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Convergence of successive approximation for a free-boundary problem in fluid-structure interaction

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Abstract

Fluid-structure-interaction problems have prominence in aerospace engineering and many other scientific and engineering disciplines. An essential property of these problems is that the interface between the fluid and the structure constitutes a free boundary. Iterative solution methods for free-boundary problems are typically based on a partitioned solution procedure: (1) the boundary-value problem(s) is (are) solved with a subset of the free-boundary conditions imposed, and (2) the free boundary is adjusted to relax the remaining free-boundary condition. This iterative procedure is referred to as successive approximation, subiteration or Picard iteration. In the present work we investigate the convergence properties of successive approximation for a model fluid-structure interaction problem, viz., the piston problem. We establish that the iteration operator is nonnormal. An important consequence of this nonnormality is that the successive approximation process can diverge before convergence occurs. The initial divergence can cause failure of the computational method despite formal stability. As such, the nonnormality induces a profound degradation in the robustness and efficiency of the subiteration method. Numerical experiments are presented to illustrate the theoretical results.

1 Introduction

The numerical solution of fluid-structure interaction problems is of great importance in a multitude of physical and engineering disciplines, e.g., aerospace engineering [1, 2], bio-mechanics [3, 4] and civil engineering [5]. An essential property of these problems is that the position of the interface between the fluid and

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the structure is not known *a priori*. It is interconnected with the state variables of the fluid and the structure through interface conditions. The number of these interface conditions is one more than the number of boundary conditions required by the separate initial-boundary-value problems for the fluid and the structure. These characteristics render fluid-structure interaction a free-boundary problem.

An essential complication in the numerical treatment of free-boundary problems is the interdependence of the state variables and their domain of definition. Consequently, implicit solution methods, such as Newton's method, require so-called shape derivatives, viz., the derivatives of the residuals of the initial-boundary-value problems with respect to the free-boundary position. Therefore, solution methods for free-boundary problems typically implement an iterative process based on partitioning: (1) the initial-boundary-value problem(s) is (are) solved with a subset of the free-boundary conditions imposed, and (2) the free boundary is adjusted to relax the remaining free-boundary condition. This method is often referred to as successive approximation, subiteration or Picard iteration.

The essential disadvantage of partitioned methods pertains to their stability and convergence behaviour, which generally depend sensitively on the parameters in the problem; see, e.g., Refs. [6–8]. In this work we investigate the properties of successive approximation for a model fluid-structure-interaction problem.

The contents of this paper are organised as follows: Section 2 contains a problem statement. In Section 3 we derive a suitable model problem which serves as a basis for the ensuing analysis. Section 4 establishes that the iteration operator associated with the successive approximation method is *nonnormal* and examines the implications for the convergence behaviour. Section 5 present numerical experiments to illustrate the preceeding theory. Section 6 contains concluding remarks.

2 Problem statement

In this paper we restrict ourselves to a prototypical fluid-structure-interaction problem, viz., the piston problem. Below we present a concise classical formulation of the problem. For an elaboration and a variational formulation see [9].

Let x and t be a spatial and a temporal coordinate, respectively. Consider an open bounded domain $\Omega_\ell := \{(x,t) : 0 < t < T; 0 < x < \ell(t)\}$, bounded by the *interface* $\Gamma_\ell := \{(x,t) : x = \ell(t); 0 \le t \le T\}$ and the fixed boundary $\partial\Omega_\ell \setminus \Gamma_\ell$. The piston problem comprises the Euler equations of gas dynamics in connection with a simple harmonic oscillator, specified by

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0\,, \qquad (x,t) \in \Omega_\ell\,, \tag{1}$$

with $\mathbf{u} := (\rho, \rho v, E)^T$, $\mathbf{f}(\mathbf{u}) := (u_2, u_2^2/u_1 + p(\mathbf{u}), [p(\mathbf{u}) + u_3]\, u_2/u_1)^T$ and $p(\mathbf{u}) := (g-1)(u_3 - u_2^2/[2\,u_1])$, and

$$M\,\ddot{z} + K\,z = \theta(t) - p^0\,, \qquad 0 < t < T\,, \tag{2}$$

respectively. In (1) ρ, v and E denote the density, velocity and total internal energy of the fluid, respectively, and g is a constant, typically $g = 1.4$. In (2) z designates the piston displacement from its equilibrium position, θ represents the force exerted by the fluid on the structure through the interface, p^0 is the ambient pressure and M and K represent mass and stiffness of the structure, respectively.

The Euler equations and the harmonic oscillator are interconnected by the interface conditions:

$$u_2\big(\ell(t)\big) = u_1\big(\ell(t)\big)\,\dot{\ell}(t), \quad \ell(t) = \ell^0 + z(t), \quad p\big(\mathbf{u}(\ell(t),t)\big) = \theta(t), \tag{3}$$

with ℓ^0 a given positive constant. The first two conditions express the impermeability of the interface and identify the interface position and the piston position. These are called the kinematic conditions. The third condition implies equilibrium of the forces exerted on the interface by the fluid and the structure and is called the dynamic condition. One can infer that the interface conditions induce an interdependence between $\mathbf{u}$ and its domain of definition Ω_ℓ through ℓ and, moreover, that the number of interface condition exceeds the number of required boundary conditions by one. Hence, the interface Γ_ℓ constitutes a free boundary.

The complementary initial and boundary conditions for the fluid-structure system are provided by:

$$u_2(0,t) = 0, \quad \mathbf{u}(x,0) = \mathbf{u}^0(x), \quad z(0) = z^0, \quad \text{and} \quad \dot{z}(0)\dot{z}^0, \tag{4}$$

with $\mathbf{u}^0(x)$ the prescribed initial conditions for the fluid and z^0 and $\dot{z}^0$ the prescribed initial displacement and velocity for the structure. Note that the first condition in (4) renders $x = 0$ a rigid impermeable boundary.

The interconnection between the state variables and their domain of definition forms a profound complication in the numerical treatment of fluid-structure-interaction problems. This complication can be bypassed through an iterative solution procedure, often referred as *successive approximation*: Given an initial approximation of the structure position $z_0(t)$, for $j = 1, 2, \ldots$

(S1) Solve the kinematic condition: find $\ell_j(t)$ such that $\ell_j(t) = \ell^0 + z_{j-1}(t)$.

(S2) Solve the fluid problem on the domain Ω_{ℓ_j} subject to the boundary condition $u_2(\ell_j(t)) = u_1(\ell_j(t))\,\dot{\ell}_j(t)$ on Γ_{ℓ_j} to obtain $\mathbf{u}_j$.

(S3) Solve the dynamic condition: find $\theta_j(t)$ such that $\theta_j(t) = p(\mathbf{u}_j(\ell_j(t),t))$.

(S4) Solve the structure problem with right member $\theta_j(t) - p^0$ to obtain $z_j(t)$.

The above algorithm is in fact an implementation of the successive approximation method; different partitions of the interface conditions are possible. However, the partition in (S1)–(S4) is standard. In this paper we are concerned with the stability and convergence of (S1)–(S4) .

It is mentionable that the *partitioned* or *staggered* time-integration methods for fluid-structure-interaction problems (see Refs. [10–13]) are essentially identical to the above iterative method, with the provision that the process is not repeated.

3 Model problem

To set up the linear model problem, we consider the piston problem and we note that if the initial conditions are specified as $\mathbf{u}^0(x) = \mathbf{u}^0$ with $\mathbf{u}^0$ a constant in $\{\mathbf{u} \in \mathbb{R}^3 : 0 < u_1; p(\mathbf{u}) = p^0; u_2 = 0\}$ and $z^0 = \dot{z}^0 = 0$, then the solution is $\mathbf{u}(x,t) = \mathbf{u}^0$, $z(t) = 0$, $\ell(t) = \ell^0$, $\theta(t) = p^0$. Hence, if we provide the successive approximation method with an initial approximation $z_0(t) = 0$, then the first iteration immediately yields the actual solution. This motivates us to assume that if instead the initial approximation is specified as $z_0(t) = \epsilon z_0'(t)$, with z_0' a suitable function independent of ϵ, then the first approximation $(\ell_1, \mathbf{u}_1, \theta_1, z_1)_\epsilon$ generated by the successive approximation method can be expanded asymptotically in ϵ. Omitting details for conciseness, we obtain $(\ell_1, \mathbf{u}_1, \theta_1, z_1)_\epsilon = (\ell^0, \mathbf{u}^0, p^0, 0) + \epsilon(\ell', \mathbf{u}', \theta', z') + O(\epsilon^2)$ as $\epsilon \to 0$, and

$$M\ddot{z}' + Kz' = -c^2 u_1'(\ell^0)\,, \quad z'(0)\,, \quad \dot{z}'(0) = 0\,, \tag{5}$$

where u_1' is determined by the linearised Euler equations

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix} \cdot \frac{\partial}{\partial x}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = 0\,, \tag{6}$$

with $c := \sqrt{gp^0/u_1^0}$ the speed of sound, and the initial and boundary conditions

$$u_1'(x,0) = 0\,, \quad x \in]0, \ell^0[\,, \qquad u_2'(x,0) = 0\,, \qquad x \in]0, \ell^0[\,, \tag{7a}$$

$$u_2'(0,t) = 0\,, \quad t \in]0, \infty[\,, \qquad u_2'(\ell^0, t) = \rho^0 \dot{z}_0'(t)\,, \quad t \in]0, \infty[\,. \tag{7b}$$

In Eqs. (5)–(7) we have assumed the perturbation $\mathbf{u}'$ to be isentropic. The generic solution to (6) is

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}(x,t) = \rho^0 \left[\hat{f}(t + x/c)\begin{pmatrix} 1 \\ -c \end{pmatrix} + \check{f}(t - x/c)\begin{pmatrix} 1 \\ c \end{pmatrix} \right]. \tag{8}$$

The complementary conditions (7) imply the following conditions on $\hat{f}$ and $\check{f}$:

$$\hat{f}(s) + \check{f}(s) = 0, \qquad -\hat{f}(s) + \check{f}(s) = 0,\ s \in [0, \ell^0/c], \tag{9a}$$

$$\check{f}(s - \ell^0/c) - \hat{f}(s + \ell^0/c) = \dot{z}_0'(s)/c,\ -\hat{f}(s) + \check{f}(s) = 0,\ s \in [0, \infty), \tag{9b}$$

From Eqs. (8)–(9) it follows straightforwardly that

$$u_1'(\ell^0, t) = -\frac{\rho^0}{c}\left(\dot{z}_0'(t) + 2 \sum_{j=1}^{\lfloor ct/2\ell^0 \rfloor} \dot{z}_0'(t - 2j\ell^0/c) \right), \tag{10}$$

where $\lfloor \cdot \rfloor$ denotes the floor (truncation to integer) function. One can infer that the integer $\lfloor ct/2\ell^0 \rfloor$ represents the number of reflections that have returned from the rigid boundary to the interface in the time interval $[0, t]$.

To cast the final expression into a convenient dimensionless form, we introduce characteristic time scales for the structure and the fluid, $T_s := 2\pi\sqrt{M/K}$ and $T_f := 2\,\ell^0/c$, respectively, and characteristic time-scale and mass ratios for the fluid-structure system, $\nu := 2\pi T_f/T_s$ and $\mu := \rho^0\ell^0/M$. Moreover, we denote by $s := 2\pi\, t/T_s$ a dimensionless time parameter. With these definitions, we obtain

$$D^2z'(s)+z'(s) = -\frac{2\,\mu}{\nu}\left(Dz'_0(s)+2\sum_{j=1}^{\lfloor s/\nu\rfloor} Dz'_0(s-j\nu)\right),\quad z'(0) = Dz'(0) = 0\,, \tag{11}$$

where D denotes differentiation with respect to the function argument. Note that the initial-value problem (11) relates consecutive approximations of the structure displacement in the successive approximation process.

Let $\mathcal{Z}$ denote a space of admissible perturbations. Eq. (11) associates a $z' \in \mathcal{Z}$ with each $z'_0 \in \mathcal{Z}$. Accordingly, we conceive of (11) as an automorphic operator on $\mathcal{Z}$, which we denote by C. Because the successive approximation procedure essentially reduces to recursion of the operator C, we refer to C as the *characteristic operator*. An important simplification of the characteristic operator is effected by discarding the sum associated to the reflections. We call the corresponding characteristic operator the *null-reflection operator* C_0. The ensuing exposition is restricted to the null-reflection operator. However, most results extend mutatis mutandis to the general subiteration operator.

4 Nonnormality of the null-reflection operator

Let us briefly review the prerequisite theory of linear operators. Consider a bounded linear operator $\mathsf{L} : \mathcal{H} \mapsto \mathcal{H}$ on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. The adjoint operator L^* is defined by the identity $\langle \mathsf{L}^*w, v\rangle = \langle w, \mathsf{L}v\rangle$ for all $v, w \in \mathcal{H}$ or a dense subspace thereof. The operator L is said to be *normal* if it commutes with its adjoint, i.e., if $\mathsf{L}^*\mathsf{L} = \mathsf{L}\mathsf{L}^*$. The norm of the operator and its spectral radius are respectively defined by

$$\|\mathsf{L}\| := \sup_{0\neq v\in\mathcal{H}} \frac{\|\mathsf{L}v\|}{\|v\|} = \sup_{0\neq v,w\in\mathcal{H}} \frac{|\langle w, \mathsf{L}v\rangle|}{\|w\|\,\|v\|}\,, \qquad \operatorname{spr}\mathsf{L} := \lim_{n\to\infty} \|\mathsf{L}^n\|^{1/n}\,, \tag{12}$$

An elementary property of a normal operator L is that $\|\mathsf{L}^n\| = \|\mathsf{L}\|^n$ and, accordingly, $\|\mathsf{L}\| = \operatorname{spr}\mathsf{L}$; see Ref. [14, I-§6.6]. Hence, for a normal operator the spectral radius $\operatorname{spr}\mathsf{L}$ delineates the norm of monomials of the operator $\|\mathsf{L}^n\|$. In particular, if $\operatorname{spr}\mathsf{L} < 1$ then $\|\mathsf{L}^n\|$ converges monotonically to 0 as n increases. For nonnormal operators the spectral radius only represents the asymptotic convergence as $n \to \infty$. If $\|\mathsf{L}\| > 1 > \operatorname{spr}\mathsf{L}$ then the convergence of $\|\mathsf{L}^n\|$ is generally nonmonotonous and transient divergence can occur before asymptotic convergence sets in; see, for instance, Ref. [15, p.28] and the plots of $\|\exp(t\mathsf{L})\|$ (which has similar properties as $\|\mathsf{L}^n\|$) versus t for various nonnormal operators in Ref. [16].

To demonstrate the nonnormality of C_0 a proper functional setting is required. Considering an open interval $\mathcal{T} := (0, T)$, an appropriate setting for C_0 is the

Sobolev space $W_2^1(\mathcal{T}) := \{u \in L_2(\mathcal{T}) : \|u\|_{W_2^1(\mathcal{T})} < \infty\}$, where the norm $\|\cdot\|_{W_2^1(\mathcal{T})}$ is induced by the inner product

$$\langle u, v\rangle_{W_2^1(\mathcal{T})} := \langle u, v\rangle + \langle Du, Dv\rangle, \tag{13}$$

with $\langle\cdot,\cdot\rangle$ the usual L_2 inner product. Omitting proof for brevity, we state that C_0 is bounded on W_2^1 as $\|\mathsf{C}_0\|_{W_2^1} \leq 2T$ and unbounded on L_2. Equipped with the inner product (13), W_2^1 is a Hilbert space.

To prove the nonnormality of the null-reflection operator, we derive its adjoint and demonstrate that there exist admissible functions that violate $\mathsf{C}_0^*\mathsf{C}_0 = \mathsf{C}_0\mathsf{C}_0^*$. Consider an open interval $\mathcal{T} := (0, T)$. Let $\mathcal{Z} := W_2^1(\mathcal{T})$, equipped with the L_2 inner product. Being a closed linear subspace of the Hilbert space L_2, the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ in L_2 is a Hilbert space. To avoid the conveyance of constants, without loss of generality we set $2\,\mu/\nu = 1$. A useful device in the derivation of the adjoint operator is the translation operator $\mathsf{T} : u \in \mathcal{Z} \mapsto u - u(0) \in \mathcal{Z}$. As $\mathsf{C}_0 u$ depends on u through Du only, C_0 is translation invariant and $\mathsf{C}_0\mathsf{T} = \mathsf{C}_0$. To determine the adjoint operator of C_0 on $\overline{\mathcal{Z}}$, we select an arbitrary $w \in \mathcal{Z}$ and associate a function ψ by

$$D^2\psi + \psi = w\,,\ \psi(T) = 0\,,\ D\psi(T) = 0\,. \tag{14}$$

Moreover, we consider an arbitrary $u \in \mathcal{Z}$ and denote by v its image under $\mathsf{C}_0\mathsf{T}$. The adjoint operator follows from the following sequences of identities:

$$\begin{aligned}\langle w, \mathsf{C}_0 u\rangle &= \langle w, \mathsf{C}_0\mathsf{T}u\rangle = \langle D^2\psi + \psi, v\rangle = v\,D\psi\big|_0^T - Dv\,\psi\big|_0^T + \langle\psi, D^2 v + v\rangle \\ &= \langle\psi, D^2 v + v\rangle = -\langle\psi, D\,\mathsf{T}u\rangle = -\mathsf{T}u\,\psi\big|_0^T + \langle D\psi, \mathsf{T}u\rangle = \langle D\psi, \mathsf{T}u\rangle\,,\end{aligned} \tag{15}$$

and

$$\langle \mathsf{C}_0^* w, u\rangle = \langle w, \mathsf{C}_0 u\rangle = \langle w, \mathsf{C}_0\mathsf{T}u\rangle = \langle \mathsf{C}_0^* w, \mathsf{T}u\rangle\,. \tag{16}$$

The third and sixth identity in (15) follow by integration by parts. The boundary terms in Eq. (15) vanish in virtue of the homogeneous initial and final conditions on v and ψ, respectively, and the property $(\mathsf{T}u)(0) = 0$ of the translation operator. From (15)–(16) the adjoint operator is readily identified as

$$\mathsf{C}_0^* : \{w \in \mathcal{Z}\} \mapsto \{D\psi \in \mathcal{Z} : D^2\psi + \psi = w\,,\ \psi(T) = 0\,,\ D\psi(T) = 0\}. \tag{17}$$

As u, w in (15)–(16) are arbitrary, we have the implication $\langle \mathsf{C}_0 u, w\rangle = \langle u, \mathsf{C}_0^* w\rangle$ for all $u, w \in \mathcal{Z}$. Because $\mathcal{Z}$ is dense in $\overline{\mathcal{Z}}$, C_0^* is the adjoint operator of C_0 on $\overline{\mathcal{Z}}$. The normality of C_0 can now be disproved straightforwardly by establishing that there exists a $u \in \mathcal{Z}$ such that $\mathsf{C}_0^*\mathsf{C}_0 u \neq \mathsf{C}_0\mathsf{C}_0^* u$. For instance, $\sin \in \mathcal{Z}$ and

$$\begin{aligned}(\mathsf{C}_0^*\mathsf{C}_0 - \mathsf{C}_0\mathsf{C}_0^*)\sin(s) = \big[&2(s-T)\cos(s-2T) - \sin(s+2T) - 2\sin(s-2T) \\ &- (4Ts - 3 - 2T^2)\sin(s) - 2s\cos(s)\big]/16 \neq 0,\end{aligned} \tag{18}$$

which completes the proof.

5 Numerical examples

To elaborate the nonnormality of the successive approximation method, we consider the null-reflection operator on the space of polynomials of degree n on the interval $\mathcal{T} := (0,T)$, denoted by $\mathcal{P}_n$. We consider an arbitrary $u \in \mathcal{P}_n$ and associate a function $v \in \mathcal{P}_n$ by the variational statement:

$$\mathsf{A}(w,v) = \mathsf{B}(w,u) \qquad \text{for all } w \in \mathcal{P}_n\,, \tag{19a}$$

$$\mathsf{A}(w,v) := \langle w, D^2 v + v\rangle + Dv\,w\big|_0 - v\,Dw\big|_0\,, \quad \mathsf{B}(w,u) := -\langle w, Du\rangle. \tag{19b}$$

Eq. (19) is a weak formulation of $v = \mathsf{C}_0 u$ with weakly imposed initial conditions. To render (19) amenable to numerical treatment, we introduce a basis $\{\varphi_0, \ldots, \varphi_n\}$ of $\mathcal{P}_n$, where φ_j is the normalised Legendre polynomial of degree j on the interval $\mathcal{T}$. Equation (19) then translates into the linear algebra problem $\mathbf{A}\cdot\mathbf{v} = \mathbf{B}\cdot\mathbf{u}$, where $A_{ij} = \mathsf{A}(\varphi_i,\varphi_j)$ and $B_{ij} = \mathsf{B}(\varphi_i,\varphi_j)$ and the vectors $\mathbf{v}$ and $\mathbf{u}$ collect the coefficients of v and u with respect to the basis of Legendre polynomials, respectively. Hence, with respect to this basis the mapping C_0 is represented by the matrix $\mathbf{C}_0 := \mathbf{A}^{-1}\cdot\mathbf{B}$.

The spectrum of C_0 on $\mathcal{P}_n$ coincides with the spectrum of $\mathbf{C}_0$ and, hence, $\operatorname{spr}\mathsf{C}_0 = \sup|\zeta|$ on $\zeta \in \mathbb{C}$ in the spectrum of $\mathbf{C}_0$. To determine the W_2^1 norm of C_0 on $\mathcal{P}_n$ we note that for any pair of functions $u, v \in \mathcal{P}_n$ it holds that $\langle u,v\rangle_{W_2^1} = \mathbf{u}^*\cdot\mathbf{W}\cdot\mathbf{v}$ with $W_{ij} = \langle\varphi_i,\varphi_j\rangle_{W_2^1}$. The matrix $\mathbf{W}$ is symmetric positive definite and, hence, it admits a Cholesky factorisation, i.e., there exists an upper triangular matrix $\mathbf{Q}$ such that $\mathbf{W} = \mathbf{Q}^*\cdot\mathbf{Q}$. It follows that $\|u\|_{W_2^1} = \|\mathbf{Q}\cdot\hat{\mathbf{u}}\|_2$ for any $u \in \mathcal{P}_n$, where $\|\cdot\|_2$ is the standard vector 2-norm. The Sobolev norm of C_0 on $\mathcal{P}_n$ can thus be determined as follows:

$$\begin{aligned}\|\mathsf{C}_0\|_{W_2^1} &= \sup_{u,w\in\mathcal{Z}_n} \frac{|\langle w, \mathsf{C}_0 u\rangle_{W_2^1}|}{\|w\|_{W_2^1}\|u\|_{W_2^1}} = \sup_{\mathbf{w},\mathbf{u}} \frac{|\mathbf{w}^*\cdot\mathbf{W}\cdot\mathbf{C}_0\cdot\mathbf{u}|}{\|\mathbf{Q}\cdot\mathbf{w}\|_2\|\mathbf{Q}\cdot\mathbf{u}\|_2} \\ &= \sup_{\bar{\mathbf{w}},\bar{\mathbf{u}}} \frac{|\bar{\mathbf{w}}^T\cdot\mathbf{Q}\cdot\mathbf{C}_0\cdot\mathbf{Q}^{-1}\cdot\bar{\mathbf{u}}|}{\|\bar{\mathbf{w}}\|_2\|\bar{\mathbf{u}}\|_2} = \|\mathbf{Q}\cdot\mathbf{C}_0\cdot\mathbf{Q}^{-1}\|_2, \end{aligned} \tag{20}$$

where $\bar{(\cdot)} := \mathbf{Q}\cdot(\cdot)$. Note that the final form in (20) is computable.

Table 1 lists the norm $\|\mathsf{C}_0\|_{W_2^1}$ for representative values of the interval length T and the polynomial degree n. The results are essentially independent of n. It is to be noted that the norm of C_0 is generally much larger than the spectral radius. Moreover, the table illustrates that both the norm and the spectral radius increase with T, which implies that convergence of the successive approximation method deteriorates with increasing T. In addition, we monitored the condition number $\kappa(\mathbf{U})$ of the matrix of eigenvectors of $\mathbf{Q}\cdot\mathbf{C}_0\cdot\mathbf{Q}^{-1}$. By the spectral mapping theorem it holds that $\|\mathsf{C}_0^m\|_{W_2^1} \le \kappa(\mathbf{U})\,(\operatorname{spr}\mathsf{C}_0)^m$ and, hence, $\kappa(\mathbf{U})$ indicates the potential divergence due to nonnormality. In all cases we obtained $\kappa(\mathbf{U}) > 10^{10}$.

To illustrate the effect of nonnormality on the convergence of successive approximation, we consider the evolution of the perturbation $u = \exp(-10s^2)$ (projected

Table 1: Norm and spectral radius of the null-reflection operator on $\mathcal{P}_n$.

	$n\backslash T$	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}
$\|\mathsf{C}_0\|_{W_2^1}$	2^6	6.37×10^{-3}	6.36×10^{-2}	6.03×10^{-1}	3.68×10^{0}	3.24×10^{1}
	2^7	6.37×10^{-3}	6.36×10^{-2}	6.03×10^{-1}	3.68×10^{0}	3.24×10^{1}
spr C_0	2^6	3.95×10^{-4}	3.95×10^{-3}	3.94×10^{-2}	3.46×10^{-1}	3.87×10^{0}
	2^7	4.16×10^{-4}	4.16×10^{-3}	4.15×10^{-2}	3.60×10^{-1}	2.04×10^{0}

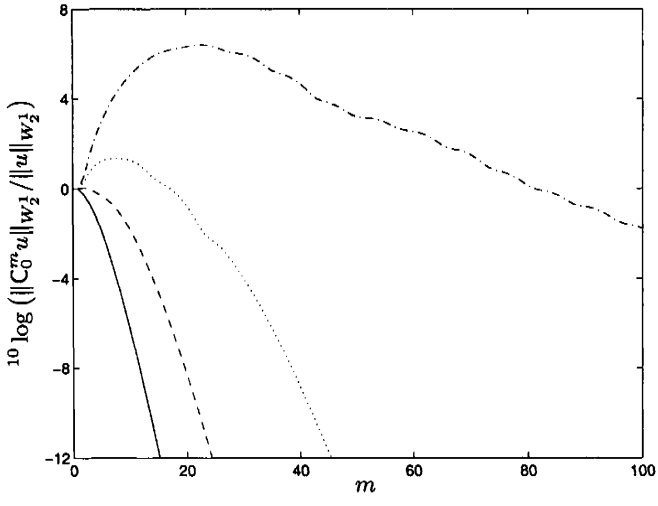

Figure 1: Example of convergence of successive approximation with $2\mu/\nu = 1$ and $T = 10^0$ (—), $T = 10^{1/2}$ (– –), $T = 10^1$ (···), and $T = 10^{3/2}$ (– ·).

onto $\mathcal{P}_n$) under recursive application of C_0. Figure 1 plots the amplification factor $\|\mathsf{C}_0^m u\|_{W_2^1}/\|u\|_{W_2^1}$ versus the iteration counter m for $n = 2^7$, $2\mu/\nu = 1$ and different lengths of the time-interval. Figure 2 displays the amplification factor for $n = 2^7$, $T = 1$ and different values of the scaling parameter $2\mu/\nu$. In all cases, the spectral radius is less than 1 and the successive approximation method is formally stable. However, monotone convergence occurs only if $\|\mathsf{C}_0\|_{W_2^1} < 1$. If $\|\mathsf{C}_0\|_{W_2^1}$ is large, then the error can increase by several orders of magnitude before convergence sets in. It is to be noted that the difference between the results in Fig. 2 emanates exclusively from a rescaling of the spectrum, as $2\mu/\nu$ appears as a multiplicative factor to the operator; cf. Eq. (11). The nonnormality of the underlying operators (in any suitable measure) is identical. This exemplifies that if the spectral radius is sufficiently small, then the effect of nonnormality on the convergence of the successive approximation method is concealed.

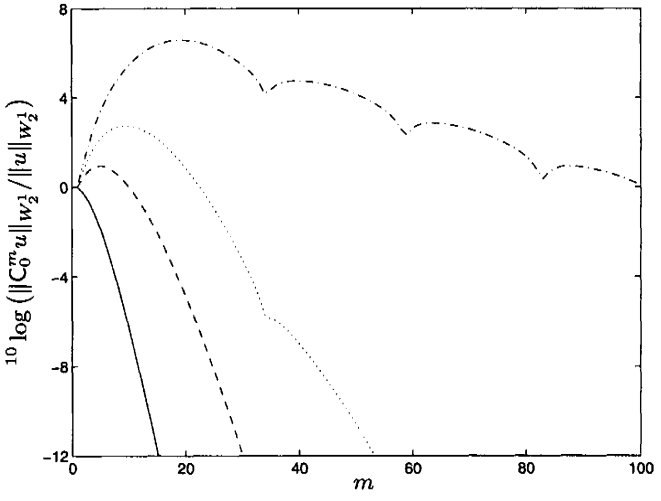

Figure 2: Example of convergence of successive approximation with $T = 1$ and $2\mu/\nu = 1$ (—), $2\mu/\nu = 5$ (– –), $2\mu/\nu = 10$ (···), and $2\mu/\nu = 20$ (– ·).

6 Conclusion

We considered the standard successive approximation method for fluid-structure interaction problems. On the basis of a model problem we established that the operator associated with the method is nonnormal. By means of numerical approximation on polynomials we showed that the appropriate norm of the operator is typically much larger than its spectral radius. Numerical examples were presented to illustrate the implications for the convergence behaviour of the successive approximation method. It was shown that the method admits divergence of the error by many orders of magnitude, despite formal stability. In practice, this can imply a profound degradation of the robustness and stability of the successive approximation method. We conjecture that the nonnormality of the successive approximation method extends with the necessary modifications to other fluid-structure-interaction problems.

References

[1] Farhat, C., Geuzaine, P. & Brown, G., Application of a three-field nonlinear fluidstructure formulation to the prediction of the aeroelastic parameters of an f-16 fighter. *Computers and Fluids*, **32**, pp. 3–29, 2003.

[2] Farhat, C., Geuzaine, P. & Grandmont, C., The discrete geometric conservation law and the nonlinear stability of ALE schemes for the solution of flow problems on moving grids. *J Comput Phys*, **174**, pp. 669–694, 2001.

[3] Heil, M., Stokes flow in an elastic tube – a large-displacement fluid-structure interaction problem. *Int J Num Meth Fluids*, **28**, pp. 243–265, 1998.

Transactions on Modelling and Simulation vol 36, © 2003 WIT Press, www.witpress.com, ISSN 1743-355X

[4] Bathe, M. & Kamm, R., A fluid–structure interaction finite element analysis of pulsatile blood flow through a compliant stenotic artery. *Journal of Biomechanical Engineering*, **121**, pp. 361–369, 1999.

[5] Robertson, I., Sherwin, S. & Bearman, P., Prediction of flutter instability due to cross winds of the second forth road bridge. *Computational Fluid Dynamics '01: Proceedings of the ECCOMAS CFD Conference 2001*, Swansea, Wales, U.K., 2001. On CD-rom.

[6] Silliman, W. & Scriven, L., Separating flow near a static contact line: Slip at a wall and shape of a free surface. *J Comput Phys*, **34**, pp. 287–313, 1980.

[7] Brummelen, E. & Segal, A., Numerical solution of steady free-surface flows by the adjoint optimal shape design method. *Int J Num Meth Fluids*, **41**, pp. 3–27, 2003.

[8] Brummelen, E., Raven, H. & Koren, B., Efficient numerical solution of steady free-surface Navier–Stokes flow. *J Comput Phys*, **174**, pp. 120–137, 2001.

[9] Brummelen, E., Hulshoff, S. & Borst, R., Energy conservation under incompatibility for fluid-structure interaction problems. *Comput Methods Appl Mech Engrg*, 2003. (accepted for publication). Preprint available at `http://www.em.lr.tudelft.nl/~brummelen/publications/CMAME-02-413.pdf`.

[10] Piperno, S., Farhat, C. & Larrouturou, B., Partitioned procedures for the transient solution of coupled aeroelastic problems - part I: Model problem, theory and two-dimensional application. *Comput Methods Appl Mech Engrg*, **124**, pp. 79–112, 1995.

[11] Piperno, S. & Farhat, C., Partitioned procedures for the transient solution of coupled aeroelastic problems - part II: Energy transfer analysis and three-dimensional applications. *Comput Methods Appl Mech Engrg*, **190**, pp. 3147–3170, 2001.

[12] Piperno, S., Explicit/implicit fluid/structure staggered procedures with a structural predictor and fluid subcycling for 2D inviscid aeroelastic simulations. *Int J Num Meth Fluids*, **25**, pp. 1207–1226, 1997.

[13] Felippa, C., Park, K. & Farhat, C., Partitioned analysis of coupled mechanical systems. *Comput Methods Appl Mech Engrg*, **190**, pp. 3247–3270, 2001.

[14] Kato, T., *Perturbation Theory for Linear Operators*. Number 132 in Grundlehren der Mathematischen Wissenschaft, Springer: Berlin, corr. pr. 2nd ed. edition, 1984.

[15] Greenbaum, A., *Iterative Methods for Solving Linear Systems*, volume 17 of *Frontiers in Applied Mathematics*. SIAM: Philadelphia, 1997.

[16] Trefethen, L., Pseudospectra of linear operators. *SIAM Rev*, **39(3)**, pp. 383–406, 1997.