On the relationship between k-planar and k-quasi planar graphs

On the Relationship between k-Planar and k-Quasi Planar Graphs

Patrizio Angelini[1], Michael A. Bekos[1], Franz J. Brandenburg[2], Giordano Da Lozzo[3], Giuseppe Di Battista[4], Walter Didimo[5], Giuseppe Liotta[5], Fabrizio Montecchiani[5], and Ignaz Rutter[6]

[1] Universität Tübingen, Germany, {angelini,bekos}@informatik.uni-tuebingen.de
[2] University of Passau, Germany, brandenb@fim.uni-passau.de
[3] University of California, Irvine, CA USA, gdalozzo@uci.edu
[4] Roma Tre University, Italy, gdb@dia.uniroma3.it
[5] Universitá degli Studi di Perugia, Italy, {walter.didimo, giuseppe.liotta, fabrizio.montecchiani}@unipg.it
[6] TU Eindhoven, The Netherlands, i.rutter@tue.nl

Abstract. A graph is k-planar ($k \geq 1$) if it can be drawn in the plane such that no edge is crossed more than k times. A graph is k-quasi planar ($k \geq 2$) if it can be drawn in the plane with no k pairwise crossing edges. The families of k-planar and k-quasi planar graphs have been widely studied in the literature, and several bounds have been proven on their edge density. Nonetheless, only trivial results are known about the relationship between these two graph families. In this paper we prove that, for $k \geq 3$, every k-planar graph is $(k+1)$-quasi planar.

1 Introduction

Drawings of graphs are used in a variety of application domains, including software engineering, circuit design, computer networks, database design, social sciences, and biology (see e.g. [15,16,25,28,38,40]). The aim of a graph visualization is to clearly convey the structure of the data and their relationships, in order to support users in their analysis tasks. In this respect, and independent of the specific domain, there is a general consensus that graph layouts with many edge crossings are hard to read, as also witnessed by several user studies on the subject (see e.g. [24,35,36,43]). This motivation has generated lots of research on finding bounds on the number of edge crossings in different graph families (see e.g. [34,37,42]) and on the problem of automatically computing graph layouts with as few crossings as possible (see e.g. [4,12]). We recall that, although it is linear-time solvable to decide whether a graph admits a planar drawing (i.e. a drawing without edge crossings) [10,23], minimizing the number of edge crossings is a well-known NP-hard problem [21].

An emerging research area, informally recognized as *beyond planarity* (see e.g. [22,26,29]), concentrates on different models of graph planarity relaxations, which allow edge crossings but forbid specific configurations that would affect

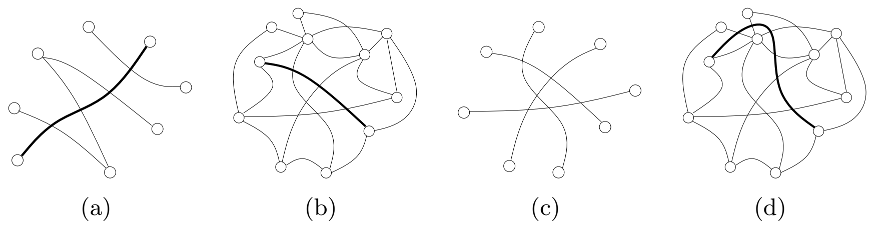


Fig. 1: (a) A crossing configuration that is forbidden in a 3-planar topological graph. (b) A 3-planar topological graph. (c) A crossing configuration that is forbidden in a 4-quasi planar topological graph. (d) A 4-quasi planar topological graph obtained from the one of Figure (b) by suitably rerouting the thick edge.

the readability of the drawing too much. Forbidden crossing configurations can be, for example, a single edge that is crossed too many times [33], a group of mutually crossing edges [20,39], two edges that cross at a sharp angle [17], a group of adjacent edges crossed by another edge [14], or an edge that crosses two independent edges [5,9,27]. Different models give rise to different families of "beyond planar" graphs. Two of the most popular families introduced in this context are the *k-planar* graphs and the *k-quasi planar* graphs, which are usually defined in terms of *topological graphs*, i.e., graphs with a geometric representation in the plane with vertices as points and edges as Jordan arcs connecting their endpoints. A topological graph is k-planar ($k \geq 1$) if no edge is crossed more than k times, while it is k-quasi planar ($k \geq 2$) if it can be drawn in the plane with no k pairwise crossing edges. Figure 1a shows a crossing configuration that is forbidden in a 3-planar topological graph. Figure 1b depicts a 3-planar topological graph that is not 2-planar (e.g., the thick edge is crossed three times). Figure 1c shows a crossing configuration that is forbidden in a 4-quasi planar topological graph. Figure 1d depicts a 4-quasi planar topological graph that is not 3-quasi planar. A graph is *k-planar* (*k-quasi planar*) if it is isomorphic to some k-planar (k-quasi planar) topological graph. Clearly, by definition, k-planar graphs are also $(k+1)$-planar and k-quasi planar graphs are also $(k+1)$-quasi planar. This naturally defines a hierarchy of k-planarity and a hierarchy of k-quasi planarity. Also, the class of 2-quasi planar graphs coincides with that of planar graphs. Note that, in the literature, 3-quasi planar graphs are also called *quasi planar*.

The k-planarity and k-quasi planarity hierarchies have been widely explored in graph theory, graph drawing, and computational geometry, mostly in terms of edge density. Pach and Tóth [33] proved that a k-planar *simple* topological graph with n vertices has at most $1.408\sqrt{k}n$ edges. We recall that a topological graph is simple if any two edges cross in at most one point and no two adjacent edges cross. For $k \leq 4$, Pach and Tóth [33] also established a finer bound of $(k+3)(n-2)$ on the edge density, and prove its tightness for $k \leq 2$. For $k = 3$, the best known upper bound on the edge density is $5.5n - 11$, which is tight up to an additive constant [6,30].

Concerning k-quasi planar graphs, a 20-year-old conjecture by Pach, Shahrokhi, and Szegedy [32] asserts that, for every fixed k, the maximum number of edges in a k-quasi planar graph with n vertices is $O(n)$. However, linear upper bounds have been proven only for $k \leq 4$. Agarwal et al. [3] were the first to prove that 3-quasi planar simple topological graphs have a linear number of edges. This was generalized by Pach et al. [31], who proved that *all* 3-quasi planar graphs on n vertices have at most $65n$ edges. This bound was further improved to $8n - O(1)$ by Ackerman and Tardos [2]. For 3-quasi planar simple topological graphs they also proved a bound of $6.5n - 20$, which is tight up to an additive constant. Ackerman [1] also proved that 4-quasi planar graphs have at most a linear number of edges. For $k \geq 5$, several authors have shown super-linear upper bounds on the edge density of k-quasi planar graphs (see, e.g., [13,19,20,32,41]). The most recent results are due to Suk and Walczak [39], who proved that any k-quasi planar simple topological graph on n vertices has at most $c_k n \log n$ edges, where c_k is a number that depends only on k. For k-quasi planar topological graphs where two edges can cross in at most t points, they give an upper bound of $2^{\alpha(n)^c} n \log n$, where $\alpha(n)$ is the inverse of the Ackermann function, and c depends only on k and t.

Despite the many papers mentioned above, the relationships between the hierarchies of k-planar and k-quasi planar graphs have not been studied yet and only trivial results are known. For example, due to the tight bounds on the edge density of 3-planar and 3-quasi planar simple graphs, it is immediate to conclude that there are infinitely many 3-quasi planar graphs that are not 3-planar. Also, it can be easily observed that, for $k \geq 1$, every k-planar graph is $(k+2)$-quasi planar. Indeed, if a k-planar graph G were not $(k+2)$-quasi planar, any topological graph isomorphic to G would contain $k+2$ pairwise crossing edges; but this would imply that any of these edges is crossed at least $k+1$ times, thus contradicting the hypothesis that G is k-planar.

Contribution. In this paper we focus on simple topological graphs and prove the first non-trivial inclusion relationship between the k-planarity and the k-quasi planarity hierarchies. We show that every k-planar graph is $(k+1)$-quasi planar, for every $k \geq 3$. In other words, we show that every k-planar simple topological graph can be redrawn so to become a $(k+1)$-quasi planar simple topological graph ($k \geq 3$). For example, the simple topological graph of Figure 1b is 3-planar but not 4-quasi planar. The simple topological graph of Figure 1d, on the other hand, is a 4-quasi planar graph obtained from the one of Figure 1b by rerouting an edge (but it is no longer 3-planar).

The proof of our result is based on the following novel methods: (i) A general purpose technique to "untangle" groups of mutually crossing edges. More precisely, we show how to reroute the edges of a k-planar topological graph in such a way that all vertices of a set of $(k+1)$ pairwise crossing edges lie in the same connected region of the plane. (ii) A global edge rerouting technique, based on a matching argument, used to remove all forbidden configurations of $(k+1)$ pairwise crossing edges from a k-planar simple topological graph, provided that these edges are "untangled".

Fig. 2: (a) An untangled 3-crossing; all vertices belong to the same face of the arrangement (the outer face). (b) A tangled 3-crossing; the circled vertices and the solid vertices belong to distinct faces of the arrangement.

The remainder of the paper is structured as follows. In Section 2 we give some basic terminology and observations that will be used throughout the paper. Section 3 describes our general proof strategy. Section 4 and Section 5 provide details about methods (i) and (ii), respectively. Conclusions and open problems are in Section 6.

2 Preliminaries

We only consider graphs with neither parallel edges nor self-loops. Also, we will assume our graphs to be connected, as our results immediately carry over to disconnected graphs. A *topological graph* G is a graph drawn in the plane with vertices represented by points and edges represented by Jordan arcs connecting the corresponding endpoints. In notation and terminology, we do not distinguish between the vertices and edges of a graph, and the points and arcs representing them, respectively. Two edges *cross* if they share one interior point and alternate around this point. Graph G is *almost simple* if any two edges cross at most once. Graph G is *simple* if it is almost simple and no two adjacent edges cross each other. Graph G divides the plane into topologically connected regions, called *faces*. The unbounded region is the *outer face*. Note that the boundary of a face can contain vertices of the graph and crossing points between edges.

If G and G' are two isomorphic graphs, we write $G \simeq G'$. A graph G' is k-planar (k-quasi planar) if there exists a k-planar (k-quasi planar) topological graph $G \simeq G'$.

Given a subgraph X of a graph G, the *arrangement of* X, denoted by A_X, is the arrangement of the curves corresponding to the edges of X. We denote the vertices and edges of X by $V(X)$ and $E(X)$, respectively. A *node* of A_X is either a vertex or a crossing point of X. A *segment* of A_X is a part of an edge of X that connects two nodes, i.e., a maximal uncrossed part of an edge of X. A *fan* is a set of edges that share a common endpoint. A set of k vertex-disjoint mutually crossing edges in a topological graph G is called a *k-crossing*. A k-crossing X is *untangled* if in the arrangement A_X of X all nodes corresponding to vertices in $V(X)$ are incident to a common face. Otherwise, it is *tangled*. For example,

(a) (b)

Fig. 3: The rerouting operation for dissolving untangled $(k+1)$-crossings. (a) An untangled $(k+1)$-crossing X. (b) The rerouting of the dashed edge (u,v) around the marked vertex w. The arrangement A'_X is thin red, the removed nodes and segments are gray. Note that the dashed curve is part of A'_X.

the 3-crossing in Figure 2a is untangled, whereas the 3-crossing in Figure 2b is tangled. We observe the following.

Observation 1 *Let $G=(V,E)$ be a k-planar simple topological graph and let X be a $(k+1)$-crossing in G. An edge in $E(X)$ cannot be crossed by any other edge in $E \setminus E(X)$. In particular, for any two $(k+1)$-crossings $X \neq Y$ in G, $E(X) \cap E(Y) = \emptyset$ holds.*

Proof. Each edge e in a $(k+1)$-crossing X crosses each of the remaining k edges in $E(X)$. Since graph G is k-planar, edge e is not crossed by any other edge in $E \setminus E(X)$. □

3 Edge Rerouting Operations and Proof Strategy

We introduce an edge rerouting operation that will be a basic tool for our proof strategy. Let G be a k-planar simple topological graph and consider an untangled $(k+1)$-crossing X in G. Without loss of generality, the vertices in $V(X)$ lie in the outer face of A_X.

Let $e=\{u,v\} \in E(X)$ and let $w \in V(X) \setminus \{u,v\}$. Denote by A'_X the arrangement obtained from A_X by removing all nodes corresponding to vertices in $V(X) \setminus \{u,v,w\}$, together with their incident segments, and by removing edge (u,v). The operation of *rerouting* $e=\{u,v\}$ *around* w consists of redrawing e sufficiently close to the boundary of the outer face of A'_X, choosing the routing that passes close to w, in such a way that e crosses the fan incident to w, but not any other edge in $E \setminus E(X)$. See Figure 3b for an illustration. More precisely, let D be a topological disk that encloses all crossing points of X and such that each edge in $E(X)$ crosses the boundary of D exactly twice. Then, the rerouted edge keeps unchanged the parts of e that go from u to the boundary of D and from v to the boundary of D. We call the unchanged parts of a rerouted edge its *tips* and the remaining part, which routes around w, its *hook*.

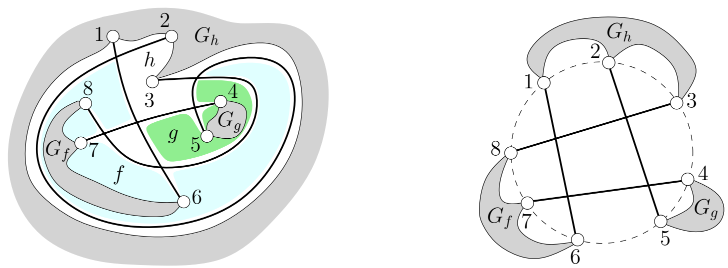
Lemma 1. *Let G be a k-planar simple topological graph and let X be an untangled $(k+1)$-crossing in G. Let $G' \simeq G$ be the topological graph obtained from G by rerouting an edge $e = \{u, v\} \in E(X)$ around a vertex $w \in V(X) \setminus \{u, v\}$. Let d be the edge of $E(X)$ incident to w. G' has the following properties: (i) Edges e and d do not cross; (ii) The edges that are crossed by e in G' but not in G form a fan at w; (iii) G' is almost simple.*

Proof. Conditions (i) and (ii) immediately follow from the definition of the rerouting operation and from the fact that edge e can be drawn arbitrarily close to the boundary of the outer face of A'_X. Since G is simple, in order to prove that G' is almost simple, we only need to show that edge e does not cross any other edge more than once. The only part of e that is drawn in G' differently than in G is the one between the intersection points of e and the boundary $B(D)$ of the topological disk D that (a) encloses all crossing points of X and such that (b) each edge in $E(X)$ crosses the boundary of D exactly twice. By (b) and by the definition of the rerouting operation, the two crossing points between an edge $e' \in E(X)$ and $B(D)$ alternate with the two crossing points between any edge $e' \neq e'' \in E(X)$ and $B(D)$ along $B(D)$. Hence, by redrawing edge e sufficiently close to any of the two parts of $B(D)$ between the two intersection points of edge e and $B(D)$, we encounter each edge in $E(X) \setminus \{e\}$ exactly once. Thus, edge e crosses all the edges in $E(X) \setminus \{e, d\}$ exactly once. This concludes the proof. □

Lemma 1 does not guarantee that graph G' is simple. Indeed, if the edge (u, w) or the edge (v, w) existed in G, then the rerouted edge $e = (u, v)$ would cross such an edge. We will show in Section 5 how to fix this problem by redrawing (u, w) and (v, w).

We are now ready to describe our general strategy for transforming a k-planar simple topological graph G into a simple topological graph $G' \simeq G$ that is $(k+1)$-quasi planar. The idea is to pick from each $(k+1)$-crossing X in G an edge e_X and a vertex w_X not adjacent to e_X, and to apply the rerouting operation *simultaneously* for all pairs (e_X, w_X), i.e., rerouting e_X around w_X. This operation, which we call *global rerouting*, is well-defined since the $(k+1)$-crossings are pairwise edge-disjoint by Observation 1.

There are however several constraints that have to be satisfied in order for such a global rerouting to have the desired effect. First of all, as mentioned above, the rerouting operation can only be applied to untangled $(k+1)$-crossings. Thus, as a first step, we will show that, in a k-planar simple topological graph, all tangled $(k+1)$-crossings can be removed, leaving the resulting graph simple and k-planar. More precisely, given a tangled $(k+1)$-crossing X, it is possible to redraw the whole graph such that either at least two edges of X do not cross or X becomes an untangled $(k+1)$-crossing, and, further, any two edges cross only if they crossed before the redrawing. The technical details for this operation are described in Section 4. Notice that, even assuming that all $(k+1)$-crossings are untangled, there are further problems that can occur when performing all the rerouting operations independently of each other. Specifically, the resulting topological graph G' may be non-simple and/or the rerouting may create new $(k+1)$-crossings. We explain how to overcome these issues in Section 5.



(a) (b)

Fig. 4: Illustration of the untangling procedure in the proof of Lemma 2: (a) A 3-planar simple topological graph with a 4-crossing X (thicker edges). (b) The topological graph resulting from the procedure that untangles X.

4 Removing Tangled $(k+1)$-Crossings

The proof of the next lemma describes a technique to "untangle" all $(k+1)$-crossings in a k-planar simple topological graph. This technique is of general interest, as it gives more insights on the structure of k-planar simple topological graphs.

Lemma 2. *Let G be a k-planar simple topological graph. There exists a k-planar simple topological graph $G' \simeq G$ without tangled $(k+1)$-crossings.*

Proof. We first show how to untangle a $(k+1)$-crossing X in a k-planar simple topological graph G by neither creating new $(k+1)$-crossings nor introducing new crossings.

Let X be a tangled $(k+1)$-crossing and let A_X be its arrangement. For each face f of A_X, denote by V_f the subset of vertices of $V(X)$ incident to f. Since in X all vertices have degree 1, the sets V_f form a partition of $V(X)$.

For each inner face f of A_X, denote by G_f the subgraph of G consisting of the vertices of V_f, and of the vertices and edges of G that lie in the interior of f. Refer to Figure 4a for an illustration. Since G is k-planar and X is a $(k+1)$-crossing, there exists no crossing between a segment in G_f and a segment not in G_f. Therefore, the boundary of f corresponds to the boundary of a topological disk D_f such that G_f is k-planarly embedded inside D_f: only the vertices of V_f lie on the boundary of D_f. For the external face h, graph G_h consists of the vertices of V_h, and of the vertices and edges of G that lie outside A_X. In this case, the topological disk D_h is obtained after a suitable inversion of G_h, if needed. We can rearrange and deform each D_f such that: (*i*) the part of the boundary of D_f that contains all the vertices of V_f lies on a circle C; (*ii*) for each face $g \neq f$ of A_X, disks D_f and D_g do not intersect; (*iii*) the interior of circle C is empty. Then, the $k+1$ edges of X are redrawn as straight-line segments inside C. This construction implies that X is untangled (and some of its edges may not cross anymore). Also, each subgraph G_f remains topologically equivalent to its initial drawing. Thus, two edges cross after the transformation only if they crossed before, which ensures that the resulting graph is simple and no new $(k+1)$-crossing is created.

We iteratively apply the above transformation to each subgraph G_f of the new topological graph until all $(k+1)$-crossings are untangled. This concludes the proof. $\square$

An illustration of the untangling procedure described in the proof of Lemma 2 is given in Figure 4. Figure 4a shows an example of a 3-planar simple topological graph G with a tangled 4-crossing X; the edges of X are thicker. Faces f, g, and h are the three faces of A_X whose union contains the vertices of $V(X)$. Subgraphs G_f, G_g, and G_h are schematically depicted. Figure 4b shows G after the transformation that untangles X.

5 Removing Untangled $(k+1)$-Crossings

Let G be a k-planar simple topological graph with $k \geq 3$. By Lemma 2, we may assume that G has no tangled $(k+1)$-crossings. In Section 5.1, we show how to transform G into a (possibly not almost simple) $(k+1)$-quasi planar topological graph $G' \simeq G$. Then, in Section 5.2, we describe how to make G' simple without introducing $(k+1)$-crossings.

5.1 Obtaining $(k+1)$-quasi planarity

We first show the existence of a global rerouting such that no two edges of G are rerouted around the same vertex (Lemma 5). Note that this is a necessary condition for almost simplicity; see Figure 5a. Then, we show that any global rerouting of G with this property yields a topological graph G' with no $(k+1)$-crossings (Lemma 9).

The existence of this global rerouting is proved by defining a bipartite graph composed of the vertices of G and of its $(k+1)$-crossings, and by showing that a matching covering all the $(k+1)$-crossings always exists. A bipartite graph with vertex sets A and B is denoted by $H = (A \cup B, E \subseteq A \times B)$. A *matching from A into B* is a set $M \subseteq E$ such that each vertex in A is incident to exactly one edge in M and each vertex in B is incident to at most one edge in M. For a subset $A' \subseteq A$, we denote by $N(A')$ the set of all vertices in B that are adjacent to a vertex in A'. We recall that, by Hall's theorem, graph H has a matching from A into B if and only if, for each set $A' \subseteq A$, it is $|N(A')| \geq |A'|$. Let G be a k-planar simple topological graph and let S be the set of $(k+1)$-crossings of G. We define a bipartite graph $H = (A \cup B, E)$ as follows. For each $(k+1)$-crossing $X \in S$, set A contains a vertex $v(X)$ and set B contains the endpoints of $E(X)$ (that is, $B = \bigcup_{X \in S} V(X)$). Also, set E contains an edge between a vertex $v(X) \in A$ and a vertex $v \in B$ if and only if $v \in V(X)$. We have the following.

Lemma 3. *Graph $H = (A \cup B, E)$ is a simple bipartite planar graph. Also, each vertex in A has degree $2k+2$.*

Proof. The graph is simple and bipartite, by construction. Also, for each $(k+1)$-crossing X, vertex $v(X) \in A$ is incident to the $2k+2$ vertices in B belonging to $V(X)$.

We prove that H is also planar by showing that a planar embedding of H can be obtained from G as follows. First, we remove from G all the vertices and edges that are not in any $(k+1)$-crossing. Then, for each $(k+1)$-crossing X of G, we remove the portion of G in the interior of a topological disk D that encloses all crossing points of X and such that each edge in $E(X)$ crosses the boundary of D exactly twice (as defined in Section 3) and add vertex $v(X)$ inside D. Finally, for each vertex v in $V(X)$, let e_v be the edge in X incident to v and let p_v be the intersection point between D and e_v in G. We complete the drawing of edge $(v(X), v)$ by adding a curve between $v(X)$ and p_v in the interior of D without introducing any crossings. The resulting topological graph is planar. □

Lemma 4. *Graph H has a matching from A into B.*

Proof. Let $A' \subseteq A$ and let H' be the subgraph of H induced by $A' \cup N(A')$. Since the vertices in A have degree $2k+2$, by Lemma 3, we have $|E(H')| = (2k+2)|A'|$. Also, since H (and thus H') is bipartite planar, by Lemma 3, we have $|E(H')| \leq 2(|A'| + N(A')) - 4$. Thus, $|N(A')| \geq k|A'| + 2 > |A'|$, and Hall's theorem applies. □

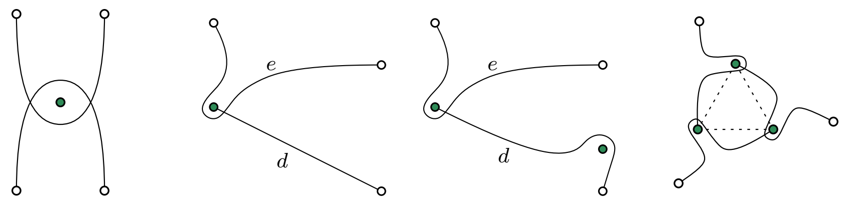
Lemma 5. *Let G be a k-planar simple topological graph. It is possible to execute a global rerouting on G such that no two edges are rerouted around the same vertex.*

Proof. Let $S = \{X_1, X_2, \ldots, X_h\}$, with $h > 0$, be the set of $(k+1)$-crossings of G. By Lemma 4, it is possible to assign a vertex $v_i \in V(X_i)$ to each $(k+1)$-crossing X_i in such a way that no two distinct $(k+1)$-crossings are assigned the same vertex. The statement follows by considering a global rerouting such that, for each $(k+1)$-crossing X_i, any edge in X_i not incident to v_i is rerouted around v_i. □

Denote by G' the topological graph obtained from G by executing a global rerouting as in Lemma 5. We prove that G' has no $(k+1)$-crossings. To this aim, we first give the conditions under which new $(k+1)$-crossings may arise in G' (Lemmas 6–8).

Lemma 6. *Let e and d be two edges that cross in G' but not in G. Then, one of e and d has been rerouted around an endpoint of the other.*

Proof. Since e and d do not cross in G, we may assume that one of them, say e, has been rerouted. Suppose first that the hook of e crosses d. We claim that e has been rerouted around an endpoint of d. In fact, if d has not been rerouted, then the claim is trivially true; see Figure 5b. On the other hand, if d has been



(a) (b) (c) (d)

Fig. 5: (a) Two edges rerouted around the same vertex. (b)–(c) Different cases of edges that do not cross before a global rerouting operation but cross afterwards. The vertices used for rerouting are filled green. (d) A 3-crossing arising from redrawing three edges.

rerouted, then the crossing with e must be on a tip of d, and not on its hook, since no two edges have been rerouted around the same vertex in the global rerouting; see Figure 5c. Thus, the claim follows. Suppose now that a tip of e crosses d. Then, this crossing must be with the hook of d, and the same argument applies to prove that d has been rerouted around an endpoint of e. □

Lemma 7. *Every non-rerouted edge e is crossed by at most three rerouted edges in G'. Further, if e is crossed by exactly three rerouted edges, then two of them have been rerouted around distinct endpoints of e.*

Proof. Since at most one edge has been rerouted around each vertex, by construction, it suffices to prove that there exists at most one rerouted edge crossing e that has not been rerouted around an endpoint of e.

For this, note that any edge with this property crosses e also in G, by Lemma 6, and thus it belongs to the same $(k+1)$-crossing as e. Since, by construction, only one edge per $(k+1)$-crossing is rerouted, the statement follows. □

Lemma 8. *If G' contains a $(k+1)$-crossing X', then X' contains at most one edge that has not been rerouted.*

Proof. Assume to the contrary that a $(k+1)$-crossing X' in G' contains at least two non-rerouted edges e and e'.

We first claim that there exists an edge d of $E(X')$ that does not cross e in G. If e has less than k crossings in G, then the claim trivially follows. If e has k crossings in G but it is not part of a $(k+1)$-crossing, then none of the edges crossing e in G can be part of a $(k+1)$-crossing in G, as otherwise they would have k crossings in the $(k+1)$-crossing and an additional crossing with e. Thus, the claim follows also in this case. Finally, if e is part of a $(k+1)$-crossing X in G, the claim follows from the fact that the edges of X do not form a $(k+1)$-crossing in G', due to a rerouting of one of its edges.

Thus, edge d has been rerouted around an endpoint of e, by Lemma 6, which means that d is part of a $(k+1)$-crossing in G containing neither e nor e'.

Hence, d and e' do not cross in G. We prove that they do not cross in G' either, a contradiction to the assumption that X' is a $(k+1)$-crossing. Namely, by Lemma 1, all new crossings of d with non-rerouted edges are on its hook; however, since e and e' cross in G (which is simple), they do not share any endpoint, and the statement follows. □

Altogether the above lemmas can be used to prove the following.

Lemma 9. *Graph G' does not contain any $(k+1)$-crossing.*

Proof. Assume for a contradiction that G' contains a $(k+1)$-crossing X'. By Lemma 8, X' contains at most one non-rerouted edge.

Suppose that X' contains one of such edges e. By Lemma 7, there are at most three rerouted edges crossing e in G'. If they are less than 3, then the claim follows, as $k \geq 3$. If they are three, say d, h, l, then by Lemma 7, two of them, say d and h, have been rerouted around (distinct) endpoints of e. Thus, d and h do not cross in G, by Observation 1, as they belong to different $(k+1)$-crossings. Hence, they can cross in G' only if one of them has been rerouted around an endpoint of the other, by Lemma 6. This is not possible since neither d nor h share an endpoint with e, as G is simple.

Suppose that X' contains only rerouted edges. Let e be any edge of X' and let w be the vertex used for rerouting e. Since at most one edge in X' can be incident to w and since $k \geq 3$, there are two edges d, h in X' that have been rerouted around distinct endpoints of e. As in the previous case, we can prove that d and h do not cross. □

For $k = 2$, Lemma 9 does not hold, as some 3-crossings may still appear after the global rerouting; see Figure 5d for an illustration and refer to Section 6 for a discussion.

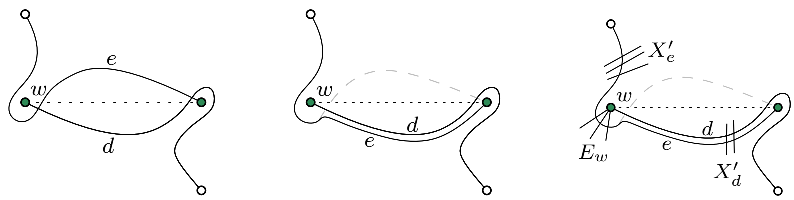
5.2 Obtaining simplicity

Lemmas 2, 5, and 9 imply that, for $k \geq 3$, any k-planar simple topological graph G can be redrawn such that the resulting topological graph $G' \simeq G$ contains no $(k+1)$-crossings and no two edges are rerouted around the same vertex. However, the graph G' may be not simple, and even not almost simple. We first show how to remove from G' pairs of edges crossing more than once, without introducing $(k+1)$-crossings, thus resulting in a $(k+1)$-quasi planar almost-simple graph (Lemma 11). Then we show how to remove crossings between edges incident to a common vertex, still without introducing $(k+1)$-crossings (Lemma 12). We will use the following auxiliary lemma.

Lemma 10. *Graph G' is almost-simple if and only if there is no pair of edges such that each of them is rerouted around an endpoint of the other.*

Proof. Clearly the condition is necessary for almost simplicity; see Figure 6a.

For the sufficiency, suppose that there exist two edges e and d crossing twice in G'. By Lemma 6 and by the simplicity of G, at least one of them, say e,



(a) (b) (c)

Fig. 6: (a) A double crossing between two edges e, d, due to a global rerouting. (b) Solving the configuration in (a) by redrawing e. (c) Edges crossing e after the transformation.

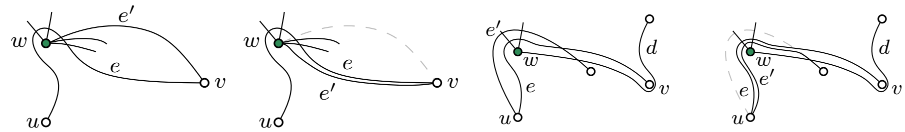
has been rerouted. By Lemma 1, this rerouting did not introduce two crossings between e and d, if d has not been rerouted, since the rerouting of a single edge leaves the graph almost simple. Thus, we may assume that also d has been rerouted. This implies that e and d belong to different $(k+1)$-crossings of G; so, they do not cross in G, by Observation 1. Hence, by Lemma 6, at least one of them has been rerouted around an endpoint of the other, say e around d. This introduces a single crossing between e and d, namely between the hook of e and a tip of d. Thus, the other crossing must be between the hook of d and a tip of e, again by Lemma 6 and by the fact that in G' no two edges are rerouted around the same vertex, and so there is no crossing between the hooks of two edges. The statement follows. □

Lemma 11. *There is a $(k+1)$-quasi planar almost simple topological graph $G^* \simeq G'$.*

Proof. We may assume that G' is not almost simple, as otherwise the statement would follow with $G^* = G'$. By Lemma 10, there exist pairs of edges in which each of the two edges has been rerouted around an endpoint of the other; see Figure 6a. For each pair e, d, we remove the two crossings by redrawing one of the two edges, say e, by following d between the two crossings. More precisely, we redraw the tip of e crossed by the hook of d by following the tip of d crossed by the hook of e, without crossing it; see Figure 6b. In the following we prove that the graph G^* obtained by applying this operation for all the pairs does not contain new $(k+1)$-crossings and is almost simple.

Observe first that each edge tip is involved in at most one pair, since no two edges are rerouted around the same vertex. Thus, no tip of an edge is transformed twice in G^* and no two transformed edges cross each other. Hence, if a $(k+1)$-crossing exists in G^*, then it contains exactly one transformed edge. We prove that this is not the possible.

Let e be an edge that has been redrawn due to a double crossing with an edge d, and let X_e and X_d be the $(k+1)$-crossings of G containing e and d, respectively. The edges crossing e in G^* are (see Figure 6c): (i) a set X'_d of edges



(a) (b) (c) (d)

Fig. 7: (a), (c) Topological graphs that are not simple. (b), (d) Avoiding the non-simplicity in (a) and (c) by redrawing one of the two edges.

in X_d crossing the tip of d that has been used to redraw e; (ii) a set X'_e of edges in X_e crossing the tip of e not crossed by d; (iii) a set E_w of edges incident to the vertex w around which e has been rerouted (and thus cross the hook of e). Note that X'_d contains all the edges that cross e in G^* and not in G'. These edges do not cross those in X'_e, since they are non-rerouted edges belonging to distinct $(k+1)$-crossings of G. Also, they do not cross edges in E_w, since X_d does not contain any edge incident to w, other than d. Finally, the edges in X'_d are at most $k-1$, since X_d contains $k+1$ edges and at least two of them do not cross e, namely d and the edge incident to the endpoint of e around which d has been rerouted. Thus, the edges in X'_d are not involved in any $(k+1)$-crossing with e. To see that the same holds for the edges in X'_e and in E_w, note that any $(k+1)$-crossing in G^* involving these edges and e would also appear in G', which is however $(k+1)$-quasi planar.

To prove that G^* is almost simple, we show that the edges in X'_d are crossed only once by e. Recall that none of these edges crosses e in G'. Also, since each tip is involved in at most one transformation, all the edges in X'_d cross the tip of d (and hence the tip of e that has been redrawn) only once. On the other hand, it could be that also the other tip of e has been transformed by following the tip of an edge h and that this transformation introduced a new crossing between e and d'. But then d' crosses both d and h in G', and hence by Lemma 6 also in G. This is however not possible, since both d and h have been rerouted, and hence they belong to different $(k+1)$-crossings in G. □

Lemma 12. *There is a $(k+1)$-quasi planar simple topological graph $\overline{G} \simeq G^*$.*

Proof. We may assume that G^* is not simple, as otherwise the statement would follow with $\overline{G} = G^*$. Let $e = (u, v)$ and $e' = (u, w)$ be two crossing edges that share an endpoint u. Since G is simple, at least one of them has been redrawn, say e.

We distinguish two cases, based on whether (i) e has rerouted but not transformed afterwards, or (ii) e has also been transformed, due to a double crossing.

In case (i), edge e crosses e' with its hook, see Figure 7a. We redraw e' by following e till reaching u, as in Figure 7b. This guarantees that e and e' no longer cross and that e' does not cross any edge twice, since e' crosses only edges that cross the tip of e incident to v. Also, no $(k+1)$-crossing is introduced. Indeed, any new $(k+1)$-crossing should contain e', but then also e would be part of this $(k+1)$-crossing, which is impossible since e and e' do not cross and G^* does not contain $(k+1)$-crossings.

In case (ii), let d be the edge that crosses e twice in G'; note that d has not been transformed, see Figure 7c. Recall that, by Lemma 10, e and d have been rerouted one around an endpoint of the other. Suppose that the endpoint of e around which d has been rerouted is v, the case in which it is u can be treated analogously. This implies that e' crosses a tip of d, and therefore e' and d are part of a $(k+1)$-crossing in G, namely the one that caused the rerouting of d. We redraw the part of e' from u to its crossing point with e by following e, without crossing it, and leave the rest of e' unchanged, as in Figure 7d. This guarantees that e and e' do no cross any longer, and that any new crossing of e' is with an edge that also crosses e. As in case (i), this implies that e' does not cross any edge twice and that no new $(k+1)$-crossing has been generated. □

The next theorem summarizes the main result of the paper.

Theorem 2. *Let G be a k-planar simple topological graph. Then, there exists a $(k+1)$-quasi planar simple topological graph $\overline{G}$ such that $\overline{G} \simeq G$.*

Proof. First recall that, by Lemma 2, we can assume that G does not contain any tangled $(k+1)$-crossing. We apply Lemma 5 to compute a global rerouting for G in which no two edges are rerouted around the same vertex. By Lemma 9, the resulting topological graph $G' \simeq G$ is $(k+1)$-quasi planar. Also, by Lemma 11, if G' is not almost simple, then it is possible to redraw some of its edges in such a way that the resulting topological graph $G^* \simeq G'$ is almost simple and remains $(k+1)$-quasi planar. Finally, by Lemma 12, if G^* is not simple, then it can be made so, again by redrawing some of its edges, while maintaining $(k+1)$-quasi-planarity. This concludes the proof that there exists a $(k+1)$-quasi planar simple topological graph $\overline{G} \simeq G$. □

6 Conclusions and Open Problems

We proved that, for any $k \geq 3$, the family of k-planar graphs is included in the family of $(k+1)$-quasi planar graphs. This result represents the first non-trivial relationship between the k-planar and the k-quasi planar graph hierarchies, and contributes to the literature that studies the connection between different families of beyond planar graphs (see, e.g. [8,9,11,18]). Several interesting problems remain open. Among them:

– The main open question is whether 2-planar graphs are quasi planar. The reason why our technique does not apply to the case of $k = 2$ is mainly due to the possible existence of three rerouted edges that are pairwise crossing after a global rerouting (as in Figure 5d). A conceivable approach to overcome this issue is by matching more than one vertex to each $(k+1)$-crossing, in order to execute a global rerouting that does not create forbidden configurations. In fact, within the lines of Lemma 4, we proved that up to k vertices can be reserved for each $(k+1)$-crossing. Nonetheless, it is not clear how to control which vertices of a $(k+1)$-crossing are assigned to it in the matching, which makes it difficult to exploit these extra vertices. Note that it was recently proved that optimal 2-planar graphs are (3-)quasi planar [7]. We recall that an n-vertex 2-planar graph is optimal if it has $4n-8$ edges.

- For $k \geq 3$, one can also ask whether the family of k-planar graphs is included in the family of k-quasi planar graphs. For $k = 2$ the answer is trivially negative, as 2-quasi planar graphs coincide with the planar graphs. On the other hand, optimal 3-planar graphs are known to be (3-)quasi planar [7]. We recall that an n-vertex 3-planar graph is optimal if it has $5.5n-11$ edges. For sufficiently large values of k, one can even investigate whether every k-planar simple topological (sparse) graph G is $f(k)$-quasi planar, for some function $f(k) = o(k)$.
- One can study *non-inclusion* relationships between the k-planar and the k-quasi planar graph hierarchies, other than those that are easily derivable from the known edge density results. For example, for any given $k > 3$, can we establish an integer function $h(k)$ such that some $h(k)$-planar graph is not k-quasi planar?

Acknowledgements. The research in this paper started at the Dagstuhl Seminar 16452 "Beyond-Planar Graphs: Algorithmics and Combinatorics". We thank all participants, and in particular Pavel Valtr and Raimund Seidel, for useful discussions on the topic.

References

1. E. Ackerman. On the maximum number of edges in topological graphs with no four pairwise crossing edges. *Discrete Comput. Geom.*, 41(3):365–375, 2009.
2. E. Ackerman and G. Tardos. On the maximum number of edges in quasi-planar graphs. *J. Comb. Theory, Ser. A*, 114(3):563–571, 2007.
3. P. K. Agarwal, B. Aronov, J. Pach, R. Pollack, and M. Sharir. Quasi-planar graphs have a linear number of edges. *Combinatorica*, 17(1):1–9, 1997.
4. P. Angelini, G. Da Lozzo, G. Di Battista, F. Frati, M. Patrignani, and V. Roselli. Relaxing the constraints of clustered planarity. *Comput. Geom.*, 48(2):42–75, 2015.
5. M. A. Bekos, S. Cornelsen, L. Grilli, S. Hong, and M. Kaufmann. On the recognition of fan-planar and maximal outer-fan-planar graphs. In C. A. Duncan and A. Symvonis, editors, *GD 2014*, volume 8871 of *LNCS*, pages 198–209. Springer, 2014.
6. M. A. Bekos, M. Kaufmann, and C. N. Raftopoulou. On the density of non-simple 3-planar graphs. In Y. Hu and M. Nöllenburg, editors, *Graph Drawing and Network Visualization*, volume 9801 of *LNCS*, pages 344–356. Springer, 2016.
7. M. A. Bekos, M. Kaufmann, and C. N. Raftopoulou. On optimal 2- and 3-planar graphs. In B. Aronov and M. Katz, editors, *SoCG 2017*, LNCS. Springer, 2017. To appear.

8. C. Binucci, M. Chimani, W. Didimo, M. Gronemann, K. Klein, J. Kratochvíl, F. Montecchiani, and I. G. Tollis. Algorithms and characterizations for 2-layer fan-planarity: From caterpillar to stegosaurus. *J. Graph Algorithms Appl.*, 21(1):81–102, 2017.
9. C. Binucci, E. Di Giacomo, W. Didimo, F. Montecchiani, M. Patrignani, A. Symvonis, and I. G. Tollis. Fan-planarity: Properties and complexity. *Theor. Comput. Sci.*, 589:76–86, 2015.
10. K. S. Booth and G. S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *J. Comput. Syst. Sci.*, 13(3):335–379, 1976.
11. F. J. Brandenburg, W. Didimo, W. S. Evans, P. Kindermann, G. Liotta, and F. Montecchiani. Recognizing and drawing IC-planar graphs. *Theor. Comput. Sci.*, 636:1–16, 2016.
12. C. Buchheim, M. Chimani, C. Gutwenger, M. Jünger, and P. Mutzel. Crossings and planarization. In R. Tamassia, editor, *Handbook on Graph Drawing and Visualization*, pages 43–85. Chapman and Hall/CRC, 2013.
13. V. Capoyleas and J. Pach. A Turán-type theorem on chords of a convex polygon. *J. Comb. Theory, Ser. B*, 56(1):9–15, 1992.
14. O. Cheong, S. Har-Peled, H. Kim, and H. Kim. On the number of edges of fan-crossing free graphs. *Algorithmica*, 73(4):673–695, 2015.
15. G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing*. Prentice Hall, Upper Saddle River, NJ, 1999.
16. W. Didimo and G. Liotta. Mining graph data. In *Graph Visualization and Data Mining*, pages 35–64. Wiley, 2007.
17. W. Didimo and G. Liotta. The crossing-angle resolution in graph drawing. In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 167–184. Springer, 2013.
18. P. Eades and G. Liotta. Right angle crossing graphs and 1-planarity. *Discrete Appl. Math.*, 161(7-8):961–969, 2013.
19. J. Fox and J. Pach. Coloring k_k-free intersection graphs of geometric objects in the plane. In *SoCG 2008*, pages 346–354. ACM, 2008.
20. J. Fox, J. Pach, and A. Suk. The number of edges in k-quasi-planar graphs. *SIAM J. Discrete Math.*, 27(1):550–561, 2013.
21. M. R. Garey and D. S. Johnson. Crossing number is NP-complete. *SIAM J. Algebraic Discrete Methods*, 4(3):312–316, 1983.
22. S. Hong and T. Tokuyama. Algorithmics for beyond planar graphs. NII Shonan Meeting Seminar 089, November 27 - December 1 2016.
23. J. E. Hopcroft and R. E. Tarjan. Efficient planarity testing. *J. ACM*, 21(4):549–568, 1974.
24. W. Huang, S. Hong, and P. Eades. Effects of sociogram drawing conventions and edge crossings in social network visualization. *J. Graph Algorithms Appl.*, 11(2):397–429, 2007.
25. M. Jünger and P. Mutzel, editors. *Graph Drawing Software*. Springer, 2003.
26. M. Kaufmann, S. Kobourov, J. Pach, and S. Hong. Beyond planar graphs: Algorithmic and combinatorics. Dagstuhl Seminar 16452, November 6 - 11 2016.
27. M. Kaufmann and T. Ueckerdt. The density of fan-planar graphs. *CoRR*, abs/1403.6184, 2014.
28. M. Kaufmann and D. Wagner, editors. *Drawing Graphs, Methods and Models*, volume 2025 of *LNCS*. Springer, 2001.

29. G. Liotta. Graph drawing beyond planarity: some results and open problems. In *ICTCS 2014.*, volume 1231 of *CEUR Workshop Proceedings*, pages 3–8. CEUR-WS.org, 2014.
30. J. Pach, R. Radoicic, G. Tardos, and G. Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. *Discrete & Computational Geometry*, 36(4):527–552, 2006.
31. J. Pach, R. Radoičić, and G. Tóth. Relaxing planarity for topological graphs. In *Discrete Comput. Geom.*, volume 2866 of *LNCS*, pages 221–232. Springer, 2003.
32. J. Pach, F. Shahrokhi, and M. Szegedy. Applications of the crossing number. *Algorithmica*, 16(1):111–117, 1996.
33. J. Pach and G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997.
34. J. Pach and G. Tóth. Thirteen problems on crossing numbers. *Geombinatorics*, 9:194–207, 2000.
35. H. C. Purchase. Effective information visualisation: a study of graph drawing aesthetics and algorithms. *Interacting with Computers*, 13(2):147–162, 2000.
36. H. C. Purchase, D. A. Carrington, and J. Allder. Empirical evaluation of aesthetics-based graph layout. *Empirical Software Engineering*, 7(3):233–255, 2002.
37. M. Schaefer. The graph crossing number and its variants: A survey. *Electron. J. Combin.*, DS21:1–100, 2014.
38. K. Sugiyama. *Graph Drawing and Applications for Software and Knowledge Engineers.* World Scientific, 2002.
39. A. Suk and B. Walczak. New bounds on the maximum number of edges in k-quasi-planar graphs. *Comput. Geom.*, 50:24–33, 2015.
40. R. Tamassia, editor. *Handbook of Graph Drawing and Visualization.* Chapman and Hall/CRC, 2013.
41. P. Valtr. On geometric graphs with no k pairwise parallel edges. *Discrete Comput. Geom.*, 19(3):461–469, 1998.
42. I. Vrto. Crossing numbers bibliography. available online at www.ifi.savba.sk/∼imrich.
43. C. Ware, H. C. Purchase, L. Colpoys, and M. McGill. Cognitive measurements of graph aesthetics. *Information Visualization*, 1(2):103–110, 2002.