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by

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# A Nonlinear Flux Approximation Scheme for the Viscous Burgers Equation

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**Abstract** We present a nonlinear flux approximation scheme for the spatial discretization of the viscous Burgers equation. We derive the numerical flux function from a local two-point boundary value problem (BVP), which results in a nonlinear equation that depends on the local boundary values and the diffusion constant. The flux scheme is consistent and stable (does not introduce any spurious oscillations), as demonstrated by the numerical results.

**Key words:** Complete flux scheme, nonlinear local BVP, viscous Burgers equation  
**MSC (2010):** 65M08, 34B15

## 1 Introduction

In this proceeding we present a nonlinear flux approximation scheme for the spatial discretization of the viscous Burgers equation. The Burgers equation is an ideal test problem, as its spatial discretization can be carried over to the convective and viscous fluxes involved in the Navier-Stokes equations. The expression for the flux is derived from a local two-point BVP and is inspired by [5], where a local BVP is solved to derive an integral representation of the complete flux for the convection-diffusion-reaction equation. The resulting flux is expressed as a sum of a *homogeneous* part, which depends on the Péclet number (local balance of convection and diffusion) and an *inhomogeneous* part depending on the effects of the source term (associated with the reaction). Note that the homogeneous flux approximation is

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similar to the approximation methods described in [2, 3]. In this contribution, we focus only on homogeneous nonlinear BVPs for the flux approximation.

In the vanishing viscosity limit, the viscous Burgers equation is a singularly perturbed problem. Moreover, the nonlinearity of the flux does not allow us to express the homogeneous flux as linear combination of the convective and the viscous part, which makes it cumbersome to have a consistent numerical flux. In this paper, we extend the local BVP method for nonlinear problems, such that the resulting numerical flux is consistent, i.e., reduces to the correct flux in the limit case. A generalization for nonlinear local two-point BVPs can be found in [1], where the authors show i) the solvability of some auxiliary local nonlinear two point BVPs, and ii) the convergence of the discrete scheme to a weak solution of the continuous problem.

The paper is organized as follows: in Sec. 2 we formulate the local BVP for the flux approximation. Sec. 3 gives details of the derivation for the numerical fluxes. In Sec. 4 we compare the nonlinear scheme with the linearized homogeneous flux scheme described in [5] as well as with other standard methods. Sec. 5 gives the concluding remarks.

## 2 Flux from local two-point BVP

Consider the one-dimensional viscous Burgers equation

$$u_t + f(u, u_x)_x = 0, \quad f(u, u_x) := \frac{1}{2}u^2 - \nu u_x, \quad (1)$$

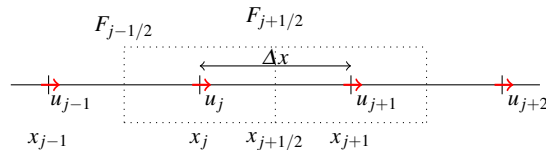
defined on  $\Omega(\subset \mathbb{R}) \times (0, T)$ , where  $\nu(\geq 0)$  is the diffusion coefficient. The spatial discretization of the Burgers equation using a finite-volume method requires the approximation of the flux function  $f(u, u_x)$  at each interface between two control volumes. The semi-discrete formulation of equation (1) is given by

$$\Delta x \dot{u}_j + F_{j+1/2} - F_{j-1/2} = 0, \quad \dot{u} := u_t, \quad (2)$$

where  $F_{j+1/2} \approx f(u, u_x)|_{x=x_{j+1/2}}$ , see Figure 1. The derivation of the flux  $F_{j+1/2}$  is based on the following model BVP, in which we ignore the time dependence of the velocity:

$$f_x = \left(\frac{1}{2}u^2 - \nu u_x\right)_x = 0, \quad x \in (x_j, x_{j+1}), \quad (3a)$$

$$u(x_j) = u_j = u_L, \quad u(x_{j+1}) = u_{j+1} = u_R. \quad (3b)$$



**Fig. 1** Spatial discretization for the one-dimensional Burgers equation

The solution of the nonlinear BVP (3) provides us the numerical flux function  $\mathcal{F}(u_L, u_R, v/\Delta x)$ , which is constant on the interval  $(x_j, x_{j+1})$ . Thus, the numerical flux at the interface of the control volume  $F_{j+1/2} = \mathcal{F}(u_L, u_R, v/\Delta x)$ . Using the normalized coordinate  $\sigma$ ,  $\sigma \in [0, 1]$  and the parameter  $\varepsilon$ , defined by

$$\sigma := \frac{x - x_j}{\Delta x}, \quad \varepsilon := \frac{v}{\Delta x},$$

the BVP (3) can be expressed as

$$\left(\frac{1}{2}u^2 - \varepsilon u\sigma\right)_\sigma = 0, \quad \sigma \in (0, 1), \quad (4a)$$

$$u(0) = u_L, \quad u(1) = u_R. \quad (4b)$$

Further, it can be shown that the above BVP has a monotonic solution.

**Lemma 1.** *The nonlinear local boundary value problem (4) has a strictly monotonic solution.*

*Proof.* Any solution  $u$  of the problem can be represented as

$$u(\sigma) = u_L + (u_R - u_L) \frac{\Lambda(\sigma)}{\Lambda(1)}, \quad u'(\sigma) = (u_R - u_L) \frac{\lambda(\sigma)}{\Lambda(1)}, \quad (\cdot)' := \frac{d}{d\sigma},$$

for  $\sigma \in [0, 1]$ , where the functions  $\lambda, \Lambda : [0, 1] \rightarrow \mathbb{R}$  are given by

$$\lambda(\sigma) := \exp\left(\frac{1}{\varepsilon} \int_0^\sigma u(\eta) d\eta\right) \quad \text{and} \quad \Lambda(\sigma) := \int_0^\sigma \lambda(\xi) d\xi.$$

For  $u_L > u_R$ ,  $u'(\sigma) < 0$  causing  $u(\sigma)$  to be a monotonically decreasing function. Similarly, if  $u_L < u_R$ , then  $u'(\sigma) > 0$ , causing  $u$  to be monotonically increasing for  $\sigma \in [0, 1]$ .  $\square$

### 3 The numerical flux function

We now derive expressions for the numerical flux function using the BVP (4). As a consequence of Lemma 1 we consider the cases:  $u_L > u_R$  and  $u_L < u_R$ .

#### 3.1 The case $u_L > u_R$

The solution of the BVP (4) in this case results in a (strictly) decreasing function, i.e.,  $u_\sigma < 0$ . Using the left boundary condition  $u(0) = u_L$ , we get that the numerical flux at the interface,  $F_{j+1/2}$  is given by

$$F_{j+1/2} = f(0) = \frac{1}{2}u_L^2 - \varepsilon u_\sigma(0). \quad (5)$$

Alternatively, the flux can be determined using the right boundary condition

$$F_{j+1/2} = \frac{1}{2}u_R^2 - \varepsilon u_\sigma(1). \quad (6)$$

Since  $u_\sigma < 0$ , we conclude that  $F_{j+1/2} > 0$ , therefore there exists a  $c \in \mathbb{R}$ , such that

$$F_{j+1/2} = \frac{1}{2}u^2 - \varepsilon u_\sigma = \frac{1}{2}u_L^2 - \varepsilon u_\sigma(0) = \frac{1}{2}u_R^2 - \varepsilon u_\sigma(1) = \frac{1}{2}c^2, \quad (7)$$

with  $|c| \geq \max(|u_L|, |u_R|)$ . The above relation gives us a first-order differential equation

$$\frac{du}{d\sigma} = \frac{1}{2\varepsilon}(u^2 - c^2), \quad (8)$$

which needs to satisfy both  $u(0) = u_L$  and  $u(1) = u_R$ . Integrating the differential equation and equating the left boundary solution with the right boundary solution results in the nonlinear equation for the unknown  $c$  with parameter  $u_L, u_R$  and  $\varepsilon$

$$H^+(c) := \log \left| \frac{(u_L + c)(u_R - c)}{(u_L - c)(u_R + c)} \right| - \frac{c}{\varepsilon} = 0. \quad (9)$$

Thus,  $F_{j+1/2}$  is given by the non-trivial roots of the function  $H^+(c)$ , which is an odd function. We restrict ourselves to  $c > 0$ . Note that the nonlinear equation (9) can also be expressed as

$$e^{-c/2\varepsilon} |(u_L + c)(u_R - c)| - e^{c/2\varepsilon} |(u_L - c)(u_R + c)| = 0. \quad (10)$$

Let  $s = (u_L + u_R)/2$ , then for  $s \geq 0$ , we get that  $u_L \geq |u_R|$  and the non-trivial solution of equation (10) satisfies  $c \geq u_L \geq |u_R|$ . In the inviscid limit  $\varepsilon \rightarrow 0$ , for  $s \geq 0$  equation (10) reduces to

$$e^{c/2\varepsilon}(c - u_L)(c + u_R) = 0 \Rightarrow c = u_L.$$

Similarly for  $s < 0$ , we have  $u_R < 0$ , implying  $c \geq -u_R \geq |u_L|$  and the limit case solution is then given by  $c = -u_R (> 0)$ . Thus, the numerical flux in the inviscid limit is given by

$$F_{j+1/2} = \begin{cases} \frac{1}{2}u_L^2, & \text{if } s \geq 0, \\ \frac{1}{2}u_R^2, & \text{if } s < 0, \end{cases} \quad (11)$$

which is actually the Godunov flux for the inviscid Burgers equation. Moreover, if  $u_L = u_R = u$ , then  $u_\sigma = 0$  and the numerical flux is given by  $F_{j+1/2} = \mathcal{F}(u, u) = \frac{1}{2}u^2 = f(u)$ , for constant  $u$ . Hence the numerical flux function  $\mathcal{F}$  is *consistent* with the continuous flux function  $f$ .

### 3.2 The case $u_L < u_R$

From Lemma 1 we conclude that  $u_\sigma > 0$  for  $u_L < u_R$ . Thus  $F_{j+1/2} = u^2/2 - \varepsilon u_\sigma$  is positive if  $\varepsilon u_\sigma < u^2/2$  and negative if  $\varepsilon u_\sigma > u^2/2$ . Therefore, we split the derivation of the numerical flux into two cases, depending on the sign of the flux.

**Case 1: Positive flux**

If the flux is positive, then the numerical flux is evaluated as for the case  $u_L > u_R$  and is given by roots of the function  $H^+(c)$ , defined in equation (9), with  $c \in (0, M)$ ,  $M := \min(|u_L|, |u_R|)$ .

**Case 2: Negative flux**

If the flux is negative, then there exists a  $c \in \mathbb{R}$ , such that

$$F_{j+1/2} = \frac{1}{2}u^2 - \varepsilon u_\sigma = -\frac{1}{2}c^2.$$

This relation gives rise to the first-order differential equation

$$\frac{du}{d\sigma} = \frac{1}{2\varepsilon}(u^2 + c^2), \quad \sigma \in (0, 1), \quad (12)$$

with the boundary conditions (4b). Integrating the first-order differential equation and equating the left boundary solution with the right boundary gives us another nonlinear equation for  $c$ , i.e.,

$$H^-(c) := \arctan\left(\frac{u_R}{c}\right) - \arctan\left(\frac{u_L}{c}\right) - \frac{c}{2\varepsilon} = 0. \quad (13)$$

As before, the numerical value of  $F_{j+1/2} = -c^2/2$  is given by the non-trivial roots of the function  $H^-(c)$ .

We now formulate the conditions for which  $H^+(c)$  and  $H^-(c)$  have non-trivial roots.

**Lemma 2.** *For  $0 < u_L < u_R$ , if the inequality*

$$\frac{1}{u_L} - \frac{1}{u_R} > \frac{1}{2\varepsilon}, \quad (14)$$

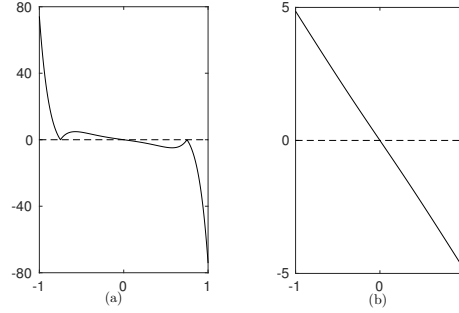
*holds then  $H^-(c)$  has a non-trivial solution, otherwise  $H^+(c)$  has a non-trivial solution.*

*Proof.* Let  $\alpha_-(c) := \arctan(u_R/c) - \arctan(u_L/c)$  and  $\beta_-(c) := c/2\varepsilon$ , such that  $H^-(c) := \alpha_-(c) - \beta_-(c)$ . Using  $\arctan(1/z) = -\arctan(z) + \pi \operatorname{sgn}(z)/2$ ,  $\alpha_-(c)$  can be expressed as

$$\alpha_-(c) = \arctan\left(\frac{c}{u_L}\right) - \arctan\left(\frac{c}{u_R}\right) + \frac{\pi}{2}(\operatorname{sgn}(u_R) - \operatorname{sgn}(u_L)).$$

For the following we consider the case  $u_L u_R > 0$ . Using the fact that  $\alpha_-(c)$  is an odd function we restrict ourselves to the case  $c > 0$ . For  $0 < u_L < u_R$ ,  $\alpha_-(c)$  has a maximum at  $c = \sqrt{u_L u_R} (< u_R)$ . Clearly  $H^-(c)$  has a non-trivial root whenever  $\alpha_-(c) = \beta_-(c)$ , i.e., the two functions intersect for  $c > 0$ , which is possible only if  $\alpha'_-(0) > \beta'_-(0)$ , or,





**Fig. 2**  $H^+(c)$  (a) and  $H^-(c)$  (b), for  $u_L = 0.75$ ,  $u_R = 1.0$  and  $\varepsilon = 0.1$ .

$$\alpha'_-(0) = \frac{1}{u_L} - \frac{1}{u_R} > \beta'_-(0) = \frac{1}{2\varepsilon}, \quad (\cdot)' = \frac{d}{dc}.$$

Thus, if the above condition holds then  $H^-(c)$  has a non-trivial solution, which satisfies  $\sqrt{u_L u_R} < c < u_R$ .

Next, we investigate the condition under which  $H^+(c)$  has a non-trivial root. Let

$$z(c) := \frac{(u_R - u_L)c}{u_L u_R - c^2},$$

such that  $z \in [0, 1]$  with  $z(0) = 0$ ,  $z(u_L) = 1$ . Clearly, for  $c \in (0, u_L)$  we have  $z(c) > 0$ . Using  $z(c)$  we can rewrite equation (9) as

$$H^+(c) = \log\left(\frac{1+z(c)}{1-z(c)}\right) - \frac{c}{\varepsilon} = 2 \operatorname{Artanh}(z(c)) - \frac{c}{\varepsilon}, \quad c \in (0, u_L).$$

Further, let  $\alpha_+(c) := 2 \operatorname{Artanh}(z(c))$  and  $\beta_+(c) := c/\varepsilon$ , such that  $H^+(c) = \alpha_+(c) - \beta_+(c)$ . For  $c \in (0, u_L)$ ,  $\alpha_+(c)$  is an increasing function, thus  $H^+(c)$  has a non-trivial root only if  $\alpha'_+(0) < \beta'_+(0)$ . The derivative  $\alpha'_+(c)$  is given by

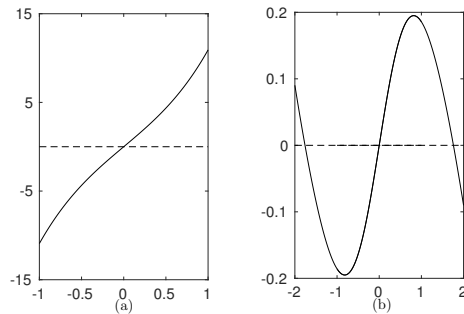
$$\alpha'_+(c) = 2(u_R - u_L) \frac{1}{1-z^2(c)} \frac{u_L u_R + c^2}{(u_L u_R - c^2)^2}.$$

The condition  $\alpha'_+(0) < \beta'_+(0)$  translates to

$$\frac{1}{u_L} - \frac{1}{u_R} < \frac{1}{2\varepsilon}. \quad (15)$$

□

Figure 2 shows the plots of the functions  $H^+(c)$  and  $H^-(c)$ , for  $u_L = 0.75$ ,  $u_R = 1$  and  $\varepsilon = 0.1$  satisfying (15). Hence,  $H^-(c)$  does not have a non-trivial root, unlike  $H^+(c)$  which has a non-trivial root at  $c = 0.749 (\approx u_R)$ . In Figure 3, for  $u_L = 1$ ,



**Fig. 3**  $H^+(c)$  (a) and  $H^-(c)$  (b), for  $u_L = 1$ ,  $u_R = 10$  and  $\varepsilon = 1$ .

$u_R = 10$  and  $\varepsilon = 1$ , condition (14) is satisfied. Thus,  $H^+(c)$  does not have a non-trivial root, whereas  $H^-(c)$  has a non-trivial root, at  $c = 1.7597$ .

#### 4 Numerical results

We compare the proposed nonlinear local BVP scheme with the upwind scheme and the homogeneous flux scheme described in [4, 5]. In the homogeneous flux scheme, the numerical flux  $F_{j+1/2}^{\text{lin}}$  is derived from a *linearized* homogeneous local two-point BVP and is given by

$$F_{j+1/2}^{\text{lin}} = \varepsilon(B(-P)u_L - B(P)u_R),$$

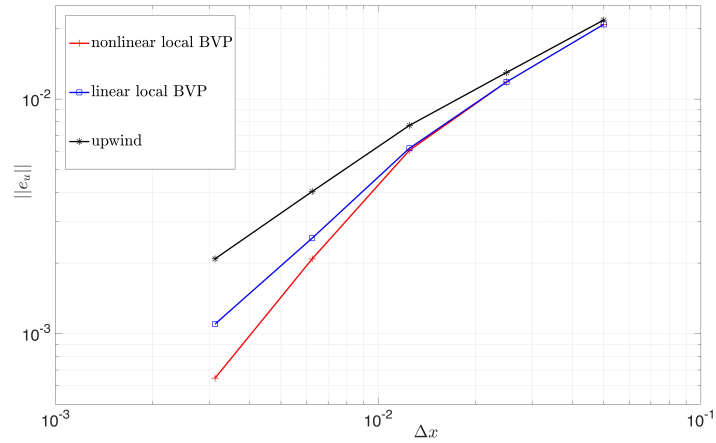
where  $B(z) := z/(e^z - 1)$  is the Bernoulli function and  $P := U_{j+1/2}/2\varepsilon$ , is the grid Péclet number. The interface velocity  $U_{j+1/2} = (u_L + u_R)/2$  is given by the central approximation. The availability of an analytical solution to the viscous Burgers equation defined on  $(0, 1) \times (0, T)$ ,  $T \in (0, 1]$  provides us a reference solution to compare the schemes:

$$u^{\text{ref}}(x, t) = 1 + \frac{1}{2} \tanh\left(\frac{1}{4\nu}\left(x - 0.1 - \frac{1}{2}t\right)\right). \quad (16)$$

We use the explicit fourth-order Runge-Kutta scheme for the temporal discretization with  $\Delta t = 10^{-3}$ . Figure 4 shows the convergence of the error  $e_u := u - u^{\text{ref}}$  for  $\nu = 10^{-3}$  over a family of uniform grids. It can be seen that the nonlinear local BVP scheme is more accurate compared to the upwind and the linearized local BVP scheme. Further, Richardson extrapolation shows that for finer grids the nonlinear local BVP scheme exhibits second-order convergence.

#### 5 Conclusion

In this paper, we have presented a flux approximation scheme for the viscous Burgers equation, in which the numerical flux function is given by the solution of a nonlinear local two-point BVP. The resulting numerical flux is shown to be consis-



**Fig. 4** Convergence of the 1-norm of the error  $e_u$  for  $v = 10^{-3}$  for the proposed nonlinear local BVP scheme, homogeneous linear local BVP scheme and the upwind scheme for a family of grids ( $\Delta x = 0.1 \times 2^{-i}$ ;  $i = 1, 2, 3, 4, 5$ ).

tent with the Godunov method in the inviscid limit and is more accurate than the linearized approximation of the flux.

In the future, we plan to extend the scheme by including source terms and also the time derivative into the local BVP, and by then solving the inhomogeneous BVP, to get the nonlinear complete-flux scheme.

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