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### A branching process on a hypercubic lattice

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# A BRANCHING PROCESS ON A HYPERCUBIC LATTICE

BACHELOR'S THESIS

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### **Abstract**

In this thesis we look at a branching process on an inhomogeneous  $d$ -dimensional hypercubic lattice. Particles get offspring in each of the  $d$  directions with probability  $p$ , except for particles on a defect hyperplane, these particles get offspring in the direction of the hyperplane with probability  $q$ . We find the critical values of the parameters and describe the behaviour of the critical case. Some useful results on simple branching process are considered as well.

## **Preface**

This is my Bachelor's thesis on a particular branching process. It is the result of several weeks work done at the end of the three year long Industrial and Applied Mathematics Bachelor program at the Eindhoven University of Technology.

The first section of this thesis contains several useful results on simple branching processes that will be used in the second section. In this second section the setting is described and the problem is analyzed. The results in the first section can be found elsewhere in literature, in particular in the works included in the list of references, while the second section contains original research.

I would like to thank my supervisor Remco van der Hofstad for all the help he gave during the course of my thesis.

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# 1 Simple Branching Processes

A simple branching process is a stochastic process that can be used to model populations. Consider a population of particles that can produce offspring of similar particles in each generation. During a generation each particle produces other particles based on some probability distribution independently of the other particles of that generation. So the amount of particles in each generation depends solely on the number of particles of the previous generation and some probability distribution of the offspring that one particle produces. Suppose  $Z_n$  is the amount of particles in generation  $n$ , suppose  $Z_0 = 1$  and let  $X$  denote the distribution of the offspring of one particle. The stochastic process  $\{Z_n, n = 0, 1, \dots\}$  is a Markov chain because  $Z_{n+1}$  only depends on  $Z_n$ .

## 1.1 Extinction probability

An interesting question that arises is: what is the probability that the population goes extinct, in other words what is the probability that  $Z_n = 0$  for some  $n$ ? Let  $\pi_0$  denote the probability that the population dies out. The entire population dies out if every branch of a generation dies out, so  $\pi_0$  satisfies:

$$\pi_0 = \sum_{k=0}^{\infty} \mathbb{P}(X = k) \pi_0^k. \quad (1)$$

The right hand side of the above equation is the generating function of  $X$  evaluated at  $\pi_0$ , so it can be rewritten using the generating function  $f(s)$  of  $X$ :

$$\pi_0 = f(\pi_0). \quad (2)$$

With the help of the above equation we can prove the following theorem about the extinction probability of the branching process.

**Theorem 1.1.** *Let  $\{Z_n, n = 0, 1, \dots\}$  be a simple branching process with offspring distribution  $X$  and generating function  $f(s)$ . Then the extinction probability  $\pi_0 = 1$  if  $\mathbb{E}[X] \leq 1$  and  $\pi_0$  is the smallest root of the equation  $s = f(s)$  on the interval  $[0, 1]$  if  $\mathbb{E}[X] > 1$ .*

*Proof.* It turns out to be useful to get some trivial cases out of the way first. Suppose  $X \equiv 0$ , then obviously  $\pi_0 = 1$ . Now let  $\mathbb{P}(X = 0) = p_0$  and  $\mathbb{P}(X = 1) = 1 - p_0$ . Then  $f(s) = p_0 + (1 - p_0)s$  and if  $p_0 > 0$ , (2) has only one solution  $\pi_0 = 1$ . If  $p_0 = 0$  then  $X \equiv 1$  and the branching process survives with certainty, so  $\pi_0 = 0$ , which is the smallest root of the equation  $s = f(s)$  on the interval  $[0, 1]$ .

Now suppose  $\mathbb{P}(X > 1) > 0$ , then the generating function  $f(s)$  is strictly convex and (2) has at most two solutions.  $f(s)$  is sketched below for two distinct cases  $\mathbb{E}[X] \leq 1$  and  $\mathbb{E}[X] > 1$ . If

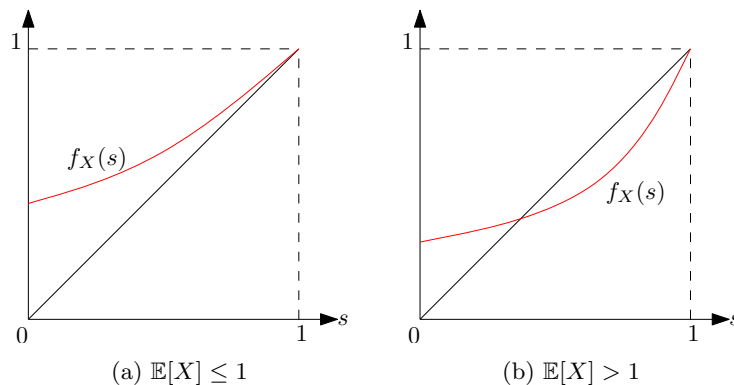


Figure 1: Generating function of  $X$

$\mathbb{E}[X] = f'(1) \leq 1$  then (2) has only one solution  $\pi_0 = 1$ . If  $\mathbb{E}[X] = f'(1) > 1$  then there is an

interval  $(\varepsilon, 1)$  where  $f(s)$  is smaller than  $s$ . Therefore using the intermediate value theorem and  $f(0) \geq 0$ , (2) has another solution besides  $\pi_0 = 1$ .

Let  $f_n(s)$  be the  $n$ 'th iterate of  $f(s)$ , so  $f_n(s) = f[f_{n-1}(s)]$  and let  $a$  be the smallest root of (2). Because  $f(s)$  is increasing, we have  $0 \leq f_1(0) \leq f_2(0) \leq \dots \leq f_n(0) \leq f_n(t) \leq f_n(a) = a$ , so  $f_n(0)$  converges to a limit  $L$  for  $n \rightarrow \infty$  and because  $f$  is continuous we can change the limits in the equation  $f_{n+1}(s) = f[f_n(s)]$  to find  $L = f(L)$ .  $f_n(0) \leq a$  for all  $n$ , but  $a$  was the smallest root of (2), therefore  $L = a$  and thus  $f_n(0) \uparrow a$ .

Let  $P(i, j)$  be the  $i, j$ -transition probability:

$$P(i, j) = \mathbb{P}(Z_{n+1} = j | Z_n = i) = \begin{cases} \mathbb{P}(X = k)^{*i} & \text{if } i \geq 0 \\ \delta_{0j} & \text{if } i = 0, \end{cases} \quad (3)$$

where  $\mathbb{P}(X = k)^{*i}$  is the  $i$ -fold convolution of  $\mathbb{P}(X = k)$  and  $\delta_{ij}$  the Kronecker delta. Note that,

$$\sum_{j=0}^{\infty} P(1, j)s^j = f(s), \quad (4)$$

and for  $i > 0$

$$\sum_{j=0}^{\infty} P(i, j)s^j = f(s)^i. \quad (5)$$

Let  $P_n(i, j)$  denote the  $n$ -step transition probability. A simple branching process is a Markov process, so the transition probability satisfies the Chapman-Kolmogorov equation:

$$\begin{aligned} \sum_{j=0}^{\infty} P_{n+1}(1, j)s^j &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P_n(1, k)P(k, j)s^j \\ &= \sum_{k=0}^{\infty} P_n(1, k) \sum_{j=0}^{\infty} P(k, j)s^j \\ &= \sum_{k=0}^{\infty} P_n(1, k)f(s)^k. \end{aligned} \quad (6)$$

If we let  $g_n(s) = \sum P_n(1, j)s^j$ , we now have  $g_{n+1}(s) = g_n[f(s)]$ , and thus by induction  $g_n(s) = f_n(s)$ .

For  $f_n(0)$  we now have

$$\begin{aligned} \lim_{n \rightarrow \infty} f_n(0) &= \lim_{n \rightarrow \infty} g_n(0) \\ &= \lim_{n \rightarrow \infty} P_n(1, 0) \\ &= \lim_{n \rightarrow \infty} \mathbb{P}(Z_i = 0, \text{ for some } 1 \leq i \leq n) \\ &= \mathbb{P}(Z_i = 0, \text{ for some } i) \\ &= \pi_0, \end{aligned} \quad (7)$$

but  $f_n(0) \rightarrow a$ , so  $a = \pi_0$ . □

A different way to approach this process is to look at the expected total number of particles across all generations, the expected total progeny, how does this depend on the probability distribution  $X$ ? Let  $T$  denote the total progeny. Because every particle in the new generation can be thought of as a new branching process we find the following equation for  $\mathbb{E}[T]$  if  $\mathbb{E}[X] < 1$ :

$$\mathbb{E}[T] = 1 + \sum_{k=0}^{\infty} k\mathbb{P}(X = k)\mathbb{E}[T]. \quad (8)$$

$\mathbb{E}[T]$  can be written outside the sum, which simply leaves  $\mathbb{E}[X]$ :

$$\begin{aligned}\mathbb{E}[T] &= 1 + \mathbb{E}[X]\mathbb{E}[T] \\ \mathbb{E}[T] &= \frac{1}{1 - \mathbb{E}[X]}.\end{aligned}\tag{9}$$

This shows that  $\mathbb{E}[T] \rightarrow \infty$  if  $\mathbb{E}[X] \uparrow 1$ . If  $\mathbb{E}[X] > 1$  we know that  $\mathbb{P}(T = \infty) > 0$  and so  $\mathbb{E}[T] = \infty$ .

## 1.2 Random walk approach to branching processes

It is possible to represent a simple branching process by a random walk. This random walk representation can be useful because results from random walks can now also be applied to branching processes. We will see an example of this in this section when studying the behaviour of  $\mathbb{P}(T \geq m)$ , where  $T$  is the total progeny of a simple critical branching process.

**Theorem 1.2.** *Let  $T$  be the total progeny of a simple branching process with offspring distribution  $X$ . Let  $\{S_m | m \geq 0\}$  be the random walk with  $S_0 = 1$  and satisfying*

$$S_{m+1} = S_m + X_m - 1,\tag{10}$$

where all  $X_m$  are independent random variables and have the same probability distribution as  $X$ . Define

$$T' = \begin{cases} \infty & \text{if } S_m > 0 \quad \forall m \geq 0 \\ \min\{m | S_m = 0\} & \text{otherwise,} \end{cases}\tag{11}$$

then

$$T \stackrel{d}{=} T'.\tag{12}$$

*Proof.* We will give an intuitive proof for this theorem.  $T$  counts the total number of particles of the branching process and you can count those particles in the following way: start with the first particle and stepwise go through the population. During a step you inspect a particular particle and keep track of the number of particles still to be inspected. Let  $S_{m+1}$  be the number of particles still to be inspected after the  $m$ th step. This number satisfies

$$S_{m+1} = S_m + X_m - 1,\tag{13}$$

because the number of remaining particles decreases by one with every step and the number of offspring  $X_m$  of the  $m$ th particles is added to the number of remaining particles. Only the number of particles is relevant, because every particle is equal.  $S_0 = 1$  because at the beginning you only need to inspect the original particle. If at some point  $S_m = 0$  for the first time you know you have inspected all particles and thus  $T = m$ . If  $S_m$  never becomes zero you know you have a population of infinite size. □

We can apply the previous result to the study of  $\mathbb{P}(T \geq m)$  for the total progeny  $T$  of a simple critical branching process. This result will be of use in the next section. We have seen that a simple critical branching process goes extinct with certainty, but how quickly does  $\mathbb{P}(T \geq m)$  tend to zero?

**Corollary 1.3.** *Let  $T$  be the total progeny of a simple branching process with offspring distribution  $X$  and let  $\mathbb{E}[X] = 1$ , then*

$$\mathbb{P}(T \geq m) \sim \frac{C}{\sqrt{m}},\tag{14}$$

for some constant  $C$ .

*Proof.* From the proof of Theorem (1.2) we know

$$\mathbb{P}(T = m) = \mathbb{P}(S_1, \dots, S_{m-1} \geq 1, S_m = 0) \quad (15)$$

The hitting time theorem [5] allows us to simplify this expression:

$$\begin{aligned} \mathbb{P}(T = m) &= \frac{1}{m} \mathbb{P}(S_m = 0) \\ &= \frac{1}{m} \mathbb{P}\left(0 \leq \frac{S_m}{\sigma\sqrt{m}} < \frac{1}{\sigma\sqrt{m}}\right), \end{aligned} \quad (16)$$

where  $\sigma$  is the standard deviation of  $S_m$ . Because  $\mathbb{E}[X] = 1$ , we know that  $\mathbb{E}[S_m] = 0$ , so for large  $m$  using the local central limit theorem [2] we have

$$\begin{aligned} \mathbb{P}(T = m) &\sim \frac{1}{m} \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{1/(\sigma\sqrt{m})} e^{-x^2/2} dx \\ &\sim \frac{1}{m} \frac{1}{\sigma\sqrt{m}} \frac{1}{\sqrt{2\pi\sigma^2}} \\ &= \frac{C^*}{m^{3/2}}. \end{aligned} \quad (17)$$

Now we can sum over  $m$ :

$$\begin{aligned} \mathbb{P}(T \geq m) &\sim \sum_{k=m}^{\infty} \frac{C^*}{k^{3/2}} \\ &\sim \int_m^{\infty} \frac{C^*}{x^{3/2}} dx \\ &= \frac{C}{\sqrt{m}} \end{aligned} \quad (18)$$

□



## 2 Branching processes on a hypercubic lattice

We now look at a branching process on a  $d$ -dimensional hypercubic lattice. Every particle now also has a location on the grid. We assume that each particle gets a child in the next generations in each of the  $2d$  directions with probability  $p$ . We also assume that there is a hyperplane in the grid of dimension  $d - 1$  in which these probabilities are not  $p$ , but  $q$ . Finally we start with one particle on this defect hyperplane in the origin, so the origin lies in the hyperplane. Different particles do not influence each other in this process; it is possible to have multiple particles at one location in the same generation.

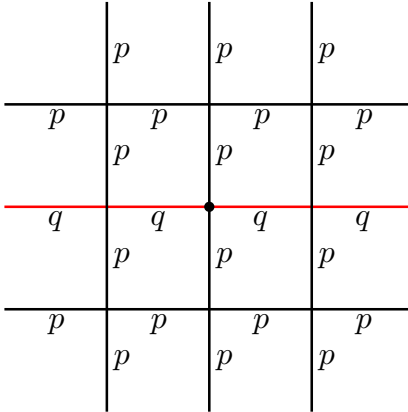


Figure 2: A branching process on a 2-dimensional cubic lattice

### 2.1 Extinction Probability

The extinction probability is again of interest in this process, in this section we will find for which values of  $p$  and  $q$  the process has a positive probability to survive. Firstly we look at the case where  $p = q$ . In that case the probabilities of getting offspring are the same everywhere in the plane, so the location of the particles no longer affects the branching process. We are left with a branching process with probability distribution  $X = \text{BIN}(2d, p)$ . The population will go extinct with certainty if  $p \leq \frac{1}{2d}$  and there will be a positive survival probability if  $p > \frac{1}{2d}$ .

Now let's assume that  $p \neq q$ . If  $\frac{1}{2d} < p < q$  there will be a positive survival probability, because for a smaller  $q$ ,  $q = p$ , there is a positive survival probability as well. Similarly if  $q < p \leq \frac{1}{2d}$  the population will go extinct, because for a larger  $q$ ,  $q = p$ , extinction is certain as well. We will look at the more interesting case where  $q > p$ , furthermore we take  $p \leq \frac{1}{2d}$  and  $\frac{1}{2d} \leq q \leq \frac{1}{2(d-1)}$ . In this case both the defect plane and the rest of the grid are needed to survive; the population would not survive in the plane or in the rest of the grid alone.

To analyse this process we will look at the random variable  $T$ , the total progeny of the branching process. Let  $X$  denote the number of particles whose parents are all particles that are not on the defect hyperplane except for the original particle. In other words  $X$  is the number of points on the lattice for which there is path of particles outside the defect hyperplane from the starting particle to that point. Let  $Y$  denote the number of particles on the defect line whose parents are all particles that are not on the defect hyperplane except for the original particle. So  $Y$  counts the points on the defect hyperplane for which there is path outside the defect hyperplane to the original particle. These three variables are related in the following way:

$$T = 1 + X + \sum_{i=1}^Y T_i, \quad (19)$$

where  $T_i$  are independent and identically distributed copies of  $T$ . The proof of the previous equation is somewhat trivial. To count the total amount of particles we first have the original particle, then there are those particles outside the defect hyperplane for which there is a path from the root to that particle outside the defect hyperplane,  $X$  and lastly the particles on the defect hyperplane for which there is a path from the root to that particle also outside the defect hyperplane,  $Y$ . For every particle from this last category a new branching process starts with a total amount of particles  $T_i$ . By taking the expectation on both sides we get

$$\mathbb{E}[T] = 1 + \mathbb{E}[X] + \mathbb{E}[Y]\mathbb{E}[T], \quad (20)$$

because  $Y$  and  $T_i$  are independent. We want to know how  $\mathbb{E}[T]$  behaves for different values of  $p$  and  $q$ , for this reason we will be looking at  $\mathbb{E}[X]$  and  $\mathbb{E}[Y]$ .

**Theorem 2.1.** *Let  $X$  denote the number of particles whose parents are all particles that are not on the defect hyperplane except for the original particle,*

(i). *if  $p < \frac{1}{2d}$ , then  $\mathbb{E}[X] < \infty$  and*

$$\mathbb{E}[X] = 2 \sum_{m=1}^{\infty} \left( \frac{p}{1 - 2p(d-1)} \right)^m \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{m} \binom{m}{\frac{m+b}{2}}. \quad (21)$$

(ii). *If  $p \geq \frac{1}{2d}$ , then  $\mathbb{E}[X] = \infty$ .*

*Proof.* For  $\mathbb{E}[X]$  we know that

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} p^n N_n, \quad (22)$$

where  $N_n$  is the number of paths of length  $n$  that start in the origin and otherwise have no points on the defect hyperplane. Call directions parallel to the defect hyperplane horizontal and the direction perpendicular to the hyperplane vertical. Let  $N_{n,m}$  be the number of such paths with  $m$  vertical steps, we find:

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} p^n \sum_{m=1}^n N_{n,m}. \quad (23)$$

These paths have  $n-m$  horizontal steps and as long as the first step is a vertical one it is irrelevant where the horizontal steps occur in the path, the restriction on the path is in the vertical direction. The  $n-m$  horizontal steps can be divided over the  $n$  steps in  $\binom{n-1}{n-m}$  ways because the first step is always in the vertical direction. Furthermore every horizontal step can be in  $2(d-1)$  directions, which gives

$$N_{n,m} = (2(d-1))^{n-m} \binom{n-1}{n-m} 2R_m, \quad (24)$$

where  $R_m$  is the number of random walks with  $m$  steps starting in 0 and that are always positive after the start. For ease of computation we only consider positive random walks and multiply by 2 for the negative random walks. According to the ballot theorem [3] we find

$$R_m = \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{m} \binom{m}{\frac{m+b}{2}}. \quad (25)$$

Combining these expressions gives

$$\mathbb{E}[X] = 2 \sum_{n=1}^{\infty} p^n \sum_{m=1}^n (2(d-1))^{n-m} \binom{n-1}{n-m} \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{m} \binom{m}{\frac{m+b}{2}}. \quad (26)$$

To simplify the previous expression we introduce  $k = n - m$  and change the summation:

$$\begin{aligned}
\mathbb{E}[X] &= 2 \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} p^{k+m} (2(d-1))^k \binom{k+m-1}{k} \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{m} \binom{m}{\frac{m+b}{2}} \\
&= 2 \sum_{m=1}^{\infty} p^m \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{m} \binom{m}{\frac{m+b}{2}} \sum_{k=0}^{\infty} (2p(d-1))^k \binom{k+m-1}{k} \\
&= 2 \sum_{m=1}^{\infty} \left( \frac{p}{1-2p(d-1)} \right)^m \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{m} \binom{m}{\frac{m+b}{2}} \sum_{k=0}^{\infty} (1-2p(d-1))^m (2p(d-1))^k \binom{k+m-1}{k}.
\end{aligned} \tag{27}$$

The last sum of the previous expression is a sum over all possible values of the negative binomial distribution with parameters  $m$  and  $2p(d-1)$ .  $2p(d-1) < 1$  if  $p < \frac{1}{2d}$ , therefore the last sum equals one in that case, so that

$$\mathbb{E}[X] = 2 \sum_{m=1}^{\infty} \left( \frac{p}{1-2p(d-1)} \right)^m \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{m} \binom{m}{\frac{m+b}{2}}. \tag{28}$$

We will now see that this sum converges for  $p < \frac{1}{2d}$  and that  $\mathbb{E}[X]$  diverges otherwise. We can continue from (25) as follows:

$$\begin{aligned}
R_m &= \frac{2^m}{m} \sum_{\substack{b=1, \\ b+m \text{ even}}}^m \frac{b}{2^m} \binom{m}{\frac{m+b}{2}} \\
&= \frac{2^m}{m} \sum_{\substack{b=1, \\ b+m \text{ even}}}^m b \mathbb{P}(S_m = b),
\end{aligned} \tag{29}$$

where  $S_m$  is an  $m$ -step symmetric random walk starting in 0. The sum above is the positive part of the sum  $\mathbb{E}[|S_m|]$  and because of symmetry the negative part is equal to the positive part, so

$$R_m = \frac{2^{m-1}}{m} \mathbb{E}[|S_m|]. \tag{30}$$

Now we can find bounds for  $\mathbb{E}[X]$  to find the values of  $p$  for which the sum converges. Substituting (30) in (23) gives

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} p^n \sum_{m=1}^n \frac{2^n (d-1)^{n-m}}{m} \binom{n-1}{n-m} \mathbb{E}[|S_m|] \\
&= \sum_{n=1}^{\infty} (2p)^n \sum_{m=1}^n \frac{(d-1)^{n-m}}{m} \binom{n-1}{n-m} \rho \mathbb{E}[|S_m| \mid |S_m| \geq 1],
\end{aligned} \tag{31}$$

where  $\rho = \mathbb{P}(S_m \neq 0)$ . We can bound  $\mathbb{E}[X]$  from below as follows:

$$\begin{aligned}
\mathbb{E}[X] &\geq \rho \sum_{n=1}^{\infty} \frac{p^n 2^n}{n} \sum_{m=1}^n (d-1)^{n-m} \binom{n-1}{n-m} \\
&= \rho \sum_{n=1}^{\infty} \frac{p^n 2^n}{n} \sum_{k=0}^{n-1} (d-1)^k \binom{n-1}{k} \\
&= 2\rho \sum_{n=1}^{\infty} \frac{p^n (2d)^{n-1}}{n},
\end{aligned} \tag{32}$$

this sum diverges for  $p \geq \frac{1}{2d}$ , so  $\mathbb{E}[X]$  diverges there as well. We need to bound the sum from above to see where it converges:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} p^n 2^n \sum_{m=1}^n (d-1)^{n-m} \binom{n-1}{n-m} \frac{\mathbb{E}|S_m|}{m} \\
&\leq \sum_{n=1}^{\infty} p^n 2^n \sum_{m=1}^n (d-1)^{n-m} \binom{n-1}{n-m} \\
&= \sum_{n=1}^{\infty} p^n 2^n \sum_{k=0}^{n-1} (d-1)^k \binom{n-1}{k} \\
&= 2 \sum_{n=1}^{\infty} p^n (2d)^{n-1},
\end{aligned} \tag{33}$$

which converges for  $p < \frac{1}{2d}$ , so  $\mathbb{E}[X]$  converges for  $p < \frac{1}{2d}$ .  $\square$

The other variable of interest is  $Y$ , the amount of children on the defect hyperplane from a single particle on the same hyperplane for which there is a path outside of the hyperplane that connects the two particles. To find a simple expression for  $\mathbb{E}[Y]$  we will need the following lemma:

**Lemma 2.2.** *Let  $|x| \leq \frac{1}{2}$ , then*

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k+1}{k+1} x^{2k+1} = \frac{1 - \sqrt{1-4x^2}}{2x}. \tag{34}$$

*Proof.* Using the binomial theorem we can find a series expansion of  $\sqrt{1-4x^2}$  which converges for  $|x| \leq \frac{1}{2}$ :

$$\begin{aligned}
\sqrt{1-4x^2} &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} 4^k x^{2k} (-1)^k \\
&= 1 - 2x^2 + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{2!} 4^2 x^4 - \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{3!} 4^3 x^6 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2} \cdot -\frac{5}{2}}{4!} 4^4 x^8 \pm \dots \\
&= 1 - 2x^2 - \sum_{k=2}^{\infty} \frac{(2k-3)!!}{k!} 2^k x^{2k},
\end{aligned} \tag{35}$$

where  $!!$  denotes the double factorial. Subtracting 1 and dividing by  $-2x$  gives

$$\begin{aligned}
\frac{1 - \sqrt{1-4x^2}}{2x} &= x + \sum_{k=2}^{\infty} \frac{(2k-3)!!}{k!} 2^{k-1} x^{2k-1} \\
&= x + \sum_{k=2}^{\infty} \frac{(2k-1)!!}{(2k-1)k!} 2^{k-1} x^{2k-1}.
\end{aligned} \tag{36}$$

We can now use the identity  $(2k-1)!! = \frac{(2k)!}{2^k k!}$ :

$$\begin{aligned}
\frac{1 - \sqrt{1 - 4x^2}}{2x} &= x + \sum_{k=2}^{\infty} \frac{(2k)!}{2(2k-1)k!k!} x^{2k-1} \\
&= x + \sum_{k=1}^{\infty} \frac{(2k+1)!(2k+2)}{2(2k+1)(k+1)!(k+1)!} x^{2k+1} \\
&= x + \sum_{k=1}^{\infty} \frac{1}{2k+1} \frac{(2k+1)!}{k!(k+1)!} x^{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k+1}{k} x^{2k+1} \tag{37}
\end{aligned}$$

□

**Theorem 2.3.** *Let  $p \leq \frac{1}{2d}$ , then*

$$\mathbb{E}[Y] = 1 + 2(d-1)(q-p) - \sqrt{1 - 4(d-1)p + 4((d-1)^2 - 1)p^2}. \tag{38}$$

*Proof.* For  $\mathbb{E}[Y]$  we can write a similar equation as (22):

$$\mathbb{E}[Y] = 2(d-1)q + \sum_{n=2}^{\infty} p^n N_n, \tag{39}$$

where now  $N_n$  is the amount of paths of length  $n$  starting in the origin and ending on the defect hyperplane and all points in between are not on the hyperplane. Let  $N_{n,m}$  denote the amount of such paths with  $m$  vertical steps, the number of steps in the vertical direction on such paths is always even. We find

$$\mathbb{E}[Y] = 2(d-1)q + \sum_{n=2}^{\infty} p^n \sum_{\substack{m=2, \\ \text{even}}}^n N_{n,m}. \tag{40}$$

These paths have  $n-m$  horizontal steps and they can occur everywhere except for the first and last step. So there are  $\binom{n-2}{n-m}$  different ways of placing the horizontal steps in the path and each horizontal step can again be in  $2(d-1)$  directions, which gives the factor  $(2(d-1))^{n-m}$ . Lastly the paths lie above or below the defect hyperplane:

$$\mathbb{E}[Y] = 2(d-1)q + \sum_{n=2}^{\infty} p^n \sum_{\substack{m=2, \\ \text{even}}}^n 2(2(d-1))^{n-m} \binom{n-2}{n-m} \hat{R}_m, \tag{41}$$

where  $\hat{R}_m$  is the number of random walks of length  $m$  that start and end in 0 and of which all other points are positive. We again use the ballot theorem to find

$$\hat{R}_m = \frac{1}{m-1} \binom{m-1}{\frac{1}{2}m}. \tag{42}$$

We conclude that

$$\mathbb{E}[Y] = 2(d-1)q + \sum_{n=2}^{\infty} p^n \sum_{\substack{m=2, \\ \text{even}}}^n \frac{2(2(d-1))^{n-m}}{m-1} \binom{n-2}{n-m} \binom{m-1}{\frac{1}{2}m}. \tag{43}$$

Let  $k = n - m$  be the number of horizontal steps of a path. We can change the sum from a summation firstly over the total steps  $n$  and secondly over the vertical steps  $m$  to a summation

firstly over  $m$  and secondly over the horizontal steps  $k$ :

$$\begin{aligned}
\mathbb{E}[Y] &= 2(d-1)q + \sum_{\substack{m=2, \\ \text{even}}}^{\infty} \sum_{k=0}^{\infty} p^{k+m} \frac{2(2(d-1))^k}{m-1} \binom{k+m-2}{k} \binom{m-1}{\frac{1}{2}m} \\
&= 2(d-1)q + \sum_{\substack{m=2, \\ \text{even}}}^{\infty} \frac{p^{m-1}}{m-1} \binom{m-1}{\frac{1}{2}m} 2p \sum_{k=0}^{\infty} (2(d-1)p)^k \binom{k+m-2}{k} \\
&= 2(d-1)q + \sum_{\substack{m=2, \\ \text{even}}}^{\infty} \frac{p^{m-1} \binom{m-1}{\frac{1}{2}m} 2p}{(1-2(d-1)p)^{m-1}} \sum_{k=0}^{\infty} (1-2(d-1)p)^{m-1} (2(d-1)p)^k \binom{k+(m-1)-1}{k}.
\end{aligned} \tag{44}$$

The second sum is the sum over all values of the probability mass function of the negative binomial distribution with parameters  $2(d-1)p$  and  $m-1$ . We have assumed  $p \leq \frac{1}{2d}$  so  $2(d-1)p < 1$ , therefore the probability mass function is properly defined and we find that the second sum equals 1:

$$\begin{aligned}
\mathbb{E}[Y] &= 2(d-1)q + 2p \sum_{\substack{m=2, \\ \text{even}}}^{\infty} \frac{\left(\frac{p}{1-2(d-1)p}\right)^{m-1}}{m-1} \binom{m-1}{\frac{1}{2}m} \\
&= 2(d-1)q + 2p \sum_{m=0}^{\infty} \frac{\left(\frac{p}{1-2(d-1)p}\right)^{2m+1}}{2m+1} \binom{2m+1}{m+1}.
\end{aligned} \tag{45}$$

We can now apply Lemma 2.2 on the sum because  $\frac{p}{1-2(d-1)p} < \frac{1}{2}$  for  $p < \frac{1}{2d}$ :

$$\begin{aligned}
\mathbb{E}[Y] &= 2(d-1)q + (1-2(d-1)p) \left(1 - \sqrt{1 - 4 \left(\frac{p}{1-2(d-1)p}\right)^2}\right) \\
&= 1 + 2(d-1)(q-p) - \sqrt{1 - 4(d-1)p + 4((d-1)^2 - 1)p^2}.
\end{aligned} \tag{46}$$

□

Because  $\mathbb{E}[X] = \infty$  if  $p \geq \frac{1}{2d}$  and (20) we know that  $\mathbb{E}[T] = \infty$  if  $p \geq \frac{1}{2d}$ . Furthermore if  $p < \frac{1}{2d}$ , using (20), we also find that  $\mathbb{E}[T] = \infty$  if  $\mathbb{E}[Y] \geq 1$  and  $\mathbb{E}[T] < \infty$  if  $\mathbb{E}[Y] < 1$ . So the curve  $\mathbb{E}[Y] = 1$  is of importance in determining the values of  $p$  and  $q$  for which the expected total progeny is finite. This curve is shown in figure 3 for  $d = 2$ ;  $\mathbb{E}[Y]$  is finite underneath the curve and infinite on and above the curve.

Besides the expected total progeny the survival probability is of interest as well. The population will not survive if  $\mathbb{E}[T] < \infty$ , but if  $\mathbb{E}[T]$  is infinite there are still two possibilities. If  $p > \frac{1}{2d}$  there will be a positive survival probability, because a normal branching process with offspring distribution  $\text{BIN}(2d, p)$  would have a chance to survive and  $p < q$ . The next theorem states for what values of  $q$  the survival probability will be positive for  $p \leq \frac{1}{2d}$ .

**Theorem 2.4.** *Let  $p \leq \frac{1}{2d}$ , then*

- (i).  $\mathbb{P}(T = \infty) = 0$  if  $\mathbb{E}[Y] \leq 1$ ,
- (ii).  $\mathbb{P}(T = \infty) > 0$  if  $\mathbb{E}[Y] > 1$ .

*Proof.* The total number of particles satisfies

$$\begin{aligned}
T &= \sum_{n=1}^{T_Y} 1 + X_n \\
&= T_Y + \sum_{n=1}^{T_Y} X_n,
\end{aligned} \tag{47}$$

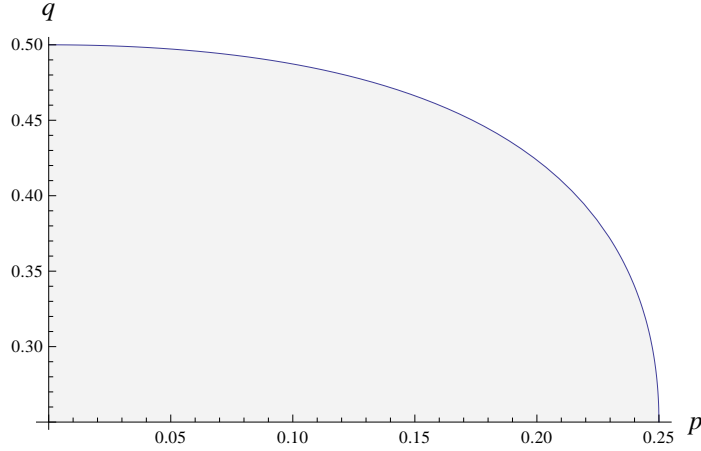


Figure 3: The curve  $\mathbb{E}[Y] = 1$  for  $d = 2$

where all  $X_n$  are i.i.d. copies of  $X$  and  $T_Y$  is the total progeny of a simple branching process with offspring distribution  $Y$ . Because  $p \leq \frac{1}{2d}$ ,

$$\mathbb{P}(X = \infty) \leq \mathbb{P}(T' = \infty), \quad (48)$$

where  $T'$  is the total progeny of a simple branching process with binomial distributed offspring with parameters  $2d$  and  $\frac{1}{2d}$ .  $\mathbb{P}(T' = \infty) = 0$ , and thus  $\mathbb{P}(T = \infty) = \mathbb{P}(T_Y = \infty)$ . Because  $T_Y$  is the total progeny of a branching process we find that  $\mathbb{P}(T_Y = \infty) > 0$  if and only if  $\mathbb{E}[Y] > 1$ .  $\square$

The previous theorem again shows the importance of the curve  $\mathbb{E}[Y] = 1$ , shown in figure 3. The survival probability is zero below this curve and strictly positive above the curve. On the curve extinction occurs with certainty, so  $\mathbb{P}(T \geq m) \rightarrow 0$  as  $m \rightarrow \infty$ . The next theorem gives insight into the speed of this convergence.

**Theorem 2.5.** *Let  $p \leq \frac{1}{2d}$  and  $\mathbb{E}[Y] = 1$ , then*

$$P(T \geq m) \sim \frac{C}{\sqrt{m}}. \quad (49)$$

*Proof.* Firstly we look at the case where  $p = \frac{1}{2d}$ , then, because  $\mathbb{E}[Y] = 1$ ,  $p = q$  and we are left with a simple branching process with offspring distribution  $\text{BIN}(2d, \frac{1}{2d})$ . For this process we can apply Corollary (1.3). We now assume that  $p < \frac{1}{2d}$ , this implies that  $\mathbb{E}[X] < \infty$ . Similar to the proof of Theorem (2.4), we characterize the population as follows:

$$T = \sum_{n=1}^{T_Y} 1 + X_n, \quad (50)$$

where  $T_Y$  is again the total progeny of a simple branching process. To find a lower bound on  $\mathbb{P}(T \geq m)$  we can write

$$\begin{aligned} \mathbb{P}(T \geq m) &\geq \mathbb{P}(T_Y \geq m) \\ &\sim \frac{C_1}{\sqrt{m}}, \end{aligned} \quad (51)$$

for some constant  $C_1$  and using Corollary (1.3). The upperbound requires some more work, conditioning on  $T_Y$  gives

$$\mathbb{P}(T \geq m) = \mathbb{P}\left(T \geq m, T_Y \leq \frac{1-\varepsilon}{1+\mathbb{E}[X]}m\right) + \mathbb{P}\left(T \geq m, T_Y > \frac{1-\varepsilon}{1+\mathbb{E}[X]}m\right), \quad (52)$$

for some small  $\varepsilon > 0$ . An upper bound for the first term is

$$\mathbb{P}\left(T \geq m, T_Y \leq \frac{1-\varepsilon}{1+\mathbb{E}[X]}m\right) \leq \mathbb{P}\left(\sum_{i=1}^{\frac{(1-\varepsilon)m}{1+\mathbb{E}[X]}} (1+X_i) \geq m\right). \quad (53)$$

We can use large deviations on this bound, but in order to do so we first need to show that  $X$  has a finite generating function. For  $X$  we have

$$X \leq T', \quad (54)$$

where  $T'$  is the total progeny of a simple branching process with binomial distributed offspring with parameters  $2d$  and  $p$ . Let  $B \sim \text{BIN}(2d, p)$ , then  $T = 1 + \sum_i^B T_i$ , where  $T_i$  are i.i.d. copies of  $T$  and the generating function  $G_T(s)$  of  $T$  satisfies

$$\begin{aligned} G_T(s) &= \mathbb{E}[s^T] \\ &= s\mathbb{E}[G_T(s)^B] \\ &= s(pG_T(s) + 1 - p)^{2d}. \end{aligned} \quad (55)$$

So  $G_T(s)$  is the root of polynomial equation and is therefore finite. This implies that the generating function of  $X$  is finite as well and we can apply large deviations [6] on (53). Let

$$n = \frac{(1-\varepsilon)m}{1+\mathbb{E}[X]}, \quad (56)$$

then

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{\frac{(1-\varepsilon)m}{1+\mathbb{E}[X]}} (1+X_i) \geq m\right) &= \mathbb{P}\left(\sum_{i=1}^n (1+X_i) \geq \frac{n(1+\mathbb{E}[X])}{1-\varepsilon}\right) \\ &= \mathbb{P}\left(\sum_{i=1}^n (X_i - \mathbb{E}[X]) \geq \frac{\varepsilon n(1+\mathbb{E}[X])}{1-\varepsilon}\right) \\ &\sim e^{-mC_2}, \end{aligned} \quad (57)$$

for some constant  $C_2 > 0$ . The second term of (52) can be bounded from above in the following way

$$\begin{aligned} \mathbb{P}\left(T \geq m, T_Y > \frac{1-\varepsilon}{1+\mathbb{E}[X]}m\right) &\leq \mathbb{P}\left(T_Y \geq \frac{1-\varepsilon}{1+\mathbb{E}[X]}m\right) \\ &\sim \frac{C_3}{\sqrt{m}}, \end{aligned} \quad (58)$$

for some constant  $C_3$  and again using Corollary (1.3). Finally,

$$\begin{aligned} \mathbb{P}(T \geq m) &\leq \mathbb{P}\left(\sum_{i=1}^n (X_i - \mathbb{E}[X]) \geq \frac{\varepsilon n(1+\mathbb{E}[X])}{1-\varepsilon}\right) + \mathbb{P}\left(T_Y \geq \frac{1-\varepsilon}{1+\mathbb{E}[X]}m\right) \\ &\sim e^{-mC_2} + \frac{C_3}{\sqrt{m}} \\ &\sim \frac{C}{\sqrt{m}}. \end{aligned} \quad (59)$$

□



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