MASTER

Distance transitive graphs of type F4

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Master's thesis

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by

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Abstract

Distance transitive graphs are a class of extremely symmetric graphs. One way to describe these graphs is by a given automorphism group G, a given maximal subgroup H that stabilizes a vertex v of the graph Γ, and an $r \in G$ that defines an edge (namely $v \sim vr$) of the graph.

A lot of research has been done on the possible combinations of automorphism groups and subgroups which can result in a distance transitive graph, mostly by Arjeh M. Cohen, Martin Liebeck and Jan Saxl. But there are some open cases, that still need to be investigated.

In this thesis, we examine some of these cases, namely with $G = F_4(q)$ and $H = D_4(q).S_3$ or $H = B_4(q)$.

Samenvatting

Afstands-transitieve grafen zijn een klasse van uiterst symmetrische grafen. Een mogelijke manier om deze afstandstransitieve grafen te beschrijven is door een gegeven automorfismegroup G, een gegeven maximale ondergroep H van G die bovendien de stabilisator is van een punt v in de graaf Γ, en een $r \in G$, die een willekeurige kant van Γ (namelijk $v \sim vg$) definieert.

Vele studies zijn al gedaan naar de mogelijke combinaties van G en H om een afstands-transitieve graaf te bekomen, vooral door Arjeh M. Cohen, Martin Liebeck and Jan Saxl. Er zijn echter nog wat open gevallen.

In deze thesis onderzoek ik enkele van deze gevallen, namelijk met $G = F_4(q)$ en $H = D_4(q).S_3$ of $H = B_4(q)$.

Acknowledgements

This Masters thesis is the result of 10 months of research at the department of Discrete Algebra and Geometry at the Eindhoven University of Technology (TU/e). With this thesis I will end my study of Industrial and Applied Mathematics, and I will receive the degree of Master of Science.

The subject of my research, Distance Transitive Graphs, was introduced to me by prof. dr. Arjeh M. Cohen, who became my supervisor on this project.

I would like to take some time to thank everybody who, in one way or another, helped me during my 10 months of research. First of all there is my supervisor. He helped me every time I had a question or problem. And always in the most patient and precise way, in spite of his very busy schedule. After meeting with him, I always saw light at the end of the tunnel, no matter how dark it was before the meeting. The same honors should go to dr. F.G.M.T. Cuijpers too, who I could also ask any question when Arjeh Cohen was unavailable.

For all my Magma problems, dr. Scott H. Murray was always there to help me out. And not only the Magma problems, but also for every other problem I wanted to discuss with him.

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Introduction

> *Beauty depends on size as well as symmetry. No very small animal can be beautiful, for looking at it takes so small a portion of time that the impression of it will be confused. Nor can any very large one, for a whole view of it cannot be had at once, and so there will be no unity and completeness.*
>
> Aristotle (384 B.C.322 B.C.), Greek philosopher, student of Plato, tutor of Alexander the Great

As with almost everything in nature, a graph can be symmetric. The graphs considered to be the most symmetric, in mathematical terms, are known as distance-transitive graphs. This implies that for any 2 pairs of points, where both pairs have distance i from each other, we have a symmetry of the graph bringing one pair to the other. We then say that the automorphism group of the graph acts transitively on every distance set.

It has already been proven that the number of distance transitive graphs of diameter larger than 2 is finite. However not every distance transitive graph (dtg) has been found, so a project was started to find all distance transitive graphs. The proceedings of this project can be found in [6].

In our case we especially look for primitive dtgs, since these are the building stones of all dtgs. A theorem of Praeger, Saxl and Yokoyama shows that for a primitive dtg Γ and its automorphism group G, we can say that either Γ is related to a Hamming graph, Γ is affine, or G is almost simple.

There exists a construction, starting from a group G and a maximal subgroup H, resulting in a nice graph structure with G as automorphism group. Thanks to the classification of finite simple groups, we get a list of groups (and maximal subgroups) which can give rise to a dtg. A lot of work has been done in this area, however the list of all dtgs with a simple finite group as automorphism group is not entirely completed.

In this thesis, we investigate the case where the automorphism group is of the form $F_4(q)$ for $q \geq 3$. And we study it for 2 different maximal subgroups, $B_4(q)$ and $D_4(q).S_3$. With these groups we will try to construct a dtg, or try to prove that no dtg exist with this automorphism group.

In Section 1, we introduce some basic notions, that are used in the entire thesis. This introduction will deal with Lie theory, character theory and graph theory. In the next section, we give a brief summary of the classification of distance transitive graphs. In the next chapters, we first construct the 2 maximal subgroups of $F_4(q)$ mentioned above and we give a representation of them. After that we introduce a representation on the module on which the representation described earlier works. The final 2 sections deal with the actual search for dtgs.

1 Preliminaries

1.1 Construction of Lie groups of type F_4

1.1.1 Introduction to Lie theory

Definition 1.1 *Let L be a vector space. We call an operation $L \times L \to L : (x, y) \mapsto [x, y]$ a **Lie bracket** if*

- *the operation is bilinear;*
- *$[x, x] = 0$ for all $x \in L$; and*
- *$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.*

This last equation is called the Jacobi identity.

Definition 1.2 *A vector space L, with a Lie bracket is called a **Lie algebra**.*

To continue we need the following:

Definition 1.3 *Let L be a Lie Algebra. Denote $L^{(1)} := [L, L], L^{(i)} := [L^{(i-1)}, L^{(i-1)}]$, and $L_{(1)} := [L, L], L_{(i)} := [L, L^{(i-1)}]$. We say that L is **solvable** if there exist a natural number n for which $L^{(n)} = 0$.*
*We say that L is **nilpotent** if there exist a natural number m for which $L_{(m)} = 0$.*
*L is called **semisimple** if the maximal solvable ideal of L is 0.*

We define $\text{ad}_x : L \to L : y \mapsto [x, y]$.

Definition 1.4 *We say that an $x \in L$ is **nilpotent** if there exist a $k \in \mathbb{N}$ for which $(\text{ad}_x)^k = 0$. We call $x \in \text{End}L$ **semisimple** if the roots of its minimal polynomial over $\mathbb{F}$ are all distinct.*
We call $x \in L$ semisimple if ad_x is semisimple.

Proposition 1.1 (Jordan-Chevalley decomposition) *Let L be a finite dimensional vector space over $\mathbb{F}$, and $x \in \text{End}(L)$. There exist a unique decomposition of x in $x_s + x_n$, where $x_s \in \text{End}(L)$ is semisimple, $x_n \in \text{End}(L)$ is nilpotent, and they commute.*

By this proposition, we can also write, for a composition $x = x_s + x_n$ as above:

$$\text{ad}_x = \text{ad}_{x_s} + \text{ad}_{x_n}.$$

Since we can also prove there is a 1-1 correspondence between $L/Z(L)$ and ad_L, we can also say that each $x \in L$ determines unique elements $s, n \in L$ such that $\text{ad}_x = \text{ad}_s + \text{ad}_n$ is the Jordan Decomposition. This also means that $x = s+n$ and, since ad_s is semisimple, $[s, n] = 0$.

Now we look at a non-nilpotent Lie algebra L. Because of the previous proposition, we can find a $x \in L$ for which $x_s \neq 0$, by this we can look at semisimple subalgebras of L:

Definition 1.5 *A subalgebra H of a Lie algebra L is called* ***toral*** *if H consists of semisimple elements. We also call such a subalgebra a* ***torus****.*

Now we fix a maximal toral subalgebra H. Consider the set

$$L_\alpha := \{x \in L \mid [h,x] = \alpha(h)x, \quad \forall h \in H\},$$

for some $\alpha \in H^*$.

Definition 1.6 *We call an $\alpha \in H^*$ for which $L_\alpha \neq 0$, and $\alpha \neq 0$ a* ***root*** *of L relative to H. The set of roots is called the* ***root system*** *L* ***with respect to*** *H and will be denoted by Φ.*

With this definition we can prove the following propositions for L and H:

1. $C_L(H) = L_0 = H$.
2. $H \oplus \sum_{\alpha\in\Phi} L_\alpha = L$.
3. Φ spans H^*.

The proofs of these propositions will not be included in this paper, but can be found in [9].

On H^*, we can define a form $\langle \alpha, \beta \rangle := \dfrac{2(\beta|\alpha)}{(\alpha|\alpha)}$ such that Φ, with $(.|.)$ the standard inner product on $\mathbb{R}^n$, is a subset of the Euclidean space $\mathbb{R}^n$. Here n denotes the dimension of H. Now define:

$$\sigma_\alpha(\beta) := \beta - \langle \beta, \alpha \rangle \alpha.$$

Now we have the following extra properties for $\alpha, \beta \in \Phi$:

1. Φ spans $\mathbb{R}^n$, with n as above.
2. $\sigma_\alpha(\Phi) = \Phi$.
3. $-\alpha \in \Phi$, and this is the only different multiple of α in the root space.
4. $\langle \alpha, \beta \rangle \in \mathbb{Z}$.
5. Let p, q be the largest integers for which respectively $\alpha + p\beta$ and $\alpha - q\beta$ are roots. Then $\{\alpha + k\beta \mid -q \leq k \leq p\} \subset \Phi$, this subset is called the **β-root chain through α**.

Furthermore $q - p = \langle \alpha, \beta \rangle$. And now we call Φ a **root system**. Note that this abstractly defined, without the use of L.

Next we define the subgroup generated by $\{\sigma_\alpha \mid \alpha \in \Phi\}$ as the **Weyl Group** of Φ.

Now since we have an inner product of roots, we can also define a length $\| \ \|$ and an angle θ between roots. Due to these last propositions, the possibilities of angles, limits to the following table:

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Definition 1.7 *A subset Δ of Φ is called a set of* ***simple roots*** *if Δ is a basis of $\mathbb{R}^n$ and each root can be written as a linear sum of fundamental roots with strictly nonnegative or nonpositive coefficients. The height of a root is then defined as the sum of these coefficients.*

So the possible simple roots in 2 dimensions are drawn in Figure 1:

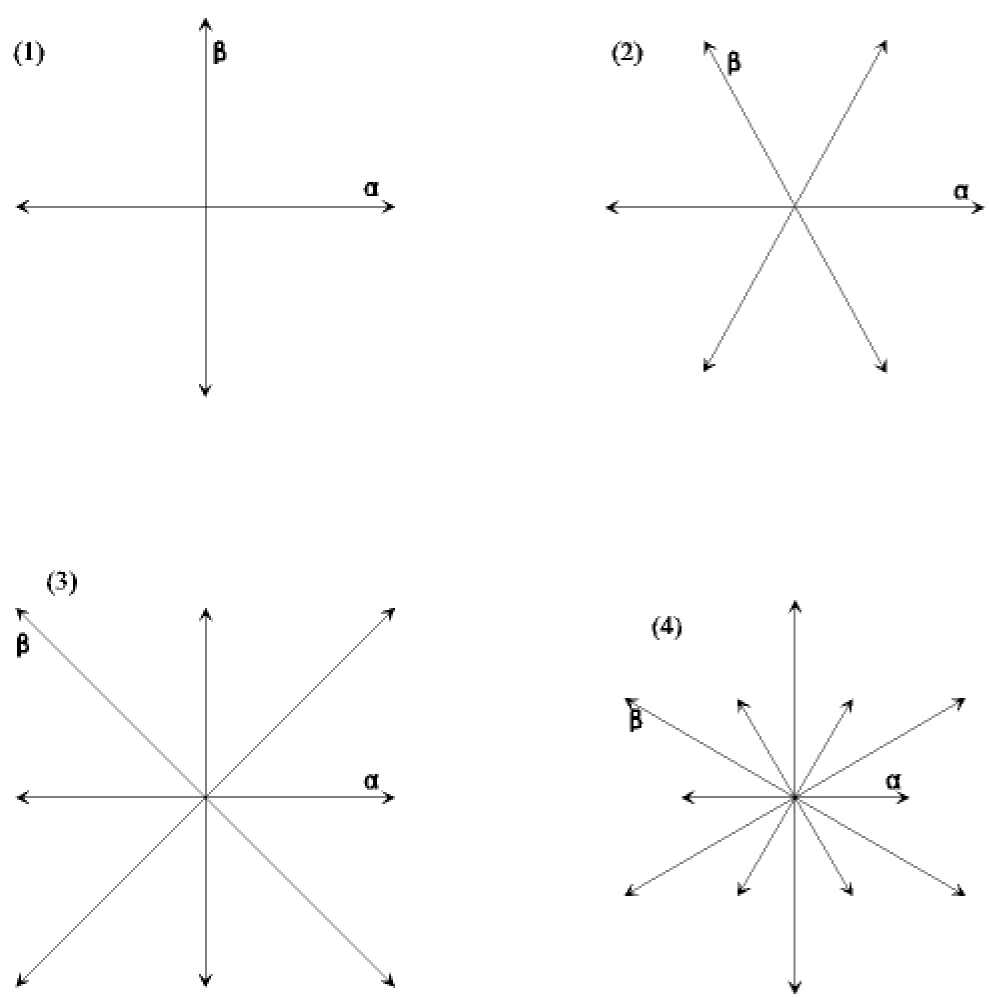


Figure 1: The possible simple roots of dimension 2

Now we fix Δ, a set of n simple roots, and order them: $(\alpha_1, ..., \alpha_n)$. We call the matrix C for which $C_{i,j} := \langle \alpha_i, \alpha_j \rangle$ the **matrix** of Φ.
With this matrix we can easily form the **Dynkin diagram** with the following conditions:

1. the diagram has n points;
2. each pair of points $\alpha_i, \alpha_j \in \Delta$ is connected with $\max\{C_{i,j}, C_{j,i}\}$ lines;
3. if there is more than 1 line between edges we add an arrow pointing to the shorter root.

So for the 4 pictures of simple roots in 2, the Dynkin diagrams are drawn in Figure 2:

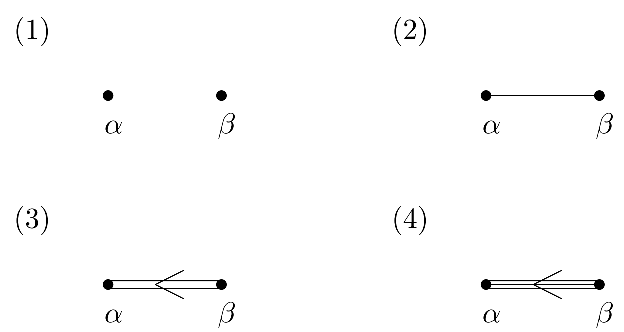


Figure 2: Dynkin diagrams with 2 fundamental roots

Theorem 1.1 (Chevalley) *It is possible to choose the root vectors $x_\alpha \in L_\alpha$, with $\alpha \in \Delta$ such that, with $h_i := [x_{\alpha_i}, x_{-\alpha_i}]$, in such a way that*

1. $[h_i, h_j] = 0 \quad \forall i, j;$
2. $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha \quad \forall i, \alpha;$
3. $[x_\alpha, x_{-\alpha}] =: h_\alpha$ *is a linear combination of* $(h_{\alpha_1}, ..., h_{\alpha_n})$;
4. *with q determined as above,* $[x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } q = 0 \\ \pm(q+1)x_{\alpha+\beta} & \text{if } q \geq 1 \end{cases}$.

Such a set $\{x_\alpha | \alpha \in \Phi\} \cup \{h_i | 1 \leq i \leq n\}$ is called a **Chevalley basis**. Now we take such a Chevalley basis and we define:

$$X_\alpha(t) := \exp(t \, \mathrm{ad}_{x_\alpha}) \quad = \quad \sum_{i=0}^{\infty} t^i \frac{(\mathrm{ad}_{x_\alpha})^i}{i!}.$$

Since ad_{x_α} is nilpotent, this sum is in fact finite.

Now we work over a fixed field $\mathbb{F}$.

Definition 1.8 *The* ***Chevalley group*** *generated by a Lie algebra L is the group generated by $\{X_\alpha(t)|\alpha \in \Phi,\ t \in \mathbb{F}\}$. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we call it a* ***Lie group****.*

It can be proved that this is contained in the group of automorphisms of the Lie algebra L.

Define

$$\begin{aligned} n_\alpha(t) &:= X_\alpha(t)X_{-\alpha}(-t^{-1})X_\alpha(t); \\ n_\alpha &:= n_\alpha(1); \\ h_\alpha(t) &:= n_\alpha(t)n_\alpha(-1). \end{aligned}$$

Take the subgroups $\mathcal{N}$, generated by n_α and $h_\alpha(t)$ and $\mathcal{H}$, generated by $h_\alpha(t)$. Then we have the following lemma:

Lemma 1.1 *The Weyl group W is isomorphic to $\mathcal{N}/\mathcal{H}$*

1.1.2 Example

Let V be a 2 dimensional vector space. Define $\mathrm{sl}(V)$ as the set of endomorphisms of V that have trace 0. It is easy to see that

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

form a basis of $\mathrm{sl}(V)$. Now we introduce the operation $[a, b] := ab - ba$, for $a, b \in SL(V)$. With this operations we see that:

$$[x, y] = h; \quad [x, h] = -2x; \quad [y, h] = 2y;$$

We can also verify the Jacobi-identity, but we leave this to the reader.
This is the Lie Algebra $\mathrm{SL}(V)$. It is easy to see that the subalgebra generated by h is a maximal toral subalgebra, so define $H := \langle h \rangle$. Now define $\alpha : H \to \mathbb{F} : ah \mapsto 2a$ and $\beta : H \to \mathbb{F} : ah \mapsto -2a$.
Then we see:

$$L_\alpha = \langle x \rangle; \qquad L_\beta = \langle y \rangle.$$

So we have 2 roots. We can also see that both x and y span a 1-dimensional Euclidean space. Both of the roots can be seen as simple roots, the Cartan matrix and Dynkin diagram in this case are trivial: we only have one simple root (say α), so the Dynkin diagram is just α, and the Cartan matrix is the matrix [2].

1.1.3 Root system of type F_4

Consider the following Dynkin diagram:

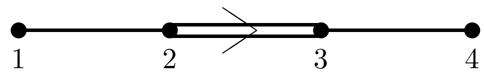


Figure 3: Dynkin diagram of F_4

This represents the 4 simple roots of L with respect to a fixed 4-dimensional toral subalgebra $\Delta := \{\alpha_1, ..., \alpha_4\} \subset \mathbb{R}^4$. This Dynkin diagram corresponds to the following Cartan matrix.

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix},$$

where $C_{ij} = \langle \alpha_i, \alpha_j \rangle$. From this we can derive that the angles between the roots are:

$$\phi_{\alpha_i, \alpha_j} = \begin{cases} 2\pi/3 & (i,j) \in \{(1,2),(3,4)\} \\ 3\pi/4 & (i,j) = (2,3) \\ \pi/2 & |i-j| \neq 1 \end{cases}$$

For the length of the roots we can deduce that α_1 and α_2 are the long roots, while α_3 and α_4 are short roots.

With the help of root chains, we now can search for the complete root system Φ in terms of the basis $\Delta = \{\alpha_1, ..., \alpha_4\}$. The positive roots of this root system are given in the following table, where $ijkl \equiv i\alpha_1 + j\alpha_2 + k\alpha_3 + l\alpha_4$.

weight	**positive roots**				#
1	**1000**	**0100**	0001	0010	4
2	**1100**	0110	0011		3
3	1110	**0120**	0111		3
4	**1120**	1111	0121		3
5	**1220**	1121	**0122**		3
6	1221	**1122**			2
7	1231	**1222**			2
8	1232				1
9	**1242**				1
10	**1342**				1
11	**2342**				1
				total	24

The roots written in bold are the long roots of this system, the others are the short roots.
For each of these 24 roots, we also have a negative root. This means that there are 48 roots α. Recall that $\dim(L_\alpha) = 1$.
To know the dimension of L, we still need to know the dimension of $L_0 = C_L(H) = H$. This dimension is exactly the number of simple roots of L, which is 4.
The total dimension of L is thus 52.

We now want to construct a Chevalley basis of L.
Let Φ be a root system with basis $\Delta = \{\alpha_1, ..., \alpha_4\}$, and let $\alpha \in \Phi$.
For a candidate Chevalley basis we take $\{h_i(\equiv h_{\alpha_i})\} \in H$ (these are independent), and $\forall \alpha \in \Phi : x_\alpha \in L_\alpha$. Then $\{x_\alpha, \alpha \in \Phi; h_i, 1 \leq i \leq k\}$ is a Chevalley basis iff:

1. $[h_i, h_j] = 0$;
2. $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$;
3. $[x_\alpha, x_\beta] = \begin{cases} c_{\alpha,\beta} x_{\alpha+\beta} & \text{if} \quad \alpha + \beta \in \Phi; \\ h_\alpha & \text{if} \quad \alpha = -\beta; \\ 0 & \text{else.} \end{cases}$
   with for $\alpha = ijkl$,
   $$h_\alpha := ih_1 + jh_2 + kh_3 + lh + 4.$$

Such a Chevalley basis always exists, but is not unique. The next step is to compute the coefficients $c_{\alpha,\beta}$. This can be done with with the following theorem of Chevalley [9]:

Theorem 1.2 *Let $\{x_\alpha, h_i : \alpha \in \Phi, 1 \leq i \leq k\}$ be a Chevalley basis of L then:*

- $[h_i, h_j] = 0$;
- $[h_i, x_\alpha] = \langle \alpha, \alpha_i \rangle x_\alpha$;
- $[x_\alpha, x_{-\alpha}] = h_\alpha$ *which is a $\mathbb{Z}$-linear combination of $h_1, ..., h_k$;*
- $[x_\alpha, x_\beta] = 0$ *if $\alpha + \beta \notin \Phi$;*
- $[x_\alpha, x_\beta] = \pm(p+1) x_{\alpha,\beta}$ *if $\alpha + \beta \in \Phi$ where $\beta - p\alpha, ..., \beta + q\alpha$ is the α-chain through β;*
- $c_{\alpha+\beta} = -c_{-\alpha-\beta}$.

I worked this out for a Chevalley basis of F_4,to construct the following multiplication table:

$C_{\alpha,\beta}$	α																							
β ↓	0001	1000	0010	0100	0011	1100	0110	0111	1110	0120	1111	0121	1120	1121	0122	1220	1221	1122	1231	1222	1232	1242	1342	2342
0001	0	0	-1	0	0	0	-1	0	-1	-1	0	-2	-1	-2	0	-1	-2	0	-1	0	0	0	0	0
1000	0	0	0	-1	0	0	-1	-1	0	-1	0	-1	0	0	-1	0	0	0	0	0	0	0	1	0
0010	1	0	0	-1	0	-1	-2	-1	-2	0	-1	0	0	0	0	0	-1	0	0	-1	-2	0	0	0
0100	0	1	1	0	1	0	0	0	0	0	0	0	-1	-1	0	0	0	-1	0	0	0	1	0	0
0011	0	0	0	-1	0	-1	-1	-2	-1	0	-2	0	0	0	0	1	1	0	2	0	0	0	0	0
1100	0	0	1	0	1	0	0	0	0	1	0	1	0	0	1	0	0	0	0	0	0	-1	0	0
0110	1	1	2	0	1	0	0	0	2	0	1	0	0	-1	0	0	0	-1	0	0	-2	0	0	0
0111	0	1	1	0	2	0	0	0	1	0	2	0	1	1	0	0	0	0	2	0	0	0	0	0
1110	1	0	2	0	1	0	-2	-1	0	0	0	1	0	0	1	0	0	0	0	0	2	0	0	0
0120	1	1	0	0	0	-1	0	0	0	0	1	0	0	0	0	0	0	-1	0	1	0	0	0	0
1111	0	0	1	0	2	0	-1	-2	0	-1	0	-1	0	0	0	0	0	0	-2	0	0	0	0	0
0121	2	1	0	0	0	-1	0	0	-1	0	1	0	0	2	0	0	-2	0	0	0	0	0	0	0
1120	1	0	0	1	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	-1	0	0	0	0
1121	2	0	0	1	0	0	1	-1	0	0	0	-2	0	0	0	0	2	0	0	0	0	0	0	0
0122	0	1	0	0	0	-1	0	0	-1	0	0	0	-1	0	0	1	0	0	0	0	0	0	0	0
1220	1	0	0	0	-1	0	0	0	0	0	0	0	0	0	-1	0	0	1	0	0	0	0	0	0
1221	2	0	1	0	-1	0	0	0	0	0	0	2	0	-2	0	0	0	0	0	0	0	0	0	0
1122	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0
1231	1	0	0	0	-2	0	0	-2	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0
1222	0	0	1	0	0	0	0	0	0	-1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
1232	0	0	2	0	0	0	2	0	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1242	0	0	0	-1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1342	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2342	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-0001	h_α	0	0	0	-1	0	0	-1	0	0	-1	-2	0	-2	-1	0	-2	-1	0	-1	-1	0	0	0
-1000	0	h_α	0	0	0	-1	0	0	-1	0	-1	0	-1	-1	0	0	0	-1	0	0	0	0	0	1
-0010	0	0	h_α	0	1	0	-2	0	-2	-1	0	-1	-1	-1	0	0	0	0	-1	0	-2	-1	0	0
-0100	0	0	0	h_α	0	1	1	1	0	0	0	0	0	0	0	-1	-1	0	0	-1	0	0	1	0
-0011	-1	0	1	0	h_α	0	0	-2	0	0	-2	-1	0	-1	-1	0	0	-1	2	0	1	1	0	0
-1100	0	-1	0	1	0	h_α	0	0	1	0	1	0	0	0	0	1	1	0	0	1	0	0	0	-1
-0110	0	0	-2	1	0	0	h_α	1	2	1	0	1	0	0	0	1	1	0	-1	0	-2	0	-1	0
-0111	-1	0	0	1	-2	0	1	h_α	0	0	2	1	0	0	1	0	1	0	2	1	1	0	1	0
-1110	0	-1	-2	0	0	1	2	0	h_α	0	1	0	1	1	0	-1	-1	0	1	0	2	0	0	1
-0120	0	0	-1	0	0	0	1	0	0	h_α	0	1	1	0	0	-1	0	0	1	0	0	-1	1	0
-1111	-1	-1	0	0	-2	1	0	2	1	0	h_α	0	0	1	0	0	-1	1	-2	-1	-1	0	0	-1
-0121	-2	0	-1	0	-1	0	1	1	0	1	0	h_α	0	2	1	0	-2	0	-1	0	1	1	-1	0
-1120	0	-1	-1	0	0	0	0	0	1	1	0	0	h_α	1	0	1	0	0	-1	0	0	1	0	-1
-1121	-2	-1	-1	0	-1	0	0	0	1	0	1	2	1	h_α	0	0	2	1	1	0	-1	-1	0	1
-0122	-1	0	0	0	-1	0	0	1	0	0	0	1	0	0	h_α	0	0	1	0	-1	-1	-1	1	0
-1220	0	0	0	-1	0	1	1	0	-1	-1	0	0	1	0	0	h_α	1	0	-1	0	0	0	-1	1
-1221	-2	0	0	-1	0	1	1	1	-1	0	-1	-2	0	2	0	1	h_α	0	1	1	-1	0	1	-1
-1122	-1	-1	0	0	-1	0	0	0	0	0	1	0	0	1	1	0	0	h_α	0	1	1	1	0	-1
-1232	0	0	-1	0	2	0	-1	2	1	1	-2	-1	-1	1	0	-1	1	0	h_α	0	1	-1	-1	1
-1222	-1	0	0	-1	0	1	0	1	0	0	-1	0	0	0	-1	0	1	1	0	h_α	1	0	-1	1
-1222	-1	0	-2	0	1	0	-2	1	2	0	-1	1	0	-1	-1	0	-1	1	1	1	h_α	1	1	-1
-1242	0	0	-1	0	1	0	0	0	0	-1	0	1	1	-1	-1	0	0	1	-1	0	1	h_α	-1	1
-1342	0	0	0	1	0	0	-1	1	0	1	0	-1	0	0	1	-1	1	0	-1	-1	1	-1	h_α	-1
-2342	0	1	0	0	0	-1	0	0	1	0	-1	0	-1	1	0	1	-1	-1	1	1	-1	1	-1	h_α
h_1	2	0	-1	0	1	0	-1	1	-1	-2	1	0	-2	0	2	-2	0	2	-1	2	1	0	0	0
h_2	0	2	0	-1	0	1	-1	-1	1	-1	1	-1	1	1	-1	0	0	1	0	0	0	0	-1	1
h_3	-1	0	2	-2	1	-2	0	-1	0	2	-1	1	2	1	0	0	-1	0	1	-2	0	2	0	0
h_4	0	-1	-1	2	-1	1	1	1	0	0	0	0	-1	-1	0	1	1	-1	0	1	0	-1	1	0

This multiplication table must be read in the following way: a digit $c := c_{\alpha,\beta}$ means

- 0 if $c = 0$;
- $cx_{\alpha+\beta}$ if $c \neq 0$ and $\alpha \neq h_i \neq \beta$;
- cx_α if $\beta = h_i$;

Now, with the knowledge that $[x, y] = -[y, x]$, $[x_\alpha, h_i] = -[x_{-\alpha}, h_i]$, $c_{\alpha+\beta} = -c_{-\alpha-\beta}$ and $[h_i, h_j] = 0$, every product can be calculated

Now we take a chevalley basis $\{x_\alpha, h_i : \alpha \in \Phi, 1 \leq i \leq k\}$, and we look at the $\mathbb{Z}$-span of this Chevalley basis. This will be a lattice (denoted by $L(\mathbb{Z})$) in L. This $L(\mathbb{Z})$ can be considered as a Lie algebra over $\mathbb{Z}$, and we have constructed a Lie algebra over $\mathbb{Z}$ of type F_4.

1.1.4 Construction of the Lie group $F_4(q)$

Definition 1.9 *An element $x \in L(\mathbb{Z})$ is an extreme element if*

$$\exists \sigma \in L(\mathbb{Z})^*, \quad \mathrm{ad}_x^2(y) = [x,[x,y]] = 2\sigma(y)x$$

Definition 1.10 *We now look at the linear transformation **exp**(a,t) of $L(\mathbb{Z})$, set by:*

$$(\exp(a,t))(v) = \sum_{i=0}^{\infty} \frac{(t\,\mathrm{ad}_a)^i(v)}{i!} \in L(\mathbb{Z})$$

for $v \in L(\mathbb{Z})$.

Lemma 1.2 *If a is an extreme element of $L(\mathbb{Z})$, then $\exp(a,t)$ is an automorphism of $L(\mathbb{Z})$.*

PROOF:
Let $y \in L(\mathbb{Z})$. Since a is an extreme element, we have an $\sigma \in L(\mathbb{Z})^*$ such that $(\mathrm{ad}_a)^2(y) = [a,[a,y]] = 2\sigma(y)a$ and $(ad_a)^m(y) = 0, \forall m \geq 3$.
So

$$(\exp(a,t))(y) = y + t[a,y] + t^2\sigma(y)a.$$

Now it is clear that $\exp(a,t)(y) \in L(\mathbb{Z})$ since it only has coefficients in $\mathbb{Z}$.
Since $\sigma \in L(\mathbb{Z})^*$ we see that $\exp(a,t)$ is a linear transformation of $L(\mathbb{Z})$.
We now have to prove that for any $x, y \in L(\mathbb{Z})$ we have

$$(\exp(a,t))([x,y]) = [(\exp(a,t))(x), (\exp(a,t))(y)].$$

Since $\exp(a,t)$ is linear we only have to prove it for the basis elements $\{x_\alpha, h_i : \alpha \in \Phi, 1 \leq i \leq k\}$.
Take $\alpha, \beta \in \Phi$, then:

$$\begin{aligned}
[(\exp(a,t))(x_\alpha), (\exp(a,t))(x_\beta)] &= [x_\alpha + t[a,x_\alpha] + t^2\sigma(x_\alpha)a, x_\beta + t[a,x_\beta] + t^2\sigma(x_\beta)a] \\
&= [x_\alpha, x_\beta] + \\
&\quad t([x_\alpha,[a,x_\beta]] - [x_\beta,[a,x_\alpha]]) + \\
&\quad t^2/2(\sigma(x_\alpha)[a,x_\beta] + 2[[a,x_\alpha],[a,x_\beta]] + \sigma(x_\beta)[x_\alpha,a]) + \\
&\quad t^3(\sigma(x_\alpha)[a,[a,x_\beta]] + \sigma(x_\beta)[[a,x_\alpha],a]) + \\
&\quad t^4\sigma(x_\alpha)\sigma(x_\beta)[a,a].
\end{aligned}$$

It is clear the coefficient of t^4 is 0. For the coefficient of t^3, we see that this is equal to $2\sigma(x_\alpha)\sigma(x_\beta)a - 2\sigma(x_\beta)\sigma(x_\alpha)a = 0$.

With the Jacobi identity we can work out the coefficient of t to $[a,[x_\alpha,x_\beta]]$. Applying the Premet identity, we see that the coefficient of t^2 is equal to $[a,[a,[x_\alpha,x_\beta]]]$. So we have:

$$\begin{aligned}[(\exp(a,t))(x_\alpha),(\exp(a,t))(x_\beta)] &= [x_\alpha,x_\beta]+t[a,[x_\alpha,x_\beta]]+t^2\sigma([x_\alpha,x_\beta])a \\ &= (\exp(a,t))([x_\alpha,x_\beta]).\end{aligned}$$

□

Now we work especially in $F_4(q)$. Let Θ be the root system of $F_4(q)$. In F_4 we have some special properties for the roots. As you can see in the in the Dynkin diagram, $F_4(q)$ only has roots of 2 different length, namely 1 and 2. This means that

$$(\alpha|\alpha)=\begin{cases}1 & \text{if } \alpha \text{ is a short root,}\\ 2 & \text{if } \alpha \text{ is a long root.}\end{cases}$$

for α and β being different roots, we have that $\alpha+\beta\in\Theta\Leftrightarrow(\alpha|\beta)\in\{-1,-2\}$

$$(\alpha|\beta)=\begin{cases}-1 & \text{if } \alpha \text{ and } \beta \text{ has the same length;}\\ -2 & \text{if } \alpha \text{ and } \beta \text{ has different length.}\end{cases}$$

Now we can introduce the following lemma.

Lemma 1.3 *Let α be a long root of Φ. Then x_α is an extreme element of L.*

PROOF:
We only have to prove the lemma for the elements x_α in the Chevalley basis. Since $h_i\in H$ we have $[x_\alpha,[x_\alpha,h_i]]=\langle\alpha,h_i\rangle[x_\alpha,x_\alpha]=0$ because of the definition of a root.
Since α is a long root, for every $\beta\neq-\alpha$ we have $\beta+2\alpha\notin\Phi$. And since $[x_\alpha,[x_\alpha,x_\beta]]\in L_{\beta+2\alpha}$, we conclude that $[x_\alpha,[x_\alpha,x_\beta]]=0$.
The only basis vector left is $x_{-\alpha}$, but then we have: $[x_\alpha,[x_\alpha,x_{-\alpha}]]=[x_\alpha,h_\alpha]$, and we can take the multiplication table to see that this is a multiple of x_α. □

We can now also enlarge this theorem to every element of the Chevalley basis.

Theorem 1.3 *Let x_γ be a root element, and m a strictly positive integer. Then $\mathrm{ad}^m_{x_\gamma}/m!$ leaves $L(\mathbb{Z})$ invariant.*

PROOF:
Since $\mathrm{ad}^m_{x_\gamma}(y+z)=\mathrm{ad}_{x_\gamma}(y)+\mathrm{ad}_{x_\gamma}(z)$, for $z,y\in L(Z)$, we have to prove that $\mathrm{ad}^m_{x_\gamma}(x_\delta)/m!$ leaves $L(\mathbb{Z})$ invariant, for x_δ being an element of the Chevalley basis.
It is clear that $\mathrm{ad}^m_{x_\gamma}(x_\alpha)=0$ if $m\geq 1$.
For h_i we get

$$\mathrm{ad}_{x_\gamma}(h_i)=[x_\gamma,h_i]=-\langle\gamma,\alpha_i\rangle x_\gamma\in L(\mathbb{Z});$$

and for $m>1$, we have

$$\mathrm{ad}^m_{x_\gamma}(h_i)=\mathrm{ad}^{m-2}_{x_\gamma}(-\langle\gamma,\alpha_i\rangle[x_\gamma,x_\gamma];)=0.$$

For $x_{-\gamma}$ we can do a similar reasoning to see that

$$\mathrm{ad}^m_{x_\gamma}(x_{-\gamma})/m! = \begin{cases} h_\alpha & \text{if} \quad m=1; \\ -x_\alpha & \text{if} \quad m=2; \\ 0 & \text{if} \quad m>2. \end{cases}$$

Now we only have to study $\frac{\mathrm{ad}^m_{x_\gamma}(x_\beta)}{m!}$ with $\beta \neq \pm\gamma$. Now let $\beta - p\gamma, \ldots, \beta + q\gamma$ be the γ-chain through β. Then it is clear that $\beta - (p+i)\gamma, \ldots (q-i)\gamma$ is the $i\gamma$-chain through β. This means that $[x_\beta, x_{i\gamma}] = \pm(p+i+1)x_{\beta+i\gamma}$. But then, for $m \leq q$,

$$\begin{aligned} ad^m_{x_\gamma}(x_\beta)/m! &= \frac{[x_\gamma,[x_\gamma,[\ldots,[x_\gamma,x_\beta]]\ldots]]}{m!}; \\ &= \frac{(p+1)(p+2)\ldots(p+m)}{m!}x_{\beta+m\gamma}; \\ &= \frac{(p+m)!}{p!m!}x_{\beta+m\gamma}; \\ &= \binom{p+m}{m}x_{\beta+m\gamma}. \end{aligned}$$

So this last is also an element in $L(\mathbb{Z})$, since a binomial is always integral. □

Now we can take F to be the automorphism group generated by every $\exp(a,t)$ with a a root element of Φ and $t \in \mathbb{F}_q$.

1.2 An introduction to character theory

This chapter is a summary of some important results from [5]. For the proofs of the lemmas we refer to that work.

Definition 1.11 *If $\mathbb{F}$ is a field and G a group, then an $\mathbb{F}$-**representation of** G is a homomorphism $\Upsilon : G \to GL(n,\mathbb{F})$ for some integer n.*

Definition 1.12 *Let Υ be an $\mathbb{F}$ -representation of G. Then the $\mathbb{F}$-**character** χ of G afforded by Υ is the function $\chi : G \to \mathbb{F}; g \mapsto \mathrm{tr}(\Upsilon(g))$.*

In the last definition tr is the well known trace function of a matrix.
On the characters χ and ζ of G, we define a inner product by:

$$\langle \chi, \zeta \rangle := \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\zeta(g)};$$

where $\overline{\zeta(g)}$ is the complex conjugate of $\zeta(g)$.

Definition 1.13 *Let G be a group acting on a set Ω. Define $\pi(g) = |\{\alpha \in \Omega | \alpha^g = \alpha\}|$ for $g \in G$. This is a character, which we call the **permutation character**.*

Now we denote $G_\alpha := \{g \in G | \alpha^g = \alpha\}$, for $\alpha \in \Omega$.

Lemma 1.4 *Let G act on Ω with permutation character π. If Ω decomposes into exactly k orbits under the action of G, then $\langle \pi, 1_G \rangle = k$.*

Lemma 1.5 *Let G act transitively on Ω with permutation character π. Suppose $\alpha \in \Omega$ and that G_α has exactly r orbits on Ω, then $\langle \pi, \pi \rangle = r$.*

Now we take a subgroup $H \subseteq G$. For a character χ of G, it is easy see that the restriction of χ to H (which we denote by χ_H) is a character of H. Let ζ be a character of H. Define

$$\zeta^G(g) := \frac{1}{|H|} \sum_{x \in G} \zeta^*(xgx^{-1});$$

where $\zeta^*(y) := \begin{cases} \zeta(y) & \text{if} \quad y \in H; \\ 0 & \text{if} \quad y \notin H. \end{cases}$.

Lemma 1.6 (Frobenius Reciprocity) *Let H be a subgroup of G. And χ and ζ characters of respectively G and H. Then*

$$\langle \zeta, \chi_H \rangle = \langle \zeta^G, \chi \rangle.$$

Lemma 1.7 *Let G act on a G-module X with permutation character χ. Then $\langle \chi, 1_G \rangle$ equals the number of G-orbits on X. This number is called the permutation rank of χ.*

To end we introduce the following:

Definition 1.14 *A group character χ is* ***multiplicity free*** *if no irreducible character occurs more than once in the decomposition of χ as a sum of irreducibles.*

1.3 Graph theory

1.3.1 An introduction

This section can be viewed as a summary of some important results of [7].

Definition 1.15 *A graph $\Gamma = (V, E)$ consists of a set of vertices V, and an edge set E, where each edge $e \in E$ is an ordered pair of V ($e := (x, y)$).*

We can represent a graph's vertices by points, and the graph's edges by arrows between the points. The graphs we are interested in have all the following properties properties

undirected: if $(x, y) \in E$ then also $(y, x) \in E$;

connected: every set of 2 points can be connected through by series of edges;

no loops: $(v, v) \notin E$ for every $v \in V$;

finite: $|V| < \infty$.

We introduce some more definitions: We say that the *valency* of a vertex v is equal to the number of edges which intersect that point: $|\{w|(v, w) \in E\}|$.
The *degree* of a graph is the maximum of all valencies.
Vertices v and w are called *adjacent* if $(v, w) \in E$.
The ordered set of edges $\{(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)\}$, for which $x_i \neq x_j$ is called a *path* (of length n) from x_0 to vertex x_n.
The *distance* $d(v, w)$ is the length of a shortest path from v to w. The *diameter* d of a graph is the maximum of the set $\{d(v, w)|(v, w) \in V \times V\}$.

Now we introduce the set

$$D_i(\Gamma) := \{(v, w) \in V \times V \mid d(v, w) = i\}.$$

It is clear that the sets $D_i(\Gamma)$, where $i = 0, \ldots, d$ form a partition of $V \times V$.
In a similar way we introduce:

$$\Gamma_i(v) := \{w \mid d(v, w) = i\}.$$

Now take two graphs $\Gamma := (V, E)$ and $\Gamma' := (V', E')$. Let $\theta : V \to V' : v \mapsto \theta(v)$ be a bijection (Note that this θ'does not always exists). If $(v, w) \in E$ implies that $(\theta(v), \theta(w)) \in E'$, than θ is called an *isomorphism* of graphs, and we say that the two graphs are isomorphic.
An isomorphism of a graph Γ into itself is called an *automorphism.* It is clear that the set of all automorphisms form a group, which we will denote as $\mathrm{Aut}(\Gamma)$. Let v^g denote the image of a vertex $v \in V$ under the automorphism $g \in G$ and $(v, w)^g = (v^g, w^g)$.

Lemma 1.8 (Orbit-Stabilizer) *Let G be a permutation group of the graph $\Gamma = (V, E)$. Take $v \in V$, then*

$$|G_v||v^G| = |G|;$$

where:

$$\begin{aligned} G_v &= \{g \in G | v^g = v\} \text{ i.e. the stabilizer of } v \text{ and} \\ v^G &= \{v^g | g \in G\} \text{ i.e. the orbit of } v. \end{aligned}$$

We say that a graph is *vertex-transitive* if $v^G = V$ and it is *edge-transitive* if $e^G = E$ where $e \in E$. Finally, we come to the definition of a distance transitivity.

Definition 1.16 *A connected graph Γ is called distance-transitive (from now on abbreviated by dtg) if $\forall i, \forall (v, w), (v', w') \in D_i(\Gamma)$, there exist a $g \in G$ such that $(v, w)^g = (v', w')$.*

In this case we also say that the automorphism group of a distance-transitive graph is a *distance-transitive group.*

To end this section we introduce some more definitions which will be used in this paper:

Definition 1.17 *A graph is called **bipartite** if the vertex set V can be written as the disjoint union of 2 sets $V_1 \cup V_2$, for which every edge e, has a vertex in V_1 and a vertex in V_2.*

Definition 1.18 *A graph Γ, with diameter d, is called **antipodal** if for every vertex v, for every $(x, y) \in \Gamma_d(v)$ we have that $d(x, y) = d$.*

Definition 1.19 *A graph Γ is called imprimitive when there is an i, for which the graph $(V, D_i(\Gamma))$ is disconnected.*
We call a graph primitive if no such i exists.

We can also relate these definitions to permutation groups. And we get:

Definition 1.20 *A permutation group G on a set X is called primitive if the only G-invariant equivalence relations ($\equiv$) on X are the 2 trivial ones defined by:*

$$\gamma \equiv \delta \text{ iff } \gamma = \delta;$$

and by

$$\gamma \equiv \delta \text{ for all } \gamma, \delta \in X.$$

Now we introduce a corollary which is proved in [1].

Corollary 1.1 *Let Γ be a dtg with automorphism group G, then Γ is imprimitive if and only if Γ is antipodal or bipartite.*

For a good and more detailed overview of distance-transitive graphs I refer to [8].

1.3.2 From Groups to Graphs

Let $\Gamma = (V, E)$ be a distance transitive graph. Let G be the automorphism group of this graph. By definition, this group works transitively on the set $D_i(v) := \{(v, w) \mid d(v, w) = i\}$ for each i. Hence G_v acts transitively on all distance sets $D_i(v)$.
Now fix $v \in V$ and define $H := G_v$, and fix an $r \in G$ such that $d(v^r, v) = 1$. Now we define the graph $\Gamma'(G, H, r)$ with:

- vertex set $H \backslash G$, the cosets of H in G;
- Hx and Hy adjacent if and only if $y \in HrHx$.

Theorem 1.4 *The graphs Γ and $\Gamma'(G, H, r)$ are isomorphic.*

PROOF: $V = \{v^g \mid g \in G\} = \{v^{Hg} \mid Hg \in H\backslash G\}$, as G is transitive on the set V. So there is a bijective correspondence between the vertex $v^{Hg} \in V$ and the coset $Hg \in H\backslash G$.
Let Hx and Hy be two vertices in $H\backslash G$. We see that $Hx \sim Hy \Leftrightarrow v^{Hx} \sim v^{Hy}$.
Since $d(v^r, v) = 1$ and H is the stabilizer of v, we have that v and v^{hr} are adjacent $\forall h \in H$. So also $v^{Hx} \equiv v^{h_1 x} \sim v^{hrh_1 x} \equiv v^{HrHx}$. We can conclude that $Hx \sim Hy$ if and only if $Hy = HrHx$, or, equivalent, $y \in HrHx$. □

With this theory we can also prove the converse. If we have a group G, a subgroup H and an $r \in G, r \notin H$, we can make a graph $\Gamma(G, H, r)$ with G as automorphism group with the vertices and edges defined above.

We can now introduce some theorems about the conditions on G and H for $\Gamma'(G, H, r)$ to be a dtg. In [6] we find the following theorem.

Theorem 1.5 *Let G be a group with a subgroup H. Then*

- $\Gamma'(G, H, r)$ *is connected iff* $\langle H, r\rangle = G$*;*
- $\Gamma'(G, H, r)$ *is undirected iff* $r^{-1} \in HrH$*.*

We have the following corollary:

Corollary 1.2 *For H a maximal subgroup of G, the graph $\Gamma'(G, H, r)$ is connected and undirected for all $r \in G$ with $r \notin H$ and $r^{-1} \in HrH$.*

1.3.3 Adjacency Diagrams

We now introduce a schematic representation of a graph. We will explain this representation with the help of the following example:

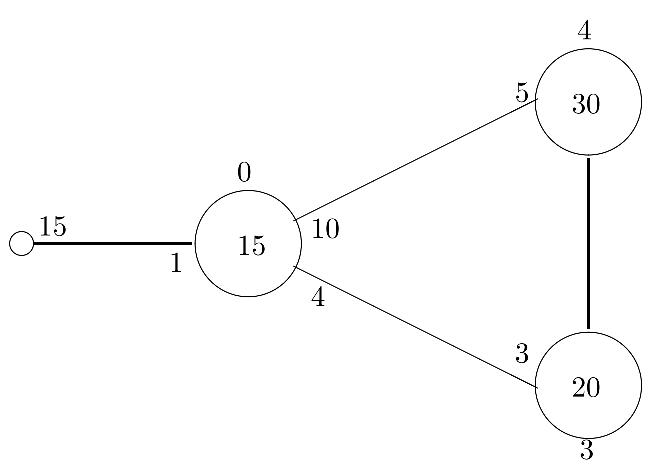
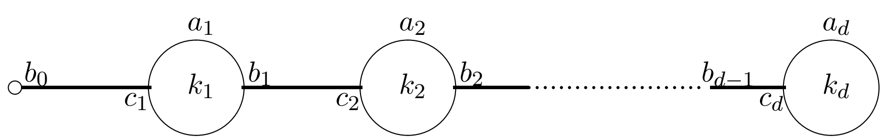


Figure 4: Example of an Adjacency Diagram

The leftmost circle represents the point $v \in V(\Gamma)$. The larger circles represent the different H-orbits, with the size of the orbit inside of the circle.
The number above the circle represents, for a point in the circle, the number of neighbors of that point in this circle. (So also the numbers of loops within this orbit.). The other number n attached to an edge from a circle I to a circle J means that every point in circle I has n adjacent points in J. With this adjacency diagram we can write down the properties of distance transitive graphs. It is clear that, if Γ is a dtg, then the adjacency diagram of Γ is a path. So from now on we write a distance transitive graph as the following diagram:



For a distance transitive graph, we have the following properties:

1. $b_0 = k_1$;
2. $c_1 = 1$;
3. $a_i + b_i + c_i = k_1$;
4. $k_{i-1}b_{i-1} = k_i c_i$.

With these properties, the adjacency diagram is determined by the values of the bs and the cs. This means that a distance transitive graph is fully determined by the array (b_i, c_j) which is called the **intersection array** of the graph.

If, furthermore, the graph of the previous diagram is distance transitive (with diameter d and degree k), there are many inequalities that the intersection array satisfies. We give a selection of the most important ones:

5. $c_0 \leq c_1 \leq c_2 \cdots \leq c_d$;
6. $k_1 = b_0 \geq b_1 \geq b_2 \cdots \geq b_{d-1}$;
7. $b_i \geq c_j$ for $i + j \leq d$;
8. $c_2 \geq k_1 - 2b_1$;
9. $\exists i, j$ such that $k_1 \leq k_2 \leq \cdots \leq k_i = \cdots = k_{i+j} \geq k_{i+j+1} \geq \cdots \geq k_d$.

The most important property to prove for the purpose of this thesis can now be derived from the previous properties:

$$\begin{aligned}
\frac{k_{d-1}}{k_1} &= \left(\frac{k_{d-1}}{k_{d-2}}\right)\left(\frac{k_{d-2}}{k_{d-3}}\right)\cdots\left(\frac{k_3}{k_2}\right)\left(\frac{k_2}{k_1}\right) \\
&\overset{(4)}{=} \left(\frac{b_{d-2}}{c_{d-1}}\right)\left(\frac{b_{d-3}}{c_{d-2}}\right)\cdots\left(\frac{b_2}{c_3}\right)\left(\frac{b_1}{c_2}\right) \\
&= \left(\frac{b_{d-2}}{c_2}\right)\left(\frac{b_{d-3}}{c_3}\right)\cdots\left(\frac{b_2}{c_{d-2}}\right)\left(\frac{b_1}{c_{d-1}}\right) \\
&\overset{(7)}{\geq} 1^{d-2} = 1.
\end{aligned}$$

So we have proven that, in a distance transitive graph, we always have $k_{d-1} \geq k_1$.
Together with $k_2 \geq k_1$, we now know that if we have the two smallest H-orbits then one of them is the first circle of the adjacency diagram.

2 Classification of distance transitive graphs

We are looking for a classification of distance transitive graphs. The starting point is the following

Theorem 2.1 *Let $\mathfrak{G}$ be a primitive dtg with automorphism group G. Assume that the degree and the diameter of the graph is greater or equal to 3, then one of the following holds:*

- *$\mathfrak{G}$ is associated with a Hamming Graph;*
- *G has an elementary abelian normal subgroup which is regular on the vertices of $\mathfrak{G}$; or*
- *there is a simple non-abelian normal subgroup N of G, such that $N \lhd G \leq Aut(N)$.*

This theorem was proved by Praeger, Saxl and Yokoyama[1].

The first and the second option of the theorem are well known [6]. However, in the third option, there are still some things to prove, especially where G is a group of Lie type. A summary of other results can also be found in [6].

We now concentrate on G being a group of Lie type, in our case especially on $F_4(q)$.
We are looking for maximal, non-abelian subgroups of $F_4(q)$. Let H be a maximal subgroup of $F_4(q)$ (meaning that if there is a subgroup H' with $H \leq H' \leq F_4(q)$ then $H' = H$ or $H = F_4(q)$). It is now clear that $N_G(H) = H$'. The following theorem is formulated in [3].

Theorem 2.2 *Let G be a group of Lie type, H be a maximal subgroup of G, B be a Borel subgroup of G, and W be the Weyl group of G. Then if the group G is multiplicity free, then $|H| > |G : B|/|W|$.*

If we use this theorem on the group $F_4(q)$, with $|W| = 2^7 3^2$, $|F_4(q)| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$ and $|B| = q^{24}(q-1)^4$, we get maximal subgroups H with order:

$$\begin{aligned} |H| &> \frac{(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{1152(q-1)^4} \\ &\approx \frac{q^{24}}{1152} + \frac{q^{23}}{288} + \frac{q^{22}}{128} + \frac{q^{21}}{72} + \frac{25q^{20}}{1152} \end{aligned} \tag{1}$$

So we have to find every subgroup of $F_4(q)$ larger than (1).

In [2] we see the following theorem for $L = F_4(q)$:

Theorem 2.3 *Let G be a group with $L \lhd G \leq Aut(L)$ and suppose M is a maximal subgroup not containing L with $|M| \geq q^{24}|G : L|$. Then $L \cap M$ is one of the following groups:*

1. *A parabolic subgroup,*

2. $B_4(q)$ ($B_4(q)$ *universal*),

3. $D_4(q).S_3$ ($D_4(q)$ *universal*),

4. ${}^3D_4(q).3$,

5. $F_4(q^{1/2})$ (q *square*),

6. ${}^2F_4(q)$ ($q = 2^{2m+1}$).

Note that (for small q), there is a gap in order of the vertex set between this theorem and the previous. Probably there are some maximal subgroups of $F_4(q)$ between these two values. However, the writer didn't study this further.

In [1] (Chapter 7), we find the following lemma:

Lemma 2.1 *If G is a distance transitive automorphism group of a graph Γ of diameter d on v vertices. Then $v \leq \sqrt{(d+1)|G|}$.*

Using chapter 1.3.2, this means that $|H| \geq \sqrt{\frac{|G|}{d+1}}$, where d is equal to the diameter of Γ, which is smaller than the number of irreducible and real characters of G.

Theorem 2.4 *Let Γ be a DTG with automorphism group G and vertex set V. Let π be the permutation character of the G-action on V. Then π is multiplicity free.*

As a summary of this section we note the following:
$\Gamma(F_4(\mathbb{F}), H, r)$ is a dtg only if H is one of the maximal subgroups mentioned above.
In this paper, we study the case of the maximal subgroups $D_4(q).S_3$ and $B_4(q)$.

3 Construction of the subgroups over any field $\mathbb{F}$

3.1 Construction of $B_4(\mathbb{F})$

Let $\mathbb{F}$ be any field. Consider the group of Lie type F_4 over $\mathbb{F}$. We have seen the Dynkin diagram in Figure 3. From this diagram, we can construct the so-called extended diagram. This means that we will insert the root of $F_4(\mathbb{F})$ with the smallest height ($:= \alpha_{48}$) in the Dynkin diagram. We then obtain the following diagram:

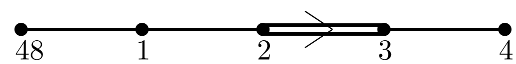


Figure 5: Extended Dynkin diagram of $F_4(\mathbb{F})$

This α_{48}, the root with the mimimum height, is of course a linear combination of the fundamental roots $\{\alpha_i | i = 1, ...4\}$. We can see that $\alpha_{48} = -2\alpha_1 - 3\alpha_2 - 4\alpha_3 - 2\alpha_4$.

Now, in the extended Dynkin diagram, there appears the Dynkin diagram $B_4(\mathbb{F})$:

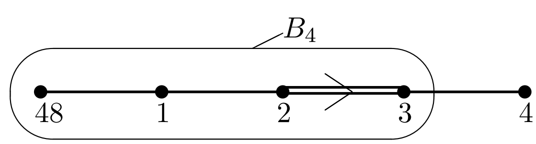


Figure 6: $B_4(\mathbb{F}) \subset F_4(\mathbb{F})$

We now consider the smallest closed subset $\Psi \subset \Phi$ which contains $\{\alpha_{48}, \alpha_1, \alpha_2, \alpha_3\}$. These 4 are the fundamental roots of Ψ and thus we obtain a root system of type B_4.

The subgroup $B_4(\mathbb{F})$ will now be generated by $\{\exp(a, t) : a \text{ a root element of } \Psi, t \in \mathbb{F}\}$.

3.2 Construction of $D_4(\mathbb{F})$

We again extend the Dynkin diagram, but this time starting from $B_4(\mathbb{F})$ as in Figure 6. We denote α_9 the extension root with minimal weight.

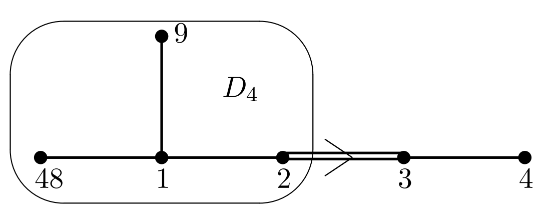


Figure 7: $D_4(\mathbb{F}) \subset B_4(\mathbb{F})$

This root is the following linear combination of simple roots:

$$\begin{aligned}\alpha_9 &= -\alpha_{48} - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 \\ &= \alpha_2 + 2\alpha_3.\end{aligned}$$

We now again make the smallest closed set of the fundamental roots $\{\alpha_9, \alpha_{48}, \alpha_1, \alpha_2\}$, and we denote this root system Θ.
The group $D_4(\mathbb{F})$ can now be generated by $\{\exp(a,t) : a \text{ a root element of } \Theta, t \in \mathbb{F}\}$.

Let it be clear that until now, we have created the subgroups:

$$D_4(\mathbb{F}) \subset B_4(\mathbb{F}) \subset F_4(\mathbb{F}).$$

3.3 Construction of $D_4(\mathbb{F}).S_3$

However, we are not directly interested in the subgroup $D_4(\mathbb{F})$ since this subgroup is not maximal. Now we are going to construct the maximal subgroup $D_4(\mathbb{F}).S_3$ of $F_4(\mathbb{F})$. To start, we again make the extended diagram of the last Dynkin diagram:

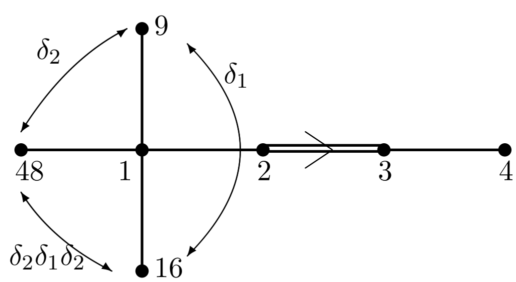


Figure 8: Diagram Automorphism of F_4

We see that there are 2 independent diagram automorphisms (δ_1, δ_2) on this Dynkin diagram. We can also easily see that these 2 automorphisms form a group isomorphic to S_3.

The next thing to do is to write these δ_i as elements of the Weyl group. This is easy to do in Magma, and we obtain the following 2 permutations:

$$\begin{aligned}\delta_1 &:= (2,9)(3,27)(4,7)(5,11)(10,13)(12,15)(17,19)(20,22) \\ &\quad (26,33)(28,31)(29,35)(34,37)(36,39)(41,43)(44,46), \\ \delta_2 &:= (3,7)(4,28)(6,10)(8,12)(9,16)(11,18)(14,20)(19,21) \\ &\quad (27,31)(30,34)(32,36)(33,40)(35,42)(38,44)(43,45).\end{aligned}$$

Since these are Weyl elements of $F_4(\mathbb{F})$, we can transform these into elements of our group $F_4(\mathbb{F})$, say D_1 and D_2
In Magma we can see that $D_i \notin D_4(\mathbb{F})$. Now we can easily define the subgroup $D_4.S_3$ as the group with generators the generators of D_4 above and $\{D_1, D_2\}$.

4 Representation of $D_4(\mathbb{F})S_3 \subset F_4(\mathbb{F})$

4.1 Representation of $D_4(\mathbb{F}) \subset F_4(\mathbb{F})$

There exists a faithful representation of $F_4(\mathbb{F})$ into $GL(26, \mathbb{F})$. If we look at the representations of the elements of $D_4(\mathbb{F}) \subset F_4(\mathbb{F})$, we see that, after a permutation of the basis, we can always write every element of $D_4(\mathbb{F})$ as:

$$\begin{pmatrix} \boxed{\text{A}} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \boxed{\text{B}} & \mathbf{0} & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \boxed{\text{C}} & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

With A,B and C in $GL(8, \mathbb{F})$.
So we have a representation of $D_4(\mathbb{F})$ consisting of 3 8-dimensional matrices and a 2×2 invariant part.

4.2 Representation of $D_4(\mathbb{F}).S_3 \subset F_4(\mathbb{F})$

The subgroup $D_4(\mathbb{F}).S_3$ is generated by the subgroup $D_4(\mathbb{F})$ and 2 Weyl elements D_1 and D_2. For the subgroup $D_4(\mathbb{F})$ we already have a representation. In Magma we can also see that that the group generated by D_1 and D_2 also can be permuted into the following form:

$$\begin{pmatrix} \boxed{\text{A}} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & \boxed{\text{B}} & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \boxed{\text{C}} & 0 \\ 0 & \dots & 0 & M \end{pmatrix}.$$

Where M is in the group generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$, and A, B and C are as above. (Note that $M \cong S_3$.)

This means that this subgroup of $F_4(\mathbb{F})$ stabilizes a 2-dimensional submodule L in the 26-dimensional module:

$$(F_4(\mathbb{F}))_L = D_4(\mathbb{F}).S_3.$$

5 Representations of the module

For the purpose of representing this module we enlarge the number of dimensions by 1. The 27-dimensional module can be written as:

$$\mathbb{K} = \left[\begin{array}{ccc|ccc|ccc} * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * \end{array}\right].$$

This means that we can write every $x \in \mathbb{K}$ as $[x^{(1)}, x^{(2)}, x^{(3)}]$.
For the following only the dimensions in the first matrix are the important ones. So we write

$$\left[\begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] := \left[\begin{array}{ccc|ccc|ccc} * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

5.1 Representation of $B_4(\mathbb{F})$-module

We denote

$$e_3 := \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Now let B be the stabilizer in G of e_3, then $B \cong B_4(\mathbb{F})$.
Now take the complementary subspace. We can write these as the direct sum of

$$V_{16} = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & * & * & * & * & * & * & 0 \\ 0 & 0 & * & * & * & * & * & * & 0 \\ * & * & 0 & 0 & 0 & 0 & * & * & 0 \end{array}\right]$$

and

$$V_{10} = \left[\begin{array}{ccc|ccc|ccc} * & * & 0 & 0 & 0 & 0 & 0 & 0 & * \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * & * & 0 & 0 & * \end{array}\right]$$

Such that $B_4(\mathbb{F})$ preserves the decomposition $\mathbb{K} = \langle e_3 \rangle \oplus V_{10} \oplus V_{16}$.

5.2 Representation of $D_4(\mathbb{F}).S_3$-module

Denote

$$\begin{aligned} e_1 &:= \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]; \\ e_2 &:= \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]; \\ e_3 &:= \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]. \end{aligned}$$

These 3 elements are stabilized by a subgroup H of $F_4(\mathbb{F})$, so we can say $L = \langle e_1, e_2, e_3 \rangle$. Then $H \cong D_4(\mathbb{F}).S_3$.
Note that e_i is stabilized by $D_4(\mathbb{F})$, and that they are interchanged by S_3.

For the following we introduce a map $f : \mathbb{K} \to \mathbb{F}$ which is a homogeneous polynomial (in the coordinates on V) of degree m. Let $\delta_x(f) : \mathbb{K} \to \mathbb{F}$ be the coefficient of t in $f(x + ty)$.

Now fix a symmetric bilinear form $(.,.)$ on $\mathbb{K}$. With this form we can define a quadratic map $\sharp : \mathbb{K} \to \mathbb{K} : x \mapsto x^\sharp$ such that $\delta_x(f)(y) = (x^\sharp, y)$.

We also need a product

$$\times : (v, w) \mapsto (v + w)^\sharp - v^\sharp - w^\sharp;$$

and the operator

$$(.,.,.) : (x, y, z) \mapsto f(x + y + z) + f(x) + f(y) + f(z) - f(x + y) - f(x + z) - f(y + z).$$

In [10] there is a construction of f and $(.,.)$, such that the product and inner product are invariant under $F_4(\mathbb{F})$. i.e.

$$\begin{aligned} (v \times w)g &= vg \times wg; \\ (gv \times gw) &= (v, w); \end{aligned}$$

for $g \in F_4(\mathbb{F})$. (Note that these invariants also mean that $v^\sharp = (vg)^\sharp$). With this product and inner product we receive the extra properties for e_i and $e := e_1 + e_2 + e_3$, namely

(i) $e_i \times e_j = e_k$;

(ii) for $x \in e_j \times V$ we have $e_i \times x = (e_k, y)e_j$;

(iii) $e_i \times e_i = 0$;

(iv) $(e_i, e_j) = 0$; $(e_i, e_j^\sharp) = 0$; $(e_i, e_i) = 1$;

(v) $e_i^\sharp = 0$;

(vi) $e^\sharp = e$; $(e, e) = 3$; $f(e) = 1$.

Now we can define $U_i := L^\perp \cap (e_i \times \mathbb{K})$ and we get the extra property

(vii) for $u_i, v_i \in U_i$ we have: $u_i \times v_i = (u_i, v_i, e_i)e_i$.

Each of these U_i are 8-dimensional submodules. They are just the 8 dimensions on which the matrices A, B and C, brought up in chapter 4.2 act. So U_i is stabilized by $D_4(\mathbb{F})$, and they are interchanged by S_3. We write these submodules as

$$U_1 = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & * & * & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 \end{array}\right];$$

$$U_2 = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & * & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & * & * & 0 & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \end{array}\right];$$

$$U_3 = \left[\begin{array}{ccc|ccc|ccc} 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & * & * & * & 0 & 0 & * \end{array}\right].$$

So we have a decomposition $\mathbb{K} = \langle e_1, e_2, e_3 \rangle \oplus U_1 \oplus U_2 \oplus U_3$. We can verify that this decomposition is an orthogonal one.

5.3 $\sharp$ and $(.,.)$ explicitly

Here we define the forms $\sharp$ and $(.,.)$ on an element of V of the form

$$x := \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix} \in \mathbb{K}.$$

Then we define $x^\sharp$ as the matrix whose i,j-entry is the i,j-minor of x and $(x,y) := \text{trace}(x,y)$.

We can expand these definitions to the entire module $\mathbb{K}$. This can be found in [10].

6 The graph $\Gamma(F_4(\mathbb{F}), B_4(\mathbb{F}), r)$

We take a finite field $\mathbb{F} := \mathbb{F}_q$, $q \geq 2$. In this section we define $G := F_4(\mathbb{F})$, B as the subgroup of G isomorphic to $B_4(\mathbb{F})$ fixing E_3, and $E_i := \langle e_i \rangle$.

6.1 Determine the points

First of all we take an $r \in G, r \notin B$. Due to maximality of the subgroup B, we know that $\langle B, r \rangle = G$. So $\Gamma(G, B, r)$ is a connected graph. Now we also take a $r \in G, r \notin B$ such that $r^{-1} \in BrB$ and denote $\Gamma := \Gamma(G, B, r)$. Because of section 1.3.2, a way to view the points of the graphs Γ is to view the G-orbit of E_3, which is stabilized by B. So $V(\Gamma) = E_3^G$.
In [11] we can see the necessary and sufficient conditions for $\langle x \rangle$ being in the G-orbit of E_3. These are: $x^\sharp = 0$ and $(x, e) \neq 0$.
Since B is the stabilizer of E_3, we can compute the size of the vertex set:

$$\frac{|F_4(q)|}{|B_4(q)|} = \frac{q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{q^{16}(q^2-1)(q^4-1)(q^6-1)(q^8-1)} = q^8(q^8+q^4+1).$$

By the construction of Γ, the set of all the neighbors of E_3 can be written as E_3BrB, but since B stabilizes E_3, we can write it as E_3rB.
We will look at the intersection array of this graph, starting from E_3. Clearly the B-orbits of $V(\Gamma)$ form a partition of $V(\Gamma)$. In the following we fill all these B-orbits in an adjacency diagram. By the lemmas in section 1.3.3 we know that, if Γ is distance-transitive, then $(E_3r)^B$ $(= E_3rB)$ should be one of the 2 smallest orbits.

6.2 Find the smallest orbits

We write

$$x_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = e_1;$$

$$x_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix};$$

and $X_i := \langle x_i \rangle$.
Clearly $X_1, X_2 \in V(\Gamma)$, since $x_i^\sharp = 0$ and $(x_i, e) \neq 0$.

Now we can compute the order of the B-orbit of X_1.
We see that the stabilizer of X_1 should stabilize E_1. But B already stabilizes E_3, because of this, it should also stabilize $E_1 \times E_3 = E_2$. This means that the stabilizer in $B_4(\mathbb{F})$ of X_1 is equal to $D_4(\mathbb{F})$. So for the size of the orbit we get:

$$|X_1^B| = \frac{|B_4(q)|}{|D_4(q)|} = \frac{q^{16}(q^2-1)(q^4-1)(q^6-1)(q^8-1)}{q^{12}(q^2-1)(q^4-1)(q^6-1)(q^4-1)} = q^4(q^4+1).$$

For the stabilizer of X_2, we can argue that this should be of size $q^{16}(q^2-1)(q^3-1)(q^4-1)$, so that the B-orbit has size $(q^3+1)(q^8-1)$.[1] So we can already say that the orbit of X_1 is smaller than the one of X_2.

We claim that X_1^B and X_2^B are the 2 smallest non-trivial B-orbits among all B-orbits of $V(\Gamma)$.

Let $Y := \langle y \rangle \in V(\Gamma)$ be such that $|Y^B| \geq \max\{|X_1^B|, |X_2^B|\}$. Then we write

$$y = \eta e_3 + y_{10} + y_{16}.$$

Since $\langle y \rangle$ should be in $V(\Gamma)$, we must have that $y^\sharp = 0$.

First let $\underline{y_{16} = 0}$.
This means $y = \eta e_3 + y_{10}$. Without loss of generality we can assume that

$$y_{10} = \begin{bmatrix} \alpha & \gamma & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

so that $y_{10}^\sharp = \alpha\beta e_3$ and $y^\sharp = \begin{bmatrix} \beta\eta & 0 & 0 \\ -\gamma\eta & \alpha\eta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$.
Since $\langle y \rangle \in V(\Gamma)$ we know that $y^\sharp = 0$, which means that $\alpha\beta = 0$ and $\eta = 0$ (another possibility is that $\gamma = \alpha = \beta = 0$, but then $y = e_3$, which is in the trivial B-orbit). But then up to conjugation by B, we may assume $y_{10} = \delta e_1$, and that $Y = E_1$.

Now let $\underline{y_{16} \neq 0}$.
It is clear that the stabilizer of a point in $V_{16} \oplus E_3$ is bigger than the stabilizer of a point in $V_{10} \oplus V_{16} \oplus E_3$. For this we can say that $y_{10} = 0$. and that y_{16} is a highest weight vector, so without loss of generality we have

$$y_{16} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =: z.$$

So $y = \alpha z + \eta e_3$, with $\alpha \neq 0$ (otherwise $y^\sharp = 0$ and $Y \notin V(\Gamma)$), from which $|Y^B| \leq |X_2^B|$ easily follows, with equality iff $\alpha = \eta = 1$, but then $y = x_2$.

So we have proven that the orbits of X_1 and X_2 are the smallest among all B-orbits.

6.3 Adjacency diagram

We now construct the beginning of the adjacency diagram, starting from the point E_3. By section 1.3.3, the points adjacent to E_3 are in one of the 2 smallest B-orbits.

[1] Why this is, is currently unclear to the writer, because, in this thesis, he didn't had time to study this. This fact, however, was given to the writer by Prof. Dr. A.M.Cohen.

Let first $X_1 \sim E_3$.

Since G is the automorphism group of the graph we know that $\forall g \in G : e_3 g \sim X_1 g$. Now we claim that there is a $g \in G$ such that $[E_3 g, X_1 g] = [X_1, X_2]$. This will mean that $X_2 \in D_2(\Gamma)$. But then, by the explanation in section 1.3.3, we can argue that Γ would have diameter 2, which is impossible.

To prove this claim we denote for every $g_1 \in \mathrm{SL}(3, \mathbb{F})$, and for every $x = [x^{(1)} \mid 0 \mid 0] \in \mathbb{K}$:

$$s_{g_1}(x) := [g_1 x^{(1)} g_1^{-1} \mid 0 \mid 0].$$

And from [10], we know that $s_{g_1} \in G$. Now take $g_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \in \mathrm{SL}(3, \mathbb{F})$ than we can see

$$\begin{aligned} s_{g_1}(e_3) &= x_1; \\ s_{g_1}(x_1) &= x_2; \end{aligned}$$

which proves the claim.

Now let $X_2 \sim E_3$. We can find a $h \in H$ such that

$$X_2 h = \left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle := X_2'.$$

Since $X_2 \sim E_3$ we can say that for every $g \in G : X_2 g \sim E_3 g$, (and the same for X_2').
Now we can make a chain:

$$E_3 \sim X_2' \sim \left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle = Y_2 \sim \left\langle \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\rangle = E_2.$$

We denote y_2 and x_2' the basis vector of respectively Y_2 and X_2'.
To see that this is indeed a chain, we look separately at the two adjacency signs which we need to prove.

For $Y_2 \sim E_2$ we can argue as above: take $g_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in \mathrm{SL}(3, \mathbb{F})$. Then we have that $s_{g_1}(e_3) = e_2$ and $s_{g_1}(x_2) = y_2$. So $E_2 \sim Y_2$.

For $X_2' \sim Y_2$, we take $g_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Now we have that $s_{g_2}(e_3) = x_2'$ and $s_{g_2}(x_2) = y_2$.

First note that E_2 is in the same B-orbit as X_1, and that Y_2 is not in that orbit, and also not in X_2^B. By the chain, we see that $d(E_2, E_3) \leq 3$, so also $d(X_1, E_3) \leq 3$. Now we use the property of the adjacency diagram of a distance transitive graph:

$$k = k_1 \leq k_2 \leq \ldots k_i = \cdots = k_{i+j} \geq \cdots \geq k_d.$$

Since X_1^B is the smallest orbit, we can say that this orbit should be the last orbit in the adjacency diagram. So the diameter of the graphΓ should be 3.

If we calculate the size of the B-orbit of Y_2, we obtain $q^4(q^8-1)(q^3+1)$. [2]Now we can see that the sum of the sizes of these three orbits is not equal to the total numbers in E_3^G, which means that the graph Γ cannot have diameter 3.

So we conclude that a graph $\Gamma(F_4(\mathbb{F}), B_4(\mathbb{F}), r)$ can never be distance transitive.

[2]This orbit-size also was given to me by Prof. Dr. A.M. Cohen

7 The graph $\Gamma(F_4(\mathbb{F}), D_4(\mathbb{F}).S_3, r)$

Again $\mathbb{F}$ is a field of the form F_q, but now with $q \geq 3$.
In this section, we define $G := F_4(\mathbb{F})$, H as the subgroup of G isomorphic to $D_4(\mathbb{F}).S_3$ stabilizing L, and H_0 the subgroup of G isomorphic to $D_4(\mathbb{F})$.

7.1 Determine the points

We have seen that H interchanges the e_is. This means that H stabilizes the plane $\pi_0 := \langle e_1, e_2, e_3 \rangle$.
Because of this the vertex set of Γ can be viewed as a G-orbit of π_0:

$$V(\Gamma) = \pi_0^G.$$

The number of points is thus

$$\frac{|F_4(q)|}{|D_4(q).S_3|} = \frac{q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)}{6q^{12}(q^2-1)(q^4-1)(q^6-1)(q^4-1)} = \frac{1}{6}q^{12}(q^4+1)(q^8+q^4+1)$$

Observe that π_0^G is just the set of planes containing $e := e_1 + e_2 + e_3$ and at least 2 points x_i for which $(x_i)^\sharp = 0$ and $(x_i, e) = 1$.

By the construction of Γ all the neighbors of π_0 belong to $\pi_0 HrH$ but, since H stabilizes π_0, we can write it as $\pi_0 rH$.
We look at the intersection array of this graph, starting from π_0. The first orbit is thus $r\pi_0^H$.
By section 1.3.3 we know that, if Γ is distance-transitive, then $r\pi_0^H$ should be one of the 2 smallest orbits.

7.2 Finding the smallest H-orbits

Now first we look at the plane:

$$\pi_1 := \langle x_1, y_1, e_3 \rangle;$$

where

$$x_1 := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$y_1 := \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is clear that $\pi_1 \in V(\Gamma)$ since $e \in \pi_1$, $x_1^\sharp = 0$, $(x, e) = 1$ and similarly for y_1. We compute the H-orbit π_1^H of this point. This orbit will now be denoted by L_1. Now, with the given product and inner product we see that

$$(x_1 \times e_i, y_1 \times e_i) = (x_1 \times e_j, y_1 \times e_j) = (x_1 \times e_i, x_1 \times e_j) = (y_1 \times e_i, y_1 \times e_j) = 0.$$

It turns out that these are invariants of the orbit. (Note that the roles of e_i can be interchanged by H). To see this we take a $h \in H$ and $i, j \in \{1,2,3\}$ such that $e_1 h = e_i$ and $e_2 h = e_j$. Then we have

$$\begin{aligned}(x_1 h \times e_i, y_1 h \times e_i) &= ((x_1 \times e_i h^{-1})h, (y_1 \times e_i h^{-1}))h;\\ &= (x_1 \times e_i h^{-1}, y_1 \times e_i h^{-1});\\ &= (x_1 \times e_1, y_1 \times e_1);\\ &= 0;\end{aligned}$$

all the other inner products above can be shown in a similar way.

Now we take an $\alpha \in \mathbb{F}$ such that $\alpha \notin \{0,1\}$ and define:

$$\begin{aligned}x_{(\alpha)} &:= \begin{bmatrix} \alpha & \alpha-\alpha^2 & 0 \\ 1 & 1-\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix};\\ y_{(\alpha)} &:= \begin{bmatrix} 1-\alpha & -\alpha+\alpha^2 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}; \text{ and}\\ \pi_{(\alpha)} &:= \langle x_{(\alpha)}, y_{(\alpha)}, e_3 \rangle.\end{aligned}$$

Again, these planes are in the G-orbits of π_0 (clearly $e \in \pi_{(\alpha)}$ and $x^\sharp_{(\alpha)} = y^\sharp_{(\alpha)} = 0$), and we have the H-orbit of π_α. This orbit is denoted by $L_{(\alpha)}$. Also for this orbit we have invariants, there exists a $i, j \in \{1,2,3\}$ such that $(x_1, e) = (y_1, e) = 1$ and

$$(x_1 \times e_i, y_1 \times e_i) = (x_1 \times e_j, y_1 \times e_j) = (x_1 \times e_i, x_1 \times e_j) = (y_1 \times e_i, y_1 \times e_j) = \alpha(1-\alpha).$$

Now we claim that the smallest H-orbits are the among these orbits. To see this we search for the biggest stabilizers.
Consider an arbitrary plane $\langle x, y, z \rangle$. Let $x_{(i)}, y_{(i)}, z_{(i)}$ be their projections on U_i. Now we look at

$$H_{\langle x,y,z \rangle} \leq H_{\{\langle x^{(1)}, y^{(1)}, z^{(1)} \rangle, \langle x^{(2)}, y^{(2)}, z^{(2)} \rangle, \langle x^{(3)}, y^{(3)}, z^{(3)} \rangle\}}.$$

We want the stabilizer to be as large as possible so, clearly, the more projections are 0, the larger the stabilizer is. So we take 2 nonzero projections, say $x^{(i)}$ and $y^{(j)}$, but then $x := l_x + x^{(i)}$, $y := l_y + y^{(j)}$ and $z := l_z$, where $l_\_ \in \langle e_1, e_2, e_3 \rangle$.

Now $e \in \langle x, y, z \rangle$, so we can write

$$\begin{aligned}e &= \delta(l_x + x^{(i)}) + \epsilon(l_y + y^{(j)}) + \zeta l_z;\\ &\in \langle e_1, e_2, e_3 \rangle + \langle \delta x^{(i)} + \epsilon y^{(j)} \rangle.\end{aligned}$$

This means that $\delta x^{(i)} + \epsilon y^{(j)} = 0$, so $U_i \supset \langle x^{(i)} \rangle = \langle y^{(j)} \rangle \subset U_j$.

So we can conclude that $i = j$ and thus that $x^{(k)} = y^{(k)} = z^{(k)} = x^{(k+1)} = y^{(k+1)} = z^{(k+1)} = 0$, for some k. Without loss of generality we take $k = 1$ and $z^{(3)} = 0$. Now we know $z \in \pi_0$, and $\langle x^{(3)} \rangle = \langle y^{(3)} \rangle$. Without loss of generality we can take $z = e_3$.

We know that e should be in the plane and $(x^{(3)})^\sharp = (y^{(3)})^\sharp = 0$. By this the only possibilities are: $x^{(3)} := x_{(\alpha)}$ and $y^{(3)} := y_{(\alpha)}$ for an $\alpha \in \mathbb{F}$. Now if $\alpha \notin \{0,1\}$ then $\langle x, y, z\rangle = \pi_{(\alpha)}$. The only problems are for $\alpha \in \{0,1\}$. But then

$$x_{(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad y_{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$
$$x_{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad y_{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Both being in the H-orbit of π_1.

This implies that the smallest orbits are in $\{L_1\} \cup \{L_{(\alpha)} : \alpha^3 \neq \alpha^2\}$.

7.3 Adjacency diagram

Lemma 7.1 *Let π' and π be two planes of type $\pi_{(\alpha)}$ or π_1. Let d be the distance between π and π'. We know that we can write $\pi = \langle x, y, z\rangle$, and $\pi' = \langle x', y', z\rangle$ such that:*

$$\pi' \cap \pi = \langle e, z\rangle;$$

and

$$(x \times x', y \times x') = (x \times y', y \times y') = (x \times x', x \times y') = (y \times x', y \times y') = k,$$

with a $k \in \mathbb{F}$.

Now take two other planes τ and τ', with $d(\tau, \tau') = d$, then we can write $\tau = \langle \tau_1, \tau_2, \tau_3\rangle$ and $\tau' = \langle \tau_1', \tau_2', \tau_3\rangle$ such that

$$\tau \cap \tau' = \langle e, \tau_3\rangle;$$

and

$$(\tau_1 \times \tau_1', \tau_2 \times \tau_1') = (\tau_1 \times \tau_2', \tau_2 \times \tau_2') = (\tau_1 \times \tau_1', \tau_1 \times \tau_2') = (\tau_2 \times \tau_1', \tau_2 \times \tau_2') = k.$$

From now on we will denote these invariants as the invariants of π' with respect to π.

PROOF:
There exist a $g \in G$ such that $\pi g = \tau$ and $\pi' g = \tau'$. Now denote $\tau_1 = xg$; $\tau_2 = yg$; $\tau_3 = zg$; $\tau_1' = x'g$; $\tau_2' = y'g$. Then we have

$$\begin{aligned} \tau \cap \tau' &= \pi g \cap \pi' g \\ &= (\pi \cap \pi')g \\ &= \langle e, z\rangle g \\ &= \langle eg, zg\rangle \\ &= \langle e, \tau_3\rangle \end{aligned}$$

and

$$\begin{array}{rcl}(\tau_1 \times \tau_1', \tau_2 \times \tau_1') & = & (xg \times x'g, yg \times x'g); \\ & = & ((x \times x')g, (y \times x')g); \\ & = & (x \times x', y \times x'); \\ & = & k.\end{array}$$

And we can do a similar reasoning as above for the other invariants. □

From now on we denote

$$z := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } w := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

.

We construct the beginning of the adjacency diagram, starting from the point π_0. By section 1.3.3 the points adjacent to π_0 should be in one of the 2 smallest H-orbits.

So first let $\pi_0 \sim \pi_1$.

Take $g_1 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Clearly $g_1 \in \mathrm{SL}(3, \mathbb{F})$. Now we have:

$$\tau_1 := s_{g_1}(\pi_1) = \left\langle e_1, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle.$$

We can also see that $s_{g_1}(\pi_0) = \pi_0$, which means that $\pi_0 \sim \tau_1$. So τ_1 is in the same H-orbit as π_1. We can also see that $\tau_1 \cap \pi_1 = \langle e \rangle$. By the lemma, we can thus say that τ_1 and π_1 are not adjacent (otherwise τ_1 and π_0 would have the same invariants with respect to π_1), so $d(\tau_1, \pi_1) = 2$.

Clearly we can also take a $h_\alpha \in H$ such that

$$h_\alpha \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

and $h_\alpha e_i = e_i$, for all i.

We denote $x'_\alpha := h_\alpha x_1$ and $y'_\alpha := h_\alpha y_1$. Then we have:

$$\begin{aligned}(x'_\alpha \times x_1, x'_\alpha \times y_1) &= (\alpha w \times z, e_1 \times e_2 - \alpha w \times z);\\ &= (\alpha e_3, e_3 - \alpha e_3);\\ &= \alpha(1-\alpha);\end{aligned}$$

$$(y'_\alpha \times x_1, y'_\alpha \times y_1) = \alpha(1-\alpha);$$

$$(x'_\alpha \times x_1, y'_\alpha \times x_1) = \alpha(1-\alpha);$$

$$(x'_\alpha \times y_1, y'_\alpha \times y_1) = \alpha(1-\alpha).$$

So $\langle x'_\alpha, y'_\alpha, e_3\rangle$ and π_1 are not adjacent (by the lemma). However, they are clearly in the same H-orbit. So we have that $d(\langle x'_\alpha, y'_\alpha, e_3\rangle, \pi_1) = 2$. By the lemma, this means that every plane at distance 2 to π_1 should have these invariants.
However $\langle x'_\alpha, y'_\alpha, e_3\rangle$ and τ have different invariants (with respect to π_1), while there distance to π_1 is for both equal to 2. So we have a contradiction with dtg.

Now let $\pi_0 \sim \pi_\alpha$.
Again we can take $g_1 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Now

$$\tau_{(\alpha)} := s_{g_1}(\pi_{(\alpha)}) = \left\langle e_1, \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & \alpha-\alpha^2 \\ 0 & 1 & 1-\alpha \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1-\alpha & -\alpha+\alpha^2 \\ 0 & -1 & \alpha \end{bmatrix} \right\rangle.$$

Again we have $s_{g_1}(\pi_0) = \pi_0$. So $\tau_{(\alpha)}$ is in the same H-orbit as $\pi_{(\alpha)}$. However $\pi_{(\alpha)} \cap \tau_{(\alpha)} = \langle e \rangle$, which means that they cannot be adjacent. So $d(\tau_{(\alpha)}, \pi_{(\alpha)}) = 2$.

Now take an element in $h \in H$ such that

$$\begin{aligned}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} h &= \begin{bmatrix} 0 & 0 & 0 \\ -\alpha^{-2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} h &= \begin{bmatrix} 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\\ e_i h &= e_i \quad \forall i.\end{aligned}$$

Then clearly

$$x_{(\alpha)} h = \begin{bmatrix} \alpha & -\alpha^2 & 0 \\ 1-\alpha^{-1} & 1-\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} := x'_{(\alpha)};$$

and

$$y_{(\alpha)}h = \begin{bmatrix} 1-\alpha & \alpha^2 & 0 \\ \alpha^{-1}-1 & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} := y'_{(\alpha)}.$$

Now denote $\upsilon_{(\alpha)} = \langle x'_{(\alpha)}, y'_{(\alpha)}, e_3\rangle$. Then $\upsilon_{(\alpha)}$ and $\pi_{(\alpha)}$ are clearly in the same H-orbit. Then we have $x_{(\alpha)} \times x'_{(\alpha)} = y_{(\alpha)} \times y'_{(\alpha)} = e_3$ and $x_{(\alpha)} \times y'_{(\alpha)} = y_{(\alpha)} \times x'_{(\alpha)} = 0$.

Now if we calculate the invariants of $\upsilon_{(\alpha)}$ with respect to $\pi_{(\alpha)}$, then we obtain

$$\begin{aligned} (x_{(\alpha)} \times x'_{(\alpha)}, x_{(\alpha)} \times y'_{(\alpha)}) &= (e_3, 0e_3); \\ &= 0. \end{aligned}$$

And the same goes for the other invariants.

Now we have found 2 planes ($\tau_{(\alpha)}$ and $\upsilon_{(\alpha)}$), both at distance 2 to $\pi_{(\alpha)}$, while one plane has invariants $\alpha(1-\alpha)$ and another has invariant 0. Now there exists in every field of order larger then 3, an element α for which $\alpha(1-\alpha) \neq 0$. So we have 2 planes with different invariants (with respect to $\pi_{(\alpha)}$). So again we have a contradiction.

So we can conclude that a graph $\Gamma(F_4(\mathbb{F}), D_4(\mathbb{F}).S_3, r)$ can never be distance transitive.

8 Summary of the results

In this thesis we have proven, for a finite field $\mathbb{F}$, there exist no distance transitive graphs with $\mathbb{F}_4(\mathbb{F})$ as an automorphism group, and $B_4(\mathbb{F})$ of $D_4(\mathbb{F}).S_3$ as the stabilizers of a vertex. Only the graph $\Gamma(F_4(\mathbb{F}_2), D_4(\mathbb{F}_2).S_3, r)$ is not covered by this paper

We did this by presuming that there exists such a distance transitive graph. Then we know the form of the adjacency diagram of the respective graphs. We first proved that the first circle in this adjacency diagram should be one of the two smallest orbits of the stabilizing subgroup. After finding representatives for these smallest orbits, we came to a contradiction in the adjacency diagram of a distance transitive graph.

With this work I contributed to the classification of primitive distance transitive graphs. However there is still more work to do. For a review I refer to [18].

A Magma code

```
/* generating the field*/

q:=5;
gf<zz>:=GF(q);




/*F_4 root datum, to create F_4 and all the necessary
subgroup in a 26 dimensional representation*/

R:=RootDatum("F4": Isogeny:="SC");
RB:=RootSubdatum(R,{1,2,3,48,16});
RD:=RootSubdatum(R,{1,2,48,9,16});
F4:=GroupOfLieType(R,gf);
B4:=GroupOfLieType(RB,gf);
D4:=GroupOfLieType(RD,gf);
gensF:=Generators(F4);
gensD := ChangeUniverse(Generators(D4),F4);
gensB := ChangeUniverse(Generators(B4),F4);
rho:=StandardRepresentation(F4);
CF:={rho(g):g in gensF};
CB:={rho(g):g in gensB};
CD:={rho(g):g in gensD};
D4Matr:=MatrixGroup<26,GF(q)|CD>;
B4Matr:=MatrixGroup<26,GF(q)|CB>;
F4Matr:=MatrixGroup<26,GF(q)|CF>;
W:=WeylGroup(F4);
CW:={rho(elt<F4|w>):w in Generators(W)};
CT:={rho(TorusTerm(F4,i,zz+1)):i in [1..4]};
WMatr:=MatrixGroup<26,GF(q)|CW>;
Torus:=MatrixGroup<26,GF(q)|CT>;
DELTA:=Stabiliser(W,{1,2,25});
Delta:=Generators(DELTA);
d1:=DELTA.1;
d2:=DELTA.2;
D1:=rho(elt<F4|d1>);
D2:=rho(elt<F4|d2>);
CD4S3:=CD join {D1,D2};
D4S3Matr:=MatrixGroup<26,GF(q)|CD4S3>;
V:=VectorSpace(GF(q),26);







/*making function to write an element of D_4 in section 4.
The function from D_4 to this form is called FindDiagonalForm
the function in the other way is called ToD4S3 */

Subm:=function(X,A,B);
Z:=ZeroMatrix(GF(q),8,8);
for i in A do
for j in B do
Z[Position(A,i),Position(B,j)]:=X[i,j];
end for;
end for;
return Z;
end function;
Row2D4:={@2,6,8,10,18,20,21,25@};
Row1D4:={@1,7,9,11,15,17,19,26@};
Row3D4:={@3,4,5,12,16,22,23,24@};
Perm1:=PermutationGroup<26|(1,12)(2,25)(16,26)(3,15)(4,17)(5,19)(7,22)(9,23)(11,24),
(1,25,16)(2,12,26)(3,15,18)(4,17,20)(5,19,21)(6,22,7)(8,23,9)(10,24,11)>;
P1:=Perm1.1;
P2:=Perm1.2;
M:=MatrixGroup<2,GF(q)|[0,1,1,0],[q-1,1,q-1,0]>;
chi:=iso<M->Perm1| <M.1,P1>,<M.2,P2>>;
DD:=[Identity(D4Matr),D1,D2,D1*D2,D2*D1,D1*D2*D1];
D:=[Submatrix(X,13,13,2,2): X in DD];

FindDiagonalForm:=function(X)
MM:=Submatrix(X,13,13,2,2);
perm:=DD[Position(D,MM^-1)];
Y:=perm*X;
Z1:=Subm(Y,Row1D4,Row1D4);
Z2:=Subm(Y,Row2D4,Row2D4);
Z3:=Subm(Y,Row3D4,Row3D4);
return [Z1,Z2,Z3];
end function;

ToD4S3:=function(A,List)
```

```
if List in M then
MM:=List;
else MM:=Matrix(GF(q),2,2,List);
end if;
perm:=DD[Position(D,MM)];
Z:=ZeroMatrix(GF(q),26,26);
for i in [1..8] do
for j in [1..8] do
Z[Row1D4[i],Row1D4[j]]:=A[1][i,j];
Z[Row2D4[i],Row2D4[j]]:=A[2][i,j];
Z[Row3D4[i],Row3D4[j]]:=A[3][i,j];
end for;
end for;
Z:=perm*Z;
Z[13,13]:=MM[1,1];
Z[13,14]:=MM[1,2];
Z[14,13]:=MM[2,1];
Z[14,14]:=MM[2,2];
return Z;
end function;
```

```
/*Enlarge the 26-dimensional representation to a 27 dimensional*/

Een:=GL(1,q)!1;
CD27:=[DiagonalJoin(c,Een): c in CD];
CF27:=[DiagonalJoin(c,Een): c in CF];
CB27:=[DiagonalJoin(c,Een): c in CB];
CD4S327:=[DiagonalJoin(c,Een): c in CD4S3];
CW27:=[DiagonalJoin(c,Een): c in CW];
CT27:=[DiagonalJoin(c,Een): c in CT];
CPar:=[DiagonalJoin(c,Een): c in UPar];
D427:=MatrixGroup<27,GF(q)|CD27>;
B427:=MatrixGroup<27,GF(q)|CB27>;
F427:=MatrixGroup<27,GF(q)|CF27>;
D4S327:=MatrixGroup<27,GF(q)|CD4S327>;
W27:=MatrixGroup<27,GF(q)|CW27>;
T27:=MatrixGroup<27,GF(q)|CT27>;
P:=MatrixGroup<27,GF(q)|CPar>;
H:=D4S327;
H0:=D427;
H2:=B427;
G:=F427;
Ngen:=#Generators(G);
Mod:=GModule(G);
V:=VectorSpace(G);
```

```
/*Finding the product invariant under F_4. This is done by just linear algebra.*/

Qvast:=[[V!0: j in [1..27]]: i in [1..27]];
for i in {1..27} do
Qvast[i,27]:=V.i;
Qvast[27,i]:=V.i;
end for;
Qvast[27,27]:=2*V.27;
Qvast[1,16]:=zz*V.2;
Qvast[16,1]:=zz*V.2;
al:={{1,16}};
z:=[1,16];
I:=z[1];
J:=z[2];
for w in W27 do
vi:=V.I*w; vj:=V.J*w;
i:=[k: k in Support(vi)][1];
j:=[k: k in Support(vj)][1];

if not({i,j} in al) then
L:=Qvast[1,16]*w;
b:=L[[k: k in Support(L)][1]];
M:=((vi[i])*(vj[j]))^-1*L;
Qvast[i,j]:=M;
Qvast[j,i]:=M;
end if;
al:=al join {{i,j}};
end for;
x:=0;
y:=0;
```

```
for i in Row1D4 do
Qvast[13,i]:=x*V.i;
Qvast[i,13]:=x*V.i;
Qvast[14,i]:=(y+zz)*V.i;
Qvast[i,14]:=(y+zz)*V.i;
end for;

for i in Row2D4 do
Qvast[13,i]:=(x-zz)*V.i;
Qvast[i,13]:=(x-zz)*V.i;
Qvast[14,i]:=(y-zz)*V.i;
Qvast[i,14]:=(y-zz)*V.i;
end for;

for i in Row3D4 do
Qvast[13,i]:=(x+zz)*V.i;
Qvast[i,13]:=(x+zz)*V.i;
Qvast[14,i]:=y*V.i;
Qvast[i,14]:=y*V.i;
end for;

product:=function(v,w)
vL:=Eltseq(v);
wL:=Eltseq(w);
Positionsv:=[i: i in [1..27]|not(vL[i] eq 0)];
Positionsw:=[i: i in [1..27]|not(wL[i] eq 0)];
Valuesv:=[vL[i]: i in Positionsv];
Valuesw:=[wL[i]: i in Positionsw];
S:=V!0;
for i in [1..#Positionsv] do
for j in [1..#Positionsw] do
S:=S+Valuesv[i]*Valuesw[j]*Qvast[Positionsv[i],Positionsw[j]];
end for;
end for;
return S;
end function;

if not(q eq 2^(Ngen/8) or q eq 3^(Ngen/8)) then
A13:={i: i in [1..Ngen]| not(V.13*G.i eq V.13 or V.13*G.i eq V.14)};
I:=Random(A13);
g:=G.I;
v13g:=V.13*g;v14g:=V.14*g;
E13g:=Eltseq(v13g); E14g:=Eltseq(v14g);
S1:=Support(v14g);
i:=Random(S1 diff {13,14});
d:=1;
j:=0;
A:=A13 diff {I};
while not (d eq 0) do
J:=Random(A);
A:=A13 diff {J};
h:=G.J;
v13h:=V.13*h;v14h:=V.14*h;
E13h:=Eltseq(v13h); E14h:=Eltseq(v14h);
S2:=Support(v13h);
j:=Random(S2 diff {13,14});
M:=KMatrixSpace(GF(q),2,2)![v13g[i],v13h[j],v14g[i],v14h[j]];
c:=VectorSpace(GF(q),2)![(2*v13g[i]*v13g[13]*Qvast[13,i])[i],(2*v13h[j]*v13h[13]*Qvast[13,j])[j]];
Sol,nulspace:=Solution(M,c);
d:=Dimension(nulspace);
end while;
a:=Sol[1]; b:=Sol[2];
Qvast[13,13]:=a*V.13+b*V.14;
Qvast[14,14]:=a*V.14+b*V.13;
Qvast[13,14]:=(a+b)*V.13+(a+b)*V.14;
Qvast[14,13]:=(a+b)*V.13+(a+b)*V.14;
end if;

for t in [1..4] do
I:=Row1D4[t];
J:=Row1D4[9-t];
Sok:={i: i in [1..Ngen]|product(V.I*G.i,V.J*G.i) eq product(V.I,V.J)*G.i  and not(V.13*G.i eq V.13 and V.14*G.i eq V.14)};
Snietok:={i: i in [1..Ngen]|not(product(V.I*G.i,V.J*G.i) eq product(V.I,V.J)*G.i)};
g:=G.Random(Snietok);
vg:=V.I*g;  wg:=V.J*g;
v14g:=V.14*g;  v13g:=V.13*g;
wg26:=(wg)[26]*V.26; vg26:=(vg)[26]*V.26;
vg1:=(vg)[1]*V.1; wg1:=(wg)[1]*V.1;
zg:=product(vg-vg1-vg26,wg-wg26-wg1)+product(vg-vg1,wg26)+product(vg1,wg-wg26)+product(vg-vg26,wg1)+product(vg26,wg-wg1);
S1:=Support(zg); S2:=Support(v13g);  S3:=Support(v14g);
i:=Random((S3 join S2) diff {13,14});
d:=2;
l:=0;
while not (d eq 0) do
k:=Random(Sok diff {l});
h:=G.k;
vh:=V.I*h; wh:=V.J*h;
v14h:=V.14*h; v13h:=V.13*h;
wh26:=(wh)[26]*V.26; vh26:=(vh)[26]*V.26;
```

```
vh1:=(vh)[1]*V.1; wh1:=(wh)[1]*V.1;
zh:=product(vh-vh1-vh26,wh-wh26-wh1)+product(vh-vh1,wh26)+product(vh1,wh-wh26)+product(vh-vh26,wh1)+product(vh26,wh-wh1);
S1h:=Support(zh);
S13h:=Support(v13h); S14h:=Support(v14h);
j:=Random([k: k in {1..26} diff {13,14}|k in S13h or k in S14h]);
M:=KMatrixSpace(GF(q),2,2)![v13g[i],v13h[j],v14g[i],v14h[j]];
c:=VectorSpace(GF(q),2)![zg[i],zh[j]];
Sol,nulspace:=Solution(M,c);
d:=Dimension(nulspace);
l:=k;
end while;
a:=Sol[1];
b:=Sol[2];
Qvast[I,J]:=a*V.13+b*V.14;
Qvast[J,I]:=a*V.13+b*V.14;
end for;

for t in [1..4] do
I:=Row2D4[t];
J:=Row2D4[9-t];
Sok:={i: i in [1..Ngen]|product(V.I*G.i,V.J*G.i) eq product(V.I,V.J)*G.i  and not(V.13*G.i eq V.13 and V.14*G.i eq V.14)};
Snietok:={i: i in [1..Ngen]|not(product(V.I*G.i,V.J*G.i) eq product(V.I,V.J)*G.i)};
g:=G.Random(Snietok);
vg:=V.I*g;  wg:=V.J*g;
v14g:=V.14*g;  v13g:=V.13*g;
wg26:=(wg)[26]*V.26; vg26:=(vg)[26]*V.26;
vg1:=(vg)[1]*V.1; wg1:=(wg)[1]*V.1;
zg:=product(vg-vg1-vg26,wg-wg26-wg1)+product(vg-vg1,wg26)+product(vg1,wg-wg26)+product(vg-vg26,wg1)+product(vg26,wg-wg1);
S1:=Support(zg); S2:=Support(v13g);  S3:=Support(v14g);
i:=Random((S3 join S2) diff {13,14});
d:=2;
l:=0;
while not (d eq 0) do
k:=Random(Snietok diff {l});
h:=G.k;
vh:=V.I*h; wh:=V.J*h;
v14h:=V.14*h; v13h:=V.13*h;
wh26:=(wh)[26]*V.26; vh26:=(vh)[26]*V.26;
vh1:=(vh)[1]*V.1; wh1:=(wh)[1]*V.1;
zh:=product(vh-vh1-vh26,wh-wh26-wh1)+product(vh-vh1,wh26)+product(vh1,wh-wh26)+product(vh-vh26,wh1)+product(vh26,wh-wh1);
S1h:=Support(zh);
S13h:=Support(v13h); S14h:=Support(v14h);
j:=Random([k: k in {1..26} diff {13,14}|k in S13h or k in S14h]);
M:=KMatrixSpace(GF(q),2,2)![v13g[i],v13h[j],v14g[i],v14h[j]];
c:=VectorSpace(GF(q),2)![zg[i],zh[j]];
Sol,nulspace:=Solution(M,c);
d:=Dimension(nulspace);
l:=k;
end while;
a:=Sol[1];
b:=Sol[2];
Qvast[I,J]:=a*V.13+b*V.14;
Qvast[J,I]:=a*V.13+b*V.14;
end for;

for t in [1..4] do
I:=Row3D4[t];
J:=Row3D4[9-t];
Sok:={i: i in [1..Ngen]|product(V.I*G.i,V.J*G.i) eq product(V.I,V.J)*G.i  and not(V.13*G.i eq V.13 and V.14*G.i eq V.14)};
Snietok:={i: i in [1..Ngen]|not(product(V.I*G.i,V.J*G.i) eq product(V.I,V.J)*G.i)};
g:=G.Random(Snietok);
vg:=V.I*g;  wg:=V.J*g;
v14g:=V.14*g;  v13g:=V.13*g;
wg26:=(wg)[26]*V.26; vg26:=(vg)[26]*V.26;
vg1:=(vg)[1]*V.1; wg1:=(wg)[1]*V.1;
zg:=product(vg-vg1-vg26,wg-wg26-wg1)+product(vg-vg1,wg26)+product(vg1,wg-wg26)+product(vg-vg26,wg1)+product(vg26,wg-wg1);
S1:=Support(zg); S2:=Support(v13g);  S3:=Support(v14g);
i:=Random((S3 join S2) diff {13,14});
d:=2;
l:=0;
while not (d eq 0) do
k:=Random(Sok diff {l});
h:=G.k;
vh:=V.I*h; wh:=V.J*h;
v14h:=V.14*h; v13h:=V.13*h;
wh26:=(wh)[26]*V.26; vh26:=(vh)[26]*V.26;
vh1:=(vh)[1]*V.1; wh1:=(wh)[1]*V.1;
zh:=product(vh-vh1-vh26,wh-wh26-wh1)+product(vh-vh1,wh26)+product(vh1,wh-wh26)+product(vh-vh26,wh1)+product(vh26,wh-wh1);
S1h:=Support(zh);
S13h:=Support(v13h); S14h:=Support(v14h);
j:=Random([k: k in {1..26} diff {13,14}|k in S13h or k in S14h]);
M:=KMatrixSpace(GF(q),2,2)![v13g[i],v13h[j],v14g[i],v14h[j]];
c:=VectorSpace(GF(q),2)![zg[i],zh[j]];
Sol,nulspace:=Solution(M,c);
d:=Dimension(nulspace);
l:=k;
end while;
a:=Sol[1];
b:=Sol[2];
Qvast[I,J]:=a*V.13+b*V.14;
```

```
Qvast[J,I]:=a*V.13+b*V.14;
end for;

product:=function(v,w)
vL:=Eltseq(v);
wL:=Eltseq(w);
Positionsv:=[i: i in [1..27]|not(vL[i] eq 0)];
Positionsw:=[i: i in [1..27]|not(wL[i] eq 0)];
Valuesv:=[vL[i]: i in Positionsv];
Valuesw:=[wL[i]: i in Positionsw];
S:=V!0;
for i in [1..#Positionsv] do
for j in [1..#Positionsw] do
S:=S+Valuesv[i]*Valuesw[j]*Qvast[Positionsv[i],Positionsw[j]];
end for;
end for;
return S;
end function;




/*Finding the invariant inner product*/

C:=ClassicalForms(F4Matr)`bilinearForm;
L:=sub<V|[V.13,V.14,V.27]>;
X1:=sub<V|[V.i: i in Row3D4 join {13,14}]>;
X2:=sub<V|[V.i: i in Row2D4 join {13,14}]>;
X3:=sub<V|[V.i: i in Row1D4 join {13,14}]>;
E1:={l: l in L|sub<V|[product(V.i,l): i in [1..26]]> eq X1};
E2:={l: l in L|sub<V|[product(V.i,l): i in [1..26]]> eq X2};
E3:={l: l in L|sub<V|[product(V.i,l): i in [1..26]]> eq X3};
a:=3;
A:=DiagonalJoin(C,Matrix(gf,1,1,[a]));
InProd:=function(x,y)
z:=(x*A,y);
return z;
end function;




/*defininining the representatives of the different orbits described in the paper*/

EE:=[[e1,e2,product(e1,e2)]: e1 in E1, e2 in E2|InProd(e1,e1) eq 1 and InProd(e2,e2) eq 1];
E:=EE[4];
e1:=E[1]; e2:=E[2]; e3:=E[3]; e:=e1+e2+e3;

plane:=function(x,y,z)
return sub<V|[x,y,z]>;;
end function;

L:=plane(e2,e1,e3);

U1:=sub<V|[product(V.i,e1): i in [1..26]|IsPerp(sub<V|[product(V.i,e1)]>,L)]>;
U2:=sub<V|[product(V.i,e2): i in [1..26]|IsPerp(sub<V|[product(V.i,e2)]>,L)]>;
U3:=sub<V|[product(V.i,e3): i in [1..26]|IsPerp(sub<V|[product(V.i,e3)]>,L)]>;

V10:=sub<V|{V.i: i in Row1D4} join {e1,e2}>;
V16:=sub<V|{V.i: i in Row2D4} join {V.i: i in Row3D4}>;

z:=V.1;
w:=V.26;

x1:=e1+z;
y1:=e2-z;
p1:=plane(x1,y1,e3);
X:=[a*e1+(a-a^2)*z+w+(1-a)*e2: a in [2..q-1]];
Y:=[(1-a)*e1-(a-a^2)*z-w+a*e2: a in [2..q-1]];
P:=[plane(X[i],Y[i],e3): i in [1..q-2]];
p2:=P[1];
p3:=P[2];
p4:=P[3];

x2:=X[1]; x3:=X[2]; x4:=X[3];
y2:=Y[1]; y3:=Y[2]; y4:=Y[3];

inprdctn:=function(x1,y1,e1,e2)
a1:=InProd(product(x1,e1),product(x1,e2));
a2:=InProd(product(y1,e1),product(y1,e2));
a3:=InProd(product(x1,e2),product(y1,e2));
a4:=InProd(product(x1,e1),product(y1,e1));
return {a1,a2,a3,a4};
end function;
```

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