Pollaczek contour integrals for the fixed-cycle traffic-light queue

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January 12, 2017

Abstract

The fixed-cycle traffic-light (FCTL) queue is the null model for intersections with static signaling, where vehicles arrive, form a queue and depart during cycles controlled by a traffic light. Classical analysis of the FCTL queue based on transform methods requires a computationally challenging step of finding the complex-valued roots of some characteristic equation. We derive a novel contour-integral expression, reminiscent of Pollaczek integrals for bulk-service queues, for the probability generating function of the steady-state FCTL queue. This representation will be the basis for effective algorithms. We show that it is straightforward to compute the queue-length distribution and all its moments using algorithms that rely on contour integrals and avoid root-finding procedures altogether.

Keywords: fixed-cycle traffic-light queue; transform methods; complex analysis
AMS 2010 Subject Classification. 60E10, 60J10, 60K25, 68M20, 90B20

1 Introduction

The fixed-cycle traffic-light (FCTL) queue is an intensively studied and widely applied stochastic model in traffic engineering \[2,5,6,10,11,15,17\]. Vehicles arrive to an intersection controlled by a traffic light and form a queue. The FCTL queue is traditionally modeled in discrete time. Time is divided into slots of unit length, and each slot corresponds to the time needed for a delayed vehicle to depart from the queue. The green and red periods, of length \(g\) and \(r\) slots, respectively, and thus the cycle length \(c = g + r\), are assumed to be fixed multiples of one slot. Those vehicles that arrive to the queue and are delayed, join the queue at the end of the slot in which they arrive.

The key feature of the FCTL queue, which distinguishes it from traditional queues, is formulated in the following assumption:

**FCTL assumption** For those cycles in which the queue clears before the green period terminates, all vehicles that arrive during the residual green period pass through the system and experience no delay whatsoever.

The FCTL assumption lets vehicles that arrive during the residual green period pass the intersection without slowing down, and therefore the discharge rate of these vehicles is larger than
the discharge rate of the delayed vehicles (one per time unit). Because of the huge difference in
discharge rates of delayed vehicles (these vehicles have to accelerate) and non-delayed vehicles,
the FCTL assumption is a sensible assumption. The FCTL assumption does, however, have some
severe consequences for the mathematical analysis of the queue length. Let $X_{k,n}$ denote the queue
length at time $k$ in cycle $n$ (time expressed in slots). Then, in cycle $n$, $X_{0,n}$ is the queue length at
the beginning of the green period, and $X_{g,n}$ the overflow defined as the queue length at the end
of the green period (and thus the beginning of the red period). Let $A_n$ denote the total number
of vehicles that arrive at the intersection in between the two measurements of the overflow $X_{g,n}$
and $X_{g,n+1}$. Thus $A_n$ are the arrivals from $X_{g,n}$ onwards in a consecutive red and green period.
Further, $A_n = A_n^d + A_n^p$, where $A_n^d$ denotes the number of delayed vehicles and $A_n^p$ the number
of vehicles that pass without delay on behalf of the FCTL assumption. The overflow queue can then
be defined as

$$X_{g,n+1} = \max\{X_{g,n} + A_n^d - g, 0\}.$$  

The fact that $A_n^d$ depends on both $X_{g,n}$ and the exact specification of when the arrivals occur makes (1.1) hard to analyze. To capture that level of detail, let $Y_{k,n}$ denote the number of vehicles
that arrive to the intersection during slot $k$ in cycle $n$. The random variables $Y_{k,n}$ are assumed
independent and identically distributed, for all $k$ and $n$. Notice that the above assumptions
together make that the queue lengths at the end of time slots can be modeled as a discrete-time
Markov chain. Using analytic techniques suitable for dealing with such Markov chains, Darroch
[5] obtained the probability generating function (PGF) of the steady-state overflow queue (the
number of vehicles waiting in front of the traffic light at the end of a green period) and the PGF
of the steady-state delay was obtained in van Leeuwaarden [15]. Hence, all information about
the distribution of the steady-state overflow queue and steady-state delay in the FCTL queue can
be obtained from the results in [5, 15], including all moments of the steady-state queue length
and delay, and the distribution of the output process (the way vehicles leave the intersection).

The classical FCTL treatment in [5, 15], however, comes with the disadvantage that numerical
evaluation is computationally cumbersome. This is because the probability generating function
(PGF) for the stationary queue length distribution contains $g - 1$ boundary terms that need to be
found separately. The traditional way of determining these remaining unknowns consists of two
steps: finding the $g - 1$ complex-valued roots and using these roots as input for a system of linear
equations whose solution gives the boundary terms. Both steps can present numerical difficulties,
but were somehow assumed unavoidable in the mathematically rigorous treatment of the FCTL
queue [5, 10, 11, 17, 15] and of related bulk-service queues [8, 13, 14].

**Paper outline.** We present in Section 2 the main result of the paper, an alternative representation
for the transform solution in terms of a contour integral. We also explain how these contour
integrals lead to algorithms that are more efficient than existing algorithms. We present these
algorithms in Section 3 and show their effectiveness. The proof of our main result is presented in
Section 4 and uses several basic notions from complex analysis. We conclude in Section 5 and
discuss the scope of application of our proof method and how the contour integrals might be used
for asymptotic analysis.

### 2 Main result

We now briefly review the standard solution method for obtaining an exact transform solution
for the steady-state FCTL queue. We then present the alternative contour integral representation.
2.1 Standard solution

Assume \( P(Y = 0) > 0, Y'(1) < 1, \) and \( Y(z) \) to be analytic in a region \(|z| < R\) with \( R > 1 \) and \( R \) maximal. The key quantity in the mathematical analysis of the FCTL queue is the steady-state overflow queue, defined as \( X_g = \lim_{n \to \infty} X_{g,n} \). Clearly, to have stability, and for \( X_g \) to be well defined,

\[
c \mathbb{E}[Y] < g. \tag{2.1}
\]

Using the kernel method and transform techniques, the PGF of \( X_g \), denoted by \( X_g(z) = \mathbb{E}[z^X] \) can be obtained using a by now classical line of reasoning. Introduce two more PGFs \( Y(z) = \mathbb{E}[z^Y] \) and \( A(z) = \mathbb{E}[z^A] = \mathbb{E}[z^T] \). Then it can be shown that \([5,15]\)

\[
X_g(z) = \frac{(z - Y(z)) \sum_{k=0}^{g-1} q_k z^k Y(z)^{s-1-k}}{z^s - A(z)} . \tag{2.2}
\]

This expression still contains \( g \) unknowns \( q_0, \ldots, q_{g-1} \), which can be found by exploiting the analytic properties of PGFs. With Rouché’s theorem, it can be shown that the denominator of (2.2) has \( g \) zeros on or within the unit circle \(|z| = 1\). Because a PGF is analytic and well-defined in \(|z| \leq 1\), the numerator of \( X_g(z) \) should vanish at each of the zeros. This gives \( g \) equations. One of the zeros equals 1, and leads to a trivial equation. However, the normalization condition \( X_g(1) = 1 \) provides an additional equation. So that summarizes the highest level of general development for FCTL queue analysis: transform techniques yield an expression for \( X_g(z) \) that in order to be evaluated demands finding \( g - 1 \) roots in the complex plane of the function \( z^s = A(z) \) and solving a set of \( g \) linear equations.

2.2 Pollaczek integral

We now turn to the alternative expression for \( X_g(z) \). Here is the main result in this paper:

**Theorem 1 (Pollaczek integral for FCTL)** There is an \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \)

\[
X_g(w) = \exp \left( \frac{1}{2\pi i} \oint_{|z| = 1 + \varepsilon} \frac{Y'(z)z - Y(z)}{z - Y(z)} \frac{w - Y(w)}{zY(w) - wY(z)} \ln \left( 1 - \frac{A(z)}{z^s} \right) dz \right), \quad |w| < 1 + \varepsilon, \tag{2.3}
\]

with principal value of the logarithm.

The formula (2.3) for \( X_g(w) \) is essentially equivalent with

\[
X_g(w) = \exp \left( \frac{1}{2\pi i} \oint_{|z| = 1 + \varepsilon} \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{(z^s - A(z))^\prime}{z^s - A(z)} dz \right), \tag{2.4}
\]

except that the validity range is more delicate due to the more complicated argument of the \( \ln \) in (2.4). Formula (2.3) follows upon manipulating (2.4) using partial integration (details in Section 4).

**Towards better numerical algorithms.** The elegant contour-integral representation requires model primitives as sole input. This representation will be the basis for effective algorithms. For the empty-queue probability and all moments we obtain explicit contour integrals by considering...
the derivatives of $X_g(w)$ at $w = 1$. These contour integrals can be evaluated numerically in a straightforward and fast manner. For calculating the entire steady-state distribution we use numerical transform inversion based on the symbolic computation of all derivatives of $X_g(w)$ at $0$. The resulting algorithms are the first to not use complex-valued roots and remarkably elegant and simple compared with existing algorithms. The contour integrals and numerical implementations are discussed in Section 3. There we also discuss how the algorithms for the overflow queue carry over to similar algorithms for the steady-state delay distribution.

**Historical notes.** Integrals of this sort go a long way back in the history of queueing theory and were first found in the ground-breaking work of Pollaczek on the classical single-server queue (see [1, 4, 7] for historical accounts). Let us point out the connection to the well known Pollaczek type integral for the discrete bulk-service queue [8]. The analysis of the FCTL queue is greatly simplified if all vehicles were delayed [3], so that all vehicles arrive while the queue length is at least one and the complicated $A_n$ random variable in (1.1) can be replaced by $A_n$. In that case (1.1) becomes a standard stochastic recursion driven by i.i.d. random variables and the FCTL queue reduces to the classical bulk-service queue, a special case of the more general single-server queue investigated by Pollaczek. Let $X_b$ denote the steady-state queue length in that bulk-service queue, defined as the solution of the stochastic equation

$$X_b \overset{d}{=} \max\{X_b + A - g, 0\}.$$  

(2.5)

Pollaczek’s result then says that (see [8] for a direct derivation)

$$X_b(w) = \exp \left( \frac{1}{2\pi i} \oint_{|z| = 1+\varepsilon} \ln \left( \frac{w-z}{1-z} \right) \frac{(z^g - A(z))'}{z^g - A(z)} \, dz \right)$$  

(2.6)

holds when $|w| < 1 + \varepsilon$ with $\varepsilon$ positive and bounded by some constant. Observe the striking similarity with (2.4). While the FCTL queue is harder to analyze than the bulk-service queue, the two contour-integral representations (2.4) and (2.6) only differ in the logarithmic function. The authors find this quite surprising themselves, particularly because there seems no way to interpret the FCTL queue as a reflected random walk (that is, a recursive structure with i.i.d. increments), while in the literature so far this seems to be a prerequisite for establishing Pollaczek-type contour integrals. Do observe that (2.6) is valid in an area that includes the unit disk while (2.4) is guaranteed only in an open set containing $[0, 1]$, see Section 4. This objection does not hold against the representation (2.3) of $X_g(w)$.

**Bernoulli arrivals.** The bulk-service queue serves as a popular approximation of the FCTL queue [3]. In fact, for Bernoulli arrivals, so per time slot one or no arrival, this approximation becomes exact. To see this, substitute $Y(z) = 1 - p + pz$ into the logarithmic function in (2.4), and observe that this gives the logarithmic function in (2.6). Obviously, when $Y$ can take values larger than one, the bulk-service queue is an approximation and yields an upper bound on the overflow queue.

**Sketch of the proof.** The proof will find a way to go from representation (2.2) to contour integrals. A significant start in this direction was made by [12], who rewrote (2.2) as

$$X_g(z) = \frac{(z - Y(z)z^{g-1} - \sum_{k=0}^{g-1} q_k \left( \frac{Y(z)}{z} \right)^{g-1-k}}{z^g - A(z)}.$$  

(2.7)
Then denote the \( g \) roots of \( z^g = A(z) \) on and within the unit circle by \( z_0, z_1, \ldots, z_{g-1} \). Now here is where the authors in [12] took an eye-opening step: instead of using the \( g \) roots in the traditional manner for finding the unknowns \( q_k \) and completing the transform (2.2), use these roots for factorizing the numerator of (2.2). Notice that this cannot be done immediately, because interpreted as a function of \( z \), the numerator is by no means a polynomial of degree \( g \) or less. However, by treating the function \( Y(z)/z \) as a variable itself, the summation in the numerator is a polynomial of degree \( g - 1 \) and can be factorized as

\[
\sum_{k=0}^{g-1} q_k \left( \frac{Y(z)}{z} \right)^{g-1-k} = q_0 \prod_{k=1}^{g-1} \left( \frac{Y(z)}{z} - \frac{Y(z_k)}{z_k} \right). \tag{2.8}
\]

After normalization using \( X_g(1) = 1 \) the factorization in (2.8) leads to the representation

\[
X_g(z) = \frac{g - A'(1)}{z^g - A(z)} \cdot \frac{z - Y(z)}{1 - Y'(1)} \cdot z^{g-1} \prod_{k=1}^{g-1} \frac{Y(z)/z - Y(z_k)/z_k}{1 - Y(z_k)/z_k}. \tag{2.9}
\]

Our proof then proceeds by interpreting (2.9) as the outcome of Cauchy’s residue theorem, the classical tool from complex analysis to evaluate line integrals of analytic functions over closed curves. An important step is to write

\[
\ln \left( z^{g-1} \prod_{k=1}^{g-1} \frac{Y(z)/z - Y(z_k)/z_k}{1 - Y(z_k)/z_k} \right) = \sum_{k=1}^{g-1} \ln \left( \frac{z Y(z_k) - z_k Y(z)}{Y(z_k) - z_k} \right), \tag{2.10}
\]

and to regard (2.10) as the sum of residues at \( z = z_k \). To construct an analytic function that, in conjunction with Cauchy’s theorem and the closed curve \( |z| = 1 + \varepsilon \), returns (2.10) and has singularities at \( z_1, \ldots, z_{g-1} \), leads us to consider the integrand in (2.4). Here, the logarithmic function

\[
\ln \left( \frac{w Y(z) - z Y(w)}{Y(z) - z} \right) \tag{2.11}
\]

follows from (2.10) and the singularities with appropriate residues are created through \( (z^g - A(z))/g(z^g - A(z)) \). After careful consideration of the analytic properties of the integrand in (2.4), we then show that Cauchy’s theorem gives (2.9) from which (2.4) follows. As said, (2.3) is obtained by manipulating (2.4), using partial integration. The formal proof of Theorem 1 presented in Section 4 contains several challenging steps, and requires among other things a proof that the function \( Y(z)/z \) is injective in a region that contains the unit disk, and a way to account for the branch cut caused by the logarithm in (2.11) being taken over negative values.

### 3 Novel FCTL algorithms

We now demonstrate how the integral representation in Theorem 1 yields efficient algorithms for effectively all stationary performance metrics, including the stationary moments and queue length distribution. The algorithms in this section are based on the representation (2.4) (but one could also take (2.3)). Note that we only expand \( X_g(w) \) at \( w = 0 \) and \( w = 1 \), so inside the validity range of (2.4).
3.1 Moments

The mean stationary overflow queue $E X_g$ is given by $X'_g(1)$ and takes the form

$$E X_g = \frac{1}{2\pi i} \oint_{|z|=1+\epsilon} \frac{Y(z) - zY'(1)}{Y(z) - zY(1)} \frac{(z^g - A(z))'}{z^g - A(z)} \, dz. \quad (3.1)$$

This result was recently obtained in [12] using a direct proof that converted the classical expression for $E X_g$ in terms of complex valued roots into the integral expression (3.1).

From the PGF $X_g(z)$ we can in principle determine all stationary moments. Define

$$f(w) := \frac{1}{2\pi i} \oint_{|z|=1+\epsilon} g(w, z) \frac{(z^g - A(z))'}{z^g - A(z)} \, dz, \quad (3.2)$$

$$g(w, z) := \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right), \quad (3.3)$$

$$h_k(w) := \begin{cases} 1 & k = 0, \\ h_{k-1}(w) f'(w) + h'_{k-1}(w) & k = 1, 2, \ldots \end{cases} \quad (3.4)$$

The moments $E[X^k_g]$ then follow from symbolically differentiating the PGF (2.4), and these derivatives can be expressed as

$$X^{(k)}_g(w) := \frac{d^k}{dw^k} X_g(w) = \frac{d^k}{dw^k} \exp \left( f(w) \right) = h_k(w) \exp \left( f(w) \right), \quad (3.5)$$

for $k = 0, 1, 2, \ldots$. Using this recursive expression, $X^{(k)}_g(w)$ can be expressed in terms of $f(w)$ and the first $k$ derivatives of $f(w)$, denoted by $f^{(1)}(w), \ldots, f^{(k)}(w)$ with

$$f^{(j)}(w) := \frac{\partial^j}{\partial w^j} \frac{1}{2\pi i} \oint_{|z|=1+\epsilon} g(w, z) \frac{(z^g - A(z))'}{z^g - A(z)} \, dz$$

$$= \frac{1}{2\pi i} \oint_{|z|=1+\epsilon} g^{(j)}(w, z) \frac{(z^g - A(z))'}{z^g - A(z)} \, dz \quad (3.6)$$

and $g^{(j)}(w, z) := \frac{\partial^j}{\partial w^j} g(w, z)$, for $j = 1, 2, \ldots, k$. After substituting $w = 1$, we can express the first $k$ moments of $X_g$ in terms of $k$ contour integrals that only involve the model primitives and the first $k$ moments of $Y$. Using $f(1) = 0$, the variance of $X_g$ given by $\text{Var}(X_g) = h_2(1) + h_1(1) - (h_1(1))^2$ takes the form

$$\text{Var}(X_g) = \frac{1}{2\pi i} \oint_{|z|=1+\epsilon} \frac{z^2 \text{Var}(Y) - zY(z)(1 + E(Y^2) - 2EY) \frac{(z^g - A(z))'}{(z - Y(z))^2}}{z^g - A(z)} \, dz. \quad (3.7)$$

3.2 Overflow and queue-length distribution

To determine the stationary distribution of the overflow queue we use that

$$P(X_g = k) = \left. \frac{1}{k!} \frac{d^k}{dw^k} X_g(w) \right|_{w=0} = \frac{1}{k!} h_k(0) \exp \left( f(0) \right). \quad (3.8)$$
First observe that
\[
\mathbb{P}(X_g = 0) = \exp(f(0)) = \exp\left(\frac{1}{2\pi i} \oint_{|z|=1+\epsilon} \ln \left(\frac{z^{\mathbb{P}(Y = 0)}}{z - Y(z)}\right) \frac{(z^g - A(z))'}{z^g - A(z)} \, dz\right). \tag{3.9}
\]
Expressions for the other probabilities \(\mathbb{P}(X_g = k)\) follow in a similar way, but require evaluating the resulting function at \(w = 0\) instead of \(w = 1\) and dividing by \(k!\). As a consequence, \(\mathbb{P}(X_g = k)\) can be expressed in terms of \(f(0), f^{(1)}(0), \ldots, f^{(k)}(0)\), again an expression that involves explicit contour integrals only.

In principle, all other stationary performance metrics follow from the distribution \(\mathbb{P}(X_g = k)\) (cf. [15]). However, some explicit results from [13] are given in terms of \(q_0, q_1, \ldots, q_g\). These are the boundary probabilities in the classical transform solution (2.2). In fact, \(q_j\) represents the probability that the queue is empty at the end of slot \(j\) in the green period. To find these probabilities, and the PGFs of the queue-length distributions at the end of all slots, we write
\[
X_0(w) = X_g(w)Y(w)^{c-g}, \quad \text{and} \quad q_0 = \exp(f(0)) Y(0)^{c-g}. \tag{3.10}
\]
We then obtain the recursions
\[
X_k(w) = Y(w)\frac{X_{k-1}(w) - q_{k-1}}{w} + q_{k-1}, \tag{3.11}
\]
\[
q_k = Y(0) \lim_{w \downarrow 0} \frac{X_{k-1}(w) - q_{k-1}}{w} + q_{k-1}, \tag{3.12}
\]
for \(k = 1, 2, \ldots, g - 1\). The limit \(w \downarrow 0\) in (3.12) yields expressions in terms of the first \(k\) derivatives of \(f(w)\) and \(Y(w)\), all of which can be evaluated symbolically, and then require numerical evaluation of contour integrals.

### 3.3 Numerical examples

To illustrate the algorithms we now show some results for the FCTL queue with \(g = 20\) and \(c = 50\). We consider Poisson arrivals with on average \(\lambda\) vehicles arriving per slot, and four scenarios: \(\lambda = 0.2\) (light traffic), \(\lambda = 0.3\) (moderate traffic), \(\lambda = 0.36\) (heavy traffic), and \(\lambda = 0.38\) (extreme traffic). These arrival rates correspond to a volume/capacity ratio \(\rho = \lambda c/g\) ranging from 0.5 to 0.95.

Figure 1(a) shows the mean queue lengths \(\mathbb{E}[X_0], \ldots, \mathbb{E}[X_g]\) through one cycle. Observe the strong cyclic behavior and the high sensitivity for \(\rho\). Figure 1(b) shows the queue-length distribution at cycle start, the moment that the traffic signal turns green and queue lengths are expected to peak. Observe the difference between operation at 75% or 90% of maximal capacity: the probability that more than 20 vehicles are waiting is only 0.002 for \(\lambda = 0.3\) and 0.32 for \(\lambda = 0.38\). Figure 1(c) depicts the distribution of the effective green time \(G\), defined in [3] as the number of slots used for departure of delayed vehicles. We have
\[
\mathbb{P}(G = k) = \begin{cases} q_0 & \text{for } k = 0, \\ q_k - q_{k-1} & \text{for } k = 1, \ldots, g - 1, \\ 1 - q_{g-1} & \text{for } k = g. \end{cases} \tag{3.13}
\]
Since only one delayed vehicle departs per slot, this can also be considered to be the distribution of the platoon length consisting of delayed vehicles departing during one cycle. Observe that
\(\mathbb{P}(G = g)\) is practically zero when \(\rho = 0.5\), but as high as 0.84 when \(\lambda = 0.36\), which means that only in 16\% of the cycles the green time is long enough to let the queue vanish.

Finally, we consider the delay distribution of an arbitrary vehicle arriving in the 10-th slot, which is during the green period. The stationary delay of a vehicle arriving in slot \(k\), denoted by \(D_{[k]}\), is defined as the number of slots between arrival and departure, not including the slot of arrival. Figure 1(d) shows \(D_{[10]}\), which can be computed directly from \(X_9\), i.e. the number of vehicles waiting at the start of the 10-th slot. If \(X_9 = 0\) we have that \(D_{[10]} = 0\); otherwise the delay can be expressed as a function of the number of vehicles present at the arrival of the tagged vehicle. This function (studied in detail in [15]) should take into account interruptions due to red periods, which explains the fragmented histograms in Figure 1(d).

4 Proof of the Pollaczek contour-integral representation

As explained briefly in Section 2 the proof of Theorem 1 exploits the factorized form (2.9) and investigates in detail the logarithmic function (2.11). We present some useful properties of the function \(Y(z)/z\), visible in both (2.9) and (2.11). We then proceed to use Cauchy’s theorem to
obtain the contour-integral representation (2.4) for the case that \(1 < w < 1 + \varepsilon\), and finally manipulate (2.4) to obtain (2.3) on the full range \(|w| < 1 + \varepsilon\).

### 4.1 Auxiliary results

Before we prove Theorem 1 we present some auxiliary results for the function \(Y(z)/z\). In [12] Theorem 1 it was shown that the function \(Y(z)/z\) is injective on the disk \(|z| \leq 1\), so that all \(Y(z_k)/z_k \neq Y(z_l)/z_l\) when \(z_k \neq z_l\). For our proof we also need injectivity, but then for the larger disk with radius \(t_0 > 1\). More specifically, let

\[
t_0 := \sup\{t \in (0, R) \mid Y'(t)t - Y(t) \leq 0\},
\]

**Lemma 2** The function \(Y(t)/t\) is strictly decreasing in \(t \in (0, t_0]\).

**Proof.** We have that

\[
\frac{Y(t)}{t} = \frac{y_0}{t} + y_1 + y_2 t + \ldots, \quad 0 < t < R,
\]

is strictly convex since \(y_0 > 0\), with derivative

\[
\left(\frac{Y(t)}{t}\right)' = \frac{Y'(t)t - Y(t)}{t^2}, \quad 0 < t < R.
\]

Since \(Y'(1) < Y(1) = 1\), we have that \(t_0 > 1\). Now consider the following cases: (i) \(y_k = 0\) for \(k = 2, 3, \ldots\), (ii) there is a \(k = 2, 3, \ldots\) such that \(y_k \neq 0\). For case (i) \(Y(t)/t = y_0t^{-1} + y_1\) is strictly decreasing in \(t > 0\) since \(y_0 > 0\). For case (ii), \(y_k > 0\) for some \(k \geq 2\), and so

\[
Y'(t)t - Y(t) = -y_0 + \sum_{k=2}^{\infty} (k-1)y_k t^k
\]

is strictly increasing in \(t \in (0, R)\). From the definition of \(t_0\), we then get that

\[
Y'(t)t - Y(t) < 0, \quad t \in (0, t_0),
\]

and so \(Y(t)/t\) is strictly decreasing in \(t \in (0, t_0)\) by (4.3).

**Lemma 3** The function \(Y(z)/z\) is injective on the open disk \(|z| < t_0\), so that for \(|z| < t_0, |w| < t_0\)

\[
\frac{Y(z)}{z} = \frac{Y(w)}{w} \Rightarrow z = w.
\]

**Proof.** In case (i), \(y_k = 0\) for \(k = 2, 3, \ldots\), we have \(Y(z)/z = y_0z^{-1} + y_1\) and the result is trivial since \(y_0 > 0\). For case (ii), there is a \(k = 2, 3, \ldots\) such that \(y_k \neq 0\), we let \(|z| < t_0, |w| < t_0\). Then

\[
\left|\frac{Y(z)}{z} - \frac{Y(w)}{w}\right| = \left|y_0 \frac{w - z}{zw} + \sum_{k=2}^{\infty} y_k \frac{z^{k-1} - w^{k-1}}{z - w}\right|
\]

\[
= |z - w| \left|\frac{y_0}{zw} + \sum_{k=2}^{\infty} y_k \frac{z^{k-1} - w^{k-1}}{z - w}\right|.
\]
Let $t := \max\{|z|,|w|\} < t_0$. Then $|\gamma_0/(zw)| \geq \gamma_0/t^2$ while
\[
\left| z^{k-1} - w^{k-1} \right| = \left| z^{k-2} + wz^{k-3} + \cdots + zw^{k-3} + w^{k-2} \right| \leq (k-1)t^{k-2}. \tag{4.8}
\]
Therefore, when $z \neq w$,
\[
\left| \frac{Y(z)}{z} - \frac{Y(w)}{w} \right| \geq \left| w - z \right| \left( \frac{\gamma_0}{t^2} - \sum_{k=2}^{\infty} (k-1)y_k t^{k-2} \right) > 0 \tag{4.9}
\]
y by (4.4) and (4.5). This proves (4.6). \hfill \Box

**Lemma 4** Let $\epsilon > 0$ be such that $1 + \epsilon < t_0$, and take $w \in (1, 1 + \epsilon)$. For $|z| < t_0$,
\[
\frac{wY(z) - zY(w)}{Y(z) - z} \in (-\infty, 0) \iff 1 \leq z \leq w. \tag{4.10}
\]

**Proof.** For $a \leq 0$,
\[
\frac{wY(z) - zY(w)}{Y(z) - z} = a \iff \frac{Y(z)}{z} = \frac{Y(w) - a}{w - a}. \tag{4.11}
\]
Since $1 < Y(w) < w$, the function $(Y(w) - a)/(w - a)$ increases from $Y(w)/w$ at $a = 0$ to 1 at $a = -\infty$ when $a$ decreases from 0 to $-\infty$. Since $Y(v)/v$ decreases strictly in $v \in [1, w]$, there is for any $a \leq 0$ a unique $v = v(a) \in [1, w]$ such that
\[
\frac{Y(v)}{v} = \frac{Y(w) - a}{w - a}. \tag{4.12}
\]
Since by Lemma 3 $Y(z)/z$ is injective in $|z| < t_0$ we get (4.10). \hfill \Box

As a consequence of Lemma 4 taking the principal value logarithm in (2.11) when $1 < w < 1 + \epsilon < t_0$, we obtain a function of $z$ that is analytic in the open disk $|z| < t_0$, with branch cut $[1, w]$.

### 4.2 Contour integral for (2.4)

We next consider the function $z^g - A(z)$ that has its zeros in $|z| \leq 1$ at $z = z_0 = 1, z_1, \ldots, z_{g-1}$, while its other zeros have modulus greater than one. Let $R_0$ be the zero outside $|z| \leq 1$ of smallest modulus; we have $R_0$ is real and larger than one. Take $\epsilon > 0$ such that $1 + \epsilon < \min\{t_0, R_0\}$ and consider the integral
\[
I(w) = \frac{1}{2\pi i} \int_{|z|=1+\epsilon} \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{(z^g - A(z))'}{z^g - A(z)} \, dz. \tag{4.13}
\]

Choose $\delta > 0$ such that $\delta < \frac{1}{2}(w-1)$ and $\delta < 1 + \epsilon - w$ while $|z_k-1| > \delta, k = 1, \ldots, g-1$. Now let $C$ be the positively oriented contour consisting of the circles $C_1(\delta)$ and $C_\infty(\delta)$ of radii $\delta$ around 1 and $w$, respectively, together with the line segments $L_+ = \{z = t + i0 \mid 1 + \delta \leq t \leq w - \delta\}$. See Fig. 2 for the positioning of the contour $C$ with its four components in the disk $|z| < 1 + \epsilon$ and relative to the zeros of $z^g - A(z)$. Then, by Cauchy's theorem,
\[
I(w) = \sum_{k=1}^{g-1} \ln \left( \frac{wY(z_k) - z_kY(w)}{Y(z_k) - z_k} \right) + \frac{1}{2\pi i} \oint_C \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{(z^g - A(z))'}{z^g - A(z)} \, dz. \tag{4.14}
\]
On the line segments $z = t \pm i0, 1 + \delta \leq t \leq w - \delta,$ we use that
\[ wY(t) - tY(w) > 0 > Y(t) - t. \] (4.15)

With the principal value choice for $\ln$, we then get
\[ \ln \left( \frac{wY(t \pm i0) - (t \pm i0)Y(w)}{Y(t \pm i0) - (t \pm i0)} \right) = \ln \left( \frac{wY(t) - tY(w)}{t - Y(t)} \right) \pm \pi i, \quad 1 + \delta \leq t \leq w - \delta. \] (4.16)

Therefore, also using that $t^g - A(t) > 0, 1 < t < 1 + \epsilon$,
\[
\begin{align*}
\frac{1}{2\pi i} \oint_{C} \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{(z^g - A(z))'}{z^g - A(z)} dz &= \left[ -\left( \ln \left( \frac{wY(t) - tY(w)}{t - Y(t)} \right) + \pi i \right) + \left( \ln \left( \frac{wY(t) - tY(w)}{t - Y(t)} \right) - \pi i \right) \right] \frac{(t^g - A(t))'}{t^g - A(t)} dt \\
&= -\int_{1+\delta}^{w-\delta} \frac{(t^g - A(t))'}{t^g - A(t)} dt + \frac{1}{2\pi i} \oint_{C_1(\delta)} + \frac{1}{2\pi i} \oint_{C_w(\delta)}.
\end{align*}
\] (4.17)

Now, since $g - A'(1) > 0$,
\[
\int_{1+\delta}^{w-\delta} \frac{(t^g - A(t))'}{t^g - A(t)} dt = \ln(t^g - A(t)) \bigg|_{1+\delta}^{w-\delta} = \ln(w^g - A(w)) - \ln((g - A'(1))\delta + O(\delta^2)) - \ln\delta + O(\delta),
\] (4.18)

where we have used that
\[ t^g - A(t) = 0 + (t^g - A(t))'_{t=1}(t - 1) + O((t - 1)^2), \quad t \to 1. \] (4.19)
As to the last integral on the last line of (4.17), we use that
\[
\begin{align*}
wY(z) - zY(w) &= (wY'(w) - Y(w))(z-w) + O(|z-w|^2), \\
Y(z) - z &= Y(w) - w + O(|z-w|), \\
z^\delta - A(z) &= w^\delta - A(w) + O(|z-w|),
\end{align*}
\]
with non-vanishing numbers \( wY'(w) - Y(w), Y(w) - w \) and \( w^\delta - A(w) \). Therefore
\[
\frac{1}{2\pi i} \oint_{C_\delta} \frac{\ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right)}{z^\delta - A(z)}\,dz = O(\delta \ln \delta), \quad \delta \downarrow 0. \tag{4.23}
\]
The middle integral on the last line of (4.17) is more delicate since both \( Y(z) - z \) and \( z^\delta - A(z) \) vanish at \( z = 1 \). For \( z = 1 + \delta e^{i\phi} \) with \( 0 < \phi < 2\pi \) and \( \delta \downarrow 0 \),
\[
\frac{wY(z) - zY(w)}{Y(z) - z} = \frac{w - Y(w) + O(|z-1|)}{1 + Y'(1)(z-1) + O(|z-1|^2)} - \frac{w - Y(w)}{1 - Y'(1)} e^{-i\phi} (1 + O(\delta)). \tag{4.24}
\]
Hence, since \( w - Y(w) > 0, 1 - Y'(1) > 0, \)
\[
\ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) = \ln \left| \frac{wY(z) - zY(w)}{Y(z) - z} \right| + i \arg \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) = \ln \left( \frac{w - Y(w)}{1 - Y'(1)} \right) - \ln \delta + i(\pi - \phi) + O(\delta). \tag{4.25}
\]
Next, as \( z \to 1, \)
\[
\frac{(z^\delta - A(z))'}{z^\delta - A(z)} = \frac{g - A'(1) + O(|z-1|)}{(g - A'(1))(z-1) + O(|z-1|^2)} = \frac{1}{z - 1} + O(1), \tag{4.26}
\]
since \( g - A'(1) > 0 \). Hence, from (4.25) and (4.26) with \( z = 1 + \delta e^{i\phi} \) and \( dz = i\delta e^{i\phi} \, d\phi \) in the integral over \( C_1, \)
\[
\frac{1}{2\pi i} \oint_{C_1} \frac{\ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) (z^\delta - A(z))'}{z^\delta - A(z)} \,i\delta e^{i\phi} \, d\phi = \ln \left( \frac{w - Y(w)}{1 - Y'(1)} \right) - \ln \delta + O(\delta), \tag{4.27}
\]
where we have also used that \( \int_0^{2\pi} (\pi - \phi) \, d\phi = 0. \)
Using (4.18), (4.23) and (4.27) in (4.17) yields

\[
\frac{1}{2\pi i} \oint_C \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{(z^g - A(z))^\prime}{z^g - A(z)} \, dz
\]

\[
= \ln \left( \frac{g - A(1)}{w^g - A(w)} \right) + \ln \delta + O(\delta) + O(\delta \ln \delta) + \ln \left( \frac{w - Y(w)}{1 - Y(1)} \right) - \ln \delta + O(\delta)
\]

\[
= \ln \left( \frac{g - A(1)}{w^g - A(w)} \right) \cdot \frac{w - Y(w)}{1 - Y'(1)} + O(\delta).
\]

(4.28)

Returning then to (4.13) - (4.14), letting \( \delta \downarrow 0 \), we see that

\[
I(w) = \ln \left[ \frac{g - A(1)}{w^g - A(w)} \cdot \frac{w - Y(w)}{1 - Y'(1)} \prod_{k=1}^{g-1} \frac{wY(z_k) - z_k Y(w)}{Y(z_k) - z_k} \right] = \ln [X_g(w)]
\]

(4.29)

by (4.9). Here we have also used that the zeros \( z_k \) are real or come in conjugate pairs so that for \( w \in (1, 1 + \epsilon) \) both \( X_g(w) \) and the product \( \prod_{k=1}^{g-1} \) in (4.29) are real and positive, with

\[
\ln \left( \prod_{k=1}^{g-1} \frac{wY(z_k) - z_k Y(w)}{Y(z_k) - z_k} \right) = \sum_{k=1}^{g-1} \ln \left( \frac{wY(z_k) - z_k Y(w)}{Y(z_k) - z_k} \right).
\]

(4.30)

This proves (2.4) for \( w \in (1, 1 + \epsilon) \).

### 4.3 Completion of the proof

The extension of the validity range of (2.4) beyond the set \( 1 < w < 1 + \epsilon \) is compromised by the appearance of the factor \( \ln[(wY(z) - zY(w))/(Y(z) - z)] \) in the integrand. The validity range can be extended to an open set containing the interval \([0, 1]\), allowing computation of moments and derivatives. To see this, let

\[
Q(z, w) = \frac{wY(z) - zY(w)}{Y(z) - z} = Y(w) \frac{1 - \frac{Y(z)/z}{Y(w)/w}}{1 - \frac{Y(z)/z}{Y(w)/w}}, \quad |z|, |w| \leq 1 + \epsilon.
\]

(4.31)

For \( 0 \leq w \leq 1 \) and \( |z| = 1 + \epsilon \),

\[
0 < Y(0) \leq Y(w) \leq 1, \quad \left| \frac{Y(z)/z}{Y(w)/w} \right| \leq \frac{Y(z)}{z} \leq \frac{Y(1 + \epsilon)}{1 + \epsilon} < 1,
\]

(4.32)

and so \( Q(z, w) \) is bounded away from \(( -\infty, 0) \) when \( 0 \leq w \leq 1 \) and \( |z| = 1 + \epsilon \). By continuity of \( Q \) as a function of \( w \), this continues to hold for \( w \) in an open set \( \Omega \) containing \([0, 1]\) and \( |z| = 1 + \epsilon \). This implies that \( \ln Q(z, w) \) is analytic in \( w \in \Omega \), with principal value \( \ln \), extending the validity of (2.4) to \( w \in \Omega \) by analyticity. We have extensive numerical evidence that the set of \( w \) for which \( Q(z, w) \not\in ( -\infty, 0) \), all \( z \) with \( |z| = 1 + \epsilon \), contains a disk around \( 0 \) with radius not significantly smaller than \( 1 + \epsilon \). This would extend the validity of (2.4) beyond the unit disk \(|w| \leq 1\).

We now re-express the integral form in (2.4) to a form that is valid for all \( w \), \( |w| < 1 + \epsilon \). We choose here \( \epsilon \) such that \( 1 + \epsilon < \min\{t_0, R_0\} =: 1 + \epsilon_0 \) as in Subsection 4.2. Let \( w \) be fixed with \( 1 < w < 1 + \epsilon \). We compute for \(|z| = 1 + \epsilon\)

\[
\frac{(z^g - A(z))^\prime}{z^g - A(z)} = \frac{g}{z} + \frac{(1 - \frac{A(z)}{z^g})^\prime}{1 - \frac{A(z)}{z^g}} = \frac{g}{z} + \frac{d}{dz} \left[ \ln \left( 1 - \frac{A(z)}{z^g} \right) \right],
\]

(4.33)
where we can choose the principal value of \( \ln \) since
\[
\left| \frac{A(z)}{z^\delta} \right| \leq \frac{A(1 + \epsilon)}{(1 + \epsilon)^{\delta}} < 1, \quad |z| = 1 + \epsilon. \tag{4.34}
\]

As in (4.16), (4.17),
\[
\frac{1}{2\pi i} \oint_{|z| = 1 + \epsilon} \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{g}{z} \, dz
= \int_{0}^{1+\delta} \frac{dz}{z} \left[ \left. \left( wY(z) - zY(w) \right) \frac{g}{z} \right|_{z = 0} \right] + \frac{g}{2\pi i} \oint_{\mathcal{C}} \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{dz}{z}
\]
\[
= g \ln w - g \int_{1+\delta}^{w-\delta} \frac{dz}{z} \ln \left( \frac{w - \delta}{1 + \delta} \right) + O(\delta \ln \delta), \tag{4.35}
\]
and this vanishes as \( \delta \downarrow 0 \). Therefore, see (4.13),
\[
I(w) = \frac{1}{2\pi i} \oint_{|z| = 1 + \epsilon} \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \frac{d}{dz} \left[ \ln \left( 1 - \frac{A(z)}{z^\delta} \right) \right] \, dz
\]
\[
= -\frac{1}{2\pi i} \oint_{|z| = 1 + \epsilon} \frac{d}{dz} \left[ \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \right] \ln \left( 1 - \frac{A(z)}{z^\delta} \right) \, dz, \tag{4.36}
\]
where we have used partial integration with continuous differentiable functions \( \ln (1 - A(z)/z^\delta) \) and \( \ln \left( (wY(z) - zY(w))/(Y(z) - z) \right) \) on the closed contour \( |z| = 1 + \epsilon \). We compute
\[
\frac{d}{dz} \left[ \ln \left( \frac{wY(z) - zY(w)}{Y(z) - z} \right) \right] = \frac{Y(z) - Y(z)}{wY(z) - zY(w)} \frac{Y'(z)z - Y(z) - Y(w) - w}{Y(z) - z}, \tag{4.37}
\]
and obtain
\[
I(w) = -\frac{1}{2\pi i} \oint_{|z| = 1 + \epsilon} \frac{Y'(z)z - Y(z)}{Y(z) - z} \frac{Y(w) - w}{wY(z) - zY(w)} \ln \left( 1 - \frac{A(z)}{z^\delta} \right) \, dz, \tag{4.38}
\]
which is valid for any \( w \in (1, 1 + \epsilon) \).

We now extend (4.38) to all \( w \) with \( |w| < 1 + \epsilon \) by using Lemma 4. Let \( 0 < \epsilon_{1} < \epsilon \). We have \( |Y(z) - z| > 0 \) when \( |z| = 1 + \epsilon \) and
\[
|wY(z) - zY(w)| > 0 \quad \text{when} \quad |z| = 1 + \epsilon, |w| \leq 1 + \epsilon_{1}, \tag{4.39}
\]
by Lemma 4 and \( Y(0) \neq 0 \). Therefore, by continuity and compactness, \((wY(z) - zY(w))/(Y(z) - z)\) is bounded away from 0 when \( |z| = 1 + \epsilon \) and \( |w| \leq 1 + \epsilon_{1} \). This implies that the right-hand side of (4.38) is analytic in \( w, |w| < 1 + \epsilon_{1} \), by analyticity of \( Y \). Since \( X_{g}(w) = \exp(I(w)) \) for \( 1 < w < 1 + \epsilon \), we then get by analyticity of \( X_{g} \) that
\[
X_{g}(w) = \exp \left( -\frac{1}{2\pi i} \oint_{|z| = 1 + \epsilon} \frac{Y'(z)z - Y(z)}{Y(z) - z} \frac{Y(w) - w}{wY(z) - zY(w)} \ln \left( 1 - \frac{A(z)}{z^\delta} \right) \, dz \right), \tag{4.40}
\]
holds for all \( w, |w| \leq 1 + \epsilon_{1} \) and any \( \epsilon_{1} \in (0, \epsilon) \). Then a simple rearrangement of the integrand in (4.40) yields Theorem 1.
5 Conclusions

We have presented novel formal solutions and algorithms for the FCTL queue in the form of contour integrals. Theorem 1 presents the contour-integral representation for the PGF of the overflow queue. From this PGF, essentially all relevant information about the stationary behavior of the FCTL can be obtained, by taking derivatives at one for the moments, derivatives at zero for the distribution, and by using simple recursions to obtain the queue lengths at all moments within the cycle and the stationary delay distribution. As a consequence, with Theorem 1, numerically evaluating the FCTL queue only involves calculating elementary contour integrals. This is in sharp contrast with existing algorithms for the FCTL queue, and many related bulk-service queues, which typically rely on complex roots that need to be determined numerically. In fact, in the order of hundreds of papers on this topic make use of the complex roots and hence the relatively cumbersome numerical schemes. The only exception in this regard is the recent work [12], which derives the contour integral representation for the mean overflow queue in (3.1) and provided a great stimulus for pursuing this direction.

The FCTL queue belongs to a large class of cyclic queueing models related to vehicle dispatching with uncertain arrivals and bulk service [13, 14, 16]. A broad variety of transportation and manufacturing systems can be modeled in this way, including batch production systems, bulk movements of goods in a factory, truck shipments and bus transportation. Within this class, many different rules can be considered that apply to customer arrivals and vehicle departures within a cycle. Think of vehicle-cancellation policies that hold a vehicle until the queue length reaches a specified threshold. The FCTL assumption can also be viewed as a special rule that influences the dynamics within a cycle. In [12] slight modifications of the FCTL assumption are considered. As of now, our proof method is not directly applicable to the settings in [12, 13, 14, 16], although there is certainly hope for future progress.

Another possible thread for future research relates to asymptotics. In classical queueing theory, a prominent line of research is related to heavy traffic, an asymptotic regime in which the traffic intensity approaches 100%. Next to more probabilistic methods such as weak convergence techniques and coupling, another way to obtain heavy-traffic results is through the asymptotic evaluation of Pollaczek-type integrals; see e.g. [9, 4] for single-server queues and [8] for classical bulk-queues. Now that Pollaczek-type integrals for the FCTL queue are available, it is worthwhile to explore the possibilities for heavy-traffic analysis.

Acknowledgments

We thank the authors of [12] for sharing an early version of their work. This work is supported by the NWO Gravitation Networks grant 024.002.003. The work of JvL is further supported by an NWO TOP-GO grant and by an ERC Starting Grant.

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