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EXISTENCE OF WEAK SOLUTIONS FOR A PSEUDO-PARABOLIC SYSTEM COUPLING CHEMICAL REACTIONS, DIFFUSION AND MOMENTUM EQUATIONS*

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Abstract. We study the weak solvability of a nonlinearly coupled system of parabolic and pseudo-parabolic equations describing the interplay between mechanics, chemical reactions, diffusion and flow in a mixture theory framework. Our approach relies on suitable discrete-in-time energy-like estimates and discrete Gronwall inequalities. In selected parameter regimes, these estimates ensure the convergence of the Rothe method for the discretized partial differential equations.

Key words. System of nonlinear parabolic and pseudo-parabolic equations, reaction-diffusion, weak solutions, existence, Rothe method.

AMS subject classifications. Primary, 35A01, 35K51; Secondary, 35D30, 35K70, 74D05, 74F10, 74F20, 74F25

1. Introduction.

We investigate the existence of weak solutions to a system of partial differential equations coupling chemical reaction, momentum transfer and diffusion, cast in the framework of mixture theory [4]. We use the Rothe method [17, 30] as main tool. For simplicity, we restrict ourselves to a model with a single non-reversible chemical reaction in a one-dimensional bounded spatial domain $[0, 1]$ enclosed by unlimited (or instantly replenished) reservoirs of the reacting chemicals. The chemical reaction is of the N -to-1-type with the reacting chemicals consisting out of solids and a single fluid, while the produced chemical is a solid. New mathematical challenges arise due to the strong nonlinear coupling between all unknowns and their transport fluxes.

Evolution systems, in which chemical reactions, momentum transfer, diffusion and stresses interplay, occur practically in every physical or biological system where there is enough knowledge to describe completely the balances of masses and forces; see e.g. [7, 10, 13, 26]. In all these situations, the interest lies in capturing the effects flows have on deformations, deformations and chemical reactions on structures, and structures on chemical reactions and flow. In biology, such a system is used, for instance, to better understand and eventually forecast the plant growth and development [26]. In structural engineering, one wants to delimit the durability of a concrete sample exposed to ambient corrosion, for example sulfate attack in sewer pipes [13]. Our initial interest in this topic originates from mathematical descriptions of sulfate corrosion [2]. We have realized that the mathematical techniques used for a system describing sulfate attack [when within a porous media (concrete) sulfuric acid reacts with slaked lime to produce gypsum], could be equally well applied to some more general systems sharing similar features (e.g. types of flux couplings and nonlinearities).

At a general level, the system outlined in this paper is a combination of parabolic

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equations of diffusion-drift type with production terms by chemical reactions and pseudo-parabolic stress equations based on viscoelastic terms. On their own, both parabolic equations (cf. e.g. [11, 19, 20]), and pseudo-parabolic equations (see e.g. [3, 12, 14, 28, 29, 31]) are well-understood from mathematical and numerical analysis perspectives. However, coupling these objects leads to systems of equations with a less understood structure. Many systems in the literature seem similar to ours at a first glance. A coupling remotely resembling our case appears in [1], but with different nonlinear terms, others like those in [1, 7] do not have the pseudo-parabolic part, [10, 13] refer to a different domain situation, while in [26] higher-order derivatives are involved.

Due to the strong coupling present in our system, we chose to investigate in this paper the simplest case: a one-dimensional bounded domain, benefiting this way of an easier control of the nonlinearities by relying on the embedding $H^1 \hookrightarrow L^\infty$ within a decoupling strategy of the model equations inspired by the method of Rothe. The study of the multidimensional case will be done elsewhere.

We apply our techniques to a general system introduced in Section 2, which covers e.g. mathematical models describing sulfate attack on concrete. In this section, we also introduce a set of assumptions based on which the existence of weak solutions can be proven. In our setting, the parabolic equations contain only coupling terms consisting of time-derivative terms of the unknowns of the pseudo-parabolic equations, while the pseudo-parabolic equations contain only coupling terms through Lipschitz-like non-linearities coupling back to the parabolic part of the system. In Section 3, we apply a time discretization decoupling the evolution system, inspired by the method of Rothe, such that the Lipschitz functions are evaluated at a different time-slice than the unknowns involved in the pseudo-parabolic part. The decoupled pseudo-parabolic equations can now be solved given the solution of the parabolic system posed at the previous time slice, while the new parabolic part can be solved with the just obtained solution of the pseudo-parabolic equations. The discrete-in-time a priori energy-like estimates are derived in Section 4 by testing the discretized system with suitable functions leading to quadratic terms and then by applying the discrete Gronwall lemma to the resulting quadratic inequalities. Based on these a-priori estimates, we show in Section 5 that our assumptions stated in Section 2 are valid in certain parameter regions. Furthermore, based on our a-priori estimates, we prove in Section 6 that the linear interpolation functions of the solutions to the discrete system converge strongly to a weak solution of the original system.

2. Description of the system.

We define our system on a time-space domain $[t_0, T] \times [0, 1]$, where t_0 is the initial time and T is the final time defined at a later stage by Claim 3. The unknowns of our system are two vector functions, $\phi : ([t_0, T] \times [0, 1])^d \rightarrow \mathbf{R}^d$ and $w : ([t_0, T] \times [0, 1])^{d-1} \rightarrow \mathbf{R}^{d-1}$, and one scalar function $v : [t_0, T] \times [0, 1] \rightarrow \mathbf{R}$. The vector ϕ consists of the volume fractions of the d chemical components active in a chemical reaction mechanism of redox type. The vector w refers to the displacements of the mixture components with respect to a reference coordinate system. The scalar function v denotes the velocity of the single chemical fluid. We identify the different components of the vectors with the different chemicals and use the following convention. The subscript 1 is related to the produced chemical. The subscript d is related to the fluid. All the other subscripts are related to the remaining solid chemicals. The unknowns interplay in the following

system of the evolution equations: For $l \in \{1, \dots, d\}$, $m \in \{1, \dots, d-1\}$, we have

$$(1a) \quad \partial_t \phi_l - \delta_l \partial_z^2 \phi_l + I_l(\phi) \partial_z (\Gamma(\phi)v) + \sum_{m=1}^{d-1} \sum_{i,j=0}^1 \partial_z^i \left(B_{lijm}(\phi) \partial_t^j w_m \right) = G_{\phi,l}(\phi)$$

$$(1b) \quad \partial_z (\Gamma(\phi)v) + \sum_{m=1}^{d-1} \sum_{j=0}^1 \partial_z \left(H_{jlm}(\phi) \partial_t^j w_m \right) = G_v(\phi)$$

$$(1c) \quad \begin{aligned} \partial_t w_m - D_m \partial_z^2 w_m - \gamma_m \partial_z^2 \partial_t w_m + F_m(\phi)v \\ + \sum_{j=1}^{d-1} \sum_{\substack{i+n=0 \\ i,n \geq 0}}^1 \partial_z \left(E_{minj}(\phi) \partial_z^i \partial_t^n w_j \right) = G_{w,m}(\phi) \end{aligned}$$

with functions $I_l, \Gamma, B_{lijm}, H_{jlm}, F_m, E_{minj}, G_{\phi,l}, G_v, G_{w,m} \in W^{1,\infty}((0,1)^d)$ and constants $\delta_l, D_m, \gamma_m \in \mathbf{R}_+$. Furthermore, abuse notation with $\|f(\cdot)\|_{W^{1,\infty}((0,1)^d)} \leq f \in \mathbf{R}_+$. Notice that this system must satisfy the conditions $\sum_{l=1}^d \phi_l = 1$, the fundamental equation of fractions, which allows for the removal of the $l = d-1$ component of Equation (1a).

We assume the volume fractions are insulated at the boundary, thus implying $\partial_z \phi = 0$ at the boundaries $z = 0$ and $z = 1$. The boundary at $z = 0$ is assumed to be fixed, while the boundary at $z = 1$ has a displacement $W(t) = h(t) - 1$, where $h(t)$ is the height of the reaction layer at the present time t . The Rankine-Hugoniot relation, see e.g. [24], states that the velocity of a chemical at a boundary is offset from the velocity, U , of the boundary by influx or outflux of the chemical, i.e.

$$\begin{cases} \phi_m (U - \partial_t w_m) \cdot \hat{n} = \hat{J}_m \mathcal{L}(\phi_{m,res} - \phi_m) \\ \phi_d (U - v) \cdot \hat{n} = \hat{J}_d \mathcal{L}(\phi_{d,res} - \phi_d) \end{cases}$$

In general the function $\mathcal{L}(\cdot)$ denotes the concentration jump across the boundary. However, we assume the boundary to be semi-permeable in such a way that only influx can occur. Hence $\mathcal{L}(f) := f\mathcal{H}(f) = f_+$ denotes the positive part of f . Furthermore, we assume that the fluid reservoir is at the boundary $z = 1$: $\phi_{d,res}$ is positive at $z > 1$, but 0 at $z < 0$. The produced chemical does not have any reservoir at the boundaries. Therefore $\phi_{1,res} = 0$ at both $z < 0$ and $z > 1$. The other chemicals have a reservoir below the $z = 0$ boundary: $\phi_{m,res}$ is positive at $z < 0$ and 0 at $z > 1$ for $1 < m < d$. We generalize the Rankine-Hugoniot relations by replacing ϕ_m with $H_{1m}(\phi)$ and ϕ_d with $\Gamma(\phi)$.

The influx due to the Rankine-Hugoniot relations shows that the displacement $w_m|_{z=1}$ will not be equal to the boundary displacement $W(t)$. This will result in stresses, which we incorporate with a Robin boundary condition at these locations [25, section 5.3]. Collectively for all $t \in [t_0, T]$ these boundary conditions are

$$(2a) \quad \begin{cases} \partial_z \phi_l|_{z=0} = 0, & l \neq d-1 \\ \partial_z \phi_l|_{z=1} = 0, & l \neq d-1 \end{cases}$$

$$(2b) \quad \begin{cases} w_1|_{z=0} = 0 \\ \partial_z w_1|_{z=1} = A_1 (w_1|_{z=1} - W(t)) \\ H_{1m}(\phi) \partial_t w_m|_{z=0} = \hat{J}_m \mathcal{L}(\phi_{m,res} - \phi_m|_{z=0}), & 1 < m < d \\ \partial_z w_m|_{z=1} = A_m (w_m|_{z=1} - W(t)), & 1 < m < d \\ v|_{z=0} = 0 \\ \Gamma(\phi) (\partial_t h(t) - v)|_{z=1} = \hat{J}_d \mathcal{L}(\phi_{d,res} - \phi_d|_{z=1}) \end{cases}$$

It is worth noting that in the limit $|A_m| \rightarrow \infty$ one formally obtains the Dirichlet boundary conditions, which are the natural boundary conditions for this system from a physical perspective.

The initial conditions describe a uniform and stationary equilibrium solution at $t = t_0$:

$$(3) \quad \phi_l(t_0, z) = \phi_{l0} \quad \text{and} \quad w_m(t_0, z) = 0 \quad \text{for all } z \in [0, 1].$$

The collection of Equations (1a)–(1c), (2a), (2b), and (3) forms our continuous system. Notice that this continuous system needs the pseudo-parabolic terms: our existence method (of weak solutions) fails when we choose $\gamma_m = 0$. Moreover, the height function $h(t)$ in Equation (2b) cannot be chosen freely. Integration of Equation (1b) in both space and time together with Equation (2b) will yield an ODE of $h(t)$, and, in special cases, a closed expression of $h(t)$. Furthermore, the initial conditions (3) do not contain a description of $v(t_0, z)$, because this function can be calculated explicitly: Equation (3) together with Equations (1b) and (1c) define a subsystem of the variables $\frac{\partial w_m}{\partial t}(t, z)$, $v(t, z)$ on $(t, z) \in \{t_0\} \times (0, 1)$ with boundary conditions of Equation (2b). This subsystem can be written in the form on to which Theorem 4 can be applied, which shows that there exists a unique solution of this subsystem in $(C^2(0, 1))^d$ iff both $\Gamma(\phi(t_0, z)) \neq 0$ and $H_{1m}(t_0, 0) \neq 0$ are satisfied.

Our existence proof relies on the following set of assumptions:

ASSUMPTION 1. $d \geq 2$, $\hat{J}_d, \hat{J}_m \geq 0$ and $\phi_{d,res}, \phi_{m,res} \in [0, 1]$ for $1 \leq m < d$ and $\sum_{l=1}^d \phi_{l,res} = 1$.

All chemical reactions, where $d - 1$ chemicals react to form 1 chemical, are allowed. The reacting solid chemicals ($1 < m < d$) flow into the domain from a reservoir at $z < 0$, while the reacting liquid chemical flows into the domain from a reservoir at $z > 1$. Moreover, the reservoir volume fractions are in the physical range $[0, 1]$.

ASSUMPTION 2. For all $0 < \alpha < 1/d$ introduce \mathcal{I}_α as $(\alpha, 1 - (d - 1)\alpha)$. Then

$$\Gamma_\alpha := \inf_{\phi \in \mathcal{I}_\alpha^d} \Gamma(\phi) \in \mathbf{R}_+ \quad \text{and} \quad H_\alpha := \min_{1 \leq m < d} \inf_{\phi \in \mathcal{I}_\alpha^d} H_{1m}(\phi) \in \mathbf{R}_+.$$

The velocity $v(t, z)$ is now guaranteed to be bounded if the other velocities $\partial_i w$ are bounded. Moreover, $v(t_0, z)$ is now given by Theorem 4.

Next to these assumptions we have additional, which will be introduced pointwise at the appropriate moment. The pointwise introduced assumptions only list necessary conditions at that moment. If a more stringent condition is needed, then a new assumption will be introduced. For completeness, we list here the reasons for introducing the additional assumptions in their most stringent form.

Assumption 9 is guaranteeing the pseudo-parabolicity. Assumption 17 guarantees that the initial volume fractions are physical and non-zero. Assumption 21 guarantees that a $L^2(t_0, T; H^1(0, 1))$ bound for v^k can be found. Assumption 25 gives the necessary upper bound for the time discretization time step Δt , for which Claim 3 can be proven.

Next to these assumptions, we want our solutions to be physical at almost every time t . So, the volume fractions must lie in $[0, 1]$ and the velocity $v(t)$ must be both essentially bounded and of bounded variation, which implies that both v and $\partial_z v$ must be functions in the Bochner space $L^2(t_0, T; L^2(0, 1))$. However, the volume fractions ϕ cannot become 0 without creating problems for the original Rankine-Hugoniot boundary conditions or allowing singularities in the domain implying $\phi \in ((0, 1))^d$.

To this end, we introduce a new time interval for which all of these constraints hold and we claim that such an interval exists.

CLAIM 3. *There exists a time domain $[t_0, T]$, a velocity $V > 0$ and a volume fraction $\phi_{\min} \in (0, 1/d)$ such that*

$$T = \inf \left\{ t \left| \begin{array}{l} \int_{t_0}^t \int_0^1 |v(s, z)|^2 dz ds > V^2, \\ \int_{t_0}^t \int_0^1 |\partial_z v(s, z)|^2 dz ds > V^2, \\ \inf_{z \in (0,1)} \left\{ \min_{1 \leq l \leq d} \phi_l(t, z) \right\} < \phi_{\min} \end{array} \right. \right\},$$

which, in line with [Assumption 1](#), implies

$$\phi_l(t, z) \in \overline{\mathcal{I}_{\phi_{\min}}} = [\phi_{\min}, 1 - (d-1)\phi_{\min}] \subset (0, 1)$$

for all $(t, z) \in [t_0, T] \times [0, 1]$ and $1 \leq l \leq d$.

This claim guarantees that all chemicals are omnipresent, while velocities and deformations remain bounded. This claim can be related to the theories on parabolic and pseudo-parabolic equations. The claim, which will be proven in [Section 5](#), mimics the necessary $L^2(0, 1)$ and $H^1(0, 1)$ regularity of the coefficients in the parabolic and pseudo-parabolic equation theory. Combining the claim with [Assumption 2](#) directly introduces the constants $\Gamma_{\phi_{\min}}$ and $H_{\phi_{\min}}$ respectively as lower bounds of $\Gamma(\phi)$ and all $H_{1m}(\phi)$ for $\phi \in \overline{\mathcal{I}_{\phi_{\min}}}^d$.

In this paper we shall prove the existence of a volume fraction $\phi_{\min} \in (0, 1/d)$, a velocity $V > 0$ and a nonempty time interval $[t_0, T]$ such that a weak solution of the continuous system exists if [Assumptions 1, 2, 9, 17, 21](#) and [25](#) are satisfied. We will prove this statement in 4 steps:

- First, we discretise the continuous system in time with a regular grid of step size Δt , apply a specific Euler scheme and prove that this new discretised system can be solved iteratively in the classical sense at each time slice.
- Second, we make a weak version of the discretised system and prove that there exists a weak solution of the continuous system. This will be done by choosing specific test functions such that we obtain quadratic inequalities. By application of Young's inequality and using the Gronwall lemmas we obtain the energy-like estimates called the a priori estimates, which are step size Δt independent upper bounds of the Sobolev norms of the weak solutions.
- Third, we prove [Claim 3](#) by showing that the upper bounds of the a priori estimates are increasing functions of $T - t_0$ and V that have to satisfy specific upper bounds in order to guarantee the validity of [Claim 3](#). Then, in certain parameter regions, regions in $(T - t_0, V)$ -space exist for which [Claim 3](#) holds.
- Fourth, we will introduce temporal interpolation functions $\bar{u}(t) = u^k$ and $\hat{u}(t) = u^{k-1} + (t - t_{k-1})\mathcal{D}_{\Delta t}^k(u)$ to justify the existence of functions on $[t_0, T] \times [0, 1]$ and we will show that the weak limit for $\Delta t \downarrow 0$ exists and is a weak solution of the continuous system.

3. The discretised system and its classical solution.

We discretise time with regular temporal grid $t_k = t_0 + k\Delta t$ for $\Delta t > 0$. This discretization is applied to [Claim 3](#) with the infimum replaced with a minimum, the time t replaced with the discrete time t_k and the time integrals replaced with Riemann sums. Consequently, the time T is now dependent on the discretization. To highlight

this fact we introduce the notation $T_{\Delta t}$ for the time T of [Claim 3](#) with regular temporal grid $t_k = t_0 + k\Delta t$.

We discretise the continuous system in such a way that the equations become linear elliptic equations with respect to evaluation at time t_k , and only contain two time evaluations: at t_k and t_{k-1} . The time derivative $\partial_t u$ is replaced with the standard first order finite difference $\mathcal{D}_{\Delta t}^k(u) := \frac{u^k - u^{k-1}}{\Delta t}$, where $u^k(z) := u(t_k, z)$. The discretised system takes the form

$$(4a) \quad \mathcal{D}_{\Delta t}^k(\phi_l) - \delta_l \partial_z^2 \phi_l^k + I_l(\phi^{k-1}) \partial_z (\Gamma(\phi^{k-1}) v^{k-1}) \\ + \sum_{m=1}^{d-1} \sum_{i=0}^1 \partial_z^i (B_{li0m}(\phi^{k-1}) w_m^{k-1} + B_{li1m}(\phi^{k-1}) \mathcal{D}_{\Delta t}^k(w_m)) = G_{\phi,l}(\phi^{k-1})$$

$$(4b) \quad \sum_{m=1}^{d-1} \partial_z (H_{0m}(\phi^{k-1}) w_m^{k-1} + H_{1m}(\phi^{k-1}) \mathcal{D}_{\Delta t}^k(w_m)) \\ + \partial_z (\Gamma(\phi^{k-1}) v^k) = G_v(\phi^{k-1})$$

$$(4c) \quad \mathcal{D}_{\Delta t}^k(w_m) - D_m \partial_z^2 w_m^k - \gamma_m \partial_z^2 \mathcal{D}_{\Delta t}^k(w_m) + F_m(\phi^{k-1}) v^{k-1} \\ + \sum_{j=1}^{d-1} \sum_{i=0}^1 \partial_z (E_{mi0j}(\phi^{k-1}) \partial_z^i w_j^{k-1} + E_{m01j}(\phi^{k-1}) \mathcal{D}_{\Delta t}^k(w_j)) = G_{w,m}(\phi^{k-1})$$

with boundary conditions

$$(5a) \quad \begin{cases} \partial_z \phi_l^k \Big|_{z=0} = 0, & l \neq d-1 \\ \partial_z \phi_l^k \Big|_{z=1} = 0, & l \neq d-1 \end{cases}$$

$$(5b) \quad \begin{cases} H_{1m}(\phi^{k-1}|_{z=0}) \mathcal{D}_{\Delta t}^k(w_m) \Big|_{z=0} = \hat{J}_m \mathcal{L}(\phi_{m,res} - \phi_m^{k-1}|_{z=0}) \\ \partial_z w_m^k \Big|_{z=1} = A_m(w_m^k|_{z=1} - \mathcal{W}^k) \\ v^k \Big|_{z=0} = 0 \\ \Gamma(\phi^{k-1}|_{z=1}) (\mathcal{D}_{\Delta t}^k(\mathcal{W}) - v^{k-1}) \Big|_{z=1} = \hat{J}_d \mathcal{L}(\phi_{d,res} - \phi_d^{k-1}|_{z=1}) \end{cases},$$

where $\mathcal{W}^k := W(t_k)$, and $\hat{J}_1 = 0$ in accordance with the first boundary condition of [\(2b\)](#), and initial conditions [\(3\)](#).

A powerful property of this discretised system is its sequential solvability at time t_k : the existence of a natural hierarchy in attacking this problem. First we obtain results for [Equation \(4c\)](#), then we use these results to obtain similar results for both [Equations \(4a\) and \(4b\)](#). Moreover, the structure of the discretised system is that of an elliptic system. Hence, the general existence theory for elliptic systems can be extended directly to cover our situation:

THEOREM 4. *Let $\Omega \subset \mathbf{R}$ bounded, $A_{\pm}, B_{\pm}, C_{\pm} \in M(\mathbf{R}^n, \mathbf{R}^n)$, D a diagonal positive definite matrix of size $n \times n$, and $E_{ij}, f_i \in (L^2(\Omega))^n$ for all $i, j \in \{1, \dots, n\}$. There exists a unique solution $u \in (C^2(\Omega))^n$ of the system*

$$\begin{cases} \partial_z^2 u - \partial_z(E(z)u) - Du = f & \text{on } (x_-, x_+) =: \Omega \\ A_{\pm} u(x_{\pm}) + B_{\pm} u'(x_{\pm}) = C_{\pm}. \end{cases}$$

Moreover, if the conditions

$$\left| \begin{array}{cc} A_+ + B_+ E(x_+) & B_+ \\ A_- + B_- E(x_-) & B_- \end{array} \right| \neq 0 \quad \text{and} \quad \min_{z \in \Omega} |\text{Tr}(E(z))| > 0$$

are satisfied, then the solution u is given by

$$\begin{pmatrix} u(z) \\ U(z) \end{pmatrix} = \Psi(z) \int_{x_-}^z \Psi(s) \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds + \begin{pmatrix} A_+ + B_+ E(x_+) & B_+ \\ A_- + B_- E(x_-) & B_- \end{pmatrix}^{-1} \begin{pmatrix} C_+ \\ C_- \end{pmatrix} \\ + \begin{pmatrix} A_+ + B_+ E(x_+) & B_+ \\ 0 & 0 \end{pmatrix} \Psi(x_+) \int_{x_-}^{x_+} \Psi(s) \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds$$

with

$$U(z) = \partial_z u - E(z)u,$$

and with $\Psi(z) \in M(\mathbf{R}^n, \mathbf{R}^n)$ for all $z \in \Omega$ satisfying

$$\begin{cases} \partial_z \Psi(z) = \begin{pmatrix} E(z) & \mathbb{I}_n \\ D^{-1} & 0 \end{pmatrix} \Psi(z) & \text{for } z \in \Omega \\ \Psi(x_-) = \mathbb{I}_n \end{cases}$$

Proof. See [27, p.130] for the general calculus result or see Chapter 6 of [11] for the elliptic theory result. In specific: rewrite system in terms of u and U . This system has a fundamental matrix $\Psi(z)$ yielding the above solution after satisfying boundary conditions. \square

COROLLARY 5. *Let $\Delta t > 0$. For all $t_0 < t_k$ such that $\phi^{k-1} \in (0, 1)^d$ there exists a unique solution $u^k := (\phi^k, w^k, v^k) \in (C^2(0, 1))^{2d-1} \times C^1(0, 1)$ of the system (4a)–(4c) with boundary conditions (5a) and (5b) and initial conditions (3).*

Proof. The result follows with induction with respect to $k \geq 0$ from applying **Theorem 4** and using both $I_l, \Gamma, B_{lijm}, H_{jm}, F_m, E_{minj}, G_{\phi,l}, G_v, G_{w,m} \in W^{1,\infty}((0, 1)^d)$ and $\delta_l, D_m, \gamma_m \in \mathbf{R}_+$. \square

This result shows that there exists a solution of the discrete system even if the solution does not satisfy **Assumptions 1, 2, 9, 17, 21** and **25** and **Claim 3**. The solution might therefore be non-physical. Furthermore these solutions might not have a weakly convergent limit as $\Delta t \downarrow 0$. We will use a weak solution framework to show the existence of physical solutions for which the weak convergence as $\Delta t \downarrow 0$ does exist.

4. A priori estimates.

The estimates in this section rely on the validity of **Claim 3**. This validity will be proven in **Section 5**. Notice that from this moment onwards the notation $l \neq d - 1$ denotes $l \in \{1, \dots, d - 2, d\}$.

We create a weak form of the discretised system by multiplying the equations with a function in $H^1(0, 1)$, integrating over $(0, 1)$ and applying the boundary conditions where needed. To this end we test **Equation (4a)** with ϕ_l^k and $\mathcal{D}_{\Delta t}^k(\phi_l)$, and **Equation (4c)** with w_m^k and $\mathcal{D}_{\Delta t}^k(w_m)$. In **Appendix A** these tests are evaluated in detail

and it is shown there how we obtain the following quadratic inequalities:

$$\begin{aligned}
(6) \quad & \frac{1}{2} \mathcal{D}_{\Delta t}^k \left(\sum_{m=1}^{d-1} \|w_m\|_{L^2}^2 + (\gamma_m + D_m) \|\partial_z w_m\|_{L^2}^2 \right) \\
& + \sum_{m=1}^{d-1} \left[\left(1 + \frac{\Delta t}{2}\right) \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}^2 + \left[\gamma_m \left(1 + \frac{\Delta t}{2}\right) + D_m \frac{\Delta t}{2} \right] \|\mathcal{D}_{\Delta t}^k(\partial_z w_m)\|_{L^2}^2 \right] \\
& \leq K_{w0} + \sum_{m=1}^{d-1} \left[K_{w1m} \|w_m^k\|_{L^2}^2 + K_{w2m} \|\partial_z w_m^k\|_{L^2}^2 + K_{w3m} \|w_m^{k-1}\|_{L^2}^2 + K_{w4m} \|\partial_z w_m^{k-1}\|_{L^2}^2 \right. \\
& \quad \left. + K_{w5m} \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}^2 + K_{w6m} \|\mathcal{D}_{\Delta t}^k(\partial_z w_m)\|_{L^2}^2 \right] + K_{w7} \|v^{k-1}\|_{L^2}^2 + K_{w8} \|\partial_z v^{k-1}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
(7) \quad & \mathcal{D}_{\Delta t}^k (\|\phi_l\|_{L^2}^2) + 2\delta_l \|\partial_z \phi_l^k\|_{L^2}^2 + \Delta t \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 \\
& \leq K_{a\phi 0} + K_{a\phi 1} \|\partial_z v^{k-1}\|_{L^2}^2 + K_{a\phi 2l} \|\phi_l^k\|_{L^2}^2 + \sum_{n \neq d-1} [K_{a\phi 3ln} \|\partial_z \phi_n^{k-1}\|_{L^2}^2] \\
& \quad + \sum_{m=1}^{d-1} \sum_{i=0}^1 \left[K_{a\phi(4+i)m} \|\partial_z^i w_m^{k-1}\|_{L^2}^2 + K_{a\phi(6+i)m} \|\mathcal{D}_{\Delta t}^k(\partial_z^i w_m)\|_{L^2}^2 \right],
\end{aligned}$$

and

$$\begin{aligned}
(8) \quad & \mathcal{D}_{\Delta t}^k \left(\sum_{l \neq d-1} \|\partial_z \phi_l\|_{L^2}^2 \right) + \sum_{l \neq d-1} \left[\frac{2}{\delta_l} \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 + \Delta t \|\mathcal{D}_{\Delta t}^k(\partial_z \phi_l)\|_{L^2}^2 \right] \\
& \leq K_{b\phi 0} + K_{b\phi 1} \|\partial_z v^{k-1}\|_{L^2}^2 + \sum_{l \neq d-1} \left[K_{b\phi 2l} \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 + K_{b\phi 3l}^{k-1} \|\partial_z \phi_l^{k-1}\|_{L^2}^2 \right] \\
& \quad + \sum_{m=1}^{d-1} \sum_{i=0}^1 \left[K_{b\phi(4+i)m} \|\partial_z^i w_m^{k-1}\|_{L^2}^2 + K_{b\phi(6+i)m} \|\mathcal{D}_{\Delta t}^k(\partial_z^i w_m)\|_{L^2}^2 \right].
\end{aligned}$$

The constants K_{index} for $x \in \{a, b\}$ can be found in [Appendix A](#) as Equations (48a)–(48i) and Equations (49a)–(49i).

We are now able to apply two versions of the Discrete Gronwall lemma. The 1st version (of [Lemma 6](#)) will be applied to Equations (6) and (7), while the 2nd version (of [Lemma 7](#)) will be applied to [Equation \(8\)](#).

LEMMA 6 (1st Discrete Gronwall lemma). *Suppose $h \in (0, H)$. Let (x^k) , (y^{k+1}) and (z^k) for $k = 0, 1, \dots$ be sequences in \mathbf{R}_+ satisfying*

$$(9) \quad y^k + \frac{x^k - x^{k-1}}{h} \leq A + z^{k-1} + Bx^k + Cx^{k-1} \quad \text{and} \quad \sum_{j=0}^{k-1} z^j h \leq Z$$

for all $k = 1, \dots$ with constants A, B, C and Z independent of h satisfying

$$A > 0, \quad Z > 0, \quad B + C > 0, \quad \text{and} \quad BH \leq 0.6838,$$

then

$$(10a) \quad x^k \leq \left(x^0 + Z + A \frac{C + 1.6838B}{C + B} kh \right) e^{(C+1.6838B)kh} \quad \text{and}$$

$$(10b) \quad \sum_{j=1}^k y^j h \leq (x^0 + Z + Ahk) e^{(C+1.6838B)kh}.$$

Proof. We rewrite Equation (9) such that x^k is on the left-hand side and x^{k-1} is on the right-hand side. We can discard the y^k term since it is always positive. The partial sum of the geometric series yields

$$x^k \leq \left(x^0 + \frac{Z}{1 + Ch} + \frac{A}{B + C} \right) \left(\frac{1 + Ch}{1 - Bh} \right)^k - \frac{A}{B + C}$$

from which we obtain Equation (10a) by applying the inequalities

$$(11) \quad 1 + a \leq e^a \leq 1 + ae^a \text{ for } a \geq 0 \quad \text{and} \quad \frac{1}{1 - a} \leq e^{a+a^2} \text{ for } 0 \leq a \leq 0.6838.$$

With $Bh - 1 \leq BH - 1 < 0$ we rewrite Equation (9) into

$$\sum_{j=1}^k y^j h \leq Ahk + Z + (1 + Ch)x^0 + (C + B)h \sum_{j=1}^{k-1} x^j.$$

We insert Equation (10a) for x^j and use $j < k$ for the factor in brackets. Then the sum over exponentials can be seen as a partial sum of a geometric series, yielding

$$\sum_{j=1}^k y^j h \leq x^0 + Z + Ahk + (C + B)h \left(x^0 + Z + A \frac{C + 1.6838B}{C + B} kh \right) \frac{e^{(C+1.6838B)kh} - 1}{e^{(C+1.6838B)h} - 1}.$$

With $1/(e^a - 1) \leq 1/a$ for $a \geq 0$ one immediately obtains Equation (10b). \square

LEMMA 7 (2nd Discrete Gronwall lemma). *Let $c > 0$ and $(y_k), (g_k)$ be positive sequences satisfying*

$$(12) \quad y_k \leq c + \sum_{0 \leq j < k} g_j y_j \quad \text{for } k \geq 0,$$

then

$$(13) \quad y_k \leq c \exp \left(\sum_{0 \leq j < k} g_j \right) \quad \text{for } k \geq 0.$$

Proof. The proof can be found in [15]. \square

We introduce a set of new constants, which aid us in the application of the Gronwall lemmas:

DEFINITION 8.

$$\begin{aligned}
A &:= \max_{t_0 \leq t_k \leq T_{\Delta t}} K_{w0}^k, \\
\frac{B}{2} &:= \max_m \left\{ K_{w1m}, \frac{K_{w2m}}{\gamma_m + D_m} \right\}, \\
\frac{C}{2} &:= \max_m \left\{ K_{w3m}, \frac{K_{w4m}}{\gamma_m + D_m} \right\}, \\
\tilde{C}^2 &:= \frac{\left((K_{w7} + K_{w8})V^2 + A \frac{C+1.6838B}{C+B} (T_{\Delta t} - t_0) \right) e^{(C+1.6838B)(T_{\Delta t} - t_0)}}{\min_m \{1 - K_{w5m}, \gamma_m - K_{w6m}\}}, \\
g_{bk} &:= \max_{l \neq d-1} K_{b\phi 3l}^k \Delta t, \\
K_{\phi 3l4} &:= \sum_{m=1}^{d-1} \frac{K_{\phi 3l2m} + K_{\phi 3l3m}}{\min\{1 - K_{w5m}, \gamma_m - K_{w6m}\}}, \\
c_b &:= K_{b\phi 0}(T_{\Delta t} - t_0) + K_{b\phi 1}V^2 + \max_m \left\{ K_{b\phi 4m}, \frac{K_{b\phi 5m}}{\gamma_m + D_m} \right\} \tilde{C}^2 (T_{\Delta t} - t_0) \\
&\quad + \max_m \left\{ \frac{K_{b\phi 6m}}{1 - K_{w5m}}, \frac{K_{b\phi 7m}}{\gamma_m - K_{w6m}} \right\} \tilde{C}^2, \\
D_b &:= \max_{l \neq d-1} \left\{ 2K_{\phi 3l1}V^2 + K_{\phi 3l4}\tilde{C}^2 \right\}.
\end{aligned}$$

The constants K_{index} not introduced here can be found in Equations (48a)–(48i) and Equations (49a)–(49i) of [Appendix A](#). Moreover, introduce the constants

$$\begin{aligned}
A_a &:= K_{a\phi 0} + c_b \exp(D_b) \max_{l,j \neq d-1} K_{a\phi 3lj} + \max_m \left\{ K_{a\phi 4m}, \frac{K_{a\phi 5m}}{\gamma_m + D_m} \right\} \tilde{C}^2, \\
Z_a &:= K_{a\phi 1}V^2 + \max_m \left\{ \frac{K_{a\phi 6m}}{1 - K_{w5m}}, \frac{K_{a\phi 7m}}{\gamma_m - K_{w6m}} \right\} \tilde{C}^2, \\
D_a &:= \max_{l \neq d-1} K_{a\phi 2l}.
\end{aligned}$$

With these new notations, we obtain several simple expressions for the upper bounds by applying the Gronwall inequalities. However these expressions are only valid if the following assumptions are met.

ASSUMPTION 9. Let B and C be given by [Definition 8](#) and \mathcal{M}_{m01j} as introduced in [Equation \(43b\)](#). Assume $B + C > 0$ and

$$\begin{aligned}
\mathcal{M}_{m11j} + \mathcal{M}_{j11m} + \mathcal{M}_{m01j}\sqrt{\gamma_m} = \\
\gamma_m |A_m| + \gamma_j |A_j| + E_{m01j}(1 + \sqrt{\gamma_m}) + E_{j01m} < \frac{2\gamma_m}{d-1}.
\end{aligned}$$

The second condition guarantees the pseudo-parabolicity, because it guarantees the conditions $1 > K_{w5m}$ and $\gamma_m > K_{w6m}$ in [Corollary 11](#).

ASSUMPTION 10. Let B be given by [Definition 8](#). Assume $H = 0.6838/B$.

With these two assumptions the Gronwall inequalities imply the following upper bounds:

COROLLARY 11. Let $\Delta t \in (0, H)$. Introduce the sequences

$$\begin{aligned} x^k &= \sum_{m=1}^{d-1} [\|w_m^k\|_{L^2}^2 + (\gamma_m + D_m) \|\partial_z w_m^k\|_{L^2}^2] \\ y^k &= \sum_{m=1}^{d-1} \left[(1 - K_{w5m}) \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}^2 + (\gamma_m - K_{w6m}) \|\mathcal{D}_{\Delta t}^k(\partial_z w_m)\|_{L^2}^2 \right] \\ z^k &= K_{w7} \|v^k\|_{L^2}^2 + K_{w8} \|\partial_z v^k\|_{L^2}^2. \end{aligned}$$

The inequality

$$x^k \leq \left((K_{w7} + K_{w8})V^2 + A \frac{C + 1.6838B}{C + B} (t_k - t_0) \right) e^{(C+1.6838B)(t_k - t_0)}$$

is valid for all $t_k \in [t_0, T_{\Delta t}]$. Furthermore, $1 > K_{w5m}$ and $\gamma_m > K_{w6m}$ and the inequality

$$\sum_{j=1}^k y^j \Delta t \leq ((K_{w7} + K_{w8})V^2 + A(t_k - t_0)) e^{(C+1.6838B)(t_k - t_0)}$$

is valid for all $t_k \in [t_0, T_{\Delta t}]$.

Proof. We first need to show that we can choose the η -indices such that $1 > K_{w5m}$ and $\gamma_m > K_{w6m}$, as otherwise we have insufficient information to bound all terms. There are $d - 1$ couplings between K_{w5m} and K_{w6m} through $\eta_{\mathcal{M}m01j1}$. These couplings only give $\mathcal{M}_{m01j} < 2\sqrt{\gamma_m}/(d - 1)$ and $\eta_{\mathcal{M}m01j1} = 1/\sqrt{\gamma_m}$ as conditions. However K_{w6m} is coupled to itself as well through $\eta_{\mathcal{M}m11j1}$. With $\eta_{\mathcal{M}m11j1} = 1$ it is immediately clear that the constraint of [Assumption 9](#) allows one to choose the remaining η -indices in K_{w5m} and K_{w6m} freely to satisfy the inequalities. Hence, $1 > K_{w5m}$ and $\gamma_m > K_{w6m}$ is guaranteed by [Assumption 9](#).

Next we apply [Lemma 6](#) to [Equation \(6\)](#) in the form of [Equation \(9\)](#) with x^k , y^k and z^k as above. With use of the identity $k\Delta t = t_k - t_0$ and with the initial condition $w_m^0(z) = 0$ for $z \in (0, 1)$, yielding $x^0(z) = 0$, the two inequalities of [Corollary 11](#) directly follow from [Lemma 6](#). \square

For the next result, we choose $\eta_{index} > 0$ such that $2 > K_{b\phi 2l}\delta_l$ holds.

COROLLARY 12. Let $t_0 < t_k \leq T_{\Delta t}$. Introduce the sequence

$$\begin{aligned} x_b^k &= \sum_{l \neq d-1} \|\partial_z \phi_l^k\|_{L^2}^2 \\ y_b^k &= 2 \sum_{l \neq d-1} \left[\left(\frac{2}{\delta_l} - K_{b\phi 2l} \right) \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 \right]. \end{aligned}$$

The inequalities

$$\begin{aligned} x_b^k &\leq c_b \exp(D_b) \\ \sum_{j=1}^k y_b^j \Delta t &\leq c_b (1 + D_b \exp(D_b)) \end{aligned}$$

are valid for all $t_k \in (t_0, T]$.

Proof. We can always choose $\eta_{index} > 0$ in $K_{\phi 2l}$ such that $2 > \delta_l K_{\phi 2l}$ holds. With the newly introduced notations c_b , D_b and g_{bn} , and with the initial condition $\partial_z \phi^0(z) = 0$ for $z \in (0, 1)$, yielding $x_b^0 = 0$, we can sum inequality (8) and rewrite it into the inequality

$$x_b^k \leq x_b^k + \sum_{j=0}^{k-1} y_b^j \Delta t \leq c_b + x_b^0 + \sum_{j=0}^{k-1} g_{bj} x_b^j = c_b + \sum_{j=0}^{k-1} g_{bj} x_b^j.$$

With $\sum_{j=0}^{k-1} g_{bj} \leq D_b$, direct application of Lemma 7 yields the bound of x_b^k . Substituting this upper bound for x_b^k will directly yield the upper bound of $\sum_{j=1}^k y_b^j \Delta t$. \square

We need an extra upper bound on H to guarantee the successful application of the Discrete Gronwall lemma for ϕ_l^k .

ASSUMPTION 13. *Let B and D_a be given by Definition 8. Assume*

$$H = \min \left\{ \frac{0.6838}{B}, \frac{0.6838}{D_a} \right\}.$$

Now an upper bound for $\|\phi_l^k\|_{L^2}$ can be determined.

COROLLARY 14. *For $\Delta t \in (0, H)$ the sequence $\|\phi_l^k\|_{L^2}^2$ satisfies the inequality*

$$\|\phi_l^k\|_{L^2}^2 \leq (\phi_{l0}^2 + Z_a + 1.6838 A_a (t_k - t_0)) e^{1.6838 D_a (t_k - t_0)}$$

for all $t_k \in [t_0, T_{\Delta t}]$ and all $l \neq d - 1$, if Assumptions 9 and 13 hold.

Proof. We insert the bounds from Corollaries 11 and 12 in Equation (7), and we use the new notations $x_a^k := \|\phi_l^k\|_{L^2}^2$, and

$$z_a^k := K_{a\phi 1} \|\partial_z v^{k-1}\|_{L^2}^2 + \sum_{m=1}^{d-1} \sum_{i=0}^1 K_{a\phi(6+i)m} \|\mathcal{D}_{\Delta t}^k(\partial_z^i w_m)\|_{L^2}^2,$$

to obtain

$$\mathcal{D}_{\Delta t}^k(x_a) \leq A_a + z_a^k + D_a x_a^k$$

from Equation (9). Once more apply Lemma 6 to obtain the result of Corollary 14. \square

5. Proof of Claim 3.

The a priori estimates of the previous section depend heavily on Claim 3. This claim restricts the time interval $(t_0, T_{\Delta t})$ for which the physical volume fraction restriction $\phi_j^k(z) \in [\phi_{\min}, 1 - (d-1)\phi_{\min}]$ for all $j \neq d-1$, and the physical velocity restrictions $\sum_{j=0}^k \|v^j\|_{L^2}^2 \Delta t \leq V^2$, and $\sum_{j=0}^k \|\partial_z v^j\|_{L^2}^2 \Delta t \leq V^2$ are valid. We need to prove that the interval $(t_0, T_{\Delta t})$ is non-empty. On closer inspection, we see that Claim 3 can be proven with upper bounds only.

LEMMA 15. *Let $t_0 \leq t_k = k\Delta t \leq T_{\Delta t}$. Let \mathbf{P}_d be the set of cyclic permutations of $(1, \dots, d)$. The constraints $\phi_l^k(z) \in [\phi_{\min}, 1 - (d-1)\phi_{\min}]$ for $l \neq d-1$, $\sum_{j=0}^k \|v^j\|_{L^2}^2 \Delta t \leq$*

V^2 , and $\sum_{j=0}^k \|\partial_z v^j\|_{L^2}^2 \Delta t \leq V^2$ are implied by

$$\sum_{j \neq d} \|\phi_{\alpha_j}^k\|_{H^1}^2 \leq \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2 \text{ for all } \alpha \in \mathbf{P}_d,$$

and

$$\sum_{j=0}^k \|\partial_z v^j\|_{L^2}^2 \Delta t \leq V^2.$$

Proof. The 3rd condition of Equation (5b) allows the application of the Poincaré inequality, which in one-dimensional space immediately gives the inequality $\|v^j\|_{L^2} \leq \|\partial_z v^j\|_{L^2}$. With this relation we can reduce the number of constraints on v^k from two to one.

For the constraints on ϕ_l^k we randomly pick an $\alpha \in \mathbf{P}_d$ and start with reversing Young's inequality on $\sum_{j \neq d} \|\phi_{\alpha_j}^k\|_{H^1}^2 \leq \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2$, which yields $\sum_{j \neq d} \|\phi_{\alpha_j}^k\|_{H^1} \leq \frac{1 - \phi_{\min}}{C_\infty}$. This inequality is transformed by the Sobolev embedding theorem into $\sum_{j \neq d} \|\phi_{\alpha_j}^k\|_{L^\infty} \leq 1 - \phi_{\min}$. With $\phi_{\alpha_j}^k \in C^2(0, 1)$ for $j \neq d$, which is shown in Theorem 4, we can change the L^∞ norm into a proper supremum on $(0, 1)$. Hence we obtain $\inf_{z \in (0, 1)} \phi_{\alpha_d}^k \geq \phi_{\min}$ from the identity $1 = \sum_{1 \leq l \leq d} \phi_l^k$. Since α was chosen randomly, we conclude that this result holds for all $\alpha \in \mathbf{P}_d$. Hence, $\min_{1 \leq l \leq d} \inf_{z \in (0, 1)} \phi_l^k(z) \geq \phi_{\min}$.

With the d infima established it is immediately clear that the d suprema follow automatically from the same identity. \square

The volume fraction identity $1 = \sum_{l=1}^d \phi_l^k$ cannot be used directly to satisfy the desired bound of Lemma 15. It would use a circular argument involving ϕ_{\min} : proving the lower bound ϕ_{\min} by a more stringent upper bound found in Lemma 15 and proving that upper bound with the volume fraction identity and the lower bound ϕ_{\min} .

Indirectly, the volume fraction identity can be used as is shown in Appendix A, where a differential equation for ϕ_{d-1}^k is constructed and tested such that new inequalities (50) and (51) are found.

Thus, from these new inequalities (50) and (51), Lemma 6 together with Corollaries 11, 12 and 14 yields an upper bound

$$(20) \quad \|\phi_{d-1}^k\|_{H^1}^2 \leq \mathcal{P}(T_{\Delta t} - t_0, V)$$

The function $\mathcal{P}(x, y)$ has $\mathcal{P}(0, y) = \phi_{(d-1)0}^2$ and it is a strictly increasing continuous function. Hence, for all cyclic permutations α of $(1, \dots, d)$ there exist strictly increasing continuous functions $P_\alpha(T_{\Delta t} - t_0, V)$ with $P_\alpha(0, 0) = \sum_{j=1}^{d-1} \phi_{\alpha_j 0}^2$, such that

$$\sum_{j=1}^{d-1} \|\phi_{\alpha_j}^k\|_{H^1}^2 \leq P_\alpha(T_{\Delta t} - t_0, V)$$

for all $t_0 \leq t_k \leq T_{\Delta t}$.

With this upper bound at hand, we are now able to show the existence of a non-empty time interval (t_0, T) for which the volume fraction condition of Claim 3 holds. Before we proceed, we need to introduce a geometric solid, the Steinmetz solid.

DEFINITION 16. Introduce a d -dimensional solid cylinder with central axis l_c and radius r as the set of points at a distance at most r from the line l_c . Following [16, 18] we introduce the d -dimensional Steinmetz solid $\mathcal{S}_d(r)$ as the intersection of d d -dimensional solid cylinders with radius r such that the axes l_c intersect orthogonally at the origin. In particular we assume the orientation of the Steinmetz solid to be such that the cylinder axes l_c are spans of Cartesian basis elements in \mathbf{R}^d . Hence

$$\mathcal{S}_d(r) = \left\{ (x_1, \dots, x_d) \in \mathbf{R}^d \left| \sum_{j \neq i} x_j^2 \leq r^2 \text{ for all } 1 \leq i \leq d \right. \right\}.$$

With the Steinmetz solid we obtain conditions for ϕ_{\min} , V and $T - t_0$ to obtain $\phi_l^k(z) \in (\phi_{\min}, 1 - (d-1)\phi_{\min})$ for all l and all $t_0 \leq t_k \leq T_{\Delta t}$. For $k = 0$ these conditions impose constraints on the initial volume fractions. Therefore we need an extra assumption.

ASSUMPTION 17. Let $\phi^0 = (\phi_{10}, \dots, \phi_{d0}) \in \mathcal{S}$, where

$$\mathcal{S} = \mathcal{S}_d\left(\frac{1 - \phi_{\min}}{\sqrt{2}C_\infty}\right) \cap \left\{ \phi^0 \in \mathbf{R}^d \left| \sum_{l=1}^d \phi_l^0 = 1 \right. \right\} \cap [\phi_{\min}, \infty)^d,$$

with $\mathcal{S}_d(\cdot)$ the Steinmetz solid as defined in Definition 16 and assume

$$\phi_{\min} \begin{cases} < 1 - \frac{\sqrt{2(d-1)}C_\infty}{d} & \text{for } 2 < d \leq 5, \\ \leq \frac{1}{1 + \sqrt{2(d-1)}C_\infty} & \text{for } 5 < d. \end{cases}$$

Denote the set of cyclic permutations of $(1, \dots, d)$ with \mathbf{P}_d .

LEMMA 18. There exists an open simply connected region $S \subset \mathbf{R}_+^2$ with $(0, 0) \in \bar{S}$, such that

$$\begin{aligned} (x, y) \in S &\Rightarrow P_\alpha(x, y) < \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2 \text{ for all } \alpha \in \mathbf{P}_d, \\ (x, y) \in \partial S &\Rightarrow P_\alpha(x, y) \leq \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2 \text{ for all } \alpha \in \mathbf{P}_d, \\ (x, y) \notin \bar{S} &\Rightarrow P_\alpha(x, y) > \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2 \text{ for at least one } \alpha \in \mathbf{P}_d, \end{aligned}$$

if Assumption 17 holds.

Proof. Remark that the largest hypercube contained in the Steinmetz solid $\mathcal{S}_d(r)$ has edges of length $2r/\sqrt{d-1}$, which is equal to the set of points with maximum norm at most $r/\sqrt{d-1}$. We have the value $r = \frac{1 - \phi_{\min}}{\sqrt{2}C_\infty}$, for which the interior of the hypercube is given by the inequality $\max_{1 \leq l \leq d} \{\phi_{l0}\} < \frac{1 - \phi_{\min}}{\sqrt{2(d-1)}C_\infty}$. The upper bound for all volume fractions is $1 - \phi_{\min}$ and this is a bound on the hypercube, which implies $d \geq 1 + 1/(2C_\infty^2) = 5/4$. The Steinmetz solid exists only for $r > 0$. However, the volume fractions have a lower bound $\phi_{\min} > 0$, which gives the inequality $\frac{1 - \phi_{\min}}{\sqrt{2(d-1)}C_\infty} \geq \phi_{\min}$ and therefore the condition $\phi_{\min} \leq \frac{1}{1 + \sqrt{2(d-1)}C_\infty}$. Another bound follows from the

lowest value for the maximum. The condition

$$\frac{1 - \phi_{\min}}{\sqrt{2(d-1)C_\infty}} > \frac{1}{d} = \min_{\sum_{l=1}^d \phi_{l0}=1} \max_{1 \leq l \leq d} \{\phi_{l0}\}$$

implies the condition $\phi_{\min} < 1 - \frac{2\sqrt{d-1}C_\infty}{d}$. The lower bound must satisfy $\phi_{\min} > 0$, which yields $d > 2 = C_\infty^2 + C_\infty \Re \sqrt{C_\infty^2 - 2}$ and conveniently satisfies the $d \geq 5/4$ condition. The two upper bounds for ϕ_{\min} are not equal. The last bound is the lesser of the two for $d < 2C_\infty^2 + 1 = 5$, while the second bound is the lesser of the two for $d > 2C_\infty^2 + 1 = 5$. Hence, the minimum of the two upper bounds for ϕ_{\min} guarantees that there exists a subset of initial conditions in $[\phi_{\min}, 1 - (d-1)\phi_{\min}]^d$ such that for all $\alpha \in \mathbf{P}_d$ we have $P_\alpha(0,0) < \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2$. Because all $P_\alpha(x, y)$ are strictly increasing and continuous functions, we immediately obtain simply connected open sets S_α for each $P_\alpha(x, y)$ for which $(0, 0) \in S_\alpha$ hold with the properties

$$\begin{aligned} (x, y) \in S_\alpha &\Rightarrow P_\alpha(x, y) < \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2, \\ (x, y) \in \partial S_\alpha &\Rightarrow P_\alpha(x, y) = \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2, \\ (x, y) \notin \overline{S_\alpha} &\Rightarrow P_\alpha(x, y) > \frac{1}{2} \left(\frac{1 - \phi_{\min}}{C_\infty} \right)^2. \end{aligned}$$

Hence, $S = \bigcap_{\alpha \in \mathbf{P}_d} S_\alpha$. \square

Now, we only need to prove that we can choose a value $V > 0$ for which [Claim 3](#) holds. To this end we use the function $Q_{\Delta t}(T_{\Delta t} - t_0, V^2)$ introduced in [Appendix A](#) as an upper bound to $\sum_{j=0}^k \|\partial_z v^j\|_{L^2}^2 \Delta t$. Hence, we need to prove $Q_{\Delta t}(T_{\Delta t} - t_0, V^2) \leq V^2$ for all $\Delta t \in (0, H)$.

ASSUMPTION 19. *Assume*

$$\frac{4d-2}{\Gamma_{\phi_{\min}}^2} \max_m \left\{ \frac{H_{1m}^2}{\gamma_m - K_{w6m}} \right\} (K_{w7} + K_{w8}) < 1.$$

LEMMA 20. *If [Assumption 19](#) holds, then there exist $H^*, V^* > 0$ and an open simply connected region $\mathcal{R} \subset \mathbf{R}_+^2$ with $(0, V^*) \in \overline{\mathcal{R}}$ such that*

$$\begin{aligned} (x, y) \in \mathcal{R} &\Rightarrow Q_{H^*}(x, y) < y^2, \\ (x, y) \in \partial \mathcal{R} &\Rightarrow Q_{H^*}(x, y) = y^2, \\ (x, y) \notin \overline{\mathcal{R}} &\Rightarrow Q_{H^*}(x, y) > y^2. \end{aligned}$$

Furthermore $\overline{\mathcal{R}} \subset \{(x, y) \in \mathbf{R}_+^2 \mid Q_{\Delta t}(x, y) \leq y^2\}$ for all $\Delta t \in (0, H^*)$.

Proof. Using [Corollaries 11, 12](#) and [14](#) we obtain:

$$\begin{aligned} Q_{\Delta t}(x, y) &= \|\partial_z v^0\|_{L^2}^2 \Delta t + \frac{4d-2}{\Gamma_{\phi_{\min}}^2} \left[8\Gamma^2 C_\infty^2 (d-1) c_b(x, y) \exp(D_b(x, y)) y^2 \right. \\ &\quad \left. + G_v^2 x + x \max_m \left\{ \frac{H_{0m}^2}{\gamma_m + D_m}, 4(d-1) C_\infty^2 H_{0m}^2 c_b(x, y) \exp(D_b(x, y)) \right\} \hat{C}^2(x, y) \right. \\ &\quad \left. + \max_m \left\{ \frac{H_{1m}^2}{\gamma_m - K_{w6m}}, \frac{4(d-1) C_\infty^2 H_{1m}^2 c_b(x, y) \exp(D_b(x, y))}{1 - K_{w5m}} \right\} \hat{C}^2(x, y) \right], \end{aligned}$$

with

$$\begin{aligned}\hat{C}^2(x, y) &:= \left((K_{w7} + K_{w8})y^2 + A \frac{C + 1.6838B}{C + B} x \right) \exp((C + 1.6838B)x) \\ c_b(x, y) &:= c_{b1}x + c_{b2}y^2 + c_{b3}x\hat{C}^2(x, y) + c_{b4}\hat{C}^2(x, y) \\ D_b(x, y) &:= \max_{l \neq d-1} \{D_{b1l}y^2 + D_{b2l}\hat{C}^2(x, y)\}.\end{aligned}$$

Introduce the constants

$$\begin{aligned}\hat{c}_b &:= c_{b2} + c_{b4}(K_{w7} + K_{w8}), \\ \hat{D}_b &:= \max_{l \neq d-1} \{D_{b2l} + D_{b3l}(K_{w7} + K_{w8})\}\end{aligned}$$

satisfying $c_b(0, y) = \hat{c}_b y^2$ and $D_b(0, y) = \hat{D}_b y^2$.

Since $Q_{\Delta t}(x, y)$ is differentiable with strictly increasing positive derivative in x and y and strictly positive derivative in Δt , it is sufficient to prove that there exists a region $y \in (y^{**}, y^{***})$ for which $Q_{H^*}(0, y) < y^2$. Because of the positive derivatives, this result guarantees for all $\Delta t \in (0, H^*)$ that there exists an open simply connected region such that $Q_{\Delta t}(x, y) < y^2$ in the region and $Q_{\Delta t}(x, y) \leq y^2$ on the boundary of the region.

The function $Q_{\Delta t}(0, y)$ satisfies the inequality

$$Q_{\Delta t}(0, y) \leq \mathbf{Q}_{\Delta t}(y) = Q_0 \Delta t + Q_1 y^2 + Q_2 y^4 \exp(Q_3 y^2)$$

with

$$\begin{aligned}Q_0 &= \|\partial_z v^0\|_{L^2}^2 \\ Q_1 &= \frac{4d-2}{\Gamma_{\phi_{\min}}^2} \max_m \left\{ \frac{H_{1m}^2}{\gamma_m - K_{w6m}} \right\} (K_{w7} + K_{w8}) \\ Q_2 &= \frac{(4d-2)(4d-4)}{\Gamma_{\phi_{\min}}^2} C_\infty^2 \hat{c}_b \max_m \left\{ 2\Gamma^2 + \frac{H_{1m}^2}{1 - K_{w5m}} (K_{w7} + K_{w8}) \right\} \\ Q_3 &= \hat{D}_b.\end{aligned}$$

If $Q_1 < 1$, as is assumed in [Assumption 19](#), then

$$\left. \frac{\partial \mathbf{Q}_{\Delta t}(y)}{\partial y^2} \right|_{y=0} = Q_1 < 1.$$

By the strictly increasing derivative in y there exists for every Δt small enough a unique point y^* such that $\left. \frac{\partial \mathbf{Q}_{\Delta t}(0, y)}{\partial y^2} \right|_{y=y^*} =: \mathbf{Q}'_{\Delta t}(0, y^*) = 1$, which implies

$$Q_1 = 1 - Q_2 (y^*)^2 (2 + Q_3 (y^*)^2) \exp(Q_3 (y^*)^2)$$

and

$$\mathbf{Q}_{\Delta t}(0, y^*) = Q_0 \Delta t + (y^*)^2 - Q_2 (y^*)^4 (1 + Q_3 (y^*)^2) \exp(Q_3 (y^*)^2).$$

Hence choose

$$H^* = \frac{Q_2}{Q_0} (y^*)^4 (1 + Q_3 (y^*)^2) \exp(Q_3 (y^*)^2),$$

such that every $\Delta t < H^*$ is small enough to satisfy [Assumption 19](#). Bolzano's theorem immediately gives the existence of an intersection point between $\mathbf{Q}_{H^*}(y)$ and y^2 .

Hence by the strictly positive derivative there exists a unique point $y^{**} \in (0, y^*)$ such that $\mathbf{Q}_{H^*}(y^{**}) = (y^{**})^2$ and $\mathbf{Q}'_{H^*}(y^{**}) < 1$. Moreover by the strictly positive derivative larger than 1 there exists a unique point $y^{***} > y^*$ such that $\mathbf{Q}_{H^*}(y^{***}) = (y^{***})^2$ and $\mathbf{Q}'_{H^*}(y^{***}) > 1$. Hence we have found the interval (y^{**}, y^{***}) for which $Q_{H^*}(0, y) \leq \mathbf{Q}_{H^*}(y) < y^2$. \square

To be more precise about the implications of $Q_1 < 1$ we need a new assumption.

ASSUMPTION 21. *Assume*

$$\frac{(d - \frac{1}{2})(d + 3)}{\Gamma_{\phi_{\min}}^2} \left[\sum_{m=1}^{d-1} \left(\frac{F_m^2}{1 - \sum_{j=1}^{d-1} \frac{\mathcal{M}_{j01m}}{2\sqrt{\gamma_j}}} + \frac{\eta_{\mathcal{M}vm2}}{2} \right) \right] \times$$

$$\times \max_{1 \leq m < d} \left\{ \frac{H_{1m}^2}{\gamma_m - \frac{\mathcal{M}_{vm}^2}{2\eta_{\mathcal{M}vm2}} - \sum_{j=1}^{d-1} \left(\frac{\mathcal{M}_{m01j}\sqrt{\gamma_m}}{2} + \frac{\mathcal{M}_{m11j}}{2} + \frac{\mathcal{M}_{j11m}}{2} \right)} \right\} < 1.$$

while still satisfying

$$\min_m \left\{ \gamma_m - \frac{\mathcal{M}_{vm}^2}{2\eta_{\mathcal{M}vm2}} - \sum_{j=1}^{d-1} \left(\frac{\mathcal{M}_{m01j}\sqrt{\gamma_m}}{2} + \frac{\mathcal{M}_{m11j}}{2} + \frac{\mathcal{M}_{j11m}}{2} \right) \right\} > 0.$$

The connection between [Assumption 21](#) and [Assumption 19](#) is now given by the next lemma.

LEMMA 22. *Let [Assumption 9](#) be satisfied, then [Assumptions 19](#) and [21](#) are equivalent.*

Proof. [Assumption 9](#) implies that the condition $1 > K_{w5m}$ can be made to hold by choosing appropriate η -indices values. This enforces an upper bound on $\eta_{Lm2} > 0$, which yields a lower bound for $1/\eta_{Lm2}$ and therefore a lower bound for K_{w7} . The constant K_{w8} is only coupled to K_{w5m} , K_{w6m} or K_{w7} via $\eta_{\mathcal{M}vm2}$ in K_{wm6} . [Assumption 9](#) implies that the condition $\gamma_m > K_{w6m}$ can be made to hold by choosing appropriate η -indices. However $\eta_{\mathcal{M}vm2}$ is now not freely determinable. Hence, we need to keep in mind that $\gamma_m > K_{w6m}$ still needs to be valid. In the proof of [Lemma 6](#) it was shown that $\eta_{\mathcal{M}m01j1}$ equals $1/\sqrt{\gamma_m}$, while $\eta_{\mathcal{M}m11j1}$ equals 1. A lower bound of K_{w6m} is then given by

$$\frac{\mathcal{M}_{vm2}^2}{2\eta_{\mathcal{M}vm2}} + \sum_{j=1}^{d-1} \left(\frac{\mathcal{M}_{m01j}\sqrt{\gamma_m}}{2} + \frac{\mathcal{M}_{m11j}}{2} + \frac{\mathcal{M}_{j11m}}{2} \right).$$

A larger lower bound needs a value for freely determinable η_{index} . Therefore, the lower bounds for both $K_{w7} + K_{w8}$ and K_{w6m} yield a necessary and sufficient condition for determining η_{index} while still satisfying [Assumption 19](#). \square

We introduce a set of new constants.

DEFINITION 23. Use the notation of the proof of [Lemma 20](#). Introduce

$$\tilde{y} := \sqrt{\frac{1}{\hat{D}_b} W_0 \left(\min_l \left\{ \frac{\left(\frac{1}{2} \left(\frac{1-\phi_{\min}}{C_\infty} \right)^2 - \sum_{j \neq l} \phi_{l0}^2 \right) \hat{D}_b}{(d-1-\delta_{d-1,l}) \hat{Z}_a + (1+(d-1)\delta_{d-1,l}) \hat{c}_b + \hat{\mathcal{P}} \delta_{d-1,l}} \right\} \right)}$$

and

$$\tilde{z} := \frac{2}{3} W_0 \left(\frac{3Q_3}{4Q_2} (1-Q_1) \right),$$

where

$$\begin{aligned} \hat{Z}_a &:= K_{a\phi 1} + \frac{\max \left\{ \frac{K_{a\phi 6m}}{1-K_{w5m}}, \frac{K_{a\phi 7m}}{\gamma_m - K_{w6m}} \right\}}{\min \{1 - K_{w5m}, \gamma_m - K_{w6m}\}} (K_{w7} + K_{w8}), \\ \hat{\mathcal{P}} &:= 5 + 2d(K_{w7} + K_{w8}), \end{aligned}$$

and $W_0(\cdot)$ denotes the standard product log branch, the inverse of $x \exp(x)$, through the origin.

The value y^* might not be expressible in standard functions preventing any explicit calculation of H^* . We can however determine another upper bound for Δt , which can be calculated explicitly.

LEMMA 24. Let \tilde{z} be as in [Definition 23](#). Then $\tilde{z} < Q_3(y^*)^2$ holds and the identity

$$H^{**} = \frac{Q_2}{Q_0 Q_3^2} \tilde{z}^2 (1 + \tilde{z}) \exp(\tilde{z}) \quad \text{implies}$$

$$H^{**} < H^* = \frac{Q_2}{Q_0} (y^*)^4 (1 + Q_3(y^*)^2) \exp(Q_3(y^*)^2).$$

Proof. Let $X = Q_3(y^*)^2$, then the identity for H^* in [Lemma 20](#) becomes $H^* = \frac{Q_2}{Q_0 Q_3^2} X^2 (1 + X) \exp(X)$ with $X > 0$ satisfying

$$\frac{1-Q_1}{Q_2} Q_3 = X(2+X) \exp(X) < \frac{4}{3} \cdot \frac{3}{2} X \exp\left(\frac{3}{2} X\right).$$

Introduce $\tilde{z} < X$ as $\frac{1-Q_1}{Q_2} Q_3 = \frac{4}{3} \cdot \frac{3}{2} \tilde{z} \exp\left(\frac{3}{2} \tilde{z}\right)$, then $\tilde{z} = \frac{2}{3} W_0\left(\frac{3}{4} \frac{1-Q_1}{Q_2} Q_3\right)$. Thus the explicitly determinable upper bound $H^{**} = \frac{Q_2}{Q_0 Q_3^2} \tilde{z}^2 (1 + \tilde{z}) \exp(\tilde{z})$ implies $H^{**} < H^*$. \square

The previous lemma showed another upper bound constraint for the time step Δt . Yet another constraint is necessary to obtain the intersection between S and \mathcal{R} , leading to the following assumption.

ASSUMPTION 25. Let Q_0, Q_1, Q_2 and Q_3 be as in the proof of [Lemma 20](#). Let B be given by [Definition 8](#), let $K_{a\phi 2l}$ be given by [Equation \(49c\)](#), and let \tilde{z} and \tilde{y} be given by [Definition 23](#). Assume

$$H < \min \left\{ \frac{(1-Q_1)^2}{4Q_2 Q_0}, \frac{\tilde{y}^4 Q_2}{Q_0}, \frac{Q_2 \tilde{z}^2}{Q_0 Q_3^2} (1 + \tilde{z}) \exp(\tilde{z}), \frac{0.6838}{B}, \frac{0.6838}{D_a} \right\}.$$

The intersection between S and \mathcal{R} must yield the admissible values for $T_{\Delta t} - t_0$ and V for which [Claim 3](#) is satisfied. However it is not yet clear whether such an intersection exists. This issue is addressed in the next lemma.

LEMMA 26. *The intersection $S \cap \mathcal{R}$ is nonempty, if [Assumptions 17, 19 and 25](#) hold with the constants \tilde{y} , \tilde{z} , \hat{Z}_a , \hat{c}_b , \hat{D}_b and $\hat{\mathcal{P}}$ of [Definition 23](#).*

Proof. We have already proven in [Lemma 18](#) that independent of Δt there exists an interval $(0, \hat{y})$ such that $P_\alpha(0, y) < \frac{1}{2C_\infty^2}(1 - \phi_{min})^2$ for all $y \in (0, \hat{y})$ and all $\alpha \in \mathbf{P}_d$. However, we have not determined the explicit value of \hat{y} . Furthermore, we have already proven there exists an interval (y^{**}, y^{***}) with $y^{**} > 0$ in which that $Q_{H^*}(y) < y^2$. Let $Q_{\Delta t}(0, y) \leq \mathbf{Q}_{\Delta t}(y) = Q_0\Delta t + Q_1y^2 + Q_2y^4 \exp(Q_3y^2) < y^2$ for all $y \in (y_{\Delta t}^{**}, y_{\Delta t}^{***})$ and for all $\Delta t \in (0, H^{**})$. The intersection $S \cap \mathcal{R}$ can then be proven to be nonempty if $y_{\Delta t}^{**} < \hat{y}$ for $\Delta t < H^{**}$ small enough. Thus we need to determine an upper bound for $y_{\Delta t}^{**}$ and a lower bound for \hat{y} . We know from [Lemma 20](#) that $y_{\Delta t}^{**}$ satisfies $Q_0\Delta t + Q_1(y_{\Delta t}^{**})^2 + Q_2(y_{\Delta t}^{**})^4 \exp(Q_3(y_{\Delta t}^{**})^2) = (y_{\Delta t}^{**})^2$. By replacing $\exp(Q_3y^2)$ in $\mathbf{Q}_{\Delta t}(y)$ by 1 we obtain a function with a less increasing derivative. Hence we obtain an upper bound for $y_{\Delta t}^{**}$ by calculating $Q_0\Delta t + Q_1y^2 + Q_2y^4 = y^2$, which yields

$$y^2 = \frac{1 - Q_1}{2Q_2} - \sqrt{\frac{(1 - Q_1)^2}{4Q_2^2} - \frac{Q_0\Delta t}{Q_2}} \leq \sqrt{\frac{Q_0\Delta t}{Q_2}}.$$

if

$$\Delta t \leq \frac{(1 - Q_1)^2}{4Q_2Q_0}$$

Introduce $y_{\Delta t}$ as an upper bound to $y_{\Delta t}^{**}$, then we can choose

$$y_{\Delta t} = \sqrt[4]{\frac{Q_0\Delta t}{Q_2}}.$$

The upper bound of \tilde{y} requires first a detailed description of $\mathcal{P}(0, y)$. After applying Cauchy-Schwartz inequality and Cauchy's inequality to [Equation \(50\)](#) and inserting [Corollaries 11, 12 and 14](#) we observe that all terms with $\|w_m^k\|_{H^1}$ and $\|\phi_n^k\|_{H^1}$ for $n \neq d - 1$ will yield an upper bound of the order $T_{\Delta t} - t_0 = x \downarrow 0$. Furthermore all terms $\|\phi_{d-1}^k\|_{L^2}$ without prefactors containing $\|\mathcal{D}_{\Delta t}^k(w_m)\|_{H^1}$ or $\|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}$ will yield an exponential with and exponent containing the prefactor $T_{\Delta t} - t_0 = x \downarrow 0$. Hence only the terms $\|v^k\|_{H^1}$, $\|\mathcal{D}_{\Delta t}^k(w_m)\|_{H^1}$ or $\|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}$ with possibly a prefactor $\|\phi_{d-1}^k\|_{L^2}$, which can be set equal to 1, will lead to upper bounds with factors $V^2 = y^2$. Therefore we obtain

$$\|\phi_{d-1}^k\|_{L^2}^2(0, y) \leq (2C_\infty^2 + 1 + 2d(K_{w7} + K_{w8})) y^2 =: \hat{\mathcal{P}}y^2.$$

Moreover, [Equation \(51\)](#) with [Corollary 12](#) yields

$$\|\partial_z \phi_{d-1}^k\|_{L^2}^2(0, y) \leq (d - 1) \sum_{l \neq d-1} \|\partial_z \phi_l^k\|_{L^2}^2(0, y) \leq (d - 1)\hat{c}_b y^2 \exp(\hat{D}_b y^2).$$

Hence, we have

$$\mathcal{P}(0, y) = \hat{\mathcal{P}}y^2 + (d - 1)\hat{c}_b y^2 \exp(\hat{D}_b y^2).$$

Similarly, by inserting the result of [Corollaries 12 and 14](#), we obtain

$$\mathcal{P}_\alpha(0, y) = \begin{cases} \hat{\mathcal{P}}y^2 + d\hat{c}_b y^2 \exp(\hat{D}_b y^2) + \sum_{j=1}^{d-1} \phi_{\alpha_j 0}^2 + (d-2)\hat{Z}_a y^2 & \text{if } d-1 \neq \alpha_d \\ \hat{c}_b y^2 \exp(\hat{D}_b y^2) + \sum_{l \neq d-1} \phi_{l0}^2 + (d-1)\hat{Z}_a y^2 & \text{if } d-1 = \alpha_d \end{cases}$$

Now, take $y^2 \exp(\hat{D}_b y^2)$ as an upper bound of y^2 to obtain a lower bound \tilde{y} of \hat{y} . This yields immediately the value of \tilde{y} as stated in [Definition 23](#). Thus for Δt smaller than $\tilde{y}^4 Q_2 / Q_0$ we observe an intersection of $S \cap \mathcal{R}$ if S exists. The existence of S , as shown in [Lemma 20](#), gives another upper bound for Δt and therefore one must take the minimum of the two. \square

THEOREM 27 (Existence of a weak solution to the discretised system with Δt independent bounds). *Let [Assumptions 1, 2, 9, 17, 21 and 25](#) hold. Then there exists $T - t_0, V \in S \cap \mathcal{R}$ independent of Δt such that there exists a solution (ϕ^k, v, w^k) of the discretised system satisfying*

$$\begin{aligned} \sum_{j=0}^k \|v^j\|_{L^2}^2 \Delta t &\leq V^2 \\ \sum_{j=0}^k \|\partial_z v^j\|_{L^2}^2 \Delta t &\leq V^2 \\ \phi^k &\in [\phi_{\min}, 1 - (d-1)\phi_{\min}]^d \\ \|\phi_1^k\|_{H^1}, \dots, \|\phi_d^k\|_{H^1} &\leq C \\ \sum_{j=1}^k \|\phi_1^j\|_{H^2}^2 \Delta t, \dots, \sum_{j=1}^k \|\phi_d^j\|_{H^2}^2 \Delta t &\leq C \\ \sum_{j=1}^k \|\mathcal{D}_{\Delta t}^k(\phi_1^j)\|_{L^2}^2 \Delta t, \dots, \sum_{j=1}^k \|\mathcal{D}_{\Delta t}^k(\phi_d^j)\|_{L^2}^2 \Delta t &\leq C \\ \|w_1^k\|_{H^2}, \dots, \|w_{d-1}^k\|_{H^2} &\leq C \\ \sum_{j=1}^k \|\mathcal{D}_{\Delta t}^k(w_1^j)\|_{H^1}^2 \Delta t, \dots, \sum_{j=1}^k \|\mathcal{D}_{\Delta t}^k(w_{d-1}^j)\|_{H^1}^2 \Delta t &\leq C \end{aligned}$$

for all $t_0 \leq t_k \leq T \leq T_{\Delta t}$ and for all $\Delta t \in (0, H)$ with $C > 0$ independent of Δt .

Proof. The existence of $T - t_0$ and V for which [Claim 3](#) holds, has been shown in [Lemma 26](#). Furthermore, the strictly increasing derivatives of both \mathcal{P}_α and $\mathcal{Q}_{\Delta t}$ with respect to Δt show that elements of the region $S \cap \mathcal{R}$ can always be chosen for $\Delta t < H$. The Δt independent bounds is a consequence of [Theorem 35](#) and [Claim 3](#) in combination with the bounds obtained in [Corollaries 11, 12 and 14](#). The H^2 norms follow directly from [Theorem 35](#) in [Appendix A](#) applied to Equations [\(4a\)](#) and [\(4c\)](#). \square

6. Interpolation functions and their time continuous limit.

In this section we will construct interpolation function on $(t_0, T) \times (0, 1)$ for our variables (ϕ, v, w) , and investigate their limits for $\Delta t \downarrow 0$.

[Theorem 27](#) shows that there exists a constant C and Sobolev spaces X_j, Y_j, Z_j such

that

$$\begin{aligned} \sup_{0 \leq k \leq K} \|u^k\|_{X_j}^2 &\leq \mathcal{C} < \infty \\ \sum_{k=0}^K \|u^k\|_{Y_j}^2 \Delta t &\leq \mathcal{C} < \infty \\ \sum_{k=1}^K \|\mathcal{D}_{\Delta t}^k(u)\|_{Z_j}^2 \Delta t &\leq \mathcal{C} < \infty \end{aligned}$$

for all $u \in (\phi, v, w)$ with $K\Delta t = T_{\Delta t} - t_0$.

These bounds guarantee that interpolation functions $\hat{u}(t) := u^{k-1} + (t - t_{k-1})\mathcal{D}_{\Delta t}^k(u)$ and $\bar{u}(t) := u^k$ lie in the desired Bochner spaces.

LEMMA 28. *Let H be given by [Assumption 25](#), let $0 < \Delta t < H$ be fixed, $\mathcal{C} > 0$ and let X be a Sobolev space. The following implications hold:*

$$\begin{aligned} \sup_{0 \leq k \leq K} \|u^k\|_X^2 \leq \mathcal{C} < \infty &\Rightarrow \begin{cases} \hat{u}_{\Delta t} \in L^\infty(t_0, T_{\Delta t}; X) \\ \bar{u}_{\Delta t} \in L^\infty(t_0, T_{\Delta t}; X) \end{cases} \\ \sum_{k=0}^K \|u^k\|_X^2 \Delta t \leq \mathcal{C} < \infty &\Rightarrow \begin{cases} \hat{u}_{\Delta t} \in L^2(t_0, T_{\Delta t}; X) \\ \bar{u}_{\Delta t} \in L^2(t_0, T_{\Delta t}; X) \end{cases} \\ \sum_{k=1}^K \|\mathcal{D}_{\Delta t}^k(u)\|_X^2 \Delta t \leq \mathcal{C} < \infty &\Rightarrow \begin{cases} \frac{\partial \hat{u}_{\Delta t}}{\partial t} \in L^2(t_0, T_{\Delta t}; X) \\ \frac{\bar{u}_{\Delta t}(t) - \bar{u}_{\Delta t}(t - \Delta t)}{\Delta t} \in L^2(t_0 + \Delta t, T_{\Delta t}; X) \end{cases} \end{aligned}$$

Proof. The measurability of $\bar{u}_{\Delta t}$ and $\hat{u}_{\Delta t}$ is easily established since the piecewise constant functions are measurable and dense in the set of piecewise linear functions. Furthermore, the construction of $\bar{u}_{\Delta t}$ and $\hat{u}_{\Delta t}$ yields

$$\begin{aligned} \int_{t_0}^{T_{\Delta t}} \|\hat{u}_{\Delta t}\|_X^2(t) dt &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|\hat{u}_{\Delta t}\|_X^2(t) dt \leq 2 \sum_{k=0}^K \|u^k\|_X^2 \Delta t, \\ \int_{t_0}^{T_{\Delta t}} \|\bar{u}_{\Delta t}\|_X^2(t) dt &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|\bar{u}_{\Delta t}\|_X^2(t) dt \leq \sum_{k=1}^K \|u^k\|_X^2 \Delta t, \end{aligned}$$

and $\operatorname{ess\,sup}_{t \in (t_0, T_{\Delta t})} \|\hat{u}_{\Delta t}\|_X(t) = \operatorname{ess\,sup}_{t \in (t_0, T_{\Delta t})} \|\bar{u}_{\Delta t}\|_X(t) = \sup_{t_k \in [t_0, T_{\Delta t}]} \|u^k\|_X$.

From the definition of derivative, it is easily seen that $\hat{u}_{\Delta t}$ has a strong derivative for a.e. $t \in (0, T_{\Delta t}]$, since we have

$$\lim_{h \downarrow 0} \left\| \frac{\hat{u}_{\Delta t}(t+h) - \hat{u}_{\Delta t}(t)}{h} - \frac{u^k - u^{k-1}}{\Delta t} \right\| = 0$$

for $t \in (t_{k-1}, t_k)$ with $t_0 < t_k \leq T_{\Delta t}$. Hence we obtain

$$\begin{aligned} \int_{t_0}^{T_{\Delta t}} \|\partial_t \hat{u}_{\Delta t}\|_X^2(t) dt &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \|\partial_t \hat{u}_{\Delta t}\|_X^2(t) dt \\ &\leq \sum_{k=1}^K \|\mathcal{D}_{\Delta t}^k(u)\|_X^2 \Delta t. \\ \int_{t_0+\Delta t}^{T_{\Delta t}} \left\| \frac{\bar{u}_{\Delta t}(t) - \bar{u}_{\Delta t}(t - \Delta t)}{\Delta t} \right\|_X^2 dt &= \sum_{k=1}^K \int_{t_{k-1}}^{t_k} \left\| \frac{\bar{u}_{\Delta t}(t) - \bar{u}_{\Delta t}(t - \Delta t)}{\Delta t} \right\|_X^2 dt \\ &\leq \sum_{k=1}^K \|\mathcal{D}_{\Delta t}^k(u)\|_X^2 \Delta t. \quad \square \end{aligned}$$

With the bounded norms, independent of Δt , and the Δt -independent time-interval $[t_0, T]$ with $T \leq T_{\Delta t}$ the weak convergence in Δt of functions $\hat{u}_{\Delta t}$ and $\bar{u}_{\Delta t}$ defined on $(t_0, T) \times (0, 1)$ is guaranteed by the Eberlein-Smulian theorem, as stated in [6].

Weak convergence of products of functions defined on $(t_0, T) \times (0, 1)$ is guaranteed by strong convergence of all but one function in the product. The strong convergence is given by the Lions-Aubin-Simon lemma, originally stated in [32]. We use a version of [9] with slight modifications as stated in [8].

LEMMA 29 (Lions-Aubin-Simon). *Let X , B , and Y be Banach spaces such that the embedding $X \hookrightarrow B$ is compact and the embedding $B \hookrightarrow Y$ is continuous. Furthermore, let either $1 \leq p < \infty$, $r = 1$ or $p = \infty$, $r > 1$, and let (u_τ) be a sequence of functions, that are constant on each subinterval (t_{k-1}, t_k) , satisfying*

$$(25a) \quad \|u_\tau\|_{L^p(0, T; X)} \leq C_0,$$

$$(25b) \quad \|u_\tau(t) - u_\tau(t - \tau)\|_{L^r(t_0 + \tau, T; Y)} \leq C_0 \tau^\alpha$$

for $\alpha = 1$ and for all $\tau > 0$, where $C_0 > 0$ is a constant that is independent of τ . If $p < \infty$, then (u_τ) is relatively compact in $L^p(0, T; B)$. If $p = \infty$, then there exists a subsequence of (u_τ) that converges in each space $L^q(0, T; B)$, $1 \leq q < \infty$, to a limit that belongs to $C^0([0, T]; B)$.

Moreover we cannot replace $\alpha = 1$ with $\alpha \in (0, 1)$.

Compactness results related to $H^r(\Omega)$ with $r \geq 0$ can be found in [22]. For one-dimensional bounded Ω compactness results can be found in section 8.2 of [5].

THEOREM 30. *Let $s \in \mathbf{R}$. If $\Omega \subset \mathbf{R}^n$ is bounded and has a $(n - 1)$ -dimensional infinitely differentiable boundary Γ with Ω being locally on one side of Γ , then the injection $H^s(\Omega) \hookrightarrow H^{s-\epsilon}(\Omega)$ is compact for every $\epsilon > 0$.*

Let $\Omega \subset \mathbf{R}$ be bounded, then for all $m > 0$ integer and $p \in (0, \infty]$ we have

$$(26) \quad W^{m+1, p}(\Omega) \hookrightarrow C^m(\bar{\Omega}) \hookrightarrow W^{m, p}(\Omega).$$

We conclude that there exists a subsequence $(\Delta t) \downarrow 0$ for which we have both weak and strong convergence (but in different functions spaces) of both $\hat{u}_{\Delta t}$ and $\bar{u}_{\Delta t}$. A priori these limits \hat{u} and \bar{u} are not necessarily the same, however with the strong convergence we show that the limits are identical.

LEMMA 31. *Let $\bar{u}_{\Delta t} \rightarrow \bar{u}$ strongly in $L^2(0, T; X)$, $\hat{u}_{\Delta t} \rightharpoonup \hat{u}$ weakly in $L^2(0, T; X)$ and let $\sum_{l=1}^K \|\mathcal{D}_{\Delta t}^l(u)\|_X^2 \Delta t \leq \mathcal{C} < \infty$ with \mathcal{C} independent of Δt , then $\hat{u} = \bar{u}$.*

Proof. Based on strong convergence, we have $\int_{t_0}^T \|\bar{u}_{\Delta t}(t) - \bar{u}(t)\|_X^2 dt \rightarrow 0$ as $\Delta t \downarrow 0$. From the construction of both $\hat{u}_{\Delta t}$ and $\bar{u}_{\Delta t}$ we have $\bar{u}_{\Delta t}(t) - \hat{u}_{\Delta t}(t) = \frac{t_k - t}{\Delta t}(u^k - u^{k-1})$, from which we obtain

$$\int_{t_0}^{T_{\Delta t}} \|\bar{u}_{\Delta t}(t) - \hat{u}_{\Delta t}(t)\|_X^2 dt \leq \sum_{k=1}^K \|\mathcal{D}_{\Delta t}^k(u)\|_X^2 (\Delta t)^2 \leq C \Delta t \downarrow 0$$

Thus by the triangle inequality we obtain $\int_{t_0}^T \|\hat{u}_{\Delta t}(t) - \bar{u}(t)\|_X^2 dt \rightarrow 0$. Hence, $\hat{u}_{\Delta t} \rightarrow \bar{u}$ strongly in $L^2(t_0, T; X)$ as $\Delta t \downarrow 0$. Now take an arbitrary $\psi \in L^2(t_0, T; X)$, then we have

$$\left| \int_0^T \int_0^1 (\hat{u} - \bar{u}) \psi dx dt \right| \leq \left| \int_0^T \int_0^1 (\hat{u}_{\Delta t} - \hat{u}) \psi dx dt \right| + \left| \int_0^T \int_0^1 (\hat{u}_{\Delta t} - \bar{u}) \psi dx dt \right| \rightarrow 0$$

by the weak convergence of $\hat{u}_{\Delta t}$ to \hat{u} and the strong convergence of $\hat{u}_{\Delta t}$ to \bar{u} . We have chosen ψ arbitrarily in $L^2(t_0, T; X)$. Hence, $\hat{u} = \bar{u}$ in $L^2(t_0, T; X)$. \square

At this point, we have shown the existence of a weak and strong limit to the discrete functions u_l^k on the interior of $[t_0, T] \times [0, 1]$. Furthermore, we have initial conditions given by Equation (3), while the final conditions at time $t = T$ exist by the construction of the interior functions and the determination of the interval $[t_0, T]$. We have not yet shown that the limit function on $(t_0, T) \times (0, 1)$ has boundary values on the lateral boundary $(t_0, T) \times \{0, 1\}$ that satisfy the boundary conditions of the continuous system. First we show that we can apply the trace theorem to identify unique boundary values for functions defined on the interior, and that these trace functions on the boundary satisfy compatibility relations. For convenience we follow the notation of Bochner spaces as stated in Lions and Magenes [23] by introducing $Y(I, X)$ for normed spaces $X, Y(I)$ of interval I as the space of functions $u(t) \in X$ satisfying $\|u\|_X \|Y(I)\| < \infty$, and by introducing the spaces

$$(27) \quad H^{r,s}(I \times \Omega) := L^2(I, H^r(\Omega)) \cap H^s(I, L^2(\Omega))$$

for $r, s > 0$ where we will use the notation $Q := I \times \Omega$ with $\Omega \subset \mathbf{R}^n$ for the joint domain and $\Sigma = I \times \Gamma$ with $\Gamma := \partial\Omega$ for the lateral boundary. Bochner space theory, as found in [23], shows the existence of trace functions and global compatibility relations, which we summarized in Theorem 32.

THEOREM 32. *Let $u \in H^{r,s}(Q)$ with $r, s \geq 0$.*

If $0 \leq j < r - \frac{1}{2}$ integer, then $\partial_\nu^j u \in H^{\mu_j, \nu_j}(\Sigma)$, where ∂_ν^j is the j th order normal derivative on Σ , oriented toward the interior of Q ; $\frac{\mu_j}{r} = \frac{\nu_j}{s} = \frac{r-j-\frac{1}{2}}{r}$ and where $u \rightarrow \partial_\nu^j u$ are continuous linear mappings of $H^{r,s}(Q) \rightarrow H^{\mu_j, \nu_j}(\Sigma)$.

Introduce functions $f_i(x) = \partial_t^i u(x, 0)$ and $g_j(\tilde{x}, t) = \partial_{x'}^j u((x', 0), t)$ for $(x', 0), x \in \mathbf{R}^n$ for some local indexation of the coordinates on Γ .

Introduce the product space F of elements

$$(28) \quad \{f_i, g_j\} \in F = \prod_{i < s - \frac{1}{2}} H^{p_i}(\Omega) \times \prod_{j < r - \frac{1}{2}} H^{\mu_j, \nu_j}(\Sigma)$$

with $\frac{p_i}{r} = \frac{s-i-\frac{1}{2}}{s}$ and $\frac{\mu_j}{r} = \frac{\nu_j}{s} = \frac{r-j-\frac{1}{2}}{r}$.

Let F_0 be the vector subspace of F which satisfies $\partial_t^i g_j(x', 0) = \partial_{x_n}^j f_i(x', 0)$ and

$\int_0^T \int_\Gamma |\partial_t^i g_j(x', \sigma^r) - \partial_{x_n}^j f_i(x', \sigma^s)|^2 dx' \frac{d\sigma}{\sigma} < \infty$.
 If $r, s > 0$ and $0 \leq \frac{j}{r} + \frac{i}{s} < 1 - \frac{1}{2}(\frac{1}{r} + \frac{1}{s})$ integers, then the map $u \mapsto \{f_i, g_j\}$ is a continuous linear surjection of $H^{r,s}(Q) \rightarrow F_0$.

Together with the Lions-Aubin-Simon lemma, which states that the strong limit is an element of $L^2(t_0, T; C^j(0, 1))$ for some integer j , we can show that the trace functions satisfy the boundary conditions, essentially because the trace functions are a limit of the trace functions of the interpolation functions for which the boundary conditions do apply.

LEMMA 33. Let $a \in \{0, 1\}$. Let $u_{\Delta t}$ be either $\hat{u}_{\Delta t}$ or $\bar{u}_{\Delta t}$. Let $u_{\Delta t}|_{z=a}$ be the single-sided trace of $u_{\Delta t}$. For $r, s, j, i, s - i \geq 0$ and $r - j - 1/2 > 0$ we have

$$u_{\Delta t} \in H^{r,s}(Q) \Rightarrow \partial_\nu^j \partial_t^i u_{\Delta t}|_{z=a} \in H^{r-j-\frac{1}{2}, \frac{r-j-1/2}{r}(s-i)}(\Sigma) = H^{\frac{r-j-1/2}{r}(s-i)}(t_0, T)$$

with weak limit $u|_{z=a}$ in $H^{\frac{r-j-1/2}{r}(s-i)}(t_0, T)$. Additionally, if $r \geq j + 1/2$, $s \geq i + 1/2$ then there exists a subsequence (Δt) such that $\partial_\nu^j \partial_t^i u_{\Delta t}|_{z=a}$ converges weakly in $L^2(t_0, T)$ to $\partial_\nu^j \partial_t^i u|_{z=a} = \partial_t^i \partial_\nu^j u|_{z=a} \in L^2(t_0, T)$. Moreover, if $r \geq j + 1$, $s \geq i + 1$, then there exists a subsequence (Δt) such that $\partial_\nu^j \partial_t^i u_{\Delta t}|_{z=a}$ converges strongly in $L^2(t_0, T)$ to $\partial_\nu^j \partial_t^i u|_{z=a} = \partial_t^i \partial_\nu^j u|_{z=a} \in C^0(t_0, T)$.

If on the other hand $\partial_\nu^j \partial_t^i u_{\Delta t}|_{z=a} = C \in \mathbf{R}$, then it converge strongly to $\partial_\nu^j \partial_t^i u|_{z=a} = C$ regardless of the function space on $[t_0, T]$ or the sequence (Δt) .

Proof. Theorem 32 immediately gives the appropriate spaces for $u_{\Delta t}|_{z=a}$. Furthermore, Lemma 29 shows that weak convergence in $H^{j+1, i+1}(Q)$ implies the existence of a subsequence (Δt) with weak convergence of $\partial_z^j \partial_t^i u_{\Delta t}$ in $L^q(t_0, T; C^0([0, 1]))$ for each $1 \leq q < \infty$ to $\partial_z^j \partial_t^i u \in C^0(\bar{Q})$ and strong convergence of $u_{\Delta t}$ to u in $H^i(t_0, T; C^j([0, 1]))$. The continuity $C^i([t_0, T]; C^j([0, 1]))$ shows that the trace operator is both an evaluation $z = a$ and a limit from $z \in (0, 1)$ to $z = a \in \{0, 1\}$. Additionally, the continuity shows that the derivatives commute. The strong convergence on the boundary follows from the following inequalities:

$$\begin{aligned} \lim_{\Delta t \downarrow 0} \left\| \partial_\nu^j \partial_t^i u_{\Delta t}|_{z=a} - \partial_\nu^j \partial_t^i u|_{z=a} \right\|_{L^2(t_0, T)} &\leq \lim_{\Delta t \downarrow 0} \left\| \sup_{\bar{z} \in [0, 1]} |\partial_z^j \partial_t^i u_{\Delta t}(\bar{z}) - \partial_z^j \partial_t^i u(\bar{z})| \right\|_{L^2(t_0, T)} \\ &= \lim_{\Delta t \downarrow 0} \left\| \partial_z^j \partial_t^i u_{\Delta t} - \partial_z^j \partial_t^i u \right\|_{L^2(t_0, T; C^0([0, 1]))} \\ &\leq \lim_{\Delta t \downarrow 0} \|u_{\Delta t} - u\|_{H^j(t_0, T; C^i([0, 1]))} = 0 \end{aligned}$$

The strong convergence of $\partial_\nu^j \partial_t^i u_{\Delta t}$ in $L^2(t_0, T)$ states that the weak convergence of $\partial_\nu^j \partial_t^i u_{\Delta t}$ in $L^2(t_0, T)$, in particular the one induced by the weak convergence of $\partial_z^j \partial_t^i u_{\Delta t}$ in $L^q(t_0, T; C^0([0, 1]))$ with $q = 2$, has the same limit as the strong convergence. Hence, the strong limit must be in $C^0(t_0, T)$.

The weak convergence of $\partial_\nu^j \partial_t^i u_{\Delta t}|_{z=a} \in L^2(t_0, T)$ to $\partial_\nu^j \partial_t^i u|_{z=a} = \partial_t^i \partial_\nu^j u|_{z=a} \in L^2(t_0, T)$ can be easily seen. The weak convergence itself is a direct application of Lemma 29 and Theorems 30 and 32, while the commuting derivatives are a consequence of the weak derivative structure itself.

The strong convergence for the case $\partial_\nu^j \partial_t^i u_{\Delta t}|_{z=a} = C \in \mathbf{R}$ is trivial. \square

We will now show the weak convergence of the semi-discrete approximations to a weak solution of the continuous system.

THEOREM 34. *If the conditions of [Theorem 27](#) are satisfied, then there exist constants $T - t_0 > 0$ and $V > 0$ such that there exist functions:*

$$\begin{aligned} \phi_l &\in L^2(t_0, T; H^2([0, 1])) \cap L^\infty(t_0, T; H^1(0, 1)) \cap C^0([t_0, T]; C^0([0, 1])) \\ &\quad \cap H^1(t_0, T; L^2(0, 1)) \\ v &\in L^2(t_0, T; H^1(0, 1)) \\ w_m &\in L^\infty(t_0, T; H^2(0, 1)) \cap C^0([t_0, T]; C^1([0, 1])) \cap H^1(t_0, T; H^1(0, 1)) \\ \mathcal{W} &\in H^1(t_0, T) \end{aligned}$$

for all $l \in \{1, \dots, d\}$ and $m \in \{1, \dots, d-1\}$ satisfying [Claim 3](#), and

$$(32a) \quad \left(\partial_t \phi_l + I_l(\phi) \partial_z(\Gamma(\phi)v) + \sum_{m=1}^{d-1} \sum_{i,j=0}^1 \partial_z^i (B_{lijm}(\phi) \partial_t^j w_m) - G_{\phi,l}(\phi), \psi_{1l} \right)_Q = -(\delta_l \partial_z \phi_l, \partial_z \psi_{1l})_Q,$$

$$(32b) \quad \left(\partial_z(\Gamma(\phi)v) + \sum_{m=1}^{d-1} \sum_{j=0}^1 \partial_z (H_{jm}(\phi) \partial_t^j w_m) - G_v(\phi), \psi_2 \right)_Q = 0,$$

$$(32c) \quad \begin{aligned} &(\partial_t w_m + F_m(\phi) w_m - G_{w,m}(\phi), \psi_{3m})_Q \\ &+ \left(\sum_{j=1}^{d-1} \sum_{\substack{i+n=0 \\ i,n \geq 0}}^1 E_{minj}(\phi) \partial_z^i \partial_t^n w_j - D_m \partial_z w_m - \gamma_m \partial_z \partial_t w_m, \psi_{3m} \right)_\Sigma \\ &= \left(\sum_{j=1}^{d-1} \sum_{\substack{i+n=0 \\ i,n \geq 0}}^1 E_{minj}(\phi) \partial_z^i \partial_t^n w_j - D_m \partial_z w_m - \gamma_m \partial_z \partial_t w_m, \partial_z \psi_{3m} \right)_Q, \end{aligned}$$

where $(f, g)_Q := \int_{t_0}^T \int_0^1 f(z, t) g(z, t) dz dt$, $(f, g)_\Sigma := \int_{t_0}^T f(1, t) g(1, t) - f(0, t) g(0, t) dt$, $\psi_2 \in L^2(Q)$ and $\psi_{1l}, \psi_{3m} \in H^{1,0}(Q)$, and

$$(33a) \quad \int_{t_0}^T (\partial_z w_m|_{z=1} - A_1 (w_m|_{z=1} - \mathcal{W})) \Psi_{1m} dt = 0,$$

$$(33b) \quad \int_{t_0}^T \left(\partial_t w_m|_{z=0} - \hat{J}_m \frac{\mathcal{L}(\phi_{m,res} - \phi_m|_{z=0})}{H_{1m}(\phi|_{z=0})} \right) \Psi_{2m} dt = 0,$$

and

$$(33c) \quad \int_{t_0}^T \left(\partial_t \mathcal{W} - v_3|_{z=1} - \hat{J}_d \frac{\mathcal{L}(\phi_{d,res} - \phi_d|_{z=1})}{\Gamma(\phi|_{z=1})} \right) \Psi_3 dt = 0,$$

where $\Psi_{1m}, \Psi_{2m}, \Psi_3 \in L^2(t_0, T)$ for all $1 \leq m < d$.

Proof. [Lemmas 28, 29](#) and [31](#), [Theorems 27](#) and [30](#), and the Eberlein-Smulian theorem show that [Claim 3](#) is valid for a time domain (t_0, T) and an upper bound $V > 0$. Moreover, they show for this time domain that there exists a subsequence of

(Δt) converging to 0 such that both the linear and nonlinear terms of Equations (4a)–(4c) converge weakly to Equations (32a)–(32c), since both $\hat{\phi}_{l,\Delta t}$ and $\bar{\phi}_{l,\Delta t}$ converge strongly in $L^2(t_0, T; C^1(0, 1))$ to ϕ_l for $l \in \{1, \dots, d\}$, and all other variables have the necessary weak convergence.

The boundary conditions (33a)–(33c) follow immediately from Theorem 32 together with Lemmas 29 and 33 and Theorem 30 applied to Equations (5a) and (5b), since $\mathcal{L}(\cdot) \in W^{1,\infty}(0, 1)$ and Assumption 2 imply $\mathcal{L}(\cdot)/H_{1m}(\cdot), \mathcal{L}(\cdot)/\Gamma(\cdot) \in W^{1,\infty}(0, 1)$. \square

7. Conclusion. We have proven the existence of physical weak solutions of the continuous system given by Equations (1a)–(1c) on the domain $[t_0, T] \times (0, 1)$ with boundary conditions (2a) and (2b) and initial conditions (3) and satisfying Assumptions 1, 2, 9, 17, 21 and 25 by applying the Rothe method to the time discrete system given by Equations (4a)–(4c) with boundary conditions (2a) and (2b) and initial conditions (3), when the time interval size $T - t_0$ and a velocity V are chosen such that $(T - t_0, V) \in S \cap \mathcal{R} \neq \emptyset$ with S and \mathcal{R} respectively as introduced in Lemmas 18 and 20.

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Appendix A. Derivation of quadratic inequalities.

The discrete Gronwall inequalities, as stated in Lemmas 6 and 7, apply to quadratic inequalities only. By testing the discrete system with suitable test functions one can obtain these quadratic inequalities with a cubic term. These cubic terms can often be transformed into quadratic terms by partial integration and application of Claim 3, which allows Lemma 6 to be applied. However, in a single case we cannot transform the cubic term into quadratic terms. In this case the cubic term can be modified to fit the framework of Lemma 7.

We test Equation (4a) successively with ϕ_l^k and $\mathcal{D}_{\Delta t}^k(\phi_l)$, which gives us, with use of $\partial_z \phi_l^k(0) = \partial_z \phi_l^k(1) = 0$,

$$\begin{aligned}
 (34) \quad & \frac{1}{2} \left[\mathcal{D}_{\Delta t}^k (\|\phi_l\|_{L^2}^2) + \Delta t \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 \right] + \delta_l \|\partial_z \phi_l^k\|_{L^2}^2 \\
 & \leq 2I_l \Gamma \sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} \|v^{k-1}\|_{L^2} + I_l \Gamma \|\partial_z v^{k-1}\|_{L^2} \|\phi_l^k\|_{L^2} + G_{\phi,l} \|\phi_l^k\|_{L^2} \\
 & \quad + \sum_{m=1}^{d-1} \left[\sum_{i=0}^1 (B_{li0m} \|\partial_z^i w_m^{k-1}\|_{L^2} + B_{li1m} \|\mathcal{D}_{\Delta t}^k(\partial_z^i w_m)\|_{L^2}) \|\phi_l^k\|_{L^2} \right. \\
 & \quad \left. + 2 \sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} (B_{l10m} \|w_m^{k-1}\|_{L^2} + B_{l11m} \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}) \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (35) \quad & \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 + \frac{\delta_l}{2} \left[\mathcal{D}_{\Delta t}^k \left(\|\partial_z \phi_l\|_{L^2}^2 \right) + \Delta t \|\mathcal{D}_{\Delta t}^k(\partial_z \phi_l)\|_{L^2}^2 \right] \\
 & \leq I_l \Gamma \left[2 \sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} \|v^{k-1}\|_{L^\infty} + \|\partial_z v^{k-1}\|_{L^2} \right] \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2} + G_{\phi,l} \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2} \\
 & \quad + \sum_{m=1}^{d-1} \left[\sum_{i=0}^1 (B_{li0m} \|\partial_z^i w_m^{k-1}\|_{L^2} + B_{li1m} \|\mathcal{D}_{\Delta t}^k(\partial_z^i w_m)\|_{L^2}) \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2} \right. \\
 & \quad \left. + 2 \sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} (B_{l10m} \|w_m^{k-1}\|_{L^\infty} + B_{l11m} \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^\infty}) \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2} \right].
 \end{aligned}$$

Furthermore, we test [Equation \(4c\)](#) successively with w_m^k and $\mathcal{D}_{\Delta t}^k(w_m)$, in the evaluation of which we use the following results, derived from

1. the first boundary condition of [\(5b\)](#), which yields the estimate

$$\left| \mathcal{D}_{\Delta t}^k(w_m(0)) \right| \leq \frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}} \quad \Rightarrow \quad |w_m^k(0)| \leq \frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}}(t_k - t_0)$$

2. the second boundary condition of [\(5b\)](#), leading to

$$\begin{aligned}
 |\partial_z w_m^k(1)| & \leq |A_m| |w_m^k(1) - \mathcal{W}^k| \\
 & \leq |A_m| \left(\frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}}(t_k - t_0) + \|\partial_z w_m^k\|_{L^2} + |\mathcal{W}^k| \right)
 \end{aligned}$$

3. the third boundary condition of [\(5b\)](#), which gives us

$$(36) \quad |v^k(z)| \leq \|\partial_z v^k\|_{L^2}$$

4. the fourth boundary condition of [\(5b\)](#), resulting in

$$(37a) \quad \left| \mathcal{D}_{\Delta t}^k(\mathcal{W}) \right| \leq \|\partial_z v^{k-1}\|_{L^2} + \frac{\hat{J}_d \phi_{d,res}}{\Gamma_{\phi_{\min}}} \quad \text{and}$$

$$\begin{aligned}
 (37b) \quad |\mathcal{W}^k| & \leq |\mathcal{W}^0| + \sum_{n=0}^{k-1} \|\partial_z v^n\|_{L^2} \Delta t + \frac{\hat{J}_d \phi_{d,res}}{\Gamma_{\phi_{\min}}}(t_k - t_0) \\
 & \leq |\mathcal{W}^0| + V \sqrt{t_k - t_0} + \frac{\hat{J}_d \phi_{d,res}}{\Gamma_{\phi_{\min}}}(t_k - t_0).
 \end{aligned}$$

5. a first integral of [\(4c\)](#), written in short hand notation as

$$\mathcal{D}_{\Delta t}^k(w_m) - \partial_z \mathbb{S}_m^k = G_{w,m}(\phi^{k-1}) - F_m(\phi^{k-1})v^{k-1},$$

where

$$\begin{aligned}\mathbb{S}_m^k &= \mathbb{S}_{m0}^k + \mathbb{S}_{m1}^k \\ \mathbb{S}_{m0}^k &= \sum_{j=1}^{d-1} [E_{m00j}(\phi^{k-1})w_j^{k-1} + E_{m01j}\mathcal{D}_{\Delta t}^k(w_j)] \\ \mathbb{S}_{m1}^k &= D_m\partial_z w_m^k + \gamma_m\mathcal{D}_{\Delta t}^k(\partial_z w_m) - \sum_{j=1}^{d-1} E_{m10j}(\phi^{k-1})\partial_z w_j^{k-1},\end{aligned}$$

stating that

$$\mathbb{S}_m^k(1) = \mathbb{S}_m^k(0) + \int_0^1 (\mathcal{D}_{\Delta t}^k(w_m) - G_{w,m}(\phi^{k-1}) + F_m(\phi^{k-1})v^{k-1}) dz.$$

With this and the preceding results, we notice that $\mathbb{S}_{m0}^k(0)$, $\mathbb{S}_{m1}^k(1)$ and $\mathbb{S}_m^k(1) - \mathbb{S}_m^k(0)$ are "known" (i.e. can be expressed in known constants and their L^2 -norms of the variables and their ∂_z -derivatives). All this gives us for the stockterm $(\mathbb{S}_m^k w_m^k)|_0^1$, occurring in the equation resulting from the test of (4c) with w_m^k ,

$$\begin{aligned}(\mathbb{S}_m^k w_m^k)|_0^1 &= [\mathbb{S}_m^k(1) - \mathbb{S}_m^k(0)] w_m^k(0) + [\mathbb{S}_{m0}^k(0) + \mathbb{S}_{m1}^k(1)][w_m^k(1) - w_m^k(0)] \\ &\quad + [\mathbb{S}_{m0}^k(1) - \mathbb{S}_{m0}^k(0)] [w_m^k(1) - w_m^k(0)],\end{aligned}$$

where

$$\begin{aligned}|\mathbb{S}_{m0}^k(1) - \mathbb{S}_{m0}^k(0)| &\leq \sum_{j=1}^{d-1} \left[|E_{m00j}(\phi^{k-1}(1)) (w_j^k(1) - w_j^k(0))| \right. \\ &\quad + |(E_{m00j}(\phi^{k-1}(1)) - E_{m00j}(\phi^{k-1}(0))) w_j^k(0)| \\ &\quad + |E_{m01j}(\phi^{k-1}(1)) (\mathcal{D}_{\Delta t}^k(w_j(1)) - \mathcal{D}_{\Delta t}^k(w_j(0)))| \\ &\quad \left. + |(E_{m01j}(\phi^{k-1}(1)) - E_{m01j}(\phi^{k-1}(0))) \mathcal{D}_{\Delta t}^k(w_j(0))| \right] \\ &\leq \sum_{j=1}^{d-1} \left[E_{m00j} \left(\|\partial_z w_j^k\|_{L^2} + \frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}}(t_k - t_0) \right) \right. \\ &\quad \left. + E_{m01j} \left(\|\mathcal{D}_{\Delta t}^k(\partial_z w_j)\|_{L^2} + \frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}} \right) \right],\end{aligned}$$

and with use of the fundamental theorem of calculus to rewrite the boundary terms. Notice that this is justified by [Theorem 35](#), which guarantees the existence of an absolutely continuous representative satisfying the fundamental theorem of calculus.

THEOREM 35. *Let I be an open, but possibly unbounded, interval. Let $u \in L^p(I)$ with $p \in (1, \infty]$, then $u \in W^{1,p}(I)$ iff u is of bounded variation, i.e., there exists a constant C such that for all $\phi \in C_c^1(I)$ ¹ we have the inequality $|\int_I u\phi'| \leq C\|\phi\|_{L^{p'}(I)}$. Furthermore, we can take $C = \|u'\|_{L^p(I)}$, [5].*

¹We follow the notation of Brezis in [5] by introducing $C_c^k(\Omega)$ as the space of compactly supported, k times continuously differentiable functions on Ω .

Moreover, $u \in W^{1,p}(I)$ with $p \in [1, \infty]$ iff there exists an absolutely continuous representative of u in $L^p(I)$ with a classical derivative in $L^p(I)$, [5, 21].

All this results in

$$(40) \quad \frac{1}{2} \left[\mathcal{D}_{\Delta t}^k (\|w_m\|_{L^2}^2) + \Delta t \|\mathcal{D}_{\Delta t}^k (w_m)\|_{L^2}^2 \right] + D_m \|\partial_z w_m^k\|_{L^2}^2 \\ + \frac{\gamma_m}{2} \left[\mathcal{D}_{\Delta t}^k (\|\partial_z w_m\|_{L^2}^2) + \Delta t \|\mathcal{D}_{\Delta t}^k (\partial_z w_m)\|_{L^2}^2 \right] \\ \leq L_m^k \|w_m^k\|_{L^2} + M_m^k \|\partial_z w_m^k\|_{L^2} + N_m^k (t_k - t_0)$$

and

$$(41) \quad \|\mathcal{D}_{\Delta t}^k (w_m)\|_{L^2}^2 + \gamma_m \|\mathcal{D}_{\Delta t}^k (\partial_z w_m)\|_{L^2}^2 \\ + \frac{D_m}{2} \left[\mathcal{D}_{\Delta t}^k (\|\partial_z w_m\|_{L^2}^2) + \Delta t \|\mathcal{D}_{\Delta t}^k (\partial_z w_m)\|_{L^2}^2 \right] \\ \leq L_m^k \|\mathcal{D}_{\Delta t}^k (w_m)\|_{L^2} + M_m^k \|\mathcal{D}_{\Delta t}^k (\partial_z w_m)\|_{L^2} + N_m^k$$

For brevity, we have omitted to point out the spatial domain dependence in the norms. The newly introduced functions L_m^k , M_m^k and N_m^k , are defined as

$$(42a) \quad L_m^k = F_m \|v^{k-1}\|_{L^2} + G_{w,m}$$

$$(43a) \quad M_m^k = \sum_{j=1}^{d-1} \left[E_{m01j} \|\mathcal{D}_{\Delta t}^k (w_j)\|_{L^2} + \sum_{i=0}^1 E_{mi0j} \|\partial_z^i w_j^{k-1}\|_{L^2} \right] \\ + \sum_{j=1}^{d-1} \left[E_{m10j} |A_j| \left(\frac{\hat{J}_j \phi_{j,res}}{H_{\phi_{\min}}} (t_k - t_0) + \|\partial_z w_j^{k-1}\|_{L^2} \right. \right. \\ \left. \left. + V \sqrt{t_k - t_0} + \frac{\hat{J}_d \phi_{d,res}}{\Gamma_{\phi_{\min}}} (t_k - t_0) \right) \right] \\ + \sum_{j=1}^{d-1} \left[E_{m00j} \left(2 \frac{\hat{J}_j \phi_{j,res}}{H_{\phi_{\min}}} (t_k - t_0) + \|\partial_z w_j^{k-1}\|_{L^2} \right) \right] \\ + \sum_{j=1}^{d-1} \left[E_{m01j} \left(2 \frac{\hat{J}_j \phi_{j,res}}{H_{\phi_{\min}}} + \|\mathcal{D}_{\Delta t}^k (\partial_z w_j)\|_{L^2} \right) \right] \\ + D_m |A_m| \left(\frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}} (t_k - t_0) + \|\partial_z w_m^k\|_{L^2} \right. \\ \left. + V \sqrt{t_k - t_0} + \frac{\hat{J}_d \phi_{d,res}}{\Gamma_{\phi_{\min}}} (t_k - t_0) \right) \\ + \gamma_m |A_m| \left(\frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}} + \|\mathcal{D}_{\Delta t}^k (\partial_z w_m)\|_{L^2} + \|\partial_z v^{k-1}\|_{L^2} + \frac{\hat{J}_d \phi_{d,res}}{\Gamma_{\phi_{\min}}} \right) \\ (43b) \quad = \mathcal{M}_m^k + \sum_{j=1}^{d-1} \sum_{i=0}^1 [\mathcal{M}_{mi0j} \|\partial_z^i w_j^{k-1}\|_{L^2} + \mathcal{M}_{mi1j} \|\mathcal{D}_{\Delta t}^k (\partial_z^i w_j)\|_{L^2}] \\ + \mathcal{M}_{mm} \|\partial_z w_m^k\|_{L^2} + \mathcal{M}_{vm} \|\partial_z v^{k-1}\|_{L^2}$$

$$(44a) \quad N_m^k = (G_{w,m} + F_m \|\partial_z v^{k-1}\|_{L^2} + \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}) \frac{\hat{J}_m \phi_{m,res}}{H_{\phi_{\min}}}$$

$$(44b) \quad = \mathcal{N}_{m0} + \mathcal{N}_{m1} \|\partial_z v^{k-1}\|_{L^2} + \mathcal{N}_{m2} \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}$$

Remark that the L^∞ norms are sufficient, since they can be bounded from above by [Claim 3](#) or the embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ for bounded intervals Ω with embedding constant $C_\infty = \sqrt{|\Omega| + 1/|\Omega|}$, which equals $\sqrt{2}$ for our domain $\Omega = (0, 1)$.

We combine the different quadratic inequalities to create inequalities for which we can apply Gronwall's lemmas [Lemmas 6](#) and [7](#). We add Equations [\(40\)](#) and [\(41\)](#) and apply Young's inequality with a parameter $\eta_{index} > 0$, Minkowski's inequality and the $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ embedding, leading to the new quadratic inequality:

$$(45) \quad \frac{1}{2} \mathcal{D}_{\Delta t}^k \left(\sum_{m=1}^{d-1} [\|w_m\|_{L^2}^2 + (\gamma_m + D_m) \|\partial_z w_m\|_{L^2}^2] \right) \\ + \sum_{m=1}^{d-1} \left[\left(1 + \frac{\Delta t}{2}\right) \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}^2 + \left[\gamma_m \left(1 + \frac{\Delta t}{2}\right) + D_m \frac{\Delta t}{2}\right] \|\mathcal{D}_{\Delta t}^k(\partial_z w_m)\|_{L^2}^2 \right] \\ \leq K_{w0} + \sum_{m=1}^{d-1} \left[K_{w1m} \|w_m^k\|_{L^2}^2 + K_{w2m} \|\partial_z w_m^k\|_{L^2}^2 + K_{w3m} \|w_m^{k-1}\|_{L^2}^2 + K_{w4m} \|\partial_z w_m^{k-1}\|_{L^2}^2 \right. \\ \left. + K_{w5m} \|\mathcal{D}_{\Delta t}^k(w_m)\|_{L^2}^2 + K_{w6m} \|\mathcal{D}_{\Delta t}^k(\partial_z w_m)\|_{L^2}^2 \right] + K_{w7} \|v^{k-1}\|_{L^2}^2 + K_{w8} \|\partial_z v^{k-1}\|_{L^2}^2,$$

Analogously, from Equations [\(34\)](#) and [\(35\)](#), the latter summed over l , we obtain the quadratic inequalities:

$$(46) \quad \mathcal{D}_{\Delta t}^k (\|\phi_l\|_{L^2}^2) + 2\delta_l \|\partial_z \phi_l^k\|_{L^2}^2 + \Delta t \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 \\ \leq K_{a\phi 0} + K_{a\phi 1} \|\partial_z v^{k-1}\|_{L^2}^2 + K_{a\phi 2l} \|\phi_l^k\|_{L^2}^2 + \sum_{n \neq d-1} [K_{a\phi 3ln} \|\partial_z \phi_n^{k-1}\|_{L^2}^2] \\ + \sum_{m=1}^{d-1} \sum_{i=0}^1 \left[K_{a\phi(4+i)m} \|\partial_z^i w_m^{k-1}\|_{L^2}^2 + K_{a\phi(6+i)m} \|\mathcal{D}_{\Delta t}^k(\partial_z^i w_m)\|_{L^2}^2 \right],$$

and

$$(47) \quad \mathcal{D}_{\Delta t}^k \left(\sum_{l \neq d-1} \|\partial_z \phi_l\|_{L^2}^2 \right) + \sum_{l \neq d-1} \left[\frac{2}{\delta_l} \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 + \Delta t \|\mathcal{D}_{\Delta t}^k(\partial_z \phi_l)\|_{L^2}^2 \right] \\ \leq K_{b\phi 0} + K_{b\phi 1} \|\partial_z v^{k-1}\|_{L^2}^2 + \sum_{l \neq d-1} \left[K_{b\phi 2l} \|\mathcal{D}_{\Delta t}^k(\phi_l)\|_{L^2}^2 + K_{b\phi 3l}^{k-1} \|\partial_z \phi_l^{k-1}\|_{L^2}^2 \right] \\ + \sum_{m=1}^{d-1} \sum_{i=0}^1 \left[K_{b\phi(4+i)m} \|\partial_z^i w_m^{k-1}\|_{L^2}^2 + K_{b\phi(6+i)m} \|\mathcal{D}_{\Delta t}^k(\partial_z^i w_m)\|_{L^2}^2 \right].$$

The constants $K_{w\ index}$ are given by

$$(48a) \quad K_{w0}^k := \sum_{m=1}^{d-1} \left[\frac{G_{w,m}^2}{2} \left(\frac{1}{\eta_{1m}} + \frac{1}{\eta_{2m}} \right) + \mathcal{N}_{m0} [1 + (t_k - t_0)] \right. \\ \left. + \frac{\mathcal{N}_{m1}^2}{2} \left(\frac{1}{\eta_{\mathcal{N}m11}} + \frac{t_k - t_0}{\eta_{\mathcal{N}m12}} \right) + \frac{\mathcal{N}_{m2}^2}{2} \left(\frac{1}{\eta_{\mathcal{N}m21}} + \frac{t_k - t_0}{\eta_{\mathcal{N}m22}} \right) \right. \\ \left. + \frac{(\mathcal{M}_m^k)^2}{2} \left(\frac{1}{\eta_{\mathcal{M}m1}} + \frac{1}{\eta_{\mathcal{M}m2}} \right) \right]$$

$$(48b) \quad K_{w1m} := \frac{\eta_{1m}}{2} + \frac{\eta_{Lm1}}{2},$$

$$(48c) \quad K_{w2m} := \frac{\eta_{\mathcal{M}m1}}{2} + \sum_{j=1}^{d-1} \sum_{i=0}^1 \left(\frac{\eta_{\mathcal{M}mi0j0}}{2} + \frac{\eta_{\mathcal{M}mi1j0}}{2} \right) + \frac{\eta_{\mathcal{M}mm}}{2} \\ + \frac{\mathcal{M}_{vm}^2}{2\eta_{\mathcal{M}vm1}} + \mathcal{M}_{mm} - D_m,$$

$$(48d) \quad K_{w3m} := \sum_{j=1}^{d-1} \left[\frac{\mathcal{M}_{j00m}^2}{2\eta_{\mathcal{M}j00m0}} + \frac{\mathcal{M}_{j00m}}{2} \eta_{\mathcal{M}j00m1} \right],$$

$$(48e) \quad K_{w4m} := \sum_{j=1}^{d-1} \left[\frac{\mathcal{M}_{j10m}^2}{2\eta_{\mathcal{M}j10m0}} + \frac{\mathcal{M}_{j10m}}{2} \eta_{\mathcal{M}j10m1} \right],$$

$$(48f) \quad K_{w5m} := \frac{\eta_{2m}}{2} + \frac{\eta_{Lm2}}{2} + \frac{\eta_{\mathcal{N}m21}}{2} + \frac{\eta_{\mathcal{N}m22}}{2} \\ + \sum_{j=1}^{d-1} \left[\frac{\mathcal{M}_{j01m}^2}{2\eta_{\mathcal{M}j01m0}} + \frac{\mathcal{M}_{j01m}}{2} \eta_{\mathcal{M}j01m1} \right],$$

$$(48g) \quad K_{w6m} := \frac{\eta_{\mathcal{M}m2}}{2} + \frac{\mathcal{M}_{mm}^2}{2\eta_{\mathcal{M}mm}} + \frac{\mathcal{M}_{vm}^2}{2\eta_{\mathcal{M}vm2}} + \sum_{j=1}^{d-1} \left[\frac{\mathcal{M}_{m01j}}{2\eta_{\mathcal{M}m01j1}} + \sum_{i=0}^1 \frac{\mathcal{M}_{mi0j}}{2\eta_{\mathcal{M}mi0j1}} \right] \\ + \sum_{j=1}^{d-1} \left[\frac{\mathcal{M}_{j11m}^2}{2\eta_{\mathcal{M}j11m0}} + \frac{\mathcal{M}_{m11j}}{2} \eta_{\mathcal{M}m11j1} + \frac{\mathcal{M}_{j11m}}{2\eta_{\mathcal{M}j11m1}} \right],$$

$$(48h) \quad K_{w7} := \sum_{m=1}^{d-1} \frac{F_m^2}{2} \left(\frac{1}{\eta_{Lm1}} + \frac{1}{\eta_{Lm2}} \right),$$

$$(48i) \quad K_{w8} := \sum_{m=1}^{d-1} \left(\frac{\eta_{\mathcal{N}m11}}{2} + \frac{\eta_{\mathcal{N}m12}}{2} + \frac{\eta_{\mathcal{M}vm1}}{2} + \frac{\eta_{\mathcal{M}vm2}}{2} \right),$$

where $\eta_{index} > 0$ are auxiliary constants found in Cauchy's inequality, while the constants $K_{x\phi index}$ for $x \in \{a, b\}$ are equal to

$$(49a) \quad K_{x\phi 0} := \left(\delta_{bx} \sum_{l \neq d-1} + \delta_{ax} \max_{l \neq d-1} \right) \frac{G_{\phi, l}^2}{\sigma_{x, l}^2 \eta_{xgl}},$$

$$(49b) \quad K_{x\phi 1} := \left(\delta_{bx} \sum_{l \neq d-1} + \delta_{ax} \max_{l \neq d-1} \right) \left[\frac{I_l^2 \Gamma^2}{\sigma_{x, l}^2 \eta_{xl}} + 2\delta_{ax} \sum_{n \neq d-1} \frac{I_l^2 \Gamma^2 C_p^2}{\eta_{aln}} \right],$$

$$(49c) \quad K_{x\phi 2l} := \eta_{xl} + \eta_{xgl} + \sum_{m=1}^{d-1} (\eta_{xl00m} + \eta_{xl01m} + \eta_{xl10m} + \eta_{xl11m}) \\ + 2\delta_{bx} \sum_{n \neq d-1} \left[\eta_{bln} + \sum_{j=1}^{d-1} (\eta_{bl10jn} + \eta_{bl11jn}) \right],$$

$$(49d) \quad K_{a\phi 3ln} := 2 \left[\eta_{aln} + \sum_{m=1}^{d-1} (\eta_{al10mn} + \eta_{al11mn}) \right],$$

$$(49e) \quad K_{b\phi 3l}^{k-1} := 2 \sum_{n \neq d-1} \left[\frac{I_n^2 \Gamma^2 C_\infty^2}{\delta_n^2 \eta_{bnl}} \|v^{k-1}\|_{H^1}^2 \right. \\ \left. + \sum_{m=1}^{d-1} \left(\frac{B_{n11m}^2 C_\infty^2}{\delta_l^2 \eta_{bn11ml}} \|\mathcal{D}_{\Delta t}^k(w_m)\|_{H^1}^2 + \frac{B_{n10m}^2 C_\infty^2}{\delta_l^2 \eta_{bn10ml}} \|w_m^{k-1}\|_{H^1}^2 \right) \right], \\ = K_{\phi 3l1} \|v^{k-1}\|_{H^1}^2 + \sum_{m=1}^{d-1} \left[K_{\phi 3l2m} \|\mathcal{D}_{\Delta t}^k(w_m)\|_{H^1}^2 + K_{\phi 3l3m} \|w_m^{k-1}\|_{H^1}^2 \right],$$

$$(49f) \quad K_{x\phi 4m} := \left(\delta_{bx} \sum_{l \neq d-1} + \delta_{ax} \max_{l \neq d-1} \right) \left[\frac{B_{l00m}^2}{\sigma_{x, l}^2 \eta_{xl00m}} + 2\delta_{ax} \sum_{n \neq d-1} \frac{B_{l10m}^2}{\eta_{al10mn}} \right],$$

$$(49g) \quad K_{x\phi 5m} := \left(\delta_{bx} \sum_{l \neq d-1} + \delta_{ax} \max_{l \neq d-1} \right) \frac{B_{l10m}^2}{\sigma_{x, l}^2 \eta_{xl10m}},$$

$$(49h) \quad K_{x\phi 6m} := \left(\delta_{bx} \sum_{l \neq d-1} + \delta_{ax} \max_{l \neq d-1} \right) \left[\frac{B_{l01m}^2}{\sigma_{x, l}^2 \eta_{xl01m}} + 2\delta_{ax} \sum_{n \neq d-1} \frac{B_{l11m}^2}{\eta_{al11mn}} \right],$$

$$(49i) \quad K_{x\phi 7m} := \left(\delta_{bx} \sum_{l \neq d-1} + \delta_{ax} \max_{l \neq d-1} \right) \frac{B_{l11m}^2}{\sigma_{x, l}^2 \eta_{xl11m}},$$

where $\sigma_{x, l}$ equals 1 for $x = a$ and δ_l for $x = b$ and where δ_{ij} denotes the Kronecker symbol. Furthermore, we have used the Poincaré inequality for v^{k-1} with the constant $C_p = |\Omega| = 1$.

With Equations (45)–(47) we can bound the $H^1(0, 1)$ norms of functions $w_1^k, \dots, w_{d-1}^k, \phi_1^k, \dots, \phi_{d-2}^k$ and ϕ_d^k . The $H^1(0, 1)$ norm bounds of ϕ_{d-1}^k and v^k need to be determined in a different way.

Even though one can use the identity $\sum_{l=1}^d \phi_l^k = 1$ to obtain a bound for $\|\phi_{d-1}^k\|_{L^2}$, this identity will give only the universal upper bound $1 - (d-1)\phi_{\min}$ as we have no knowledge for the lower bounds of $\|\phi_l^k\|_{L^2}$ except for that this value is greater or equal to ϕ_{\min} . Therefore, we create a differential equation for ϕ_{d-1} by applying $\sum_{l=1}^d \phi_l^k = 1$ to [Equation \(4a\)](#). Then, we test with ϕ_{d-1}^k to obtain a non-trivial inequality. Finally, we can apply [Lemma 6](#) to obtain an upper bound to the $L^2(0, 1)$ norm of ϕ_{d-1}^k . All this results in

$$\begin{aligned}
 (50) \quad & \mathcal{D}_{\Delta t}^k (\|\phi_{d-1}\|_{L^2}^2) + \Delta t \|\mathcal{D}_{\Delta t}^k (\phi_{d-1})\|_{L^2}^2 \\
 & \leq 2 \|\phi_{d-1}^k\|_{L^2} \sum_{l \neq d-1} \left[2I_l \Gamma \sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} \|v^{k-1}\|_{L^\infty} + I_l \Gamma \|\partial_z v^{k-1}\|_{L^2} \right. \\
 & \quad \left. + G_{\phi, l} + \sum_{m=1}^{d-1} \left[\sum_{i=0}^1 B_{li0m} \|\partial_z^i w_m^{k-1}\|_{L^2} + B_{li1m} \|\mathcal{D}_{\Delta t}^k (\partial_z^i w_m)\|_{L^2} \right] \right] \\
 & + 4 \sum_{l \neq d-1} \sum_{m=1}^{d-1} \left[\sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} (B_{l10m} \|w_m^{k-1}\|_{L^2} + B_{l11m} \|\mathcal{D}_{\Delta t}^k (w_m)\|_{L^2}) \right] \\
 & \quad + 2 \|\partial_z \phi_{d-1}^k\|_{L^2} \sum_{l \neq d-1} \delta_l \|\partial_z \phi_l^k\|_{L^2}
 \end{aligned}$$

Fortunately, we can use $\sum_j \phi_j^k = 1$ to obtain directly an upper bound for $\|\partial_z \phi_{d-1}^k\|_{L^2}$.

$$(51) \quad \|\partial_z \phi_{d-1}^k\|_{L^2}^2 \leq (d-1) \sum_{l \neq d-1} \|\partial_z \phi_l^k\|_{L^2}^2$$

The bounds for $\|\partial_z \phi_l^k\|_{H^1}$ follow from [Theorem 35](#) applied to [Equation \(4a\)](#) together with the previous upper bounds. The upper bounds for $\|\partial_z \phi_l^k\|_{H^1}$ equal

$$\|\partial_z \phi_l^k\|_{H^1}^2 = \|\partial_z \phi_l^k\|_{L^2}^2 + \|\partial_z^2 \phi_l^k\|_{L^2}^2$$

with

$$\begin{aligned}
 \|\partial_z^2 \phi_l^k\|_{L^2} & \leq \frac{1}{\delta_l} \left[\|\mathcal{D}_{\Delta t}^k (\phi_l)\|_{L^2} + I_l \Gamma \|\partial_z v^{k-1}\|_{L^2} \right. \\
 & \quad \left. + G_{\phi, l} + 2I_l \Gamma C_\infty \|v^{k-1}\|_{H^1} \left(\sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} \right) \right. \\
 & \quad \left. + \sum_{m=1}^{d-1} \left[\sum_{i=0}^1 (B_{li0m} \|\partial_z^i w_m^{k-1}\|_{L^2} + B_{li1m} \|\mathcal{D}_{\Delta t}^k (\partial_z^i w_m)\|_{L^2}) \right. \right. \\
 & \quad \left. \left. + 2C_\infty \left(\sum_{n \neq d-1} \|\partial_z \phi_n^{k-1}\|_{L^2} \right) (B_{l10m} \|w_m^{k-1}\|_{H^1} + B_{l11m} \|\mathcal{D}_{\Delta t}^k (w_m)\|_{H^1}) \right] \right]
 \end{aligned}$$

Therefore, the sum $\sum_{j=1}^k \|\partial_z^2 \phi_l^j\|_{L^2}^2 \Delta t$ can be bounded independent of Δt .

A similar approach to determine bounds for $\|\partial_z w_m^k\|_{H^1}$ leads to a Δt -independent

upper bound for $\|\partial_z w_m^k\|_{H^1}$ itself.

For the $H^1(0, 1)$ norm of v^k we will use the previously determined inequalities and apply them to a rewritten version of Equation (4b) leading to the following inequality.

$$\begin{aligned}
& \sum_{j=0}^k \|\partial_z v^j\|_{L^2}^2 \Delta t \\
& \leq \|\partial_z v^0\|_{L^2}^2 \Delta t + \frac{\Delta t}{\Gamma(\phi^0)^2} \|\partial_z (\Gamma(\phi^0)v^1)\|_{L^2}^2 \\
& \quad + \sum_{j=2}^k \frac{\Delta t}{\Gamma_{\phi_{\min}}^2} \left(\|\partial_z (\Gamma(\phi^{j-1})v^j)\|_{L^2} + C_\infty \Gamma \sum_{l \neq d-1} \|\partial_z \phi_l^{j-1}\|_{L^2} \|v^j\|_{H^1} \right)^2 \\
& \leq \|\partial_z v^0\|_{L^2}^2 \Delta t + \frac{d\Delta t}{\Gamma(\phi^0)^2} \left[G_v(\phi^0)^2 + \sum_{m=1}^{d-1} H_{1m}(\phi^0)^2 \|\mathcal{D}_{\Delta t}^1(\partial_z w_m)\|_{L^2}^2 \right] \\
& \quad + \sum_{j=2}^k \frac{(4d-2)\Delta t}{\Gamma_{\phi_{\min}}^2} \left(8C_\infty^2 \Gamma^2 (d-1) \left[\sum_{l \neq d-1} \|\partial_z \phi_l^{j-1}\|_{L^2}^2 \right] \|\partial_z v^j\|_{L^2}^2 + G_v^2 \right. \\
& \quad \left. + \sum_{m=1}^{d-1} \left[H_{0m}^2 \|\partial_z w_m^{j-1}\|_{L^2}^2 + H_{1m}^2 \|\mathcal{D}_{\Delta t}^j(\partial_z w_m)\|_{L^2}^2 \right. \right. \\
& \quad \left. \left. + 4C_\infty^2 (d-1) \left(\sum_{n \neq d-1} \|\partial_z \phi_n^{j-1}\|_{L^2}^2 \right) \left(H_{0m}^2 \|w_m^{j-1}\|_{H^1}^2 + H_{1m}^2 \|\mathcal{D}_{\Delta t}^j(w_m)\|_{H^1}^2 \right) \right] \right) \\
& \leq Q_{\Delta t}(T_{\Delta t} - t_0, V^2)
\end{aligned}$$

Since all previous upper bounds are strictly increasing and continuous in $T_{\Delta t} - t_0$ and V^2 , these properties hold as well for $Q_{\Delta t}(T_{\Delta t} - t_0, V^2)$.

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