$ then $\deg(i) \geq \deg(j)$. Then $\mathbf{d}(\Gamma) = (d_1, d_2, \ldots, d_n)$ is the *degree sequence* of Γ, where d_i is the degree of vertex with label i. We define its *conjugate sequence*, $\mathbf{d}'(\Gamma) = (d'_1, d'_2, \ldots, d'_n)$ where

$$d'_k = \#\{i : d_i \geq k\} \tag{1.1}$$

When we view a degree sequence $\mathbf{d}$ as a partition of the integer $\sum d_i$, then $\mathbf{d}'$ is the conjugate partition. Two such sequences can be given the same length if necessary by padding the shorter one with zeroes.

We define the *eigenvalue sequence* of graph Γ, $\lambda(\Gamma) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i are the eigenvalues of a matrix corresponding to graph Γ, in non-increasing order. This matrix is either the Laplace matrix or the adjacency matrix of Γ.

1.2 Some basic Linear Algebra

Let V be a complex vector space with finite dimension, and fix a basis. Then we can define an inner product on V by putting $(x, y) = \sum \overline{x_i} y_i = \overline{x}^\top y$ for all $x, y \in V$, where the bar denotes complex conjugation. If A is *Hermitian*, i.e., if $(Ax, y) = (x, Ay)$ for all $x, y \in V$, then all eigenvalues of A are real, and V admits an orthonormal basis of eigenvectors of A.

Proposition 1.1. *Let $\mathcal{A}$ be a collection of commuting Hermitian linear transformations on V (i.e., $AB = BA$ for $A, B \in \mathcal{A}$), then V has a orthonormal basis consisting of common eigenvectors of all $A \in \mathcal{A}$.*

Proof. We prove this by induction on $\dim V$. If each $A \in \mathcal{A}$ is a multiple of identity I, then the proposition holds. Otherwise, let $A \in \mathcal{A}$ not a multiple of identity I. If $Au = \theta u$ and $B \in \mathcal{A}$, then $A(Bu) = \theta Bu$, so B acts as a linear transformation on the eigenspace V_θ for the eigenvalue θ of A. As V_θ is a smaller vectorspace than V(since A is not a multiple of I), so by the induction hypothesis we can choose a basis consisting of common eigenvectors for each $B \in \mathcal{A}$ for each eigenspace. The union of these bases of all eigenspaces is the basis of V. □

Given a square matrix A, we regard A as a linear transformation on a finite dimensional vector space with a fixed orthonormal basis. Hence the above concepts apply. The matrix A will be Hermitian precisely when $A = \overline{A}^\top$; a real symmetric matrix is Hermitian.

Consider two sequences of real numbers: $\theta_1 \geq \cdots \geq \theta_n$, and $\eta_1 \geq \cdots \geq \eta_m$ with $n > m$. The second sequence is said to *interlace* the first one whenever,

$$\theta_i \geq \eta_i \geq \theta_{n-m+i} \text{ for } i = 1, 2, \ldots, m. \tag{1.2}$$

The interlacing is *tight* if there exists an integer $k \in [1, m]$ such that

$$\theta_i = \eta_i \text{ for } 1 \leq i \leq k \text{ and } \theta_{n-m+i} = \eta_i \text{ for } k + 1 \leq i \leq m \tag{1.3}$$

The following results on interlacing are used at many places in this text.

Lemma 1.2. *If B is a principal submatrix of a symmetric matrix A, then the eigenvalues of B interlace the eigenvalues of A.*

The Laplacian eigenvalues behave nicely when we add or remove edges [13].

Lemma 1.3. *Let Γ be a graph on n vertices and let Δ be a graph obtained by deleting an edge from Γ. The (first $n-1$) Laplacian eigenvalues of Δ interlace the Laplacian eigenvalues of Γ, that is*

$$\lambda_1(\Gamma) \geq \lambda_1(\Delta) \cdots \geq \lambda_n(\Gamma)(= \lambda_n(\Delta) = 0)$$

Let A be a real symmetric matrix and let u be a nonzero vector. The *Rayleigh quotient* of u w.r.t. A is defined as

$$\frac{u^\top A u}{u^\top u}$$

Let $u_1, \ldots, u_n$ be an orthonormal set of eigenvectors of A, say with $Au_i = \theta_i u_i$ where $\theta_1 \geq \cdots \geq \theta_n$. If $u = \sum \alpha_i u_i$ then $u^\top u = \sum \alpha_i^2$ and $u^\top A u = \sum \alpha_i^2 \theta_i$. It follows that

$$\frac{u^\top A u}{u^\top u} \geq \theta_i \text{ if } u \in \langle u_1, \ldots, u_i \rangle$$

and

$$\frac{u^\top A u}{u^\top u} \leq \theta_i \text{ if } u \in \langle u_1, \ldots, u_{i-1} \rangle^\perp = \langle u_i, \ldots, u_n \rangle$$

We now state the Courant-Fischer-Weyl min-max principle, one of the most powerful tools for the study of the spectrum of Hermitian matrices.

Theorem 1.4. *Let H be an $n \times n$ Hermitian matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\lambda_k = \max_{\dim(S)=k} \min_{0 \neq x \in S} \frac{(Hx, x)}{(x, x)} = \min_{\dim(S)=n-k+1} \max_{0 \neq x \in S} \frac{(Hx, x)}{(x, x)} \tag{1.4}$$

where $\max(\min$ *resp.*$)$ *ranges over all k-dimensional* $((n-k+1)-$*dimensional resp.*$)$ *subspaces.*

Below we state Schur's inequality, which is also related to majorization theory.

Theorem 1.5. *Let A be a real symmetric matrix with eigenvalues $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and diagonal elements $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $\sum_{i=1}^{t} d_i \leq \sum_{i=1}^{t} \theta_i$ for $1 \leq t \leq n$.*

Proof. Let B be the principal submatrix of A obtained by deleting the rows and columns containing $d_{t+1}, \ldots, d_n$. If B has eigenvalues η_i $(1 \leq i \leq t)$, then by interlacing, $\sum_{i=1}^{t} d_i = \operatorname{tr}(B) = \sum_{i=1}^{t} \eta_i \leq \sum_{i=1}^{t} \theta_i$. □

Here, we state one more result without proof from [26] regarding the continuity relationship between eigenvalues and matrix entries.

Theorem 1.6. *The eigenvalues of a matrix are continuous functions of the matrix entries.*

1.3 Laplace Matrices and their spectrum

Lemma 1.7. *Let N be a directed incidence matrix for a simple graph Γ with respect to some orientation of the edges and let L be the Laplace matrix of Γ. Then*

$$L = NN^{\top} \tag{1.5}$$

Hence L is a positive semi-definite matrix.

Proof. Let $u, v \in V\Gamma$ and $e \in E\Gamma$. Then the uv-entry of $NN^{\top}$ is:

$$\sum_{e \in E\Gamma} N_{ue} N_{ev}^{\top} = \begin{cases} -1 & u \neq v, \text{ if there is a directed edge joining } u, v, \\ \deg(u) & u = v, \\ 0 & \text{otherwise.} \end{cases}$$

As $L = NN^{\top}$, $L = L^{\top}$ and for any vector x, $x^{\top} L x = x^{\top} N N^{\top} x \geq 0$. So L is a positive semi-definite matrix. □

Similarly we can see that $|L|$ is also a positive semi-definite matrix and $L = MM^\top$ where M is the incidence matrix of Γ. Also we easily see from Lemma (1.7), that for any vector x, $x^\top Lx = \sum_{e \in E\Gamma}(x_u - x_v)^2$. Similarly, $x^\top |L| x = \sum_{e \in E\Gamma}(x_u + x_v)^2$ for u, v being end vertices of e.

The rows and columns of a matrix of size n are labeled in order from 1 to n, while L is indexed by the vertices of Γ, so that writing down L requires one to assign some numbering to the vertices. However, the Laplace spectrum of the matrix obtained does not depend on the numbering chosen. It is the spectrum of the linear transformation L on the vector space K^X of maps from X into K, where X is vertex set of the graph and K is some field containing the rationals, such as $\mathbb{R}$ or $\mathbb{C}$.

Let Γ be an undirected and simple graph with n vertices. Since L is real and symmetric, the Laplacian spectrum is real. Moreover, L is a positive-semidefinite matrix (from Lemma 1.7) and singular (L has all-1 vector as an eigenvector with eigenvalue 0), so we may denote the eigenvalues by $\lambda_1, ..., \lambda_n$, where $\lambda_1 \geq \cdots \geq \lambda_n = 0$. The sum of these eigenvalues is $tr(L)$, which is the sum of degrees of $V\Gamma$ hence equals $2|E\Gamma|$. Also $\mathrm{tr}(L) = \mathrm{tr}(|L|)$.

The Laplacian characteristic polynomial of graph Γ is that of L, that is, the polynomial $p_\Gamma(\theta)$ defined by $p_\Gamma(\theta) = \det(\theta I - L)$.

A graph Γ is called *regular* with valency k, when every vertex has precisely k neighbors. So Γ is regular of degree k precisely when its adjacency matrix has row sums k, that is when $A1 = k1$ or $AJ = kJ$ where $\mathbf{1}$ denotes all-1 vector and J denotes all-1 matrix of appropriate size. If Γ is regular with valency k, then $L = kI - A$ where I is a identity matrix of appropriate size. Hence we can easily calculate the eigenvalues of L from the eigenvalues of A.

The Laplace matrix of the complement $(\overline{\Gamma})$ of a graph Γ, $\overline{L} = nI - J - L$ (the degree of each vertex in the complement will be $(n-1) -$ degree of vertex in Γ).

So, the eigenvectors of L are also eigenvectors of J (they live in the space orthogonal to all-1 vector, $\langle \mathbf{1} \rangle^\perp$). We immediately see:

Proposition 1.8. *If we have a graph* Γ, *then the eigenvalues of* $\overline{\Gamma}$ *are* $0, n-\lambda_1, \ldots, n-\lambda_{n-1}$ *in non-decreasing order.*

If we take the disjoint union of graphs, then following proposition helps.

Proposition 1.9. *Let* Γ *be a graph with connected components* Γ_i $(1 \leq i \leq s)$. *Then the ordinary and Laplacian spectrum of* Γ *is the union of the spectra of* Γ_i *(and multiplicities are added).*

Proof. Let us denote the Laplace matrix for the union of components by L. Without loss of generality, L is a block diagonal matrix if we group the vertices of Γ_i for $(1 \leq i \leq s)$ together. If we take some eigenvector of Γ_i corresponding to an eigenvalue, then we can extend it to an eigenvector of L if we place 0 in the entries of an eigenvector which do not correspond to Γ_i. So the Laplacian of the union is the union of the Laplacians. □

1.4 Laplacian spectrum of some graphs

We discuss the Laplacian spectrum of some special graphs. All graphs in this section are simple, finite and undirected. Observe that the all-1 matrix J of order n has rank 1, and that all-1 vector $\mathbf{1}$ is an eigenvector with eigenvalue n and the spectrum is $n^1, 0^{n-1}$. (Throughout the text, we write multiplicities as exponents where that is convenient and no confusion seems likely).

1.4.1 The complete graph

A graph Γ is *complete* if all vertices are adjacent to each other. We denote the complete graph on n vertices by K_n. Its adjacency matrix is $A = J - I$ and hence $L = (n-1)I - J + I = nI - J$. The Laplacian spectrum of K_n is $n^{n-1}, 0^1$. Another way to see that is by going to the complement, a graph with n isolated vertices and the Laplacian eigenvalues of $\overline{K_n}$ are 0^n. Hence the Laplacian eigenvalues of K_n are $n^{n-1}, 0$.

1.4.2 The complete bipartite graph

A graph Γ is called *bipartite* when its vertex set can be partitioned into two disjoint sets X_1 and X_2 (color classes) such that all edges of Γ meet both X_1 and X_2. A graph Γ is called *complete bipartite graph* $K_{m,n}$ if there are edges from each vertex of X_1 to each vertex of X_2 ($|X_1| = m$ and $|X_2| = n$). The complement of $K_{m,n}$ is the disjoint union of the graphs K_m and K_n. The Laplacian eigenvalues of $\overline{K_{m,n}}$ are $n^{n-1}, m^{m-1}, 0^2$. Hence, the Laplacian eigenvalues of $K_{m,n}$ are $(m+n)^1, n^{m-1}, m^{n-1}, 0^1$.

1.4.3 The Friendship Graph

A *Friendship graph* F_n can be constructed by joining n triangles with a common vertex. These graphs are also called *dutch windmill graphs.* These graphs have $2n+1$ vertices and $3n$ edges. The complement of F_n is a isolated vertex and a n-partite graph with each color class of cardinality 2. So, the Laplacian eigenvalues of complement of F_n are $(2n)^{n-1}, (2n-2)^n, 0^2$. Hence, the Laplacian eigenvalues of F_n are $(2n+1)^1, 3^n, 1^{n-1}, 0^1$.

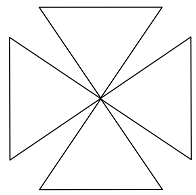

Figure 1.1: Friendship graph F_4

1.5 Connectivity and Laplacian eigenvalues

From the Laplacian spectrum of a graph, one can determine its number of spanning trees (which will be nonzero only if the graph is connected).

Theorem 1.10. *Let* Γ *be an undirected graph with at least one vertex, and Laplace matrix* L *with eigenvalues* $\lambda_1 \geq ... \geq \lambda_n = 0$. *Let* l_{xy} *be the* (x, y)-*cofactor of* L, *then the number* N *of spanning trees of* Γ *equals:*

$$N = l_{xy} = \frac{\lambda_1 \ldots \lambda_{n-1}}{n} \tag{1.6}$$

There is a strong connection between the diameter of a graph and the number of distinct eigenvalues of a graph.

Lemma 1.11. *Let* Γ *be a connected graph with diameter* d. *Then* Γ *has at least* $d + 1$ *distinct ordinary eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues.*

Proof. Let M be any nonnegative symmetric matrix with rows and columns indexed by $V\Gamma$ and such that for distinct vertices x, y we have $M_{xy} > 0$ if and only if x and y are adjacent. Let the distinct eigenvalues of M be $\theta_1, \ldots, \theta_t$. Then $(M - \theta_1 I) \ldots (M - \theta_t I) = 0$. This means M^t is linear combination of $I, M, M^2, \ldots, M^{t-1}$. But if $d(x, y) = t$ for two vertices x, y of Γ, then $(M^i)_{xy} = 0$ for $0 \leq i \leq t - 1$ and $(M^t)_{xy} > 0$, contradiction. Hence $t > d$. This applies to $M = A$, to $M = nI - L$ and to $M = |L|$, where A is the adjacency matrix, L is the Laplace matrix and $|L|$ is the signless Laplace matrix of graph Γ. □

Another concept of connectivity involving the Laplacian eigenvalues is *algebraic connectivity*. The second smallest Laplacian eigenvalue $\lambda_{n-1}(\Gamma)$ is called *algebraic connectivity* of a graph Γ. This concept was introduced by Fiedler [10]. As we know that the multiplicity of 0 as a Laplacian eigenvalue gives us the number of connected components of the graph, $\lambda_{n-1}(\Gamma) > 0$, for a connected graph.

The algebraic connectivity is monotone: it does not decrease when edges are added to the graph.

Proposition 1.12. *Let Γ and Δ be two edge-disjoint graphs on the same vertex set, and $\Gamma \cup \Delta$ their union. We have $\lambda_{n-1}(\Gamma \cup \Delta) \geq \lambda_{n-1}(\Gamma) + \lambda_{n-1}(\Delta) \geq \lambda_{n-1}(\Gamma)$.*

Proof. We have $\lambda_{n-1}(\Gamma) = \min_u\{u^\top Lu \mid (u,u) = 1, (u, 1) = 0\}$. If the minimum for Γ and Δ is at the same u, then first inequality holds with equality. □

Proposition 1.13. *Let Γ be a graph with vertex set $V\Gamma$. Suppose $D \subset V\Gamma$ is a set of vertices such that graph induced by Γ on $V\Gamma \setminus D$ is disconnected. Then $|D| \geq \lambda_{n-1}(\Gamma)$.*

Proof. By monotonicity, we may assume that Γ contains all the edges between D and $V\Gamma \setminus D$. Now a vector u which is 0 on the vertices corresponding to D and constant on each component of $V\Gamma \setminus D$ and satisfies $(u, 1) = 0$, is eigenvector of $L(\Gamma)$ with eigenvalue $|D|$. □

1.6 Laplacian integral graphs

If the Laplacain spectrum of a graph Γ consists of integers only then such graph Γ is called *Laplacian integral*. If Γ is Laplacian integral then $\overline{\Gamma}$ is also Laplacian integral.

A tree is a *star* if we have one common vertex and all other vertices are connected to the common vertex and form a stable set. So a star is complete bipartite graph $K_{1,d}$ for some d.

Lemma 1.14. *Let Γ be a tree. Then Γ is Laplacian integral if and only if Γ is a star.*

Proof. Let Γ be a Laplacian integral tree on n vertices. The number of spanning trees of a tree is 1. Let the Laplacian eigenvalues of Γ be $\lambda_1 \geq \lambda_2 \geq \cdots \leq \lambda_n = 0$. By Theorem (1.10), we can say that $\frac{\lambda_1 \ldots \lambda_{n-1}}{n} = 1$. This means that $\lambda_1 \ldots \lambda_{n-1} = n$. The number of edges in a tree on n vertices is $n-1$ so the sum of the Laplacian eigenvalues is $\sum_{i=1}^{n} \lambda_i = 2n-2$. We know that $\lambda_i \geq 1$ for all $1 \leq i \leq n-1$. We can write $\lambda_1 \ldots \lambda_{n-1} = (1 + (\lambda_1 - 1)) \ldots (1 + (\lambda_{n-1} - 1)) = n + \sum_{i,j}(\lambda_i - 1)(\lambda_j - 1) + \ldots$. This implies that $\lambda_1 = n, \lambda_2 = \cdots = \lambda_{n-1} = 1$. From Proposition (1.8), we see that $\lambda_1 \leq n$ with equality if the complement of Γ is disconnected. The only tree whose complement is disconnected is a star.

Let's now calculate the eigenvalues of a star on n vertices. Let's go the complement then we get a complete graph on $n-1$ vertices and one isolated vertex. So the eigenvalues of the complement of a star be $(n-1)^{n-2}, 0^2$.

So the Laplacian eigenvalues of a star are $n^1, 1^{n-2}, 0$. Hence Γ is Laplacian integral. □

Let us denote by G_n, the graph obtained by subdividing one edge of $K_{n-1}(n \geq 2)$. We see that G_n is a Laplacian integral graph because if we move to the complement, $\overline{G_n}$, we get a graph with two connected components, K_2 and $K_{1,n-3}$. The star is Laplacian integral and K_2 as well so $\overline{G_n}$ is Laplacian integral, then so is G_n.

$S(\Gamma)$ is the graph obtained from Γ after subdividing every edge of Γ. We state the following result by Grone-Merris [15, Thm. 1] without proof.

Theorem 1.15. *Let Γ be a connected, regular, Laplacian integral graph on n vertices. Then $S(\Gamma)$ is Laplacian Integral if and only if $\Gamma = K_n$.*

A graph Γ is *(r,s)-semiregular* if it is bipartite with a bipartition (V_1, V_2), in which each vertex of V_1 has degree r and each vertex of V_2 has degree s. The next result was proved by Mohar [24, Thm. 3.9].

Theorem 1.16. *Let Γ be a connected, (r, s)-semiregular, Laplacian integral graph. Then its line graph is Laplacian integral.*

Proof. Let's orient the edges in the direction from V_1 to V_2. Let N be the incidence matrix with respect to aforesaid orientation. The line graph of an (r, s)-semiregular graph, $\widetilde{\Gamma}$, is $(r + s - 2)$ regular. Then the Laplacian matrix of Γ, $L = NN^\top$ and $N^\top N = 2I + A(\widetilde{\Gamma})$, where $A(\widetilde{\Gamma})$ is the adjacency matrix of the line graph of Γ. So the Laplace matrix of the line graph of Γ is $L(\widetilde{\Gamma}) = (r + s - 2)I - A(\widetilde{\Gamma})$. As $NN^\top$ and $N^\top N$ have the same nonzero eigenvalues, this means that $\widetilde{\Gamma}$ is adjacency integral and as it is regular, hence Laplacian integral. □

A *cograph* is a graph that does not contain P_4 as an induced subgraph. (The graph P_4 is the path on four vertices.) It is well-known [5, Thm. 11.3.3] that cographs have the following inductive characterization.

1. K_1 is a cograph.
2. If Γ is a cograph, then $\overline{\Gamma}$ (the complement of Γ) is also a cograph.
3. If Γ and Δ are cographs, then their disjoint union $\Gamma \cup \Delta$ is also a cograph.

Every cograph can be constructed in this way.

Lemma 1.17. *The spectrum of a cograph is Laplacian integral.*

Proof. We prove by induction, we can easily see that K_1 is integral. Let's have a cograph Γ which is Laplacian integral, then by (1.8), $\overline{\Gamma}$ is also Laplacian integral. Let us have two cographs Γ and Δ which are Laplacian integral. By Proposition (1.9) we know that the eigenvalues of Laplace matrix of $\Gamma \cup \Delta$ are union of eigenvalues (with multiplicities added) of Γ and Δ. Hence $\Gamma \cup \Delta$ is Laplacian integral. □

Chapter 2

The Grone-Merris conjecture

2.1 Majorization

The theory of majorization arose during early part of 20 century from a number of apparently unrelated topics: wealth distribution [20], inequalities involving the convex functions [18], convex combinations of the permutation matrices [4], doubly stochastic matrices [1], and relation between the eigenvalue spectrum and diagonals of self-adjoint matrices [19].

For non-increasing real sequences $\mathbf{x}$ and $\mathbf{y}$ of length n, we say that $\mathbf{x}$ is *majorized* by $\mathbf{y}$ or $\mathbf{y}$ *majorizes* $\mathbf{x}$ (denoted by $\mathbf{x} \preccurlyeq \mathbf{y}$) if

$$\sum_{i=1}^{t} x_i \leq \sum_{i=1}^{t} y_i \text{ for } 1 \leq t < n, \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \tag{2.1}$$

The concept of majorization extends to the vectors by comparing the sequences obtained by sorting the elements of the vector in non-increasing order.

The spectrum of a Laplace matrix and majorization are connected by the majorization relation between the eigenvalue spectrum and diagonal elements of Hermitian matrices. The relation below is immediate from Schur's theorem (1.5)

$$\mathbf{d}(\Gamma) \preccurlyeq \lambda(\Gamma) \tag{2.2}$$

The degree sequence of a graph Γ is majorized by its conjugate degree sequence. This can be seen easily from the theorem below if A is the adjacency matrix of the graph Γ.

Theorem 2.1. *There exists a 0-1 matrix A with row and column sum vectors $\mathbf{r}$ and $\mathbf{c}$ if and only if $\mathbf{r}$ is majorized by $\mathbf{c}'$.*

Ky Fan [9] gives a majorization inequality for the sum of Hermitian matrices.

Theorem 2.2. *If S_1 and S_2 are Hermitian matrices, then*

$$\lambda(\mathbf{S_1} + \mathbf{S_2}) \preccurlyeq \lambda(\mathbf{S_1}) + \lambda(\mathbf{S_2}) \tag{2.3}$$

2.2 Split graphs

A graph Γ is called *split* if its vertex set $V\Gamma$ can be partitioned into sets C and S, such that the induced graph on C is a clique and S is a stable set. (C, S) is referred to as the *split partition* of the graph and we denote the graph by $\Gamma(C, S)$. These graphs were defined and characterized by Foldes and Hammer[11]. They were also introduced independently by A.A. Chernyak and Zh. A. Chernyak [7] under the name *polar graphs*.

It follows from the definition that the complement, and every induced subgraph of a split graph is also split.

Given a set of vertices K and two vertices x, y, not necessarily in K, we say that x *dominates* y in K, denoted by $x \succsim_K y$, if every neighbor of y in $K - x$ is also a neighbor of x. A graph Γ is said to be *triangulated* if it contains no induced cycle of size four or more.

The following theorem is from [12].

Theorem 2.3. *For every graph Γ, the following conditions are equivalent:*

(1) Γ is a split graph.

(2) both Γ and its complement $\overline{\Gamma}$ are triangulated.

(3) Γ does not contain C_4,C_5, or $2K_2$ as induced subgraphs.

Proof. **1**) $\Rightarrow$ **2**) : By definition every induced graph of a split graph is split. Hence a split graph does not contain any induced cycle of size 4 or more, since the vertex set of a cycle of size 4 or more cannot be partitioned into a clique and a stable set. Also the complement of a split graph is a split graph, hence it is triangulated too.
2) $\Rightarrow$ **3**) : Since Γ is triangulated it does not contain C_4 or C_5, and since $\overline{\Gamma}$ is triangulated and $\overline{2K_2}$ is C_4, Γ does not contain $2K_2$.
3) $\Rightarrow$ **1**) : Let K be a maximum clique in Γ with fewest possible edges in $V\Gamma - K$. Assume that, if possible , two vertices x, y in $V\Gamma - K$ are adjacent. If there are two vertices z, w in K such that $xz, yw \in E\Gamma$ but $xw, yz \notin E\Gamma$, then x, y, z, w induce a C_4. It follows that either $x \succsim_K y$ or $y \succsim_K x$. Assume without loss of generality that $x \succsim_K y$. Now, if x has two nonadjacent vertices z, w in K, then x, y, z, w induce a $2K_2$, a contradiction. If y has at

most one non-adjacent vertex z in K, then $K - z + \{x, y\}$ is a larger clique than K in Γ, a contradiction. It follows that x has exactly one non-adjacent vertex in K, say z, and y has at least one more non-adjacent vertex in K, say w. Now z must have a neighbor u in $V\Gamma - K$ that is not adjacent to x, for otherwise $K' = K - z + x$ is also a maximum clique in Γ and $V - K'$ has fewer edges than $V\Gamma - K$, a contradiction. Now $yu \in E\Gamma$ or else x, y, z, u induce a $2K_2$ in Γ. Applying same argument as x, y to u, y, we find that $u \gtrsim_K y$ and u has exactly one non-adjacent vertex w' in K. Now, x, y, u, z, w' induce a C_5, a contradiction, so x, y cannot be adjacent and hence Γ is a split graph. □

Let the size of the set C be N and let the size of the set S be M. Let $\delta(\Gamma)$ be the maximum degree of vertices in S. Clearly $\delta(\Gamma) \leq N$. The Laplace matrix of the split graph Γ is of the form:

$$L(\Gamma) = \begin{bmatrix} K + D & -A \\ -A^T & E \end{bmatrix}$$

In this notation K, denotes the Laplace matrix of the complete graph on N vertices. A is the adjacency matrix of the cross edges from the clique to the stable set. D and E are diagonal matrices with row sums of A and column sums of A respectively. The number of edges in a split graph is the sum of edges in the clique C and number of cross edges.

First we look at the eigenvalues of the *complete split graph*. In the complete split graph, we have all edges from the clique to the stable set, so there are NM cross edges in addition to the $N(N-1)/2$ edges in the clique.

Lemma 2.4. *For the complete split graph of clique size N and stable set size M, the Laplacian spectrum is*

$$\{(M + N)^N, N^{M-1}, 0^1\} \tag{2.4}$$

The eigenspace corresponding to the eigenvalue N consists of vectors of form $\begin{pmatrix} \mathbf{0}_N \\ v \end{pmatrix}$, where v is M-dimensional and $v \perp \mathbf{1}_M$; the eigenspace corresponding to eigenvalue $(M + N)$ is spanned by the (mutually orthogonal) vectors:

$$(\mathbf{0}_{i-1}, M + N - i, -\mathbf{1}_{M+N-i})^T \ , \ 1 \leq i \leq N \tag{2.5}$$

Proof. Let's take the complement of the complete split graph, we get a clique in set S and stable set in set C. There are no cross edges, so we get a clique on M vertices and N isolated vertices. So the eigenvalues of the complement are $\{M^{M-1}, 0^{N+1}\}$. So the eigenvalues of a complete split graph are $\{(M+N)^N, N^{M-1}, 0^1\}$. We know that the all-1 vector is an eigenvector with an eigenvalue 0. Definitely for $v \perp \mathbf{1}_M$, the eigenvalue is N, also such a v lives in a $(M-1)$-dimensional space and hence the multiplicity of the eigenvalue N is $M-1$. Lastly for eigenvalue $(M+N)$, we see from the structure of Laplace matrix of the complete split matrix,

$$L(\Gamma) = \begin{bmatrix} K_N + MI & -J_{N\times M} \\ -J_{M\times N} & NI, \end{bmatrix}$$

that

$$(\mathbf{0}_{i-1}, M+N-i, -\mathbf{1}_{M+N-i})^T \ , \ 1 \le i \le N \tag{2.6}$$

are N eigenvectors of Γ, hence the eigenvalue $(M+N)$ has multiplicity N. □

Lemma 2.5. *If Γ is a split graph of clique size N, then*

$$\lambda_{N-1}(\Gamma) \ge N \ge \delta(\Gamma) \ge \lambda_{N+1}(\Gamma) \tag{2.7}$$

Moreover, if $\lambda_N(\Gamma) \ge N$, then

$$\sum_{i=1}^{N} d'_i = N^2 + \mathrm{tr}(D) \tag{2.8}$$

Proof. To prove the inequalities in (2.7) involving $\lambda_{N-1}(\Gamma)$ and $\lambda_{N+1}(\Gamma)$, by Courant-Fischer-Weyl min-max principle Theorem (1.4), it suffices to find a $(N-1)$-dimensional (M-dimensional, resp.) subspace such that the action of $L(\Gamma)$ has a desired minimum (maximum, resp.) value. Let $P \subset \mathbb{R}^{M+N}$ be the $(N-1)$-dimensional subspace which consists of vectors of the form $\begin{pmatrix} u \\ \mathbf{0}_M \end{pmatrix}$ with $u \in \mathbb{R}^N$ and $u \perp \mathbf{1}_N$. So for any unit vector u,

$$\begin{aligned} \lambda_{N-1}(\Gamma) &= (L(\Gamma)\begin{pmatrix} u \\ \mathbf{0}_M \end{pmatrix}, \begin{pmatrix} u \\ \mathbf{0}_M \end{pmatrix}) \\ &= ((K+D)u, u) \\ &= N + (Du, u) \\ &\ge N \end{aligned}$$

This is true as N is an eigenvalue for K_N for every $u \perp \mathbf{1}_N$.
Let $Q \subset \mathbb{R}^{M+N}$ be the M-dimensional subspace which consists of vectors of the form $\begin{pmatrix} \mathbf{0}_N \\ u \end{pmatrix}$ with $u \in \mathbb{R}^M$. So for any unit vector u,

$$\begin{aligned}\lambda_{N+1}(\Gamma) &= (L(\Gamma)\begin{pmatrix}\mathbf{0}_N\\ u\end{pmatrix}, \begin{pmatrix}\mathbf{0}_N\\ u\end{pmatrix})\\ &= (Eu, u)\\ &\leq \delta(\Gamma)\end{aligned}$$

This establishes inequality in (2.7).

When $\lambda_N(\Gamma) \geq N$, we assert that degree of any vertex in C is at least N. Assume that our assertion is false, then we have a vertex v with degree $N-1$. This means v is adjacent to any vertex in S, so we can get a new split graph Γ' by moving vertex v from clique to coclique. The new graph has clique size $N' = N-1$ and stable set size $M' = M+1$. Applying (2.7), we get

$$\lambda_N(\Gamma) = \lambda_{N'+1}(\Gamma') \leq N' = N-1$$

This is a contradiction.

We have $d'_i = \sum_{j=1}^{M+N} \chi(d_j \geq i)$ where χ is a characteristic function with a value 1 if $d_j \geq i$ and 0 otherwise. This can also be seen as:

$$\sum_{i=1}^{N} d'_i = \sum_{i=1}^{N}\sum_{j=1}^{M+N} \chi(d_j \geq i) = \sum_{j=1}^{M+N} \min(d_j, N)$$

$$\sum_{i=1}^{N} d'_i = \sum_{j\in C} N + \sum_{j\in S} d'_j = N^2 + \operatorname{tr}(D)$$

This establishes (2.8). □

2.3 Proof of the Grone-Merris Conjecture

From Schur's inequality (1.5), we know that the Laplacian spectrum majorizes the degree sequence of the graph and from Gale-Ryser Theorem (2.1), the conjugate degree sequence majorizes the degree sequence.

In 1994, Grone and Merris [15] raised the natural question whether the Laplacian spectrum and the conjugate degree sequence are majorization comparable.

Theorem 2.6. *For any graph Γ, the Laplacian spectrum is majorized by the conjugate degree sequence*

$$\lambda(\Gamma) \preccurlyeq \mathbf{d}'(\Gamma) \tag{2.9}$$

This conjecture was proved by Hua Bai [2] . Here I use the version of the proof given by Andries Brouwer [6].

The graph Γ is a *threshold graph* if and only if it satisfies the above majorization relation with equality which implies that the conjugate degree sequence is the same as the Laplacian spectrum. This means threshold graphs are also Laplacian integral.

A threshold graph (also called *degree maximal graph* [23]) is a graph obtained from the one-vertex graph by repeatedly adding an isolated vertex or a dominating vertex (vertex adjacent to all vertices). A threshold graph is both a cograph and a split graph.

The proof of Theorem (2.6) is by reducing the problem to the case of a split graph. Then for split graphs, a continuity argument proves the crucial inequality stated in the following lemma.

Lemma 2.7. *For any split graph Γ of clique size N, if either*

$$\lambda_N(\Gamma) > N \text{ or } \lambda_N(\Gamma) = N > \delta(\Gamma) \tag{2.10}$$

then the N-th inequality of the Grone-Merris conjecture holds:

$$\sum_{i=1}^{N} \lambda_i \leq \sum_{i=1}^{N} d'_i \tag{2.11}$$

Proof. of Theorem 2.6 (assuming Lemma 2.7). Consider a counterexample to Theorem (2.6) with minimal possible t.

Step 2.8. *If Γ is a counterexample with minimal number of edges, and if x, y are vertices in Γ of degree at most t, then they are nonadjacent.*

If x is adjacent to y then let Γ' be the graph obtained from Γ by removing the edge xy. Now we reduce the degrees of x and y by 1 hence $\sum_{i=1}^{t} \#\{x|d'_x \geq i\} + 2 = \sum_{i=1}^{t} \#\{x|d_x \geq i\}$. Laplace matrices Γ and Γ' satisfy $L(\Gamma) = L(\Gamma') + H$ where H has eigenvalues 2, 0^{n-1}. By Theorem (2.2), we have $\sum_{i=1}^{t} \lambda_i \leq \sum_{i=1}^{t} \lambda'_i + 2$, and since Γ' has fewer edges than Γ we find $\sum_{i=1}^{t} \lambda_i \leq \lambda'_i + 2 \leq \#\{x|d'_x \geq i\} + 2 = \sum_{i=1}^{t} \#\{x|d_x \geq i\}$, contradiction.

Step 2.9. *There is a split graph Γ, counterexample for the same t, with clique size $N = \#\{x|d_x \geq t\}$.*

We can form a new graph by adding edges to the nonadjacent vertices x, y with degree at least t. This will not affect $\sum_{i=1}^{t} \#\{x|d_x \geq i\}$ as we are increasing degree of vertices with degree $\geq t$. We also know that by adding edges the Laplacian eigenvalues do not decrease, so the new graph is a split graph with the stated clique size.

This will be our graph Γ for rest of the proof.

Since t was chosen minimal, we have $\lambda_t > \#\{x|d_x \geq t\}(= N)$. From Lemma (2.5), we see that $N \geq t$, for $N < t$ implies $\lambda_{N+1} \geq N$, contradiction. In fact $N > t$ since if $N = t$ then $\lambda_N > N$ and Lemma 2.7 gives a contradiction.

All the vertices in the coclique of Γ have degree at most $t-1$ and all vertices in the clique have degree at least $N-1$. So $\#\{x|d_x \geq i\} = N$ for $t \leq i \leq N-1$. From Lemma (2.5), we have $\lambda_i \geq \lambda_{N-1} \geq N$ for $t \leq i \leq N-1$. Since $\sum_{i=1}^{t} \lambda_i > \sum_{i=1}^{t} \#\{x|d_x \geq i\}$ we also have $\sum_{i=1}^{N-1} \lambda_i > \sum_{i=1}^{N-1} \#\{x|d_x \geq i\}$. Now if $\lambda_N \geq N$ we contradict Lemma (2.7) since the number of vertices in Γ with degree N is $\leq N$. So $\lambda_N < N$.

We now show that (2.9) behaves nicely under complementation.

Step 2.10. *The m-th Grone-Merris inequality for a graph Γ is equivalent to the $(n-1-m)$-th Grone-Merris inequality for its complement $\overline{\Gamma}$ $(1 \leq m \leq n-1)$.*

Indeed, $\overline{\Gamma}$ has Laplacian eigenvalues $\overline{\lambda}_i = n - \lambda_{n-i}$ and conjugate degrees $\#\{x|\overline{d}_x \geq i\} = n - \#\{x|d_x \geq n-i\} = n - d'_{n-i}$ and $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \#\{x|d_x\}$. In our case $\overline{\Gamma}$, is also a split graph with clique size $M = n-N$, and by (2.10), we have $\overline{\lambda}_M = n - \lambda_N > M$ (as $\lambda_N < N$) and $\sum_{i=1}^{M} \overline{\lambda}_i > \sum_{i=1}^{M} \#\{x|\overline{d}_x \geq i\}$ (applying (2.10) to $\sum_{i=1}^{N-1} \lambda_i > \sum_{i=1}^{N-1} \#\{x|d_x \geq i\}$). This contradicts Lemma (2.7).

□

This contradiction completes the proof of Grone-Merris conjecture except that Lemma (2.7) still needs to be proved.

Proof of Lemma 2.7. Γ is the split graph referred to in section 2.2.

Step 2.11. *Suppose that the subspace W spanned by $L(\Gamma)$-eigenvectors belonging to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ is spanned by columns of $\begin{bmatrix} I \\ X \end{bmatrix}$. Then $L(\Gamma)\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} Z$ for some matrix Z, and $\sum_{i=1}^{N} \lambda_i = \operatorname{tr}(Z)$.*

If $\begin{bmatrix} U \\ V \end{bmatrix}$ has these eigenvectors as columns, then $L(\Gamma)\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} T$ where T is a diagonal matrix with eigenvalues on diagonal. Now $\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} U$, so that $L(\Gamma)\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} Z$ where $Z = UTU^{-1}$ and as Z, T are similar, $\operatorname{tr}(Z) = \operatorname{tr}(T) = \sum_{i=1}^{N} \lambda_i$.

By (2.5), $\delta(\Gamma) \geq \lambda_{N+1}(\Gamma)$ and by the hypothesis we have $\lambda_N(\Gamma) > \delta(\Gamma)$. The subspace W has a basis which forms columns of $\begin{bmatrix} I \\ X \end{bmatrix}$. But W can also contain nonzero vectors with zeros on the top. All the vectors in W have rayleigh quotient atleast λ_N. If the vectors have zeros on the top then the rayleigh quotient of the vectors is at most $\delta(\Gamma)$ as the rayleigh quotient is $u^\top Eu$ if $\begin{bmatrix} 0 \\ u \end{bmatrix} \in W$. As $\lambda_N > \delta(\Gamma)$, so W has the required form. So U is invertible.

Let's assume X to be nonpositive. We also know $\delta(\Gamma) \leq N$. For proving (2.7), it suffices to show that if $\lambda_N > N$ or $\lambda_N = N > \delta(\Gamma)$ then

$$\operatorname{tr}(Z) \leq N^2 + \operatorname{tr}(D) \tag{2.12}$$

by Lemma (2.5).

Now $L(\Gamma) \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} K + D - AX \\ -A^\top + EX \end{bmatrix}$, so $Z = K + D - AX$, and $\operatorname{tr}(Z) = \operatorname{tr}(K + D - AX) = N(N-1) + \operatorname{tr}(D) - \operatorname{tr}(AX)$. So to prove (2.12), we need to show $\operatorname{tr}(AX) \geq -N$. But since $N < n$, the eigenvectors are orthogonal to $\mathbf{1}$ so that column sums of X are -1. Now X is nonpositive, so $\operatorname{tr}(AX) \geq -N$.

We will need following lemma to prove nonpositivity of X.

Let $\alpha \in [0,1]$. Define a $(M+N) \times (M+N)$ matrix, $L^{(\alpha)}(\Gamma)$ as follows:

$$L^{(\alpha)}(\Gamma) = (1-\alpha) \begin{bmatrix} K_N + M & -J_{N\times M} \\ -J_{M\times N} & N, \end{bmatrix} + \alpha \begin{bmatrix} K_N + D & -A \\ -A^T & E \end{bmatrix} \tag{2.13}$$

Going forward we will denote $L^{(0)}(\Gamma)$ as matrix $L^{(\alpha)}(\Gamma)$ at $\alpha = 0$ and $L^{(1)}(\Gamma)$ as matrix $L^{(\alpha)}(\Gamma)$ at $\alpha = 1$.

Lemma 2.12. *If $\lambda_N > \delta(\Gamma)$, then the invariant subspace W spanned by $L(\Gamma)$-eigenvectors for λ_i, $1 \leq i \leq N$, is spanned by columns of $\begin{bmatrix} I \\ X \end{bmatrix}$ where X is nonpositive.*

Proof. We argue by continuity, viewing $L(\Gamma) = L(A)$ and $X = X(A)$ as functions of real-valued matrix A, where $0 \leq A \leq J$. (Now D has the row sums of A, and E has the column sums, and δ is the largest element of the diagonal matrix E). We write J for the $N \times M$ all-1 matrix and J_N for all-1 matrix order N, so that $JX = -J_N$ as all-1 vector is orthogonal to eigenvectors spanning W.

Our hypothesis $\lambda_N > \delta(\Gamma)$ holds for all matrices $L^{(\alpha)}(\Gamma) = L(\alpha A + (1 - \alpha)J_N) = \alpha L(\Gamma) + (1-\alpha)L^{(0)}(\Gamma)$ for $0 \le \alpha \le 1$. Indeed, let $L^{(\alpha)}(\Gamma)$ have the eigenvalues as $\lambda_i^{(\alpha)}$ so that $\lambda_N^{(0)} = M + N$ and $\lambda_{N+1}^{(0)} = \cdots = \lambda_{N+M-1}^{(0)} = N$. The matrix $L^{(\alpha)}(\Gamma)$ has lower right-hand corner as $\alpha E + (1-\alpha)NI$ so that $\delta^{(\alpha)} = \alpha\delta + (1-\alpha)N$. The space W is orthogonal to all-1 vector, so by Rayleigh quotient we know that $\lambda_N^{(\alpha)} \ge \alpha\lambda_N + (1-\alpha)N$, hence $\lambda_N^{(\alpha)} > \delta^{(\alpha)}$ for $0 < \alpha \le 1$. Also for $\alpha = 0$, since $\lambda_N = M + N$ and $\delta^{(0)} = N$. Again applying Rayleigh quotient and using our hypothesis, we get $\lambda_N^{(\alpha)} > \lambda_{N+1}^{(\alpha)}$ for $0 \le \alpha \le 1$.

By Lemma (2.4), we know the spectrum of $L^{(0)}(\Gamma)$ and hence $X(J) = -\frac{1}{M}J^\top$, as desired. Above we also found, $XZ = -A^\top + EX$ on X, that is, $X(K_N + D - AX) - EX + A^\top = 0$, that is $X(K_N + J_N + D) = XJ_N + XAX + EX - A^\top = -X(J-A)X + EX - A^\top$. It follows, since $K + J_N + D = NI + D$ is a positive diagonal matrix, that if $X \le 0$ and $A > 0$, then $X < 0$. The matrix $X(A)$ depends continuously on A (in the region where $\lambda_N > \lambda_{N+1}$) and is strictly negative when $A > 0$. Then it is nonpositive when $A \ge 0$. □

□

Chapter 3

The Brouwer Conjecture

All graphs are considered undirected, finite and simple. For a graph Γ, the notation $\Gamma[A]$ denotes the subgraph of Γ induced by the subset A of the vertices. The number n will denote the number of vertices of the graph under consideration. The set $\delta(W, W')$ for disjoint sets of vertices W, W' is the set of edges having one endpoint in W and one endpoint in W'. The abbreviation $\delta(W)$ is shorthand for $\delta(W, V\Gamma\backslash W)$.

Andries Brouwer conjectured the following [6, section 3.8].

Conjecture 3.1. *Let Γ be a graph. Then, for $k = 1, \ldots, n$, we have*

$$\sum_{i=1}^{k} \lambda_i \leq |E\Gamma| + \binom{k+1}{2}. \tag{3.1}$$

Brouwer remarks in [6] that it is easy to see that the conjecture holds for threshold graphs. In personal communication Brouwer mentions that he has checked the conjecture for all graphs on 10 vertices using the program **nauty** [22] (to enumerate all isomorphism classes of graphs) combined with the GNU Scientific Library [25] (to calculate the Laplacian eigenvalues of a graph). We have verified this result independently. W. Haemers, A. Mohammadian and B. Tayfeh-Rezaie proved Conjecture 3.1 for trees and for general graphs in the case $k = 2$ [16].

3.1 The Brouwer conjecture for split graphs

As we saw in the previous chapter, Hua Bai [2] proved the Grone-Merris conjecture.

Theorem 3.2. *Let Γ be a graph. Then, for $k = 1, \dots, n$, we have*

$$\sum_{i=1}^{k} \lambda_i \le \sum_{i=1}^{k} d_i', \tag{3.2}$$

with equality if $k = n$.

We want to investigate in which cases the conjectured Brouwer bound (3.1) is sharper than the Grone-Merris bound (3.2). To this end, we define the sequence $f_0(\Gamma), \dots, f_n(\Gamma)$ of integers by

$$f_k(\Gamma) = |E\Gamma| + \binom{k+1}{2} - \sum_{i=1}^{k} d_i'(\Gamma).$$

Note that f_k is the difference between the two bounds, and it is negative at precisely those k for which the Brouwer bound is sharper.

Lemma 3.3. *For $k = 1, \dots, n$, we have $d_k \ge k$ if and only if $d_k' \ge k$.*

Proof. This follows from $d_k \ge j$ if and only if $d_1 \ge \cdots \ge d_k \ge j$ if and only if $d_j' \ge k$. □

Lemma 3.4. *The minimum of $f_0(\Gamma), \dots, f_n(\Gamma)$ is attained at*

$$m(\Gamma) := \max\{k : 0 \le k \le n, d_k' \ge k\}. \tag{3.3}$$

Proof. Use the fact that d_i' is nonincreasing and $f_k - f_{k-1} = \binom{k+1}{2} - \binom{k}{2} - d_k' = k - d_k'$, so $f_k \le f_{k-1}$ if and only if $k \le d_k'$ if and only if $k \le m$. □

This m is precisely the *Durfee number* of the degree sequence of graph Γ.

We observe that $m(\Gamma) = 0$ implies that Γ is edgeless. Furthermore, note that $m(\Gamma)$ could equivalently have been defined as $\max\{k : 1 \le k \le n, d_k \ge k\}$ if Γ is not edgeless, due to Lemma 3.3.

Recall that a graph Γ is split if its vertices can be partitioned into two sets A, B such that $\Gamma[A]$ is complete and $\Gamma[B]$ is edgeless. It turns out that for the class of split graphs, the Brouwer bound can be derived from the Grone-Merris bound.

Theorem 3.5. *If Γ is split, then $f_m(\Gamma) = 0$.*

Proof. Since Γ is split, we can partition its vertices into sets A and B such that $\Gamma[A]$ is complete and $\Gamma[B]$ is edgeless. Now take a split partition (A, B) for which the set B is maximal, and denote $N := |A|$ and $M := |B|$.

We have that the degrees of the vertices A are all at least N. This follows, since they are all at least $N - 1$ by virtue of $\Gamma[A]$ being a clique, and if the degree of (say) v was exactly $N-1$, the pair $(A\backslash\{v\}, B\cup\{v\})$ would form another split partition for Γ whose second entry is larger. But this is not possible by the choice of (A, B). Since $\Gamma[B]$ is a coclique, the vertices B all have degree at most N.

Now we can number the vertices of A, say $A = \{1, \dots, N\}$, in descending order of degree, and we can number the vertices of B, say $B = \{N+1, \dots, N+M\}$, in descending order of degree. But now all vertices of Γ are ordered in descending order of degree. We can see that $m(\Gamma) = N$, since $d_1 \geq \cdots \geq d_N \geq N \geq d_{N+1} \geq \cdots \geq d_{N+M}$.

Denote by X the set of edges between vertices A and vertices B. We first see

$$\begin{aligned}\sum_{i=1}^{N} d_i' &= \sum_{i=1}^{N} \#\{v \in V\Gamma : \deg(v) \geq i\} \\ &= \sum_{i=1}^{N} (\#\{v \in A : \deg(v) \geq i\} + \#\{v \in B : \deg(v) \geq i\}) \\ &= N^2 + |X|.\end{aligned}$$

Then

$$\begin{aligned}f_N &= \binom{N+1}{2} + |E| - \sum_{i=1}^{N} d_i' \\ &= \binom{N+1}{2} + \binom{N}{2} + |X| - (N^2 + |X|) \\ &= \frac{(N+1)N}{2} + \frac{N(N-1)}{2} - N^2 = 0,\end{aligned}$$

which had to be shown. □

As an immediate corollary we have the following.

Corollary 3.6. *The Brouwer bound (3.1) holds for split graphs.*

Proof. Combine the Grone-Merris bound (3.2) with Theorem 3.5. □

3.2 Splittance of graphs

We now established the Brouwer conjecture for split graphs. Before we proceed, we will need to introduce the concept of the *splittance* of a graph. The splittance was first defined by Hammer and Simeone [17].

Definition 3.7 (Splittance)**.** Let $\Gamma = (V, E)$ be a (finite, undirected, simple) graph. The *splittance* $\sigma(\Gamma)$ of Γ is defined as the minimum cardinality of a set $F \subseteq \binom{V}{2}$ such that the graph $(V, E\Delta F)$ is split. (Here $E \triangle F$ is the symmetric difference between E and F.)

Essentially, the Splittance is minimal number of edges to be added or removed in order to change the graph Γ into a split graph.

It is immediate that the splittance of a split graph is equal to zero. Let us now review some important properties of the splittance that can be found in the original paper by Hammer and Simeone [17].

Lemma 3.8. *Set* $s(S) = \binom{|S|}{2} - |E(\Gamma[S])| + |E(\Gamma[S^c])|$*, where* $S^c = V\Gamma \backslash S$*. Then*

$$\sigma(\Gamma) = \min_{S \subseteq V\Gamma} s(S). \tag{3.4}$$

Proof. It is obvious that $\sigma(\Gamma) \leq s(S)$ for all subsets $S \subseteq V\Gamma$. On the other hand, let Γ' be the split graph $(V\Gamma, E\Gamma \triangle F)$ obtained from the graph Γ, where F is of minimal cardinality. Since Γ' is split, its vertices can be partitioned into sets A and B, where $\Gamma'[A]$ is a clique and $\Gamma'[B]$ is edgeless. By minimality of F, no $e \in F \cap E\Gamma$ had an endpoint in A, and no $e \in F \backslash E\Gamma$ had an endpoint in B. But $s(A)$ is equal to the number of edges to be added to $\Gamma[A]$ to make it into a complete graph, plus the number of edges to be removed from $\Gamma[B]$ to make it into an edgeless graph. So $|F| = s(A) = \sigma(\Gamma)$. □

From this we can see that the splittance of a graph can be calculated by considering all subsets of the vertices. A surprising result [17, Theorem 4] is that the degree sequence of the graph is actually enough to calculate the splittance. First fix some notation:

$$\sigma_k(\Gamma) = \frac{1}{2}\left(k(k-1) - \sum_{i=1}^{k} d_i(\Gamma) + \sum_{i=k+1}^{n} d_i(\Gamma)\right).$$

Lemma 3.9. *The sequence* $\sigma_1(\Gamma), \ldots, \sigma_n(\Gamma)$ *attains its minimum at* $m(\Gamma)$ *as defined in (3.3).*

Proof. Observe $\sigma_k - \sigma_{k-1} = (k-1) - d_k$, so this quantity is negative if and only if $d_k > k-1$ if and only if $d_k \geq k$. Since d_i is nonincreasing and $i-1$ is increasing, σ_m is the minimum value of the sequence $\sigma_1, \ldots, \sigma_n$. □

Theorem 3.10. *We have* $\sigma(\Gamma) = \sigma_m(\Gamma)$*. Furthermore, the set* F *consisting of the edges*

$$F = \{\{u, v\} : u \not\sim v, 1 \leq u < v \leq m\} \cup \{\{u, v\} : u \sim v, m+1 \leq u < v \leq n\}$$

is of minimal cardinality such that $(V\Gamma, E\Gamma \triangle F)$ *is split.*

Proof. Recall that $\delta(S)$ denotes the subset of the edges having one endpoint in S and one endpoint in $V\Gamma\backslash S$. Note that

$$\sum_{v\in S}\deg(v) = 2|E(\Gamma[S])| + |\delta(S)|$$

for all subsets S of the vertices. But

$$\begin{aligned} s(S) &= \binom{|S|}{2} - |E(\Gamma[S])| + |E(\Gamma[S^c])| \\ &= \binom{|S|}{2} - \frac{1}{2}\left(\sum_{v\in S}\deg(v) - \sum_{v\in S^c}\deg(v)\right). \end{aligned}$$

The sequence $d_1, \ldots, d_n$ is nonincreasing. Hence

$$s(S) \geq \binom{|S|}{2} - \frac{1}{2}\left(\sum_{i=1}^{|S|} d_i - \sum_{i=|S|+1}^{n} d_i\right) = \sigma_{|S|}, \tag{3.5}$$

with equality if $S = \{1, \ldots, k\}$ (assuming without loss of generality that the vertices are ordered with descending degree). But, by equation (3.4), that implies that $s(S) = \min_k \sigma(k) = \sigma_m(k)$. The fact that the set F is of minimal cardinality such that $(V\Gamma, E\Gamma\Delta F)$ is split, is now obvious from equation (3.5) where $S = \{1, \ldots, m\}$. □

We see from the above that if we define $A := \{1, \ldots, m\}$ and $B := \{m + 1, \ldots, n\}$, then adding an edge to $\Gamma[A]$ or removing an edge from $\Gamma[B]$ will decrease the splittance by 1.

3.3 The Brouwer bound is sharper for nonsplit graphs

Lemma 3.4 gives a value for k for which f_k is minimal. We saw that the minimum for f_k is zero for split graphs. For nonsplit graphs we can also compute the minimum value of f_k. The splittance of a graph will be denoted by $\sigma(\Gamma)$. The next theorem will generalize Theorem 3.5, but use the result in its proof.

Theorem 3.11. *Let* Γ *be a graph. Then*

$$f_m(\Gamma) = -\sigma(\Gamma). \tag{3.6}$$

Proof. Use induction on the splittance of the graph. We may assume that the equation (3.6) holds for all graphs having splittance less than $\sigma(\Gamma)$. We may also assume without loss of generality that the vertices are numbered $1, \dots, n$ and they are ordered with descending degree, i.e. $\deg(i) = d_i$. Now define sets $A := \{1, \dots, m\} \subseteq V\Gamma$ and $B := \{m+1, \dots, n\} \subseteq V\Gamma$. There are now three cases (they need not be mutually disjoint):

1. the graph $\Gamma[A]$ is not complete;
2. the graph $\Gamma[B]$ is not edgeless;
3. the graph $\Gamma[A]$ is complete and the graph $\Gamma[B]$ is edgeless.

The third case means that Γ is split. Then $\sigma(\Gamma) = 0$, and we are done by Theorem 3.5. So we only need to consider the first two cases.

Proof of case 1. So $\Gamma[A]$ is not complete. That means that there exist nonadjacent distinct $v, w \in A$. Define $\triangle := \Gamma + \{v, w\}$. We know that $\sigma(\triangle) = \sigma(\Gamma) - 1$. We have

$$d'_1(\Gamma) \geq \dots \geq d'_m(\Gamma) \geq m(\Gamma) \geq d'_{m+1}(\Gamma) \geq \dots d'_n(\Gamma)$$

and

$$d_1(\Gamma) \geq \dots \geq d_m(\Gamma) \geq m(\Gamma) \geq d_{m+1}(\Gamma) \geq \dots d_n(\Gamma)$$

and from $d'_i(\triangle) = d'_i(\Gamma)$ if $i \leq m$ we see that $m(\triangle) \geq m(\Gamma)$. But $d_i(\Gamma) = d_i(\triangle)$ if $i > m$ since the added edge was between vertices of $A = \{1, \dots, m\}$. That means that $m(\triangle) = m(\Gamma)$. Now set $m := m(\triangle) = m(\Gamma)$ and $d'_i := d'_i(\Gamma) = d'_i(\triangle)$ as long as $i \leq m$. So by the induction hypothesis we have in $\triangle$ that $f_m(\triangle) = |E\triangle| + \binom{m+1}{2} - \sum_{i=1}^m d'_i = -\sigma(\triangle)$.

We had $|E\triangle| = |E\Gamma| + 1$, hence

$$\begin{aligned} f_m(\Gamma) &= |E\Gamma| + \binom{m+1}{2} - \sum_{i=1}^m d'_i \\ &= |E\triangle| - 1 + \binom{m+1}{2} - \sum_{i=1}^m d'_i \\ &= f_m(\triangle) - 1 = -\sigma(\triangle) - 1 = -\sigma(\Gamma), \end{aligned}$$

and we are done with case 1.

Proof of case 2. In this case $\Gamma[B]$ is not edgeless. So take an edge of $\Gamma[B]$, say $e = \{v, w\}$. Define $\triangle := \Gamma - e$. We again know that $\sigma(\triangle) = \sigma(\Gamma) - 1$. By a similar reasoning as in the proof of case 1, we have $m(\triangle) = m(\Gamma)$ so we set $m := m(\triangle) = m(\Gamma)$. On the other hand, since the degrees of vertices v and w in Γ are both at most k, we can deduce $\sum_{i=1}^k d'_i(\triangle) = \sum_{i=1}^k d'_i(\Gamma) - 2$.

We had $|E\triangle| = |E\Gamma| - 1$, hence

$$\begin{aligned} f_m(\Gamma) &= |E\Gamma| + \binom{m+1}{2} - \sum_{i=1}^{m} d_i'(\Gamma) \\ &= |E\triangle| + 1 + \binom{m+1}{2} - (\sum_{i=1}^{m} d_i'(\triangle) + 2) \\ &= f_m(\triangle) - 1 = -\sigma(\triangle) - 1 = -\sigma(\Gamma), \end{aligned}$$

and also the proof of this case is finished.

Since we covered all cases for the induced subgraphs $\Gamma[A]$ and $\Gamma[B]$, the theorem is established. □

3.4 The Brouwer conjecture for cographs

A *cograph* is a graph that does not contain P_4 as an induced subgraph. (The graph P_4 is the path on four vertices).

We have given a inductive definition for cograph in the section Laplacian integral graphs. We will prove the Brouwer conjecture for cographs inductively. So it suffices if the Brouwer conjecture is satisfied under complement and disjoint union operation. Now we observe that K_1 satisfies the Brouwer inequality (3.1) is obvious. The following two lemmata will establish the Brouwer conjecture for cographs. The proof of the first lemma can also be found in [16], but we include it for the sake of completeness.

Lemma 3.12 ([16])**.** *If Γ satisfies the Brouwer inequality (3.1) for all $k = 1, \dots, n$, then $\overline{\Gamma}$ does as well.*

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n = 0$ be the Laplacian eigenvalues of Γ, and let $\overline{\lambda}_1 \geq \cdots \geq \overline{\lambda}_n = 0$ be the Laplacian eigenvalues of $\overline{\Gamma}$. It is easy to see that $\overline{\lambda}_i = n - \lambda_{n-i}$ for $i = 1, \dots, n-1$. But now:

$$\begin{aligned} \sum_{i=1}^{n-k-1} \overline{\lambda}_i &= n(n-k-1) - \sum_{i=k+1}^{n-1} \lambda_i \\ &= n(n-k-1) - 2|E\Gamma| + \sum_{i=1}^{k} \lambda_i \\ &= n(n-k-1) - |E\Gamma| - \binom{n}{2} + |E\overline{\Gamma}| + \sum_{i=1}^{k} \lambda_i \\ &\leq n(n-k-1) - \binom{n}{2} + |E\overline{\Gamma}| + \binom{k+1}{2} \end{aligned}$$

$$= |E\overline{\Gamma}| + \binom{n-k}{2},$$

and we are done. □

Lemma 3.13. *If Γ and $\triangle$ satisfy the Brouwer conjecture (3.1) for all k, then $\Gamma \cup \triangle$ does as well.*

Proof. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the Laplacian eigenvalues of Γ, and $\mu_1 \geq \cdots \geq \mu_m$ be the Laplacian eigenvalues of $\triangle$. The Laplacian eigenvalues of $\Gamma \cup \triangle$, say $\nu_1 \geq \cdots \geq \nu_{n+m}$, consist of the union $\{\lambda_i\} \cup \{\mu_i\}$. The k-th partial sum $\sum_{i=1}^{k} \nu_i$ of them is equal to $\sum_{i=1}^{p} \lambda_i + \sum_{i=1}^{q} \mu_i$ for some p, q satisfying $0 \leq p, q \leq k$ and $p+q = k$. We now claim that $p(p+1)+q(q+1) \leq k(k+1)$. But this is obvious, since it is equivalent to $p^2 + q^2 \leq k^2 = (p+q)^2 = p^2 + 2pq + q^2$. By assumption we have

$$\sum_{i=1}^{p} \lambda_i \leq |E\Gamma| + \binom{p+1}{2}, \tag{3.7}$$

and

$$\sum_{i=1}^{q} \mu_i \leq |E\triangle| + \binom{q+1}{2}. \tag{3.8}$$

Now combine inequalities (3.7) and (3.8) to obtain

$$\begin{aligned}\sum_{i=1}^{k} \nu_i &= \sum_{i=1}^{p} \lambda_i + \sum_{i=1}^{q} \mu_i \\ &\leq |E(\Gamma \cup \triangle)| + \frac{p(p+1)+q(q+1)}{2} \\ &\leq |E(\Gamma \cup \triangle)| + \frac{k(k+1)}{2},\end{aligned}$$

hence the lemma follows. □

Combining these two lemmata gives us the following theorem.

Theorem 3.14. *The Brouwer bound (3.1) holds for cographs.*

Corollary 3.15. *The Brouwer bound (3.1) holds for complete k-partite graphs.*

Proof. Complete k-partite graphs are cographs. □

3.5 The Brouwer conjecture for regular graphs

We now prove Brouwer Conjecture for regular graphs.

Theorem 3.16. *The Brouwer conjecture holds for regular graphs.*

Proof. Let Γ be a regular graph on n vertices with valency r. We may assume without loss of generality that $r < \frac{n}{2}$ by considering the complementary graph (Lemma (3.12)).

Let the ordinary eigenvalues be $\theta_1, \dots, \theta_n$. Then the Laplacian eigenvalues λ_i are $r - \theta_i$ for all $1 \leq i \leq n$. As the graph Γ is regular so number of edges in Γ is $\frac{nr}{2}$.

Let's have n-dimensional vectors a and b such that a have entries as ordinary eigenvalues of the graph Γ and b have first t entries as 1 and rest as 0.

By Cauchy-Schwarz, $|(a,b)|^2 \leq (a,a)(b,b)$, so $(\sum_{i=1}^{t} \theta_i)^2 \leq nrt$. We observe that above expression is true for any t ordinary eigenvalues as we can make any t positions 1 in vector b. So $\sum_{i=1}^{t} \lambda_i = rt - \sum_{i=1}^{t} \theta_i \leq rt + \sqrt{nrt}$.

We know by computations that the Brouwer conjecture holds for graphs with atmost 10 vertices so to prove the Brouwer conjecture for regular graphs, it suffices to show that below inequality holds for $n \geq 11$.

$$nr + t^2 + t - 2tr - 2\sqrt{nrt} \geq 0 \text{ for } 1 \leq r \leq \frac{n}{2}, 1 \leq t \leq n$$

The above equation is a quadratic function f in $\sqrt{r}$. Let's substitute $\sqrt{r} = x$ in the above equation,

$$f(x) = (n-2t)x^2 - 2\sqrt{nt}x + t^2 + t \geq 0 \text{ for } 1 \leq x \leq \sqrt{\frac{n}{2}}, 1 \leq t \leq n \quad (3.9)$$

Now we split the proof into three cases depending upon whether $n - 2t = 0$, $n - 2t < 0$ or $n - 2t > 0$.

Case I ($n = 2t$)
Substitute $n = 2t$ in (3.9) then $f(x) = \frac{n^2}{4} + \frac{n}{2} - \sqrt{2}nx$. As $x \leq \sqrt{\frac{n}{2}}$, we have $f(x) \geq \frac{n^2}{4} + \frac{n}{2} - \sqrt{n^3}$. Now, $f(x) \geq 0$ is now equivalent to $n \geq 12$. But since n is even so the equation (3.9) holds if $n = 2t$.

Case II ($n < 2t$)
We write (3.9) as,

$$f(x) = (n-2t)(x - \frac{\sqrt{nt}}{n-2t})^2 - \frac{nt}{n-2t} + t^2 + t \geq 0 \text{ for } 1 \leq x \leq \sqrt{\frac{n}{2}}, 1 \leq t \leq n \quad (3.10)$$

So from (3.10), the graph of function f is an inverted parabola as $n-2t<0$ with maximum value at $\frac{\sqrt{nt}}{n-2t}$. We observe that f is a decreasing function in first quadrant, hence it suffices to show that $f(\sqrt{\frac{n}{2}}) \geq 0$.

So we show $2f(\sqrt{\frac{n}{2}}) = g(n) = n^2 - n(2t+2\sqrt{2t}) + 2t + 2t^2 \geq 0$. Roots of $g(n)$ are $(t+\sqrt{2t} \pm \sqrt{2t\sqrt{2t}-t^2})$. Both the roots are complex except when $t \leq 8$. But that means that convex quadratic function $n \to n^2 - n(2t+2\sqrt{2t}) + t + t^2$ does not have any real roots, and thus it is positive everywhere. We need more investigation for the cases where $t \leq 8$.

1. If $t \leq 5$, then $n < 10$, contradiction.

2. If $t = 6$, then we should check for $n = 11$. Now checking for $r = 5$ will suffice, and we observe that $f(\sqrt{5}) \geq 0$.

3. If $t = 7$, then we should check for $n = 11, 12, 13$. Now if $n = 11$, checking for $r = 5$ will suffice and we observe that $f(\sqrt{5}) \geq 0$. Similar computations done for $n = 12, 13$ lead to $f(x) \geq 0$.

4. If $t = 8$, then $g(n) = (n-12)^2$ which implies $f(x) \geq 0$.

So equation (3.9) holds if $n < 2t$.

Case III $(n > 2t)$

As $(n-2t)(x - \frac{\sqrt{nt}}{n-2t})^2 \geq 0$, it suffices to show that $t^2 + t - \frac{nt}{n-2t} \geq 0$. This is true for $n \geq 2t+2$. As n and t are integers so now we need to prove (3.9) for $n = 2t+1$. So, we show that $f(x) = x^2 - 2\sqrt{nt}x + t^2 + t \geq 0$ for $1 \leq x \leq \sqrt{\frac{n}{2}}, 1 \leq t \leq n$. Roots of $f(x)$ are $(\sqrt{\frac{n(n-1)}{2}} \pm \frac{n-1}{2})$. We know $\sqrt{\frac{n(n-1)}{2}} + \frac{n-1}{2}$ is always greater than $\sqrt{\frac{n}{2}}$ and $\sqrt{\frac{n(n-1)}{2}} - \frac{n-1}{2}$ is less than $\sqrt{\frac{n}{2}}$ for $n \leq 10$. So for $n \geq 11$, (3.9) holds if $n > 2t$.

□

Chapter 4

The Laplacian of Simplicial Complex

4.1 Basics

Let $[n]$ denote the set $\{1, 2, \dots, n\}$. An *abstract simplicial complex* K on the vertex set $[n]$ is a collection of subsets of $[n]$ called *faces or simplices* which is closed under inclusion, that is if $F \in K$ and $F' \subset F$ then $F' \in K$. Note that we do not assume all the singletons $\{v\}$ for $v \in [n]$ are faces of K, nor do we assume the empty set $\emptyset$ is a face of K. The cardinality of face F is denoted by $|F|$ and the dimension is one less than the cardinality. The dimension of the whole complex K is the maximum dimension of a face in K. A face F is a *facet* if it is maximal in K. A complex is *pure* if all of its facets have the same dimension. The set

$$K_j = \{F \in K : \dim F = j\} \tag{4.1}$$

forms a family of $(j+1)$-subsets of $[n]$.

The *f-vector* of a simplicial complex K is the sequence

$$f(K) = (f_{-1}(K), f_0(K), \dots) \tag{4.2}$$

where $f_j(K) = |K_j|$. So $f_j(K)$ is the number of j-dimensional simplices in K.

Given an abstract simplicial complex K on the vertex set $[n]$. Let, for $i \geq -1$, $C_i(K)$ be the vector space (over the field $\mathbb{R}$) that has the elements of K_i as basis. We denote a simplex F of dimension i by $s_0 \dots s_i$ when $F = \{s_0, \dots, s_i\}$ and $s_0 < \dots < s_i$. Let us define a *boundary operator* by $\partial_i : C_i(K) \to C_{i-1}(K)$ by

$$\partial_i \; s_0 \dots s_i = \sum_j (-1)^j s_0 \dots \hat{s}_j \dots s_i \tag{4.3}$$

In the above expression, $\hat{s}_j$ means that we remove vertex s_j and we get a $i-1$ dimensional face.

Ex. Let K be $P_1P_2P_4$ then $\partial_2(K) = P_2P_4 - P_1P_4 + P_1P_2$.

As $\partial_{i-1}\partial_i = 0$ for all $i \geq 0$, we can define the *homology group* as

$$H_i(K) \;=\; \ker \partial_i \,/\, \operatorname{im} \partial_{i+1} \tag{4.4}$$

Let N_i be the matrix of ∂_i. In case of a simplicial complex K, the elements of K_{i-1} index the rows of N_i and the columns are indexed by the elements of K_i. So N_i is a $f_{i-1}(K) \times f_i(K)$ matrix. N_i generalizes the vertex-edge incidence matrix.

If we define an inner product on $C_i(K)$, that makes the basis of $C_i(K)$ orthonormal, then it allows one to identify the $\mathbb{R}$-dual space as

$$C^i(K) = Hom(C_i(K), \mathbb{R}) \cong C_i(K) \tag{4.5}$$

where $Hom(C_i(K), \mathbb{R})$ denotes the group of homomorphisms from $C_i(K)$ to $\mathbb{R}$. We can now define a *coboundary operator* as

$$\delta_i \;: C^i(K) \to C^{i+1}(K) \tag{4.6}$$

δ_i can be seen as the adjoint ∂^*_{i+1} of the boundary map ∂_{i+1} with respect to this inner product.

We can define two operators $C_i(K) \to C_i(K)$

$$L_i = \partial_{i+1}\partial^*_{i+1} = N_{i+1}N^\top_{i+1} \tag{4.7}$$

$$L'_i = \partial^*_i\partial_i = N^\top_i N_i \tag{4.8}$$

The matrices L_i generalize the Laplacian. Indeed, in case of a 1-dimensional simplicial complex (that is a graph) the ordinary Laplace matrix is just L_0 and L'_0 is the all-1 matrix J. Since $\partial_i\partial_{i+1} = 0$, we have $N_iN_{i+1} = 0$ which implies $L_iL'_i = L'_iL_i = 0$ generalizing $LJ = JL = 0$. We have $\operatorname{tr}(L_{i-1}) = \operatorname{tr}(L'_i) = (i+1)|f_i|$ as the diagonal entry of L'_i is $i+1$. This generalizes the fact that $\operatorname{tr}(L)$ is twice the number of edges, and $\operatorname{tr}(J)$ the number of vertices.

Theorem 4.1. *There is an isomorphism of vector spaces*

$$H_i(K) \;=\; \ker \partial_i \;/\; \operatorname{im} \partial_{i+1} \cong \ker(L_i + L'_i) \tag{4.9}$$

More precisely, there is an orthogonal direct sum decomposition of $C_i(K)$ with respect to the chosen inner product

$$C_i(K) = \operatorname{im} \partial_{i+1} \oplus \ker(L_i + L'_i) \oplus \operatorname{im} \partial^*_i \tag{4.10}$$

Proof. The operators L_i, L'_i are self-adjoint and positive semidefinite. As $\partial_i \partial_{i+1} = 0$ for all $i \geq 0$, we know $L_i L'_i = L'_i L_i = 0$ so they must annihilate each other's images (non-zero eigenspaces), the kernel of $L_i + L'_i$ is intersection of the nullspaces of L_i, L'_i. Consequently the eigenspace decomposition gives an orthonormal direct sum decomposition

$$C_i(K) = \operatorname{im} \partial_{i+1} \partial^*_{i+1} \oplus \ker (L_i + L'_i) \oplus \operatorname{im} \partial^*_i \partial_i \tag{4.11}$$

But as well known in Linear Algebra, we have $\operatorname{im} \partial_{i+1} \partial^*_{i+1} = \operatorname{im} \partial_{i+1}$ (In case the underlying field is $\mathbb{R}$). Now $\ker \partial_i = \operatorname{im} \partial_{i+1} \oplus \ker (L_i + L'_i)$, so we get our result. $\square$

From now on, the spectrum (eigenvalues with multiplicity) of L_i, L'_i is referred to the Laplacian spectrum of the simplicial complex. Wherever in the following text, we show the equivalence of the spectrum, we are precisely showing the equivalence of the non-zero part of the spectrum. We should see whether the spectra of both of these operators is well defined, since the definition of the boundary map ∂_i involves an ordering of vertices in the simplicial complex, and also the matrix N_i involves ordering of the sets that index its rows and columns.

However, we can easily see that these different ordering of the vertices of the sets indexing rows and columns of the boundary map will only result in ∂_i replaced by $\alpha_i \circ \partial_i \circ \beta_i$ for some operators α_i, β_i on $C_{i-1}(K)$ and $C_i(K)$ respectively. These operators can be represented by signed permutation matrices with respect to basis of simplices, so that they are invertible and satisfy $\alpha_i^\top = \alpha_i^{-1}$ and $\beta_i^\top = \beta_i^{-1}$. Consequently, this has the effect of conjugating the matrices, L_i, L'_i by α_i, β_i, respectively, which will not affect their spectra.

The degree of a vertex v in the simplicial complex K with respect to i-dimensional faces is:

$$\deg_i(K, v) = |\{e \in K_i \mid v \in e\}| \tag{4.12}$$

Similarly, the degree of some k-subset P, where $k < i$, in the simplicial complex K with respect to i-dimensional faces is:

$$\deg_i(K, P) = |\{e \in K_i \mid P \subseteq e\}| \tag{4.13}$$

Given two sets, F, F', let

$$F \triangle F' = (F \setminus F') \cup (F' \setminus F)$$

denote their symmetric difference.

The faces $F, F' \in K_i$ can both lie in the same face $F'' \in K_{i+1}$ only if $|F \triangle F'| = 2$. If there is such a F'', then we say F, F' are *upper adjacent*. These two faces are said to be *similarly oriented* if the sign of F, F' in $\partial_i(F \cup F')$ is the same and if the signs are different, then we say faces are *dissimilarly oriented.* Let $F \triangle F'$ be $\{j, k\}$ with $j < k$.

From the boundary operator formula, by inspection we know that, faces F, F' in $\partial_i(F \cup F')$ will have the same sign if the number of elements in the set $F \cap F'$ which lie between j and k is odd and different sign if the number of elements in the set $F \cap F'$ which lie between j and k is even. Let's define,

$$\epsilon(F, F') = (-1)^{(|\{l:\ j<l<k,\ l\in F\cap F'\}|)} \tag{4.14}$$

Let us write $F \sim F'$ when F and F' are distinct and upper adjacent. So (F, F')-entry of L_i is

$$(L_i)_{F,F'} = (N_{i+1}N_{i+1}^{\top})_{F,F'} = \begin{cases} -\epsilon(F, F') & F \sim F', \\ \deg_{i+1}(K, F) & F = F', \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

This matrix is the same as $\mathcal{L}_i^{UP}$ in [14].

We will similarly analyze the (F, F')-entry of L_i'. If $|F \triangle F'| = 2$, then it means that they have a face of dimension $i - 1$ in common. In this case we say F, F' are *lower adjacent*. If F, F' are upper adjacent, they are also lower adjacent. These two faces are said to be *similarly oriented* if the sign of the common face $F \cap F'$ in $\partial_i(F)$ and $\partial_i(F')$ is the same sign and if the signs are different, then we say faces are *dissimilarly oriented.* Let $F \triangle F'$ be $\{j, k\}$ with $j < k$.

From the boundary operator formula, by inspection we know that, the common face of F, F' will have the same sign in $\partial_i(F)$ and $\partial_i(F')$ if the number of elements in the set $F \cap F'$ which lie between j and k is even and different sign if the number of elements in the set $F \cap F'$ which lie between j and k is odd. Let us write $F \sim F'$ when F and F' are distinct and lower adjacent. So the (F, F')-entry of L_i' is

$$(L_i')_{F,F'} = (N_i^{\top} N_i)_{F,F'} = \begin{cases} \epsilon(F, F') & F \sim F', \\ i + 1 & F = F', \\ 0 & \text{otherwise.} \end{cases} \tag{4.16}$$

This matrix is the same as $\mathcal{L}_i^{DN}$ in [14].

The nonzero values in the spectrum of L_i which acts on $C_i(K)$, depend entirely on the $(i+1)$-dimensional faces in K. This is because an i-dimensional face of K that does not lie in any $(i+1)$-dimensional face of K is annihilated by ∂^*_{i+1}, and hence gives rise to a 0-eigenspace of L_i. Similarly the nonzero values in the spectrum of L'_i which acts on $C_i(K)$, depend entirely on the i-dimensional faces of K.

4.2 Constructions and the Effect on Spectra

A *hypergraph* is a (usually finite) collection of finite sets. If all these sets have the same cardinality m, then the hypergraph is said to be m-uniform. We will refer to these sets as the edges of the hypergraph.

Given an m-uniform hypergraph $\mathcal{H}$, there is a unique abstract simplicial complex K that has as facets the edges of $\mathcal{H}$. Let the Laplacian of this hypergraph $\mathcal{H}$ be the matrix L_{m-2} of a simplicial complex.

In this section we describe some standard constructions of uniform hypergraphs from a given m-uniform hypergraph. We also study the effect of these constructions on the Laplacian spectrum.

Let us have an m-uniform hypergraph $\mathcal{H}$. Let $\binom{[n]}{m}$ denote the *complete m-family* consisting of all m-subsets of $[n]$. We now define two new families derived from $\mathcal{H}$.

$$\begin{aligned}\mathcal{H}^c &= \binom{[n]}{m} \setminus \mathcal{H} \\ &= \{F \subseteq [n] : |F| = m,\ F \notin \mathcal{H}\}\end{aligned}$$

and

$$\mathcal{H}^* = \{[n] \setminus F : |F| = m,\ F \in \mathcal{H}\}$$

Lemma 4.2.

i. $\mathcal{H}^c$ *is an m-family,*

ii. $\mathcal{H}^*$ *is an* $(n-m)$*-family,*

iii. both of these operations are involutive : $(\mathcal{H}^c)^c = (\mathcal{H}^*)^* = \mathcal{H}$,

iv. they commute with each other : $\mathcal{H}^{c*} = \mathcal{H}^{*c}$.

For the purpose of studying the spectrum of $\mathcal{H}^*$, we will look into the operator L'_i. If there are k edges in $\mathcal{H}$, then L'_{m-1} is acting on the k-dimensional subspace $C_{m-1}(\mathcal{H})$. Let $\phi : C_{m-1}(\mathcal{H}) \to C_{n-m-1}(\mathcal{H}^*)$ be the $\mathbb{R}$-linear isomorphism defined by

$$\phi(F) = (-1)^(\sum_{i \in F} i)([n] \setminus F) \tag{4.17}$$

where F is an m-subset of $\mathcal{H}$.

Proposition 4.3. *For any m-family $\mathcal{H}$ of subsets of $[n]$ we have*

$$L'_{m-1}(\mathcal{H}) + \phi^{-1}L'_{n-m-1}(\mathcal{H}^*)\phi = n.I_k \tag{4.18}$$

Proof. We check the equation 4.18 entry by entry. The (F, F')-entry of $L'_{m-1}(\mathcal{H})$ is

$$\begin{cases} \epsilon(F, F') & F \neq F',\ |F \triangle F'| = 2, \\ m & F = F', \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the (F, F')-entry of $\phi^{-1}L'_{n-m-1}(\mathcal{H}^*)\phi$ is

$$\begin{cases} (-1)^(\sum_{i\in F} i + \sum_{i\in F'} i)\epsilon([n]\setminus F, [n]\setminus F') & F \neq F',\ |F \triangle F'| = 2, \\ (n-m) & F = F', \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see from this that the diagonal entries are as per 4.18. For the off-diagonal entries, it suffices to show that

$$(-1)^(\sum_{i\in F} i + \sum_{i\in F'} i)\epsilon(F, F') = -\epsilon([n]\setminus F, [n]\setminus F')$$

whenever $|F \triangle F'| = 2$. To see this, let $F \triangle F' = (j, k)$, and then it is easy to see that

$$(-1)^(\sum_{i\in F} i + \sum_{i\in F'} i) = (-1)^{j+k}$$

and

$$\epsilon(F, F')\epsilon([n]\setminus F, [n]\setminus F') = (-1)^{j-k-1}$$

The last equality hold because the left hand side will have all the elements between j and k.

□

Let's represent the spectrum of $L'_i(\mathcal{H})$ by $\lambda^i(\mathcal{H})$.We immediately deduce:

Theorem 4.4. *In the situation of the previous proposition, the spectra*

$$\lambda^{m-1}(\mathcal{H}) = (\lambda_1^{m-1} \geq \cdots \geq \lambda_k^{m-1})$$

$$\lambda^{m-1}(\mathcal{H}^*) = (\lambda_1^{*(m-1)} \geq \cdots \geq \lambda_k^{*(m-1)})$$

are related by

$$\lambda_i^{*(m-1)} = n - \lambda_{k+1-i}^{m-1} \tag{4.19}$$

Let us denote the complement of m-uniform hypergraph $\mathcal{H}$ by $\mathcal{H}^c$. Two $(m-1)$-subsets are adjacent if they intersect at precisely $(m-2)$ positions and they lie on a edge. As per definition of $\mathcal{H}^c$, an m-set is either an edge of $\mathcal{H}$ or of $\mathcal{H}^c$.

Secondly as the diagonal elements of $L_{m-2}(\mathcal{H})$ and $L_{m-2}(\mathcal{H}^c)$ both represent degrees of $(m-1)$-subsets in m-subsets that are edges in the respective hypergraphs, the sum of the diagonal elements of $L_{m-2}(\mathcal{H})$ and $L_{m-2}(\mathcal{H}^c)$ is $n-m+1$ as each $(m-1)$-subset is in $n-m+1$ m-subsets . So we can conclude that $L_{m-2}(\mathcal{H}) + L_{m-2}(\mathcal{H}^c) = L_{m-2}\binom{[n]}{m}$, where $L_{m-2}\binom{[n]}{m}$ is Laplace matrix of the complete m-uniform hypergraph.

From proposition 4.4 of [8] we also know all these matrices commute. So we immediately deduce,

Theorem 4.5. *In the situation of the above discussion, the spectra*

$$\lambda^{m-2}(\mathcal{H}) = (\ \lambda_1^{m-2} \geq \cdots \geq \lambda^{m-2}_{\binom{n-1}{m-1}}\ )$$

$$\lambda^{m-2}(\mathcal{H}^c) = (\ \lambda_1^{c(m-2)} \geq \cdots \geq \lambda^{c(m-2)}_{\binom{n-1}{m-1}}\ )$$

are related by

$$\lambda_i^{c(m-2)} = n - \lambda^{m-2}_{\binom{n-1}{m-1+1-i}} \tag{4.20}$$

Given any m-uniform $\mathcal{H}$, we can form a new hypergraph $\mathcal{H}^{\curlywedge}$ of one-dimension higher by an operation called *coning*. Intuitively, we form the cone of m-uniform hypergraph $\mathcal{H}$, by taking a vertex w completely separate from $\mathcal{H}$ and then forming new m-dimensional faces by adding w to every $(m-1)$-dimensional face. We will denote this operation often by $w * \mathcal{H}$, so $w * \mathcal{H} = \mathcal{H}^{\curlywedge}$.

Cones are extensively studied objects in topology and combinatorial algebra. The operation of coning on a given hypergraph results in another hypergraph, and often certain properties of the original hypergraph are predictably altered by process of coning. We will study what happens to the Laplacian spectrum after coning.

Let's assume that $\mathcal{H}$ has k edges of cardinality m. Then the hypergraph $\mathcal{H}^{\curlywedge}$ will have k edges of cardinality $m+1$. We assume that vertex $w = n+1$ is a new vertex in the pointset $[n+1]$ which is distinct from all elements of $[n]$. Now we concentrate on $L_{m-1} = N_m N_m^\top$. For an m-dimensional edge e in $\mathcal{H}^{\curlywedge}$ which we get from an $(m-1)$-dimensional edge f in $\mathcal{H}$, we observe that in $\partial_m e$, the last term is always f with a constant sign. This will be positive if $m+1$ is odd and negative otherwise. Now for rest of the terms, they have one-to-one correspondence with all $(m-1)$-subsets of $\partial_{m-1} f$. So if we rearrange the rows of N_m such that all m-subsets of $\mathcal{H}$ are at the end, then

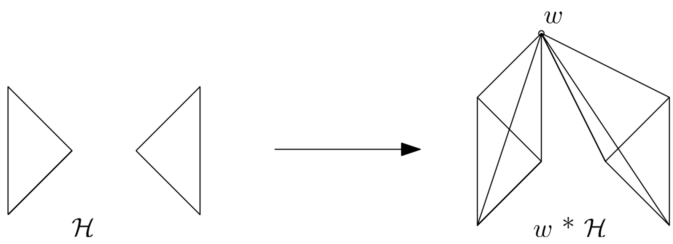

Figure 4.1: Coning of a hypergraph

we can see $N_m(\mathcal{H}^{\curlywedge}) = \begin{bmatrix} N_{m-1}(\mathcal{H}) \\ I \end{bmatrix}$ or $N_m(\mathcal{H}^{\curlywedge}) = \begin{bmatrix} N_{m-1}(\mathcal{H}) \\ -I \end{bmatrix}$, depending upon whether $m+1$ is odd or even respectively. So L_m can be written as

$$\begin{bmatrix} N_{m-1}(\mathcal{H})N_{m-1}^{\top} & N_{m-1}(\mathcal{H}) \\ N_{m-1}^{\top}(\mathcal{H}) & I \end{bmatrix} \text{ or } \begin{pmatrix} N_{m-1}N_{m-1}^{\top}(\mathcal{H}) & -N_{m-1}(\mathcal{H}) \\ -N_{m-1}^{\top}(\mathcal{H}) & I \end{pmatrix}$$

The matrices above can be converted to the below matrix by elementary operations:

$$\begin{bmatrix} N_{m-1}^{\top}(\mathcal{H})N_{m-1}(\mathcal{H}) + I & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix above means that we add 1 to each eigenvalue of $\mathcal{H}$ to get eigenvalues of $\mathcal{H}^{\curlywedge}$.

This can be generalized in case we calculate the spectrum for $i \leq m$. Then the only thing to observe is that $\mathcal{H}^{\curlywedge}$ also has i-dimensional faces inherited from $\mathcal{H}$. So N_i can be written as

$$\begin{bmatrix} N_{i-1}(\mathcal{H}) & 0 \\ I & N_i(\mathcal{H}) \end{bmatrix} \text{ or } \begin{bmatrix} N_{i-1}(\mathcal{H}) & 0 \\ -I & N_i(\mathcal{H}) \end{bmatrix}$$

depending upon whether i is odd or even respectively. So L_i can be written as

$$\begin{bmatrix} N_{i-1}(\mathcal{H})N_{i-1}^{\top}(\mathcal{H}) & N_{i-1}(\mathcal{H}) \\ N_{i-1}^{\top}(\mathcal{H}) & N_i(\mathcal{H})N_i^{\top}(\mathcal{H}) + I \end{bmatrix} \text{ or } \begin{bmatrix} N_{i-1}(\mathcal{H})N_{i-1}^{\top}(\mathcal{H}) & -N_{i-1}(\mathcal{H}) \\ -N_{i-1}^{\top}(\mathcal{H}) & N_iN_i^{\top}(\mathcal{H}) + I \end{bmatrix}$$

The matrices above can be converted to the below matrix by elementary operations:

$$\begin{bmatrix} N_{i-1}^{\top}(\mathcal{H})N_{i-1}(\mathcal{H}) + N_i(\mathcal{H})N_i^{\top}(\mathcal{H}) + I & 0 \\ 0 & 0 \end{bmatrix}$$

Let's represent spectrum of $L_{i-1}(\mathcal{H})$ by $\lambda^i(\mathcal{H})$. We may summarize the discussion above in the following theorem,

Theorem 4.6. *If $\mathcal{H}$ is an m-uniform hypergraph, $\mathcal{H}^\curlywedge$ be the cone over $\mathcal{H}$, then*

$$\lambda^i(\mathcal{H}^\curlywedge) = 1^{f_i(\mathcal{H})} + (\lambda^i(\mathcal{H}) \cup \lambda^{i-1}(\mathcal{H})) \tag{4.21}$$

4.3 Connectivity of Hypergraphs

We now try to understand when the hypergraphs can be said to be connected. There are several definitions which currently exists. Most known definition is given in [3] where a hypergraph is said to be *connected* if an intersection graph (An *intersection graph* of a hypergraph has edges as vertices and two vertices are adjacent if the corresponding edges have a nontrivial intersection) of the edges is connected.

Now in our case of m-uniform hypergraphs, we prefer to think of vertices as $(m-1)$-subsets and generalizing from graphs we can say that two vertices are adjacent if they lie on a edge. So two $(m-1)$-subsets are adjacent if they lie on a edge.

In our context we can change the definition by Berge's given above, m-uniform hypergraph is said to be *connected* if the intersection graph (A intersection graph of a m-uniform hypergraph has edges as vertices and these vertices are connected if edges intersect at exactly $m-1$ places) of the edges is connected.

Now with the above definition in hand, we can look into some examples. We can think of a 4-uniform hypergraph on 8 points with faces of the cube as the edges. We can easily see that 6 faces of cube are disconnected and as $N_3 N_3^\top$ and $N_3^\top N_3$ have same nonzero eigenvalues, so each face has a eigenvalue of 4. So cube has eigenvalues as $4^6, 0^{18}$. Similarly we can see eigenvalues of 3-uniform Fano plane as $3^7, 0^{14}$. This generalizes Proposition (1.9).

Let's define a *path* in hypergraphs. An m-uniform hypergraph is said to be a path if consecutive edges have a $(m-1)$-subset in common. For example, if we have $[n] = \{1, 2, \ldots, 6\}$, then the 4-uniform hypergraph $\mathcal{H}$ with edges $\{1,2,3,4\}, \{2,3,4,5\}, \{3,4,5,6\}$ is a path.

Proposition 4.7. *Let $\mathcal{H}$ be an m-uniform hypergraph which is a path with k edges. The spectrum $L_{m-2}(\mathcal{H})$ is*

$$\lambda_s = m + 2\cos\left(\frac{\pi s}{k+1}\right) \text{ for } s = 1, \ldots, k \tag{4.22}$$

The remaining eigenvalues are zeros.

Proof. We know that $L_{m-2}(\mathcal{H}) = N_{m-1}N_{m-1}^\top$, and that nonzero eigenvalues of $N_{m-1}N_{m-1}^\top$ and $N_{m-1}^\top N_{m-1}$ are same. Each of the columns of N_{m-1}, has m entries which are either $+1$ or -1 so the diagonal entry of $N_{m-1}^\top N_{m-1}$ is m. Consider the i-th column of N_{m-1} where $1 < i < k$. It will have an entry in a row corresponding to a common face with $i-1$ and $i+1$ columns of N_{m-1}. If m is odd, then common face entry of column i and column $i-1$ (or $i+1$) will have same sign and otherwise both of entries will have a an opposite sign (this comes from the definition of the boundary operator). The first and last column of N_{m-1} also have one entry column with their adjacent columns. Hence the matrix $N_{m-1}^\top N_{m-1}$ will be a tridiagonal matrix and hence the eigenvalues. □

Let's have one more interesting case of a 3-uniform hypergraph. An n-*flapwheel* is a simplicial complex consisting of n distinct 2-simplices, called *flaps*, whose intersection is a single 1-simplex, called the *axis*. The two vertices of the axis are called *axial vertices* and rest of the vertices of the flapwheel are called *flap vertices*.

We can easily see that for any integer $n > 0$, n-flapwheel has $n+2$ vertices.

Proposition 4.8. *Let $\mathcal{H}$ be an n-flapwheel. Then the nonzero spectrum of $L_2(\mathcal{H})$ is*

$$\{n+2,\ 2^{n-1}\} \tag{4.23}$$

Proof. We know that $L_i = N_{i+1}N_{i+1}^\top$, and that nonzero eigenvalues of $N_{i+1}N_{i+1}^\top$ and $N_{i+1}^\top N_{i+1}$ are the same. We know 2-simplices of $\mathcal{H}$ correspond to the columns of N_2, and each diagonal entry of $N_2^\top N_2$ is 3 and each off diagonal entry is 1 as all the edges meet at the axis. So $N_2^\top N_2$ can be written as $2I + J$ and the result follows. □

Now let us treat the case of the complete m-uniform hypergraph $\mathcal{H}$. Then $\mathcal{H}^c$ has no edges, so all the eigenvalues are zero. So from (4.20), eigenvalues are n or 0. The trace of $L_{m-2}(\mathcal{H})$ is $(n-m+1)\binom{n}{m-1}$ so the multiplicity of eigenvalue n is $\binom{n-1}{m-1}$ and the multiplicity of eigenvalue 0 is $\binom{n-1}{m-2}$.

Proposition 4.9. *The spectrum of $L_{m-2}(\mathcal{H})$ where $\mathcal{H}$ is a complete m-uniform hypergraph is*

$$\{n^{\binom{n-1}{m-1}}, 0^{\binom{n-1}{m-2}}\}$$

4.4 Interlacing in Hypergraphs

By Lemma (1.3), we know that if we add an edge to a graph, then the eigenvalues of Laplace matrix do not decrease. We can extend this concept to hypergraphs as well. If we add an edge to a hypergraph, then the eigenvalues do not decrease.

Theorem 4.10. *Let $\mathcal{H}$ be an m-uniform hypergraph on $[n]$ with $n > m$. Let $\mathcal{H}'$ be a hypergraph obtained from $\mathcal{H}$ by adding an edge. Then $\lambda_i(\mathcal{H}) \leq \lambda_i(\mathcal{H}')$ for all i and $\lambda_i(\mathcal{H}') \leq \lambda_{i+1}(\mathcal{H})$.*

Proof. We know that $L_i = N_{i+1}N_{i+1}^\top$, and that the nonzero eigenvalues of $N_{i+1}N_{i+1}^\top$ and $N_{i+1}^\top N_{i+1}$ are the same. Let us have k edges in $\mathcal{H}$, then $N_{i+1}^\top N_{i+1}$ is of size $k \times k$. So if we add an edge to $\mathcal{H}$, that means we are adding a column to N_{i+1} so the resulting matrix $N_{i+1}^\top N_{i+1}$ for $\mathcal{H}'$ has size $(k+1) \times (k+1)$. Use Lemma (1.2) and we are done. □

This generalizes Lemma (1.3).

One immediate consequence of interlacing and Proposition (4.9) is that for any m-uniform hypergraph $\mathcal{H}$ will have at least $\binom{n-1}{m-2}$ zeros in the spectrum.

4.5 Dominance Order and Downshifting

Let $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$ be two finite nonincreasing sequences of nonnegative real numbers. We say that $\mathbf{b}$ *dominates* $\mathbf{a}$, and we write $\mathbf{a} \trianglelefteq \mathbf{b}$ when $\sum_{i=1}^{t} a_i \leq \sum_{i=1}^{t} b_i$ for all t, and $\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i$, where missing elements are taken to zero.

For example, in this notation Schur's inequality (1.5) says that $\mathbf{d} \trianglelefteq \theta$ if $\mathbf{d}$ is sequence of diagonal elements and θ is the sequence of eigenvalues of a real symmetric matrix.

If $\mathbf{a} = (a_j)$ be a finite nonincreasing sequences of nonnegative integers, then $\mathbf{a}^\top$ denotes the sequence $(a_j^\top)$ with $a_j^\top = \#\{i \mid a_i \geq j\}$. If $\mathbf{a}$ and $\mathbf{a}^\top$ are regarded as partitions then they are conjugate, and their's Ferrer's diagram are each other's transpose.

If $\mathbf{a}$ and $\mathbf{b}$ are two nonincreasing sequences, then let $\mathbf{a} \cup \mathbf{b}$ denote the (multiset) union of both sequences, with elements sorted in nonincreasing order.

Lemma 4.11. *Let $\mathbf{a}$ and $\mathbf{b}$ are two nonincreasing integral sequences, then:*

1. $\mathbf{a}^{\top\top} = \mathbf{a}$;
2. $(\mathbf{a} \cup \mathbf{b})^\top = \mathbf{a}^\top + \mathbf{b}^\top$ *and* $\mathbf{a} + \mathbf{b} = \mathbf{a}^\top \cup \mathbf{b}^\top$;
3. $\mathbf{a} \trianglelefteq \mathbf{b}$ *if and only if* $\mathbf{b}^\top \trianglelefteq \mathbf{a}^\top$

Let's put a total order on the vertices of an m-uniform hypergraph $\mathcal{H}$ in such a way that if $x \leq y$ then $d_x \geq d_y$. Now $\mathcal{H}$ is said to be invariant under *downshifting* if whenever $\{x_1, \ldots, x_m\}$ is an edge of $\mathcal{H}$, and $\{y_1, \ldots, y_m\}$ is a m-subset with $y_i \leq x_i$ for all i, then also $\{y_1, \ldots, y_m\}$ is an edge of $\mathcal{H}$. If this hold for one total order, then it holds for any total order that is compatible with degrees.

The simplicial complexes which are invariant under downshifting are referred as *shifted simplicial complex* or the simplical complex is said to be *shifted*.

4.6 The Duval-Reiner Conjecture and some bounds

Let $\mathcal{H}$ be an m-uniform hypergraph. As per definition of the Laplace matrix (L_{m-2}) of a hypergraph, the diagonal entries are the degrees of $(m-1)$-subsets in $\mathcal{H}$ as given in 4.13. Thus we can define the nonincreasing degree sequence of $(m-1)$-subsets $\mathbf{d}^{(m-1)}(\mathcal{H})$, as below:

$$\mathbf{d}^{(m-1)}(\mathcal{H}) = (d_1^{m-1}, \ldots, d_{f_{m-2}}^{m-1})$$

Also denote the nonincreasing eigenvalue sequence of the Laplacian of the hypergraph $\mathcal{H}$, as $\lambda(\mathcal{H})$ and the degree sequence of the vertices of $\mathcal{H}$ as $\mathbf{d}(\mathcal{H})$. Then from Schur's inequality (1.5), we can easily say that:

$$\mathbf{d}^{(m-1)}(\mathcal{H}) \trianglelefteq \lambda(\mathcal{H}) \tag{4.24}$$

We have another interesting result which shows an elegant use of the Gale-Ryser Theorem (2.1). This result is taken from [8].

Lemma 4.12. *For any m-uniform hypergraph $\mathcal{H}$, we have:*

$$\mathbf{d}^{(m-1)}(\mathcal{H}) \trianglelefteq \mathbf{d}^{\top}(\mathcal{H}) \tag{4.25}$$

Proof. Let's consider a 0-1 matrix whose rows are indexed by i-subsets for some $i \geq 1$, columns are indexed by $(m-i)$-subsets, and the entry indexed by i-subset A and $(m-i)$-subset B is 1 exactly when the disjoint union of A and B is an edge in $\mathcal{H}$. Now we apply the Gale-Ryser theorem to the above matrix, we get

$$\mathbf{d}^{(i)}(\mathcal{H}) \trianglelefteq (\mathbf{d}^{(m-i)}(\mathcal{H}))^{\top} \tag{4.26}$$

Now by Lemma (4.11), we get our result for $i = 1$.

□

Duval and Reiner [8] generalized the Grone-Merris conjecture. We will refer to this conjecture as the Duval-Reiner conjecture.

Conjecture 4.13. *Let the m-uniform hypergraph $\mathcal{H}$ have degrees d_x, and Laplacian eigenvalues λ_i, ordered such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. Then*

$$\lambda \trianglelefteq \mathbf{d}^\top \tag{4.27}$$

Equality holds if and only if $\mathcal{H}$ is invariant under downshifting.

We observe that the last inequality in (4.27) holds with a equality.

Proposition 4.14.

$$\sum_{t=1}^{\infty} \lambda_t = \sum_{t=1}^{\infty} d_t^\top$$

Proof.

$$\begin{aligned}\sum_{t=1}^{\infty} d_t^\top &= \sum_{t=1}^{\infty} d_t \\ &= \sum_{i=1}^{n} |\{A \in \mathcal{H} \ : \ i \in A\}| \\ &= \sum_{B \subset [n],\ |B|=m-1} |\{A \in \mathcal{H} \ : \ B \subset A\}| \\ &= \operatorname{tr}\left(L_{m-2}(\mathcal{H})\right) \\ &= \sum_{t=1}^{\infty} \lambda_t\end{aligned}$$

□

We now prove the first inequality in (4.27).

Proposition 4.15.

$$\lambda_1 \leq d_1^\top$$

Proof. Note that $d_1^\top$ is simply the number of vertices in $[n]$ which lie in some m-subset of $\mathcal{H}$, that is the number of non-isolated vertices with respect to $\mathcal{H}$. Since the isolated vertices will only add to zero rows to the various Laplace matrices, and hence only add zeroes to their spectra, we may assume without loss of generality that $d_1^\top = n$. Hence we must show that $\lambda_1 \leq n$.

By Theorem (4.4), spectra of $\mathcal{H}$ and $\mathcal{H}^*$ are related by $\lambda_i = n - \lambda_j^*$ for some j. As all λ_i are nonnegative so we conclude that $\lambda_i \leq n$ for all i.

□

We now prove that conjecture (4.13) holds for m-uniform paths.

Proposition 4.16. *The Duval-Reiner conjecture holds for m-uniform paths.*

Proof. From Proposition (4.7), we know the Laplace spectrum of a path. Let's find out the conjugate degree sequence of an m-uniform path $\mathcal{H}$. Let $\mathcal{H}$ have k edges and $k \leq m$. Then $\mathcal{H}$ has $m + k - 1$ distinct vertices from $[n]$. So $d_i^\top = m + k - (2i - 1)$ for $1 \leq i \leq k$. Let us take $\hat{\lambda}_i = \lambda_i - m$ and $\hat{d_i^\top} = d_i^\top - m$.

Let's construct two new sequences a_s and b_s where $a_s = \sum_{i=1}^{s} \hat{\lambda}_i$ and $b_s = \sum_{i=1}^{s} \hat{d_i^\top}$. We can easily see that $\hat{\lambda}_i = -\hat{\lambda}_{k-i+1}$ and hence $a_s = a_{k-s}$. Similarly $b_s = b_{k-s}$.

Now if $\mathbf{a} \trianglelefteq \mathbf{b}$ then it implies $\lambda \trianglelefteq \mathbf{d}^\top$. Also we only need to show all inequalities hold for $1 \leq s \leq (k+1)/2$. If k is odd, $\hat{\lambda}_i$ and $\hat{d_i^\top}$ are symmetric around 0 and $\hat{\lambda}_i$ increases by 2 and $\hat{d_i^\top}$ increases by at most 2 (from 0) so all inequalities holds for $1 \leq s \leq (k+1)/2$. Now if k is even then also we have the same argument as all inequalities hold for $1 \leq s \leq k/2$.

But if $k \geq m$, then there are only m terms in the conjugate degree sequence but k eigenvalues. So we need the first m inequalities to hold. Let's say the p-th inequality holds then

$$\begin{aligned}
\sum_{i=1}^{p} d_i' &= \sum_{i=1}^{p} k + m - (2i - 1) \\
&= (m + k)p - p^2 \\
&\geq \sum_{i=1}^{p} \lambda_i \\
&\geq mp + 2\sum_{i=1}^{p} \cos\left(\frac{\pi i}{k+1}\right) \\
&\geq mp + 2p
\end{aligned}$$

It is equivalent to $k \geq p + 2$. So if $k \geq m + 2$, then all m inequalities hold. Hence the only case which has to be checked is $k = m+1$. We can easily see that the first $m-1$ inequalities are satisfied and the m-inequality is satisfied because $m \geq 2\cos\left(\frac{\pi}{m+2}\right)$.

□

The Duval-Reiner conjecture also holds for complete m-uniform graphs.

Proposition 4.17. *The Duval-Reiner conjecture holds for complete m-uniform hypergraphs with equality.*

Proof. Let's first look into the degree sequence for complete m-uniform hypergraphs. Now each vertex can be in $\binom{n-1}{m-1}$ distinct edges. So the conjugate degree sequence is $(n, n, \ldots, n, 0, \ldots, 0)$ with first $\binom{n-1}{m-1}$ entries as n. So by Proposition (4.9), we are done. This can also seen from 4.18 as complete m-uniform hypergraphs are also invariant under downshifting. □

4.7 Spectra of shifted complexes

Now we will study the spectra of shifted complexes. We will use all the terminology and results which have been established in the previous sections. The following theorem is the main result of [8].

Theorem 4.18. *For any shifted m-uniform hypergraph $\mathcal{H}$, the spectrum of its Laplacian $L_{m-2}(\mathcal{H})$, satisfies*

$$\lambda(\mathcal{H}) = \mathbf{d}^{\top}(\mathcal{H}) \tag{4.28}$$

where here $\mathbf{d}^{\top}$ is the conjugate partition of the degree sequence $\mathbf{d}$ of vertices with respect to the m-sets in $\mathcal{H}$.

This result was proved by Merris in [23] in case of graphs ($m = 2$).

Given an m-family on $[n]$, let's define a new $(m-1)$-family and m-family on the vertex set $[n] \setminus \{1\}$, respectively, the *link* and *deletion* of vertex 1 in $\mathcal{H}$.

$$\text{link}_{\mathcal{H}}1 = \{A \setminus \{1\} : \ |A| = m, \ 1 \in A \in \mathcal{H}\} \tag{4.29}$$

$$\text{del}_{\mathcal{H}}1 = \{A : \ |A| = m, 1 \notin A \in \mathcal{H}\} \tag{4.30}$$

There is always a decomposition

$$\mathcal{H} = (1 * \text{link}_{\mathcal{H}}1) \cup \text{del}_{\mathcal{H}}1 \tag{4.31}$$

where $1 * \text{link}_{\mathcal{H}}1$ denotes the m-family that contain 1.

We say that $\mathcal{H}$ is a *near-cone with apex 1* if for every set A in $\text{del}_{\mathcal{H}}1$ and every a in A one has that $(A \setminus a) \cup 1$ is a m-set in $\mathcal{H}$.

The relation between the shifted families and near-cones are given by the following proposition.

Proposition 4.19. *Let $\mathcal{H}$ be a m-family on $[n]$. Then $\mathcal{H}$ is shifted if and only if $\mathcal{H}$ is a near-cone with apex 1, and both $\text{link}_{\mathcal{H}}1$ and $\text{del}_{\mathcal{H}}1$ are shifted families with respect to $[n] \setminus \{1\}$.*

Proof. $\Rightarrow$ Let $\mathcal{H}$ be a shifted m-family. Now if we consider $A \in \text{del}_{\mathcal{H}}1$ and for $a \in A$, $(A \setminus a) \cup 1$ will be a m-subset in $\mathcal{H}$ because $\mathcal{H}$ is invariant under downshifting, so $\mathcal{H}$ is a near-cone apex 1. Again $\text{del}_{\mathcal{H}}1$ and $\text{link}_{\mathcal{H}}1$ are also invariant under downshifting because they are either m-subsets of $\mathcal{H}$ (which

is invariant under downshifting) or $(m-1)$-subsets without 1 which will also be invariant under downshifting on the point set $[n] \setminus \{1\}$.
$\Leftarrow$ Now let $x = (x_1, \ldots, x_m)$ be a m-subset $\in \mathcal{H}$. Then if $x_1 = 1$, $(x \setminus 1) \in$ $\text{link}_{\mathcal{H}}1$ and as $\text{link}_{\mathcal{H}}1$ is shifted, we are done. Now if $x_1 \neq 1$, we have $x \in \text{del}_{\mathcal{H}}1$. Now $y = (y_1, \ldots, y_m)$ such that $x \geq y$ over the $[n] \setminus \{1\}$, are in $\text{del}_{\mathcal{H}}1$ and rest y with $y_1 = 1$ are also there because of near-cone condition. ☐

By the above proposition, the following two lemmas about behavior of $\lambda(\mathcal{H})$ and $\mathbf{d}^\top(\mathcal{H})$ for the near-cones give a proof of (4.18) by induction on n.

Lemma 4.20. *If $\mathcal{H}$ is a m-family which is a near-cone with apex 1, then*

$$\mathbf{d}^\top(\mathcal{H}) \trianglerighteq 1^{|link_{\mathcal{H}}1|} + (\mathbf{d}(link_{\mathcal{H}}1)^\top \cup \mathbf{d}(del_{\mathcal{H}}1)^\top) \tag{4.32}$$

Furthermore, when $\mathcal{H}$ is shifted, the above majorization inequality is a equality.

Proof. Applying Lemma (4.11) to (4.32), we get

$$\mathbf{d}(\mathcal{H}) \trianglelefteq |\text{link}_{\mathcal{H}}1| \cup (\mathbf{d}(\text{link}_{\mathcal{H}}1) + \mathbf{d}(\text{del}_{\mathcal{H}}1)) \tag{4.33}$$

By the defintion of $\text{link}_{\mathcal{H}}1$, the vertex 1 of $\mathcal{H}$ lies in exactly $|\text{link}_{\mathcal{H}}1|$ set of $\mathcal{H}$. So the above majorization relation can be modified so as to remove $|\text{link}_{\mathcal{H}}1|$ from both sides, we get

$$\mathbf{d}'^{(}\mathcal{H}) \trianglelefteq \mathbf{d}(\text{link}_{\mathcal{H}}1) + \mathbf{d}(\text{del}_{\mathcal{H}}1) \tag{4.34}$$

where $\mathbf{d}'(\mathcal{H})$ denotes the degree sequence obtained after removing degree of vertex 1. Now let

$$D'(\mathcal{H}), D(\text{link}_{\mathcal{H}}1), D(\text{del}_{\mathcal{H}}1) \tag{4.35}$$

denote the unsorted degree sequence of the ordered vertex set $[n] \setminus \{1\}$ that corresponds after sorting to

$$\mathbf{d}'(\mathcal{H}), \mathbf{d}(\text{link}_{\mathcal{H}}1), \mathbf{d}(\text{del}_{\mathcal{H}}1) \tag{4.36}$$

From (4.31), we know,

$$D'(\mathcal{H}) = D(\text{link}_{\mathcal{H}}1) + D(\text{del}_{\mathcal{H}}1) \tag{4.37}$$

On the other hand, it is easy to see that [21, 22, Chapter 6, A.1] if d', d_1, d_2 are the sorted partitions corresponding to any three vectors D', D_1, D_2 of nonnegative integers satisfying

$$D' = D_1 \; + \; D_2 \tag{4.38}$$

then

$$d' \trianglelefteq d_1 + d_2 \tag{4.39}$$

When $\mathcal{H}$ is shifted, the unsorted degree sequences $D'(\mathcal{H}), D(\text{link}_{\mathcal{H}}1), D(\text{del}_{\mathcal{H}}1)$ coincide with $\mathbf{d}'(\mathcal{H}), \mathbf{d}(\text{link}_{\mathcal{H}}1), \mathbf{d}(\text{del}_{\mathcal{H}}1)$ as ordering of the degree sequence of a shifted family is given by linear ordering of vertices. This completes the proof. □

Lemma 4.21. *If $\mathcal{H}$ is a m-family which is a near-cone with apex 1, then*

$$\lambda(\mathcal{H}) \trianglerighteq 1^{|link_{\mathcal{H}}1|} + (\lambda(link_{\mathcal{H}}1) \cup \lambda(del_{\mathcal{H}}1)) \tag{4.40}$$

Proof. Let $\mathcal{H}'$ be the $(m-1)$-dimensional hypergraph generated by $\text{link}_{\mathcal{H}}1 \cup \text{del}_{\mathcal{H}}1$. By notation introduced earlier, $(\mathcal{H}')_j$ denotes the $(j+1)$-family of j-dimensional faces of $\mathcal{H}'$, so

$$(\mathcal{H}')_{m-1} = \text{del}_{\mathcal{H}}1 \tag{4.41}$$

and the near-cone condition of $\mathcal{H}$ implies

$$(\mathcal{H}')_{m-2} = \text{link}_{\mathcal{H}}1 \tag{4.42}$$

Now $1 * \mathcal{H}'$ is a cone over $\mathcal{H}'$. Then

$$(1 * \mathcal{H}')_{m-1} = (1 * \mathcal{H}')_{m-2} \cup (\mathcal{H}')_{m-1} = (1 * \text{link}_{\mathcal{H}}1) \cup \text{del}_{\mathcal{H}}1 = \mathcal{H} \tag{4.43}$$

Thus

$$\begin{aligned}\lambda(\mathcal{H}) &= \lambda_{m-2}(1 * \mathcal{H}') \\ &= 1^{f_{m-2}(\mathcal{H}')} + (\lambda_{m-3}(\mathcal{H}') + \lambda_{m-2}(\mathcal{H}')) \\ &= 1^{f_{m-2}(\mathcal{H}')} + (\lambda(\text{link}_{\mathcal{H}}1) + \lambda(\text{del}_{\mathcal{H}}1))\end{aligned}$$

The above equality holds because of Theorem (4.6) and that spectrum of L_i depends only upon the $(i+1)$-dimensional faces.

□

From Theorem (4.18), we can easily see that the Duval-Reiner conjecture holds for n-flapwheel complexes as they are shifted.

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