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Several insurance risk and queueing models with dependence structures

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Several insurance risk and queueing models
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Abstract

Part I of the report reviews a paper of Kwan and Yang [1] and a thesis of Emaga [2]. Both articles focus on single threshold risk models. In [1], a closed-form expression is derived for the ruin probability of a special case of the single threshold risk model by means of solving the corresponding integro-differential equation. In addition, Lundberg's Inequality is presented. In [2], another special case of the single threshold risk model is related to two dual queueing models. For instance, several relations between the ruin probability and some performance measures of the dual queueing models are obtained.

In Part II, we consider a few problems about single threshold risk models. Chapter 4 examines the effects of positive and negative correlation on the ruin probability functions and the strictness of Lundberg's Inequality. In Chapter 5, the martingale property of a certain stochastic quantity in the $M/\overleftarrow{G}_a/1$ risk model is proven not to be true. On the other hand, Lundberg's Inequality is verified for two more general risk models with dependence structures. In Chapter 6, a Rouché problem is considered in which we try to apply Rouché's theorem on the denominator of the Laplace-Stieltjes transform of the sojourn time in the $M/\overleftarrow{G}_M/1$ queueing model.

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Chapter 1

Introduction

In the past, especially in the time of economic crisis, a lot of efforts are spent on investigating the solvency of insurance companies. Many risk models are developed to describe the process in which the surplus of such a company evolves. In particular, the ruin probability of these models is of great interest for the consequences are far-reaching if an insurance company declares bankruptcy.

A risk model captures the rising and the falling of surplus due to the accumulation of premiums over time and as a result of claim demands. To simplify the process, premiums are assumed to be collected continuously, whereas the claims occur instantaneously. In the well-studied classical Cramér-Lundberg model, inter-occurrence times and claim sizes are assumed to be independent. Although this simplifying assumption leads to many elegant results and applications, it sometimes turns out to be rather restrictive and unrealistic. For this reason, more general models are considered that take into account some dependence structures. Whilst certain dependence structures retain the nice features of the classical model, others may overthrow the system.

This report consists of two parts: a literature study and a research part. In the literature study, some basic concepts of risk models are reviewed by providing the summary of two articles. Moreover, it introduces the dependence model, which the research part proceeds on and generalizes. The first article [1] is a paper of Kwan and Yang, which treats the single threshold risk model. For this model, the closed-form expression of the ruin probability is found by solving the corresponding integro-differential equation. Furthermore, Lundberg's Inequality is presented. Next, the thesis of Emaga [2] reconsiders the risk model examined by Kwan and Yang and relates it to several dual queueing models. Interesting relations are derived between the ruin probability and some performance measures of the corresponding queueing models.

In the research part, we consider several problems in the setting of risk models with dependence structures. Chapter 4 examines how the correlation between inter-arrival times and claim sizes affects the strictness of Lundberg's upper bound and the ruin probability. In particular, we distinguish between positive and negative correlation. In Chapter 5, we consider two generalizations of the single threshold model and check whether certain properties of the classical model remain to be true. In chapter 6, a Rouché problem is considered, which is suggested by a discussion in the thesis of Emaga [2]. The motivation to solve this problem is that we can use it to derive the last missing constant of a certain Laplace-Stieltjes transform in a certain queueing model.

Notation for models

Before going into the details, some notations are introduced to denote different risk and queueing models. Throughout this report the Kendall notation system is used to describe certain characteristics of models.

Queueing models

By means of the following three-part code $a/b/c$, the three main aspects of a queueing model are specified:

a : inter-arrival time distribution,

b : service distribution,

c : the number of servers; throughout the report only models consisting of one server will be considered.

a and b are either M or G . M corresponds to the exponential distribution, whereas G denotes a general distribution.

Risk models

In the similar way, risk models are denoted by a/b :

a : inter-arrival time distribution,

b : claim size distribution.

Dependence structures

Both in queueing models and risk models some dependence structures may be present between inter-arrival times and respectively service times and claim sizes. Depending on the precise dependence structure, an arrow are placed above a or b .

Symbol	Dependence structure
a/\overleftarrow{b}	b depends on the previous a
a/\overrightarrow{b}	b depends on the next a
\overrightarrow{a}/b	a depends on the next b
\overleftarrow{a}/b	a depends on the previous b

Table 1.1: Notation for dependence structures.

Threshold models

In this report, special attention is spent on a particular dependence structure involving one or more thresholds, which will be indicated by a subscript attached to b . These thresholds can either be deterministic or exponentially distributed and will be denoted by a and M respectively.

Part I

Literature study

Chapter 2

Summary of article [1]

Title: Ruin probability in a threshold insurance risk model.

Author: I.K.M. Kwan, H. Yang.

2.1 The insurance risk model

In the paper "Ruin probability in a threshold insurance risk model" of I.K.M. Kwan and H. Yang, the ruin probability of a certain risk model is considered. Some understanding of this probability is developed. Furthermore, the closed-form expression is obtained for a special case of this model.

Model assumptions

The model under consideration is the threshold risk model with a dependence structure, which will be denoted by M/\overleftarrow{G}_a . Here, the k -th claim size X_k depends on the inter-arrival time between the $(k-1)$ -th claim and the k -th claim T_k . The letter M indicates the i.i.d. inter-arrival times, which are exponentially distributed with parameter λ , whereas the \overleftarrow{G}_a stands for the general distributed claim sizes having either F_1 or F_2 as distribution function. If the previous inter-arrival time is less than a predetermined threshold a , the claim size has distribution F_1 and F_2 otherwise. This backward dependency is denoted by the arrow above G_a .

Starting from an initial surplus $U(0) = u_0$, the process $U(t)$ develops according to the following description:

$$U(t) = u_0 + ct - S(t), \quad (2.1)$$

where c is the constant premium rate, $N(t)$ the number of claims up to time t and $S(t) = \sum_{k=1}^{N(t)} X_k$ the aggregate claim size at time t .

Ruin probability

In this model, the ruin probability is defined as a function of the initial surplus: $\psi(u) = \mathbb{P}[T < \infty \mid u_0 = u]$, where $T = \inf_{t \geq 0} \{t : U(t) < 0\}$ is the time of ruin. On the other hand, the survival probability, the probability that ruin does not ever occur, is denoted by $\phi(u) = \mathbb{P}[T = \infty \mid u_0 = u]$. As these events, ruin and survival, are complementary, the relation holds that $\phi(u) = 1 - \psi(u)$.

Net profit condition

In order to prevent ruin to happen with probability one, the following net profit condition is assumed to be satisfied throughout the report:

$$ct = (1 + \theta)\mathbb{E}[S(t)], \quad \text{with } \theta > 0. \quad (2.2)$$

Lundberg's Inequality

In general, it is not always possible to find a closed-form expression for the ruin probability. In such cases, one can still try to gain some understanding about the ruin probability by considering an upper bound provided by Lundberg's Inequality:

$$\psi(u) \leq e^{-Ru}, \quad (2.3)$$

where R is the unique positive solution of $\mathbb{E}[e^{r(X_1 - cT_1)}] = 1$. This number is called the adjustment coefficient and its unique existence is ensured by the following theorem.

Theorem 2.1.

Assume that $M_i(r) = \int_0^\infty e^{rx} dF_i(x) < \infty$ for $r < r_\infty^{(i)}$ and that $\lim_{r \rightarrow r_\infty^{(i)}} M_i(r) = \infty$, where

$r_\infty^{(i)} \in \mathbb{R} \cup \{\infty\}$ for $i = 1, 2$.

Then there exists a unique solution $R > 0$ to the equation $\mathbb{E}[e^{r(X_1 - cT_1)}] = 1$.

The surplus process can also be seen as a random walk with i.i.d. increments $X_k - cT_k$. The proof of Lundberg's Inequality in [1] goes by changing of measure which follows the same arguments as in [4].

2.2 Integro-differential equations

Now the model is established, integro-differential equations for $\phi(u)$ and $\psi(u)$ are developed in this section and will be presented in the next two theorems. These will be a tool to find the closed-form solution for $\psi(u)$ in some special cases like exponential distributed claim sizes.

In the sequel, $\bar{F}_i(x)$ and $f_i(x)$ are used to denote the tail probability and the density function of $F_i(x)$ respectively for $i = 1, 2$.

Integro-differential equation for $\phi(u)$

Theorem 2.2.

The survival probability $\phi(u)$ satisfies the following integro-differential equation:

$$c \frac{d\phi(u)}{du} - \lambda\phi(u) = \lambda e^{-\lambda a} \int_0^{u+ca} [f_1(x) - f_2(x)]\phi(u+ca-x)dx - \lambda \int_0^u f_1(x)\phi(u-x)dx. \quad (2.4)$$

Sketch of proof.

1. By conditioning on the first inter-arrival time and the first claim size, an expression for $\phi(u)$ arises:

$$\begin{aligned} \phi(u) &= \int_0^a \lambda e^{-\lambda t} \int_0^{u+ct} f_1(x)\phi(u+ct-x)dxdt \\ &\quad + \int_a^\infty \lambda e^{-\lambda t} \int_0^{u+ct} f_2(x)\phi(u+ct-x)dxdt. \end{aligned} \quad (2.5)$$

2. Now let $s = u + ct$ and multiply the resulting equation by c to obtain:

$$\begin{aligned} c\phi(u) &= \int_u^{u+ca} \lambda e^{-\lambda(\frac{s-u}{c})} \int_0^s f_1(x)\phi(s-x)dxds \\ &\quad + \int_{u+ca}^\infty \lambda e^{-\lambda(\frac{s-u}{c})} \int_0^s f_2(x)\phi(s-x)dxds. \end{aligned} \quad (2.6)$$

3. The next step is to differentiate Equation (2.6) with respect to u :

$$\begin{aligned} c \frac{d\phi(u)}{du} &= -\lambda \int_0^u f_1(x)\phi(u-x)dx \\ &\quad + \lambda e^{-\lambda a} \int_0^{u+ca} f_1(x)\phi(u+ca-x)dx \\ &\quad - \lambda e^{-\lambda a} \int_0^{u+ca} f_2(x)\phi(u+ca-x)dx \\ &\quad + \frac{\lambda}{c} \int_u^{u+ca} \lambda e^{-\lambda(\frac{s-u}{c})} \int_0^s f_1(x)\phi(s-x)dxds \\ &\quad + \frac{\lambda}{c} \int_{u+ca}^\infty \lambda e^{-\lambda(\frac{s-u}{c})} \int_0^s f_2(x)\phi(s-x)dxds. \end{aligned} \quad (2.7)$$

4. The last two terms of Equation (2.7) together are together $\lambda\phi(u)$. By applying this substitution the proof is completed. ■

Integro-differential equation for $\psi(u)$

Theorem 2.3.

The ruin probability $\psi(u)$ satisfies the following integro-differential equation:

$$\begin{aligned} c\psi(u) &= \lambda \left(\int_u^\infty \bar{F}_1(x) dx + \int_0^u \psi(u-x) \bar{F}_1(x) dx \right) \\ &\quad + \lambda e^{-\lambda a} \left(\int_{u+ca}^\infty [\bar{F}_2(x) - \bar{F}_1(x)] dx \right. \\ &\quad \left. + \int_0^{u+ca} \psi(u+ca-x) [\bar{F}_2(x) - \bar{F}_1(x)] dx \right). \end{aligned} \quad (2.8)$$

Sketch of proof.

1. First, integrate both sides of Equation (2.4) from 0 to u to get:

$$\begin{aligned} &c[\phi(u) - \phi(0)] - \lambda \int_0^u \phi(y) dy \\ &= \lambda e^{-\lambda a} \int_0^u \int_0^{y+ca} [f_1(x) - f_2(x)] \phi(y+ca-x) dx dy \\ &\quad - \lambda \int_0^u \int_0^y f_1(x) \phi(y-x) dx dy. \end{aligned} \quad (2.9)$$

2. After applying a series of substitutions on Equation (2.9), the following arises:

$$\begin{aligned} &c[\phi(u) - \phi(0)] - \lambda \int_0^u \phi(y) dy \\ &= \lambda e^{-\lambda a} \int_0^{u+ca} \phi(u+ca-x) [F_1(x) - F_2(x)] dx \\ &\quad - \lambda e^{-\lambda a} \int_0^{ca} \phi(ca-x) [F_1(x) - F_2(x)] dx \\ &\quad - \lambda \int_0^u \int_0^{u-y} f_1(x) \phi(y) dx dy. \end{aligned} \quad (2.10)$$

3. By combining the last term of both sides of Equation (2.10), we get:

$$\begin{aligned} &\lambda \int_0^u \phi(y) dy - \lambda \int_0^u \int_0^{u-y} f_1(x) \phi(y) dx dy \\ &= \lambda \int_0^u \phi(y) \left(1 - \int_0^{u-y} f_1(x) dx \right) dy \\ &= \lambda \int_0^u \phi(y) \bar{F}_1(u-y) dy. \end{aligned} \quad (2.11)$$

4. Substituting Equation (2.11) in Equation (2.10) gives:

$$\begin{aligned} c[\phi(u) - \phi(0)] &= \lambda \int_0^u \phi(y) \bar{F}_1(u-y) dy \\ &\quad + \lambda e^{-\lambda a} \int_0^{u+ca} \phi(u+ca-x) [F_1(x) - F_2(x)] dx \\ &\quad - \lambda e^{-\lambda a} \int_0^{ca} \phi(ca-x) [F_1(x) - F_2(x)] dx. \end{aligned} \quad (2.12)$$

5. Finally, by taking the limit of Equation (2.12) as $u \rightarrow \infty$, an expression for $\phi(0)$ is deduced. This expression will then be put back into Equation (2.12) and by substituting $\phi(u) = 1 - \psi(u)$ Theorem 2.3 is proven. ■

2.3 Exponential claim size distribution

Now, let the claim sizes be exponentially distributed. The parameters of F_1 and F_2 are set to be β_1 and β_2 respectively. For this case, a closed-form expression for the ruin probability can be obtained by deducing a differential equation for $\psi(u)$ from the integro-differential equations established earlier. Subsequently, the characteristic equation of this differential equation is solved which gives us an expression for $\psi(u)$.

Differential equation for $\psi(u)$

Theorem 2.4.

The ruin probability $\psi(u)$ satisfies the following third order differential equation:

$$c \frac{d^3 \psi(u)}{du^3} + (c\beta_1 + c\beta_2 - \lambda) \frac{d^2 \psi(u)}{du^2} + \beta_2(c\beta_1 - \lambda) \frac{d\psi(u)}{du} - \lambda e^{-\lambda a} (\beta_1 - \beta_2) \frac{d\psi(u + ca)}{du} = 0. \quad (2.13)$$

Sketch of proof.

1. Differentiate Equation (2.4) and rearrange some terms to get:

$$\begin{aligned} c \frac{d^2 \phi(u)}{du^2} + (c\beta_1 - \lambda) \frac{d\phi(u)}{du} - \lambda e^{-\lambda a} (\beta_1 - \beta_2) \phi(u + ca) \\ = \lambda e^{-\lambda a} (\beta_2 - \beta_1) \int_0^{u+ca} \beta_2 e^{-\beta_2(u+ca-x)} \phi(x) dx. \end{aligned} \quad (2.14)$$

2. Let set $h(u)$ to be the right-hand side of Equation (2.14) and differentiate it:

$$\begin{aligned} \frac{dh(u)}{du} &= \lambda e^{-\lambda a} (\beta_2 - \beta_1) \beta_2 \left(\phi(u + ca) - \int_0^{u+ca} \beta_2 e^{-\beta_2(u+ca-x)} \phi(x) dx \right) \\ &= \lambda e^{-\lambda a} (\beta_2 - \beta_1) \beta_2 \phi(u + ca) - \beta_2 h(u). \end{aligned} \quad (2.15)$$

3. Bring $\beta_2 h(u)$ of Equation (2.15) to the left-hand side to get:

$$\frac{dh(u)}{du} + \beta_2 h(u) = \lambda e^{-\lambda a} (\beta_2 - \beta_1) \beta_2 \phi(u + ca). \quad (2.16)$$

4. Finally, by substituting the left-hand side of Equation (2.14) into $h(u)$ in Equation (2.16) and replacing $\phi(u)$ by $1 - \psi(u)$ Theorem 2.4 is proven. ■

Characteristic equation for $\psi(u)$

This third order differential Equation (2.13) can be reduced to a second order differential equation by integrating it from 0 to ∞ . Using the equalities $\psi(\infty) = 0$, $\psi'(\infty) = 0$ and $\psi''(\infty) = 0$ then gives us:

$$c \frac{d^2 \psi(u)}{du^2} + (c\beta_1 + c\beta_2 - \lambda) \frac{d\psi(u)}{du} + \beta_2(c\beta_1 - \lambda) \psi(u) - \lambda e^{-\lambda a} (\beta_1 - \beta_2) \psi(u + ca) = 0. \quad (2.17)$$

By substituting the Taylor series of $\psi(u + ca)$, $\sum_{k=0}^{\infty} \frac{(ca)^k \psi^{(k)}(u)}{k!}$, into Equation (2.17), an ordinary differential equation for $\psi(u)$ arises, which has the following characteristic equation:

$$l(x) = cx^2 + (c\beta_1 + c\beta_2 - \lambda)x + \beta_2(c\beta_1 - \lambda) - \lambda e^{\lambda a} (\beta_1 - \beta_2) e^{xca} = 0. \quad (2.18)$$

Closed-form expression for $\psi(u)$

Since $l(x)$ is an entire function, Equation (2.18) would have no more than a finite number of zeros in any vertical strip of the complex plane. Let R_i be the roots of Equation (2.18) and l_i be the corresponding multiplicities for $i = 1, \dots, n$. By some linearity arguments, it can be proven that Equation (2.17) must admit $p_i(u)e^{R_i u}$ as solution for each i , which is the sum of all solutions of the form $u^k e^{R_i u}$ for $k = 0, 1, \dots, l_i - 1$. Note that by definition, $p_i(u)$ is a polynomial of order $(l_i - 1)$. By adding up all these solutions, the expression for the ruin probability is found to be:

$$\psi(u) = \sum_{i=1}^n p_i(u)e^{R_i u}. \quad (2.19)$$

2.4 Zeros of characteristic equation

This section will offer a study about the zeros of the characteristic equation $l(x) = 0$. Each subsection is devoted to a certain type of zero, for which the existence or some existence conditions will be derived.

First of all, define the auxiliary functions $g(x)$ and $h(x)$ as follows:

$$\begin{aligned} g(x) &= cx^2 + (c\beta_1 + c\beta_2 - \lambda)x + \beta_2(c\beta_1 - \lambda) = (cx + c\beta_1 - \lambda)(x + \beta_2), \\ f(x) &= \lambda e^{-\lambda a}(\beta_1 - \beta_2)e^{xca}. \end{aligned}$$

For instance, $l(x)$ can be written as $g(x) - f(x)$ and the zeros of $l(x)$ are just the interception points of $g(x)$ and $f(x)$.

Roots with non-negative real part

As the ruin probability $\psi(u) = \sum_{i=1}^n p_i(u)e^{R_i u}$ converges to zero as u tends to infinity, all terms with non-negative real part of R_i should converge to 0 as well. This implies that these terms should not exist.

Negative real zeros

The second type is negative real zeros. It can be verified that two distinct negative real zeros exist for the characteristic equation. To see this, distinguish the following two cases.

Case $\beta_1 > \beta_2$

Suppose that $\beta_1 > \beta_2$. The net profit condition, Equation (2.2), implies $c\beta_1 - \lambda > 0$. In addition, we have $\beta_2 > 0$, therefore $g(x) = (cx + c\beta_1 - \lambda)(x + \beta_2)$ should have two negative real zeros, say ξ_1 and ξ_2 . Also, note that $f(x) > 0$ for all $x \in \mathbb{R}$, hence $f(x) > g(x)$ for $x \in (\xi_1, \xi_2)$.

Furthermore, it can be proven that $g(0) > f(0)$ holds. In Section 2.3, the net profit condition is stated as follows: $ct = (1 + \theta)\mathbb{E}[S(t)]$, for $\theta > 0$. Applying Wald's Identity on this equality gives:

$$\begin{aligned} ct &= (1 + \theta)\lambda t(\mathbb{P}[T < a]/\beta_1 + \mathbb{P}[T \geq a]/\beta_2) \\ &= (1 + \theta)\lambda t((1 - e^{-\lambda a})/\beta_1 + e^{-\lambda a}/\beta_2) \\ &> \lambda t((1 - e^{-\lambda a})/\beta_1 + e^{-\lambda a}/\beta_2). \end{aligned}$$

Dividing both sides of the above equation by t then yields $c > \lambda((1 - e^{-\lambda a})/\beta_1 + e^{-\lambda a}/\beta_2)$, which is equivalent to $g(0) = c\beta_1\beta_2 - \lambda\beta_2 > \lambda e^{-\lambda a}(\beta_1 - \beta_2) = f(0)$. As $g(x) - f(x)$ is strictly decreasing on $(-\infty, \xi_1]$ and strictly increasing on $(\xi_2, 0]$, $g(x)$ and $f(x)$ must have two negative interception points, see Figure 2.1(a).

Case $\beta_1 < \beta_2$

Suppose that $\beta_1 < \beta_2$. Again, by Wald's Identity, $g(0) > f(0)$ holds. Next, it can be shown that $g(x)$ is less than $h(x)$ at the minimum point of $g(x)$, $\frac{\lambda - c\beta_1 - c\beta_2}{2c}$, by considering $r(a) = g(\frac{\lambda - c\beta_1 - c\beta_2}{2c}) - f(\frac{\lambda - c\beta_1 - c\beta_2}{2c})$ as a function of the variable a . It is less than 0 at $a = 0$ and $r'(a) < 0$ for all $a \in \mathbb{R}^+$, therefore $g(x) - f(x)$ is always less than 0 at the minimum point of $g(x)$. Moreover, $g(x) - f(x)$ is strictly decreasing on $(-\infty, \gamma]$ and strictly increasing on $(\gamma, 0]$, where γ is the minimum point of $g(x) - f(x)$. Hence, it can be concluded that $g(x)$ and $f(x)$ have two negative interception points, see Figure 2.1(b).

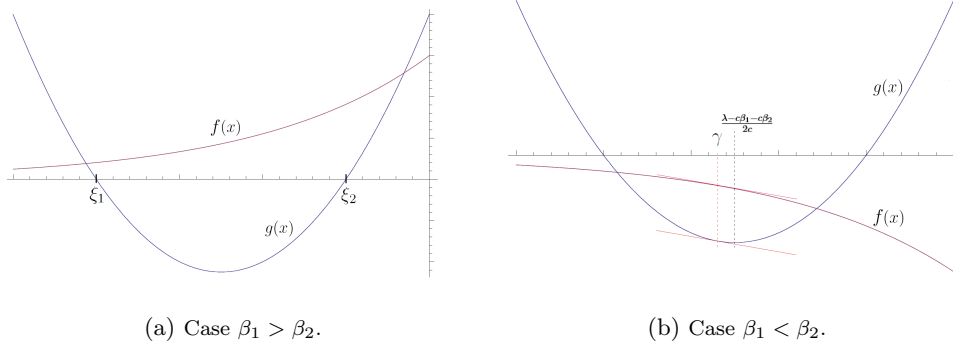


Figure 2.1: $g(x)$ and $f(x)$ on $(-\infty, 0]$.

Multiple zeros

For any multiple zero to exist, it must satisfy the second derivative condition $l'(x) = 0$. A new equation arises by subtracting $l'(x) = 0$ from $l(x) = 0$, which has a determinant always greater than zero. Therefore, any possible multiple root must be real and some more analysis offers an expression. Substituting this expression into $l'(x) = 0$ gives the following existence conditions for multiple zeros:

$$2A + \sqrt{(B^2 - 4AC)k^2 + 4A^2} - Dk^2 e^{\frac{2A - Bk + \sqrt{(B^2 - 4AC)k^2 + 4A^2}}{2A}} = 0, \quad \text{for } \beta_1 > \beta_2,$$

$$2A - \sqrt{(B^2 - 4AC)k^2 + 4A^2} - Dk^2 e^{\frac{2A - Bk - \sqrt{(B^2 - 4AC)k^2 + 4A^2}}{2A}} = 0, \quad \text{for } \beta_1 < \beta_2.$$

The following definitions are used in the formulae above:

$$\begin{aligned} A &= c, \\ B &= c\beta_1 + c\beta_2 - \lambda, \\ C &= \beta_2(c\beta_1 - \lambda), \\ D &= \lambda e^{-\lambda a}(\beta_1 - \beta_2), \\ k &= ca. \end{aligned}$$

Complex zeros

Finally, the existence of complex zeros is considered. If any complex zero $R = p + qi$ may exist, it must come in pairs with its conjugate \bar{R} , since $l(x)$ is a real analytic function. By substituting R and \bar{R} into the characteristic equation respectively and executing some algebraic manipulations, a quadratic equation appears. Now, by some similar arguments as those for the negative real roots, the possible set of complex zeros is found. For the exact expression of the complex zeros, see Section 5.1 of [1].

2.5 Explicit solution

In the previous section, it is shown that the characteristic equation $l(x) = 0$ must have two distinct negative real zeros, say R_1 and R_2 . Furthermore, there may exist some multiple and complex zeros. However, some analysis of the characteristic equation can eliminate the possibility of having any zeros other than R_1 and R_2 . The reasoning makes use of Rouché's Theorem.

First of all, note that $l(x)$ is an entire function, since $g(x)$ and $f(x)$ are both analytic over the complex plane. Moreover, there always exists a closed contour K in the left-half plane such that $|g(x)| > |f(x)|$ holds on K . In addition, this contour can be chosen to be big enough to comprise all zeros of $g(x)$ and $l(x)$. By Rouché's Theorem, $l(x) = g(x) - f(x)$ must then have two zeros inside K , since $g(x)$ has two zeros inside K . These two are exactly R_1 and R_2 .

Now the zeros of the characteristic equation are found, the expression of $\psi(u)$ becomes:

$$\psi(u) = \bar{A}e^{R_1 u} + \bar{B}e^{R_2 u}, \quad (2.20)$$

where \bar{A} and \bar{B} are two constants. The final step is to compute \bar{A} and \bar{B} . This can be done by substituting Equation (2.20) into the Equations (2.4) and (2.8) and putting $u = 0$. This will give us two equations for the two missing constants.

Chapter 3

Summary of article [2]

Title: Insurance risk and queueing models with threshold dependence.

Author: J.C.S. Emaga.

The thesis of Emaga consists of 3 parts.

1. The first part covers some properties of the ruin probability in the classical Cramér-Lundberg model with no dependency between inter-arrival times and claim sizes.
2. The second part elaborates the M/\overleftarrow{G}_a risk model in the same setting as in [1] and two special cases of it. Some findings of [1] are recapitulated and the M/\overleftarrow{G}_a risk model is related to several queueing models using dualities.
3. The last part deals with the $M/\overleftarrow{G}_M/1$ queueing model. The Laplace-Stieltjes transforms and the expectations of some important stochastic variables are computed. In particular, Chapter 6 of this report is devoted to finding the missing constant in the Laplace-Stieltjes transform of the sojourn time.

This summary will only consist of the contents which are relevant to the research part. This would be the second and the third part of the thesis, which correspond to Chapter 3 and Chapter 4 of [2]. Furthermore, certain parts of [2] are omitted for they are already covered in [1].

For the sake of the textual coherence of this report, we will follow the same notations as in the previous chapter in stead of the notations of [2].

3.1 Ruin probability of the M/\overleftarrow{M}_a risk model

In this section, explicit expressions for the (zero-delayed) ruin probability and the stationary ruin probability of a special case of the M/\overleftarrow{G}_a risk model are determined. This special case, namely M/\overleftarrow{M}_a , belongs to the same general setting as the risk model discussed in [1], but with the particularity that no claims are accepted before some predetermined threshold.

Setting of M/\overleftarrow{M}_a

Recall that the claims arrive according to a Poisson process with rate λ . Now, whenever the inter-arrival time is less than some predetermined threshold a , the claim size would simply equal zero. Otherwise, it follows the exponential distribution with parameter β_2 .

Thus, the tail probability of the claim size distributions takes the following form:

$$\begin{cases} \bar{F}_1(x) = 0 & \text{if } T_k < a, \\ \bar{F}_2(x) = e^{-\beta_2 x} & \text{if } T_k \geq a. \end{cases} \quad (3.1)$$

Ruin probability $\psi(u)$

In order to derive an expression for the ruin probability, the integro-differential equation in Theorem 2.2 is rewritten. First, the tail probabilities in Equation (3.1) are substituted in Equation (2.8):

$$c\psi(u) = \lambda e^{-\lambda a} \left(\int_{u+ca}^{\infty} e^{-\beta_2 x} dx + \int_0^{u+ca} \psi(u+ca-x) e^{-\beta_2 x} dx \right) \quad (3.2)$$

$$= \frac{\lambda}{\mu} e^{-\lambda a} e^{-\beta_2(u+ca)} + \int_0^{u+ca} \psi(u+ca-x) e^{-\beta_2 x} dx. \quad (3.3)$$

Differentiating Equation (3.3) and reordering terms give:

$$c \frac{\psi(u)}{du} + \beta_2 c \psi(u) - \lambda e^{\lambda a} \psi(u+ca) = 0. \quad (3.4)$$

Using the Taylor series of $\psi(u+ca)$, a differential equation arises:

$$c \frac{\psi(u)}{du} + \beta_2 c \psi(u) - \lambda e^{\lambda a} \sum_{k=0}^{\infty} \frac{(ca)^k \psi^{(k)}(u)}{k!} = 0. \quad (3.5)$$

Now, substituting $\psi(u) = Ae^{Ru}$ into Equation (3.5) gives us a nonlinear characteristic equation:

$$l(x) = xc - \lambda e^{-\lambda a} e^{xca} + \beta_2 c = 0. \quad (3.6)$$

It still remains to find the zeros of the characteristic equation. By decomposing the function into 2 parts in an adequate way, the equation is brought to a form on which Rouché's Theorem can be applied. This will help us to prove that the characteristic equation has exactly one negative zero. To be exact, define auxiliary functions $g(x)$ and $f(x)$ as follows:

$$g(x) = x + \beta_2, \quad (3.7)$$

$$f(x) = -\frac{\lambda}{c} e^{-\lambda a} e^{xca}. \quad (3.8)$$

$g(x)$ and $f(x)$ are analytic inside and on the left-half ball K . In addition, $|g(x)| > |f(x)|$ on K . By Rouché's Theorem, $l(x)$ has one zero inside K , since $g(x)$ has exactly one negative zero there. Let R be the zero of $l(x)$. Finally, the constant A can be found by substituting $\psi(u) = Ae^{Ru}$ into Equation (3.2). As a result, we find:

$$\psi(u) = \frac{\lambda e^{-\lambda a}}{\beta_2 c} e^{R(u+a)}. \quad (3.9)$$

Remark 3.1. Note that the derivation of the ruin probability follows the same scheme as the one in [1]. It makes sense, since this is a special case of the risk model discussed there. In particular, Equation (3.5) can be obtained from Equation (2.13) by dividing the latter equation by β_1 and taking the limit of $\beta_1 \rightarrow \infty$.

Remark 3.2. The classical Cramér-Lundberg model with exponentially distributed claim sizes can be seen as a special case of M/\overleftarrow{M}_a , where the threshold a is set to be zero. The expression of the ruin probability then simply reduces to $\psi(u) = \frac{\lambda}{\mu_2 c} e^{Ru}$.

Remark 3.3. In this rather simple case, it is already difficult to solve for the ruin probability due to the nonlinearity of the differential equation. In general, it is not always clear how to derive the ruin probability.

Laplace-Stieltjes transform of the compound inter-arrival time

Before turning to the stationary ruin probability, the Laplace-Stieltjes transform of the inter-arrival time between the non-zero claims \tilde{T}_c is derived. Each of such compound inter-arrival times consists of two parts: T_s and T_l . Here, T_s is the sum of a few small inter-arrival times T_s^i , whereas T_l is a large one. Now, the number of small inter-arrivals until a large one comes around can be seen as a geometrically distributed random variable N with success probability $p = e^{-\lambda a}$. This very fact gives rise to a way to compute the Laplace-Stieltjes transform \tilde{T}_c , see [2] p. 21 for the details of the computation:

$$\begin{aligned} \tilde{T}_c(s) &= \tilde{T}_s(s) \cdot \tilde{T}_l(s) \\ &= \mathbb{E}[e^{-s(\sum_{i=1}^N T_s^i)}] \mathbb{E}[e^{-sT_l}] \\ &= \left(\sum_{k=1}^{\infty} \mathbb{E}[e^{-s(\sum_{i=1}^k T_s^i)} \mid N = k] P[N = k] \right) \mathbb{E}[e^{-sT_l}] \\ &= \frac{\lambda}{\lambda + se^{(s+\lambda)a}}. \end{aligned} \quad (3.10)$$

Remark 3.4. $\tilde{T}_c(s)$ is also the Laplace-Stieltjes transform of the compound inter-arrival time of the dual queueing model $M/\overrightarrow{M}_a/1$. The only difference between the inter-arrival times of these two models is that the large arrival occurs at the beginning of the compound arrival in the $M/\overrightarrow{M}_a/1$ queue, due to the reversed dependency.

Remark 3.5. The mean compound inter-arrival time is now simply: $\lambda_c = \mathbb{E}[T_c] = -\tilde{T}_c'(0) = \frac{e^{\lambda a}}{\lambda}$.

Remark 3.6. Let $G(x)$ be the distribution function of T_c and $G_{eq}(x)$ its corresponding equilibrium distribution function, then the standard relation holds that $G_{eq}(x) = \frac{\int_0^x (1-G(u)) du}{\lambda_c}$. Remarkably, in the setting of M/\overleftarrow{M}_a , the latter relation simply reduces to a shift by a , i.e. $G_{eq}(x) = G(x+a)$, which can be derived from a computation on the Laplace-Stieltjes transforms:

$$\tilde{T}_{eq}(s) = \int_0^{\infty} e^{-sx} \frac{1-G(x)}{\lambda_c} dx = \frac{1-\tilde{T}_c(s)}{s\lambda_c} = \frac{\lambda}{se^{\lambda a}} \frac{se^{(\lambda+s)a}}{\lambda + se^{(\lambda+s)a}} = \tilde{T}_c(s) e^{sa}. \quad (3.11)$$

Stationary ruin probability $\psi^{(0)}(u)$

With the Laplace-Stieltjes transforms in hand, an expression for the stationary ruin probability $\psi^{(0)}(u)$ can be obtained. Note that $\psi^{(0)}(u)$ is the integral of the x -delayed ruin probability $\psi_x(u)$ with respect to $G_{eq}(x)$, where $\psi_x(u) = \bar{F}_2(u+x) + \int_0^{u+x} \psi(u+x-y)dF_2(y)$:

$$\begin{aligned}
 \psi^{(0)}(u) &= \int_0^\infty \left(\bar{F}_2(u+x) + \int_0^{u+x} \psi(u+x-y)dF_2(y) \right) dG_{eq}(x) \\
 &= \int_a^\infty \left(\bar{F}_2(u+x-a) + \int_0^{u+x-a} \psi(u+x-a-y)dF_2(y) \right) dG(x) \\
 &= \int_0^\infty \left(\bar{F}_2(u+x-a) + \int_0^{u+x-a} \psi(u+x-a-y)dF_2(y) \right) dG(x) \\
 &= \psi(u-a).
 \end{aligned} \tag{3.12}$$

The second equality follows from $G_{eq}(x) = G(x+a)$, while the third equality is due to the fact that $G(x) = 0$ for $x < a$.

3.2 Relations between M/\overleftarrow{M}_a and $G/M/1$

Following the same setting as the previous section where no claims are accepted before the deterministic threshold a , even more properties can be proven about the ruin probability if the premium rate c just equals one. Using some duality arguments, the probabilities of interest can be linked to two performance probabilities of the $G/M/1$ queue.

Duality D_1

Duality D_1 is constituted by mapping a reversed finite sequence of inter-arrival times and claim sizes of M/\overleftarrow{M}_a to the dual $M/\overrightarrow{M}_a/1$ queue. To be exact, fix some $N \in \mathbb{N}$, let T_1, T_2, \dots, T_N be the inter-arrival times of the claims and X_1, X_2, \dots, X_N the claim sizes, then the inter-arrival times and the service times of the dual queueing model would be T_N, T_{N-1}, \dots, T_1 and X_N, X_{N-1}, \dots, X_1 respectively.

In fact, this specific $M/\overrightarrow{M}_a/1$ queue is equivalent to a $G/M/1$ queue, where the inter-arrival times are governed by the Laplace-Stieltjes transform of the compound inter-arrival times $\tilde{T}_c(s)$, given by Equation (3.10).

Two performance probabilities of $G/M/1$

By results from queueing theory (see [8] p. 279) the $G/M/1$ queue obtained by duality D_1 has the following distributions for the steady-state waiting time W and workload V :

1. steady-state waiting time

$$W \sim \begin{cases} 0, & \text{with probability } 1 - \theta, \\ \exp(\eta) & \text{with probability } \theta, \end{cases} \quad (3.13)$$

2. steady-state workload

$$V \sim \begin{cases} 0, & \text{with probability } 1 - \rho, \\ \exp(\eta) & \text{with probability } \rho, \end{cases} \quad (3.14)$$

where η is the positive solution of the equation $\frac{\beta_2}{\beta_2 - \eta} \int_0^\infty e^{-\eta x} dG(x) = 1$, $\theta = \frac{\beta_2 - \eta}{\beta_2}$ and $\rho = \frac{1}{\beta_2 \lambda c} = \frac{\lambda}{\beta_2 e^{\lambda a}}$.

Two relations

By setting $R = -\eta$ two interesting equalities arise. The derivation of the first equality makes use of Equation (3.9) and Equation (3.6), which implies $\beta_2 + R = \lambda e^{-\lambda a} e^{Ra}$ for $c = 1$ and Equation (3.9). The second equality is derived from Equation (3.9).

1. zero-delayed ruin probability

$$\mathbb{P}[W > u] = \frac{\beta_2 - \eta}{\beta_2} e^{-\eta u} = \psi(u), \quad (3.15)$$

2. stationary ruin probability

$$\mathbb{P}[V > u] = \frac{\lambda}{\beta_2 e^{\lambda a}} e^{-\eta u} = \psi^{(0)}(u). \quad (3.16)$$

3.3 Relations between M/\overleftarrow{M}_a and $M/G/1$

The M/\overleftarrow{M}_a risk model can also be related to the $M/G/1$ queueing model by another duality D_2 . The main result of this section is the density of the stationary waiting time of the corresponding $M/G/1$ queue.

Duality D_2

Duality D_2 maps the inter-arrival times of claims in the M/\overleftarrow{M}_a risk model T_1, T_2, \dots to the service times of an initially empty $\overleftarrow{M}_a/M/1$ queue, whereas the claim sizes X_1, X_2, \dots are mapped to the inter-arrival times of customers.

Just like the case of D_1 , this specific $\overleftarrow{M}_a/M/1$ queueing model where no claims are accepted before a , is equivalent to a $M/G/1$ queueing model, where the service times have the same distribution as T_c of Equation (3.10).

Stationary waiting time of $M/G/1$

By some results from queueing theory, an expression for the Laplace-Stieltjes transform of the stationary waiting time in a $M/G/1$ queue can be obtained. $\widetilde{W}(s)$ is expressed in terms of the Laplace-Stieltjes transform of the service time \widetilde{T}_c as follows (see [8] p. 66):

$$\widetilde{W}(s) = \frac{(1 - \rho)s}{\beta_2 \widetilde{T}_c(s) + s - \beta_2}, \quad (3.17)$$

where $\rho = \beta_2 \lambda_c = \beta_2 \frac{e^{\lambda a}}{\lambda}$. In particular, $\rho < 1$ by the stability condition.

Density of the stationary waiting time

To derive the density of the stationary waiting time, Equation (3.17) is rewritten to a form from which we can identify the density. Let $p(x)$ for $x \geq a$ be the density function of the random variable X_p having the corresponding Laplace-Stieltjes transform: $\psi_p(s) = \frac{\lambda'}{\lambda' + s e^{(\lambda' + s)a}}$, where λ is a solution of $\lambda' = \lambda e^{a(\lambda' - \lambda - \beta_2)}$. Then Equation (3.17) can be rewritten to be:

$$\widetilde{W}(s) = (1 - \rho) \cdot 1 + \rho(1 - \rho)e^{sa}\psi_p(s - \beta_2). \quad (3.18)$$

This expression tells us that with probability $(1 - \rho)$, W equals zero and with probability ρ , W has distribution corresponding to the Laplace-Stieltjes transform $(1 - \rho)e^{sa}\psi_p(s - \beta_2)$. By some linearity and shifting arguments of Laplace-Stieltjes transforms, the corresponding density for the latter transform is found to be $(1 - \rho)e^{\beta_2(x+a)}p(x + a)$, $x \geq 0$. To be certain that this function admits a density function, the properties of density functions are checked. Obviously, $(1 - \rho)e^{\beta_2(x+a)}p(x + a) \geq 0$ holds since $p(x)$ is a density. Furthermore, note that the Laplace-Stieltjes transform and the density of the random variable $X_p - a$ are $e^{as}\psi_p(s)$ and $p(x + a)$ for $x \geq 0$. The integral over the density $(1 - \rho)e^{\beta_2(x+a)}p(x + a)$ then becomes:

$$\begin{aligned} \int_0^\infty (1 - \rho)e^{\beta_2(x+a)}p(x + a)dx &= (1 - \rho)e^{a\beta_2} \int_0^\infty e^{\beta_2 x}p(x + a)dx \\ &= (1 - \rho)e^{a\beta_2}e^{-a\beta_2}\psi_p(-\beta_2) \\ &= (1 - \rho)e^{a\beta_2}e^{-a\beta_2}(1 - \rho)^{-1} \\ &= 1. \end{aligned} \quad (3.19)$$

Here we make use of the fact that $\psi_p(-\beta_2) = (1 - \rho)^{-1}$.

3.4 $M/\overleftarrow{G}_M/1$ queueing model

The last part of [2] focusses on the $M/\overleftarrow{G}_M/1$ queueing model with a specific dependence structure between inter-arrival times and service times. Three mean performance measures are examined, namely the mean sojourn time, the mean number of customers and the mean waiting time. The main focus will be the analysis of Laplace-Stieltjes transform of the sojourn time, from which some properties of the other two performance measures can be derived.

Setting of $M/\overleftarrow{G}_M/1$

In the $M/\overleftarrow{G}_M/1$ queueing model, the customers arrive according to a Poisson process with parameter λ . The inter-arrival time between the $(k-1)$ -th and the k -th customer will be denoted by T_{k-1} . The service time of the k -th customer is X_k has the following distribution:

$$\begin{cases} \bar{F}_1(x), & \text{if } T_k < A_k, \\ \bar{F}_2(x), & \text{if } T_k \geq A_k, \end{cases} \quad (3.20)$$

where A_k are i.i.d exponentially distributed with parameter τ . The random variables corresponding to distribution F_1 and F_2 will be denoted by X_{F_1} and X_{F_2} respectively.

Stability condition

To prevent the system from exploding, the following stability condition $\mathbb{E}[T_k] > \mathbb{E}[X_k]$ is required to be satisfied. Evaluating the condition gives:

$$\frac{1}{\lambda} > \frac{\lambda}{\lambda + \tau} \mathbb{E}[X_{F_1}] + \frac{\tau}{\lambda + \tau} \mathbb{E}[X_{F_2}]. \quad (3.21)$$

Sojourn time

The sojourn time of the $(k+1)$ -th customer S_{k+1} depends on his inter-arrival time, his service time and the sojourn time of the previous customer. This recursive relation can be stated as follows:

$$S_{k+1} = (S_k - T_k)^+ + X_{k+1}. \quad (3.22)$$

It can be shown that S_k converges to a limit random variable S that satisfies the relation:

$$S \stackrel{d}{=} (S - T)^+ + X, \quad \text{as } k \rightarrow \infty. \quad (3.23)$$

Laplace-Stieltjes transform of S

It requires some work to derive the Laplace-Stieltjes transform of the sojourn time $\tilde{S}(s)$. First, Equation (3.23) is used to substitute S . The transform is then split into two terms, one for $S \leq T$ and one for $S > T$. In turn, each of them is split into two cases: $T \leq A$ and $T > A$. The computation makes extensive use of interchanging the order of integration. After elaborating each term the following expression is found:

$$\tilde{S}(s) = \mathbb{E}[e^{-sS}] = - \frac{C_1 s(\lambda + \tau - s)(\lambda + \tau)\widetilde{X}_{F_2}(s) - \lambda s(\lambda - s)(\widetilde{X}_{F_2}(s) - \widetilde{X}_{F_1}(s))C_2}{(\lambda + \tau)[(\lambda - \lambda\widetilde{X}_{F_2}(s) - s)(\lambda + \tau - s) + \lambda(\lambda - s)(\widetilde{X}_{F_2}(s) - \widetilde{X}_{F_1}(s))]}, \quad (3.24)$$

where $C_1 = \mathbb{E}[e^{-\lambda S}]$ and $C_2 = \mathbb{E}[e^{-(\lambda+\tau)S}]$. Note that C_1 and C_2 are still unknown.

Derivation of C_1

Two constants still remain to be found, C_1 and C_2 . The derivation of C_1 makes use of exploiting a zero of the numerator of $\tilde{S}(s)$, $s_0 = 0$. Now, divide both the numerator and the denominator of $\tilde{S}(s)$ by s and rewrite the equation to the following:

$$\tilde{S}(s) = -\frac{C_1(\lambda + \tau - s)(\lambda + \tau)\widetilde{X}_{F_2}(s) - \lambda(\lambda - s)(\widetilde{X}_{F_2}(s) - \widetilde{X}_{F_1}(s))C_2}{(\lambda + \tau)\left[\left(\lambda\frac{1-\widetilde{X}_{F_2}(s)}{s} - 1\right)(\lambda + \tau - s) + \lambda(\lambda - s)\left(\frac{1-\widetilde{X}_{F_1}(s)}{s} - \frac{1-\widetilde{X}_{F_2}(s)}{s}\right)\right]}. \quad (3.25)$$

A property of Laplace-Stieltjes transform is that $\lim_{s \rightarrow 0} \frac{1-\tilde{Y}(s)}{s} = \mathbb{E}[Y]$, for any random variable Y . Taking the limit of $\tilde{S}(s)$ to s_0 gives an equation for C_1 :

$$\lim_{s \rightarrow s_0} \tilde{S}(s) = -\frac{C_1(\lambda + \tau)^2}{(\lambda + \tau)\left[(\lambda\mathbb{E}[X_{F_2}] - 1)(\lambda + \tau) + \lambda^2(\mathbb{E}[X_{F_1}] - \mathbb{E}[X_{F_2}])\right]} = 1. \quad (3.26)$$

Equation (3.26) is rewritten so that $C_1 = 1 - \frac{\lambda\tau\mathbb{E}[X_{F_2}] + \lambda^2\mathbb{E}[X_{F_1}]}{\lambda + \tau}$.

Remark 3.7. C_1 is actually the stationary probability that the system is empty on arrival, because $C_1 = \mathbb{E}[e^{-\lambda S}] = \int_0^\infty e^{-\lambda x} f_S(x) dx = \int_0^\infty \mathbb{P}[T > x] f_S(x) dx = \mathbb{P}[T > S]$, where $f_S(x)$ is the density of the sojourn time.

Waiting time

As the sojourn time is the sum of the waiting time and the service time, it suggests that the waiting time W has the same distribution as $(S - T)^+$.

Laplace-Stieltjes transform of W

The Laplace-Stieltjes transform of W can be expressed in terms of C_1 and $\mathbb{E}[e^{-sS}]$ as follows:

$$\mathbb{E}[e^{-sW}] = \frac{1}{s - \lambda} \left(sC_1 - \lambda\mathbb{E}[e^{-sS}] \right). \quad (3.27)$$

The derivation of Equation (3.27) goes by conditioning on the events $T > S$ and $T < S$ and interchanging the order of integration. It also makes use of the fact that $\mathbb{E}[e^{-\lambda S}] = C_1$.

Number of customers

Let us now define the following random variables, which share the same distribution by the step argument and the PASTA property:

- N_d : the number of customers arrived during the sojourn time of a customer,
- N : the number of customer in the system in steady-state.

Generating function of N

Consider the generating function of N_d :

$$\mathbb{E}[z^{N_d}] = \int_0^\infty \sum_{n=0}^\infty z^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} dF_S(t) = \int_0^\infty e^{-(1-z)\lambda t} dF_S(t) = \mathbb{E}[e^{-(1-z)\lambda S}], \quad (3.28)$$

where $F_S(t)$ is the distribution function of S . Finally, the mean number of customers in the system is found to be $\mathbb{E}[N] = \mathbb{E}[N_d] = \frac{d(\mathbb{E}[z^{N_d}])}{dz} \Big|_{z=1} = \lambda\mathbb{E}[S]$.

Part II

Research part

Chapter 4

Correlation in threshold model

In Part I, an elaborate study is done on the single threshold risk model M/\overleftarrow{G}_a . The threshold structure has introduced correlation between inter-arrival times and claim sizes, which can either be positive or negative. In Section 2.1, we have seen that the upper bound for the ruin probability provided by Lundberg's Inequality is still valid for this model. In Section 4.1 of this chapter, it is investigated how correlation influences the strictness of the Lundberg upper bound by means of some numerical results.

In [9], another risk model with dependence structure is considered. It is found that in comparison with the independent classical setting, positive and negative correlation decreases and increases the ruin probability respectively. In Section 4.2, it is verified whether the same is true for our single threshold risk model.

Setting

First, we recapitulate the setting of the single threshold risk model. The claims arrive according to a Poisson process with rate λ and the claim sizes satisfy the following dependence structure:

$$\begin{cases} X_k \sim \exp\left(\frac{1}{\mu_1}\right) & \text{if } T_k < a, \\ X_k \sim \exp\left(\frac{1}{\mu_2}\right) & \text{if } T_k \geq a, \end{cases}$$

where $\mu_i = \frac{1}{\beta_i}$ for $i = 1, 2$. The net profit condition of this model is $c > \lambda((1 - e^{-\lambda a})\mu_1 + e^{-\lambda a}\mu_2)$. The set of all pairs of parameters (μ_1, μ_2) satisfying the net profit condition will be called the net profit region Ω .

Independence

Whenever $\mu = \mu_1 = \mu_2$, the single threshold risk model reduces to the classical model, in which inter-arrival times and claim sizes are independent. In this case, the ruin probability is just $\psi(u) = \frac{\lambda\mu}{c} e^{R_1 u}$, where $R_1 = \frac{\lambda}{c} - \frac{1}{\mu}$.

Positive and negative correlation

Inter-arrival times and claim sizes are positively and negatively correlated for $\mu_1 < \mu_2$ and $\mu_1 > \mu_2$ respectively. These two inequalities correspond to the regions above and beneath the line $\mu_1 = \mu_2$. See Figure 4.1 for how such a net profit region typically looks like.

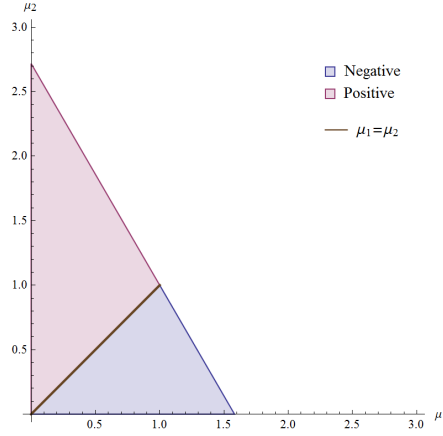


Figure 4.1: Net profit region Ω for $\lambda = c = a = 1$.

The exact ruin probability

When comparing different single threshold models, it is needed to calculate the ruin probability function given the parameters μ_1 and μ_2 . In [1], the following explicit expression for the ruin probability is derived:

$$\psi(u) = \bar{A}e^{R_1 u} + \bar{B}e^{R_2 u}. \quad (4.1)$$

In the above formula, R_1 and R_2 are the two negative zeros of the following characteristic equation:

$$l(x) = cx^2 + (c\beta_1 + c\beta_2 - \lambda)x + \beta_2(c\beta_1 - \lambda) - \lambda e^{-\lambda a}(\beta_1 - \beta_2)e^{xca} = 0, \quad (4.2)$$

whereas \bar{A} and \bar{B} are the solutions of the following two equations:

$$\begin{aligned} & \left[\lambda - cR_1 + \lambda e^{-\lambda a} e^{R_1 ca} \left(\frac{\beta_1(1 - e^{-(R_1 + \beta_1)ca})}{R_1 + \beta_1} - \frac{\beta_2(1 - e^{-(R_1 + \beta_2)ca})}{R_1 + \beta_2} \right) \right] \bar{A} \\ & + \left[\lambda - cR_2 + \lambda e^{-\lambda a} e^{R_2 ca} \left(\frac{\beta_1(1 - e^{-(R_2 + \beta_1)ca})}{R_2 + \beta_1} - \frac{\beta_2(1 - e^{-(R_2 + \beta_2)ca})}{R_2 + \beta_2} \right) \right] \bar{B} = \lambda + \lambda e^{-\lambda a} (e^{-\beta_2 ca} - e^{-\beta_1 ca}), \\ & \left[c - \lambda e^{-\lambda a} e^{R_1 ca} \left(\frac{1 - e^{-(R_1 + \beta_2)ca}}{R_1 + \beta_2} - \frac{1 - e^{-(R_1 + \beta_1)ca}}{R_1 + \beta_1} \right) \right] \bar{A} \\ & + \left[c - \lambda e^{-\lambda a} e^{R_2 ca} \left(\frac{1 - e^{-(R_2 + \beta_2)ca}}{R_2 + \beta_2} - \frac{1 - e^{-(R_2 + \beta_1)ca}}{R_2 + \beta_1} \right) \right] \bar{B} = \lambda \mu_1 + \lambda e^{-\lambda a} \left(\frac{e^{-\beta_2 ca}}{\beta_2} - \frac{e^{-\beta_1 ca}}{\beta_1} \right). \end{aligned}$$

4.1 Adjustment coefficient

One way to assess the effect of positive and negative correlation between inter-arrival times and claim sizes on the strictness of the Lundberg upper bound is, is to compare the exact ruin probability to the Lundberg upper bound. We do this by varying (μ_1, μ_2) as the adjustment coefficient R is fixed. The set of parameters with the same R as adjustment coefficient will be denoted by $\Gamma_R = \{(\mu_1, \mu_2) \in \Omega \mid \mathbb{E}[e^{R(X-cT)}] = 1\}$, where

$$\begin{aligned} \mathbb{E}[e^{R(X-cT)}] &= \int_0^a \mathbb{E}[e^{RX} \mid T < a] e^{-Rct} \lambda e^{-\lambda t} dt + \int_a^\infty \mathbb{E}[e^{RX} \mid T \geq a] e^{-Rct} \lambda e^{-\lambda t} dt \\ &= \frac{\lambda}{1 - \mu_1 R} \frac{1 - e^{-(Rc+\lambda)a}}{Rc + \lambda} + \frac{\lambda}{1 - \mu_2 R} \frac{e^{-(Rc+\lambda)a}}{Rc + \lambda}. \end{aligned} \quad (4.3)$$

In Figure 4.2, Γ_R is plotted for several values of R , while λ , c and a are fixed to be one.

When R is fixed, we would like to find the points along Γ_R . Rewriting Equation (4.2) yields an expression for μ_2 which corresponds with μ_1 :

$$\mu_2 = \frac{(c - \lambda\mu_1 + \lambda\mu_1 e^{-(Rc+\lambda)a})R - \mu_1 R^2}{R(cR - \lambda\mu_1 R - \mu_1 cR^2 + \lambda e^{-(Rc+\lambda)a})}. \quad (4.4)$$

Remark 4.1. As $R \rightarrow 0$, Γ_R tends to the boundary of the net profit region $c = \lambda((1 - e^{-\lambda a})\mu_1 + e^{-\lambda a}\mu_2)$. Furthermore, these Γ_R do not cross each other for different values of R .

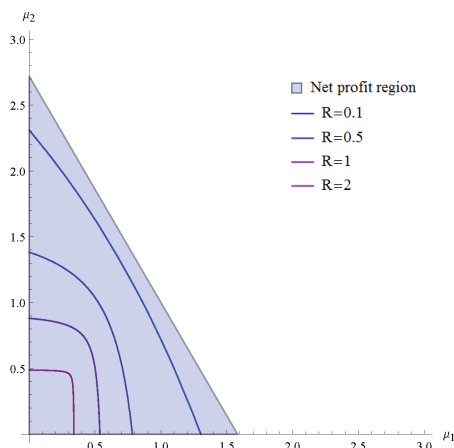


Figure 4.2: Γ_R for different values of the adjustment coefficient $R = 0.1, 0.5, 1, 2$ and $\lambda = c = a = 1$.

Numerical illustrations

In this section, the Lundberg upper bound is compared to real ruin probability derived from the Equation (4.1) for several pairs points along Γ_R . In this analysis the following parameters are fixed: $\lambda = c = a = 1$. For the convenience, R is also fixed to be one. Next, we choose five points with positive correlation and five points with negative correlation from Γ_1 , see Figure 4.3. For each (μ_1, μ_2) , the exact ruin probability function $\psi(u)$ is plotted in Figure 4.4, see also Table 4.1 for the constants of the expressions. These ruin probability functions are presented together with the Lundberg upper bound (dashed blue) and the ruin probability function corresponding with $\mu_1 = \mu_2$ (brown).

		μ_1	μ_2	R_1	R_2	A	B
Positive	1	0.4975	0.515404	-1	-1.94416	0.498165	0.00332362
	2	0.4700	0.632797	-1	-1.65356	0.448925	0.059081
	3	0.4000	0.757851	-1	-1.70487	0.264960	0.217302
	4	0.3000	0.823037	-1	-2.39369	0.154185	0.253131
	5	0.2000	0.852763	-1	-4.00901	0.124459	0.189861
Negative	1	0.5025	0.483415	-1	-2.06521	0.501323	-0.002997447
	2	0.5050	0.465506	-1	-2.14195	0.502126	-0.0056688
	3	0.5100	0.42503	-1	-2.34275	0.502153	-0.0100315
	4	0.5200	0.318606	-1	-3.12982	0.495692	-0.0144807
	5	0.5300	0.155763	-1	-6.42322	0.479276	-0.00972397

Table 4.1: The constants of the ruin probability function for several values of (μ_1, μ_2) for $\lambda = c = a = 1$.

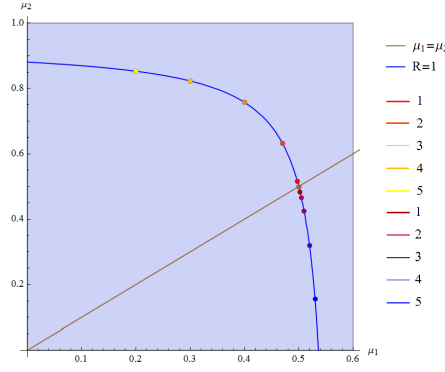


Figure 4.3: Points from Γ_1 for $\lambda = c = a = 1$.

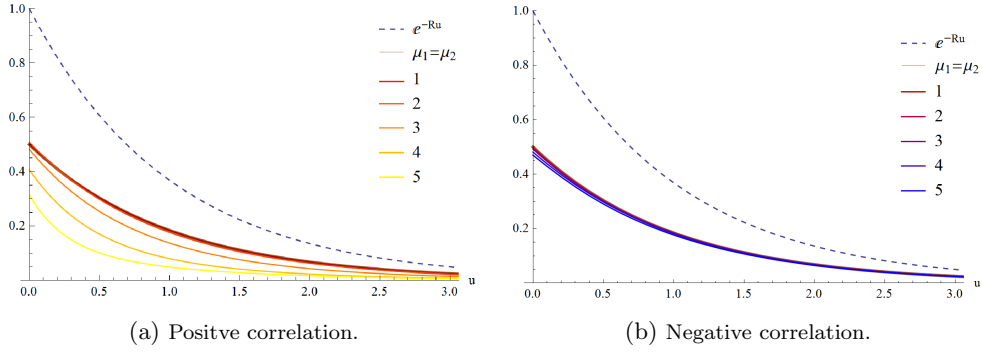


Figure 4.4: Ruin probability and the Lundberg upper bound for $R = 1$ and $\lambda = c = a = 1$.

Observations

1. When λ , c , a and R are chosen to be one, we see that -1 is always a zero of the characteristic equation (4.2). To see this, substitute $R = 1$ into Equation (4.4) to get

$$\mu_2 = \frac{1 - 2\mu_1 + \mu_1 e^{-2}}{1 - 2\mu_1 + e^{-2}}. \quad (4.5)$$

Substituting Equation (4.5) into $l(x)$ and putting x equal to -1 then yield:

$$\begin{aligned} l(-1) &= 2 - \frac{1}{\mu_1} - \frac{2}{\mu_2} + \frac{1}{\mu_1 \mu_2} - \frac{e^{-2}}{\mu_1} + \frac{e^{-2}}{\mu_2} \\ &= 2 - \frac{1}{\mu_1} - \frac{2(1 - 2\mu_1 + e^{-2})}{1 - 2\mu_1 + \mu_1 e^{-2}} + \frac{1}{\mu_1} \frac{1 - 2\mu_1 + e^{-2}}{1 - 2\mu_1 + \mu_1 e^{-2}} - \frac{e^{-2}}{\mu_1} + e^{-2} \frac{1 - 2\mu_1 + e^{-2}}{1 - 2\mu_1 + \mu_1 e^{-2}} \\ &= 0. \end{aligned} \quad (4.6)$$

2. The plots obtained from $R = 1$ are similar to the ones from other values of R . The ruin probability functions are greater and closer to the Lundberg upper bound at $u = 0$ for $R > 1$, while for $R < 1$ the opposite is true.
3. For each fixed R , the ruin probability function of $\mu_1 = \mu_2$ acts as an upper bound for the functions with correlation.
4. Again for fixed R , when considering the points along Γ_R , both for positive and negative correlations, the ruin probability functions decrease as (μ_1, μ_2) gets further away from $\mu_1 = \mu_2$.

4.2 Load

In stead of fixing the adjustment coefficient, it is also possible to set the load of the system to a constant value. The load is defined as the expected claim size divided by the expected inter-arrival time. The set of parameters with the same load ρ will be denoted by $\Upsilon_\rho = \{(\mu_1, \mu_2) \in \Omega \mid \frac{\mathbb{E}[X]}{\mathbb{E}[T]} = \rho\}$, where $\frac{\mathbb{E}[X]}{\mathbb{E}[T]} = \lambda((1 - e^{-\lambda a})\mu_1 + e^{-\lambda a}\mu_2)$. In Figure 4.5, Γ_R is plotted for several values of ρ , while λ , c and a are fixed to be one. Again, when ρ is fixed, the points along Υ_ρ can be computed by the following formula:

$$\mu_2 = \frac{e^{-\lambda a} - 1}{e^{-\lambda a}} \mu_1 + \frac{\rho}{\lambda e^{-\lambda a}}. \quad (4.7)$$

Remark 4.2. Equation (4.7) is a linear function in μ_1 , having the same slope as the one of the boundary of the net profit region.

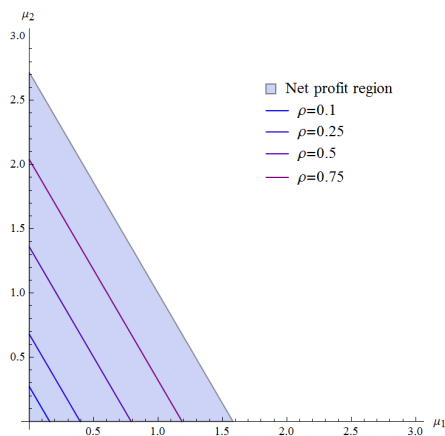


Figure 4.5: Υ_ρ for different values of the load $\rho = 0.10, 0.25, 0.50, 0.75$ and $\lambda = c = a = 1$.

Numerical illustrations

In this section, we proceed in the same fashion as in the previous section, but this time the set of parameters under consideration is Υ_ρ with $\rho = 0.5$. To simplify the computations, λ , c and a are chosen to be one. The ruin probability functions are computed by Equation (4.1) for several points on $\Upsilon_{0.5}$ and compared to the ruin probability function of $\mu_1 = \mu_2$ (brown). Now, the points are chosen to be equidistant, see Figure 4.6. The constants of the ruin probability functions can be found in Table 4.2.

		μ_1	μ_2	R_1	R_2	A	B
Positive	1	0.45	0.515404	-1.11699	-1.76632	0.432986	0.0466881
	2	0.40	0.632797	-1.15112	-1.74875	0.285277	0.173252
	3	0.35	0.757851	-1.07304	-1.97647	0.184409	0.252047
	4	0.30	0.823037	-0.965318	-2.3931	0.160024	0.25347
	5	0.25	0.852763	-0.863433	-3.02673	0.162464	0.227452
Negative	1	0.55	0.414086	-0.877558	-2.40242	0.53462	-0.0149088
	2	0.60	0.328172	-0.769933	-3.0369	0.560135	-0.0210644
	3	0.65	0.242258	-0.678573	-4.12353	0.579982	-0.0216293
	4	0.70	0.156344	-0.601405	-6.39566	0.594868	-0.0171502
	5	0.75	0.0704295	-0.536011	-14.19867	0.605871	-0.0088563

Table 4.2: The constants of the ruin probability function for several values of (μ_1, μ_2) for $\lambda = c = a = 1$.

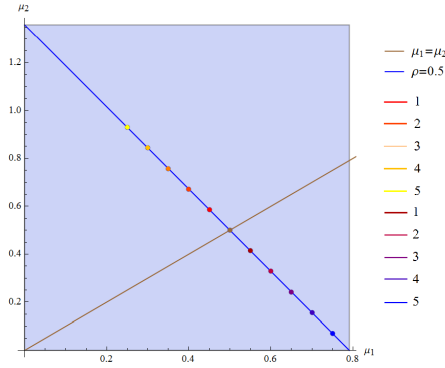


Figure 4.6: Points from $\Upsilon_{0.5}$ for $\lambda = c = a = 1$.

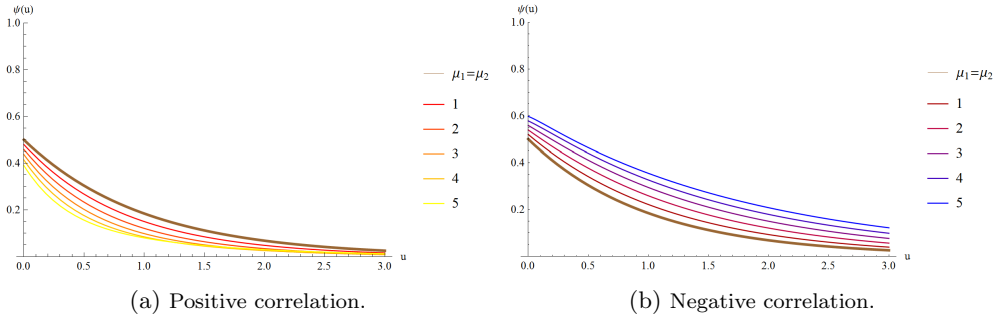


Figure 4.7: Ruin probability for $\rho = 0.5$ and $\lambda = c = a = 1$.

Observations

After experimenting with various values of ρ , it can be observed that every time positive and negative correlation decreases and increases the ruin probability function respectively, while the one of $\mu_1 = \mu_2$ stays in the middle. When considering points on Υ_ρ , as the parameters get further away from the independence case, the corresponding ruin probability get drifted further away from the independent one, see Figure 4.7.

A possible explanation for this behaviour has to do with the impact of correlation on the stochastic quantity $-(X_k - cT_k)$. The surplus process can be seen as a random walk starting from u with identical and independent increments $-(X_k - cT_k)$. Positive correlation between X_k and T_k tends to stabilize the increments, allowing the surplus to safely drift from zero. On the other hand, negative correlation causes more abrupt fallings which increase the probability of eventual ruin.

4.3 Conclusion

In the first section of this chapter, we have seen the effects of positive and negative correlation between inter-arrival times and claim sizes on the strictness of the Lundberg upper bound. When considering models with the same λ , c , a and R , correlation always decreases the ruin probability function, leading to a less strict Lundberg's Inequality.

The conjecture about the effects of correlation on the ruin probability motivated by [9] is confirmed. Positive correlation stabilizes fluctuations of the surplus, therefore the ruin probability decreases. On the contrary, negative correlation leads to the opposite effects and enlarges the ruin probability.

Chapter 5

Lundberg's Inequality in risk models with dependence structures

In Section 2.1, Lundberg's Inequality is derived for the single threshold risk model: $\psi(u) \leq e^{-Ru}$. It is interesting to investigate whether this upper bound is still applicable for more general models. Therefore, this chapter is devoted to the derivation of Lundberg's Inequality for some generalizations of the single threshold risk model. First, the model is specified. Subsequently, some sufficient conditions for the existence of the adjustment coefficient R is provided. Lundberg's Inequality is then proven by similar arguments as for the single threshold model.

In the classical model Lundberg's Inequality is rich in nice properties. Unfortunately, not all of them hold true in more general models. For instance, some fine martingales disappear when the independence assumption is dropped, which will be proven in the last part of this chapter. In section 5.3, the martingale property is proven to be true in the classical setting, while the contrary is proven for a risk model with dependence structure.

5.1 Multiple threshold risk model

A way to generalize the single threshold risk model [1] is by introducing more levels of threshold. As before, the claims arrive according to a Poisson process with rate λ to a system with initial surplus, $u_0 = u$. But now, if the previous inter-arrival time, T_k , falls into the interval $[a_i, a_{i+1})$, then the claim size X_k follows the distribution F_i for $i = 0, 1, \dots, m$. By convention, $a_{m+1} = \infty$. Following the same notations as before, this multiple thresholds risk model will be denoted by $M/\overleftarrow{G}_{\bar{a}}$, where $\bar{a} = (a_0, a_1, \dots, a_m)^T$.

Existence of the adjustment coefficient

Similar to Theorem 2.1, the existence of the moment generating function of F_i in a neighborhood around zero can ensure the existence of the adjustment coefficient R .

Theorem 5.1.

Assume that $M_i(r) = \int_0^\infty e^{rx} dF_i(x) < \infty$ for $r < r_\infty^{(i)}$ and that $\lim_{r \rightarrow r_\infty^{(i)}} M_i(r) = \infty$, where $r_\infty^{(i)} \in \mathbb{R} \cup \{\infty\}$ for $i = 0, 1, \dots, m$.

Then there exists a unique positive solution $R > 0$ to the equation:

$$M(r) = \mathbb{E}[e^{r(X_1 - cT_1)}] = 1, \quad (5.1)$$

where R is referred to as the adjustment coefficient.

Proof.

To see the finiteness of $M(r)$ in a neighborhood around zero, rewrite it in the following way, for $r < \min_i r_\infty^{(i)}$:

$$\begin{aligned} M(r) &= \mathbb{E}[e^{r(X_1 - cT_1)}] \\ &= \int_0^\infty \mathbb{E}[e^{r(X_1 - ct)} | T_1 = t] \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m \int_{a_i}^{a_{i+1}} \mathbb{E}[e^{r(X_1 - ct)} | T_1 = t] \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m \int_{a_i}^{a_{i+1}} \int_0^\infty e^{r(x-ct)} dF_i(x) \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m \int_{a_i}^{a_{i+1}} e^{-rct} \int_0^\infty e^{rx} dF_i(x) \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m \int_{a_i}^{a_{i+1}} e^{-rct} M_i(r) \lambda e^{-\lambda t} dt \\ &\leq \sum_{i=0}^m \int_{a_i}^{a_{i+1}} M_i(r) \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m M_i(r) \int_{a_i}^{a_{i+1}} \lambda e^{-\lambda t} dt \\ &\leq \sum_{i=0}^m M_i(r). \end{aligned} \quad (5.2)$$

This upper bound ensures the existence of $M(r)$ as long as all $M_i(r)$ are finite for $i = 0, 1, \dots, m$, which is the case if $r < \min_i r_\infty^{(i)}$. The uniqueness of the solution of the equation $M(r) = 1$ can be derived by combining the following facts:

- $M(0) = 1$,
- $M'(0) = \mathbb{E}[X_1 - cT_1] < 0$, by the net profit condition,
- $M''(r) = \mathbb{E}[(X_1 - cT_1)^2 e^{r(X_1 - cT_1)}] > 0$, i.e. $M(r)$ is convex,
- there exists some $r_\infty \in \mathbb{R} \cup \{\infty\}$ such that $\lim_{r \rightarrow r_\infty} M(r) = \infty$.

Starting from 1 the function $M(r)$ first drops, but eventually grows to infinity as r increases. Due to its convexity, $M(r)$ crosses the horizontal line of unit height just once, which generates a unique solution to Equation (5.1), the adjustment coefficient. ■

Lundberg's Inequality

Provided that the adjustment coefficient exists, Lundberg's Inequality can be proven to be valid for the $M/\overline{G_a}$ risk model. The proof of Lundberg's Inequality for the single threshold risk model in [1] p. 43 makes use of a change of measure, which is applicable here as well. Nevertheless, there are other ways to prove the same result. A more intuitive proof will be presented here, which follows the same scheme as a proof in [7] pp. 92-93.

Theorem 5.2.

Starting with initial surplus, $u_0 = u$, Lundberg's Inequality holds for the ruin probability $\psi(u)$ of the $M/\overline{G_a}$ risk model:

$$\psi(u) \leq e^{-Ru}, \quad (5.3)$$

where R is the adjustment coefficient defined in Theorem 5.1.

Proof.

Let $\psi_k(u)$, $k = 0, 1, 2, \dots$, be the probability that ruin occurs at or before the k -th claim. Note that the sequence $(\psi_k(u))_{k \geq 0}$ is increasing and converges to the ruin probability $\psi(u)$ as $k \rightarrow \infty$. Therefore, it suffices to prove that $\psi_k(u) \leq e^{-Ru}$ for each k . The proof will be done by induction.

For $k = 0$, distinguish two cases: $u < 0$ and $u \geq 0$. If $u < 0$, $e^{-Ru} > 1$ and also ruin is already a fact, i.e. $\psi_0(u) = 1$. In case $u \geq 0$, we have $\psi_0(u) = 0$. Thus, $\psi_0(u) \leq e^{-Ru}$ holds for both cases.

Suppose $\psi_{k-1}(u) \leq e^{-Ru}$ is true, evaluate $\psi_k(u)$ by conditioning on the first inter-arrival time and first claim size:

$$\begin{aligned} \psi_k(u) &= \int_0^\infty \mathbb{P}[\text{ruin at or before the } k\text{-th claim} | T_1 = t] \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m \int_{a_i}^{a_{i+1}} \mathbb{P}[\text{ruin at or before the } k\text{-th claim} | T_1 = t] \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m \int_{a_i}^{a_{i+1}} \int_0^\infty \mathbb{P}[\text{ruin at or before the } k\text{-th claim} | T_1 = t, X_1 = x] dF_i(x) \lambda e^{-\lambda t} dt \\ &= \sum_{i=0}^m \int_{a_i}^{a_{i+1}} \int_0^\infty \psi_{k-1}(u - x + ct) dF_i(x) \lambda e^{-\lambda t} dt \\ &\leq \sum_{i=0}^m \int_{a_i}^{a_{i+1}} \int_0^\infty e^{-R(u-x+ct)} dF_i(x) \lambda e^{-\lambda t} dt \\ &= e^{-Ru} \sum_{i=0}^m \int_{a_i}^{a_{i+1}} \int_0^\infty e^{R(x-ct)} dF_i(x) \lambda e^{-\lambda t} dt \\ &= e^{-Ru} M(R) \\ &= e^{-Ru}. \end{aligned} \quad (5.4)$$

The inequality between the fourth and the fifth line of Equation (5.4) is based on the induction hypothesis. Subsequently, the whole sum on the sixth line is found to be the same as $M(R)$, see the fourth line of Equation (5.2). Finally, $M(R) = 1$ by the definition of adjustment coefficient. Consequently, Lundberg's Inequality is proven for the multiple thresholds risk model $M/\overline{G_a}$. ■

5.2 General dependence structure

We have seen that Lundberg's Inequality is true for the multiple thresholds risk model. This fact left us wondering if the inequality would be more general. Instead of letting the claim sizes depend on the inter-arrival times discreetly, we can also consider a continuous dependence structure. The distribution of the claim size depends directly on the previous inter-arrival time, that is if $T_k = t$, then X_k has distribution F_t . Again, we will prove Lundberg's Inequality for this risk model.

This generalization may seem redundant from the modelling point of view for it is often sufficient to describe the behaviour of the claim sizes with respect to some range of inter-arrival time. However, this extension might help us understand dependence structures more fully. The multiple thresholds risk model is actually a special case of this generalization, namely if we choose $F_t(x) = F_i(x)$ for $t \in [a_i, a_{i+1})$.

Existence of the adjustment coefficient

As before, some conditions are stated to ensure the existence of the adjustment coefficient.

Theorem 5.3.

Assume that $M_t(r) = \int_0^\infty e^{rx} dF_t(x) < \infty$ for $r < r_\infty^{(t)}$ and that $\lim_{r \rightarrow r_\infty^{(t)}} M_t(r) = \infty$, where $r_\infty^{(t)} \in \mathbb{R} \cup \{\infty\}$, for $t \in \mathbb{R}$. Moreover, assume there exists a constant $\inf_{t \in \mathbb{R}} r_\infty^{(t)} > 0$ such that $\int_0^\infty M_t(r) \lambda e^{-\lambda t} dt < \infty$ for $r < \inf_{t \in \mathbb{R}} r_\infty^{(t)}$.

Then there exists a unique positive solution $R > 0$ to the equation:

$$M(r) = \mathbb{E}[e^{r(X_1 - cT_1)}] = 1, \quad (5.5)$$

where R is referred to as the adjustment coefficient.

Proof.

When examining the proof for the multiple thresholds risk model more closely, one can see that the distinction of intervals is not necessary. For $r < \inf_{t \in \mathbb{R}} r_\infty^{(t)}$:

$$\begin{aligned} M(r) &= \mathbb{E}[e^{r(X_1 - cT_1)}] \\ &= \int_0^\infty \int_0^\infty e^{r(x-ct)} dF_t(x) \lambda e^{-\lambda t} dt \\ &= \int_0^\infty e^{-rct} \int_0^\infty e^{rx} dF_t(x) \lambda e^{-\lambda t} dt \\ &= \int_0^\infty e^{-rct} M_t(r) \lambda e^{-\lambda t} dt \\ &\leq \int_0^\infty M_t(r) \lambda e^{-\lambda t} dt. \end{aligned} \quad (5.6)$$

By assumption, the last integral is finite. The same arguments about the first and the second derivative of $M(r)$ are still applicable, thus the proof is completed. ■

Lundberg's Inequality

Theorem 5.4.

Starting with initial surplus, $u_0 = u$, Lundberg's Inequality holds for the ruin probability $\psi(u)$ of the continuous dependence risk model:

$$\psi(u) \leq e^{-Ru}, \quad (5.7)$$

where R is the adjustment coefficient defined in Theorem 5.3.

Proof.

$$\begin{aligned} \psi_k(u) &= \int_0^\infty \mathbb{P}[\text{ruin at or before the } k\text{-th claim} | T_1 = t] \lambda e^{-\lambda t} dt \\ &= \int_0^\infty \int_0^\infty \mathbb{P}[\text{ruin at or before the } k\text{-th claim} | T_1 = t, X_1 = x] dF_t(x) \lambda e^{-\lambda t} dt \\ &= \int_0^\infty \int_0^\infty \psi_{k-1}(u - x + ct) dF_t(x) \lambda e^{-\lambda t} dt \\ &\leq \int_0^\infty \int_0^\infty e^{-R(u-x+ct)} dF_t(x) \lambda e^{-\lambda t} dt \quad (5.8) \\ &= e^{-Ru} \int_0^\infty \int_0^\infty e^{R(x-ct)} dF_t(x) \lambda e^{-\lambda t} dt \\ &= e^{-Ru} M(R) \\ &= e^{-Ru}. \end{aligned}$$

As before, the induction arguments of the proof of Theorem 5.2 still hold. Therefore, this generalization of Lundberg's Inequality is established. ■

5.3 Martingale property

The concept of martingale is well-studied in probability theory and has a lot of applications. In the classical Cramér-Lundberg model, where inter-arrival times and claim sizes are independent, $e^{-RU(t)}$ is found to be a continuous martingale. This section provides some findings about the martingale property if some dependence structures are taken into consideration.

Classical Cramér-Lundberg model

Let us first go back to the classical Cramér-Lundberg model and review the proof of the martingale property. The following proof comes from [7] p. 96.

Theorem 5.5.

In the classical Cramér-Lundberg model, starting with initial surplus $u_0 = u$, $e^{-RU(t)}$ is a martingale, where R is the adjustment coefficient, the solution of $M(r) = \mathbb{E}[e^{r(X_1 - cT_1)}] = 1$.

Proof.

Thanks to the independency between X_1 and T_1 , $M(R)$ can be factorized into two part: $M(R) = M_X(R)\mathbb{E}[e^{-RcT_1}] = M_X(R)\frac{\lambda}{Rc+\lambda}$, where $M_X(r) = \mathbb{E}[e^{rX_1}]$. By definition, $M(R)$ is equal to one, which yields:

$$M_X(R) - 1 = \frac{Rc}{\lambda}. \quad (5.9)$$

Furthermore, note that the aggregate claim at time t is a compound Poisson process with parameter λt due to the independence structure. Hence, $\mathbb{E}[e^{RS(t)}] = e^{\lambda t(M_X(R)-1)}$. This fact can be used to simplify $e^{-RU(t)}$.

To prove that $e^{-RU(t)}$ is a martingale, it has to be shown that it is constant in t :

$$\begin{aligned} \mathbb{E}[e^{-RU(t)}] &= \mathbb{E}[e^{-R(u+ct-S(t))}] \\ &= e^{-Ru} \left(e^{-Rc} e^{\lambda(M_X(R)-1)} \right)^t \\ &= e^{-Ru} \left(e^{-Rc} e^{\lambda \frac{Rc}{\lambda}} \right)^t \\ &= e^{-Ru} \\ &= e^{-RU(0)}. \end{aligned} \quad (5.10)$$

In addition, $U(t) < \infty$ for any finite t . Therefore, $\mathbb{E}[|e^{-RU(t)}|] < \infty$ for all t . The proof is now completed. ■

Single threshold model with constant claim sizes

In general, $e^{-RU(t)}$ is not always a martingale. We will prove that this is the case in the special threshold model. The idea is to choose the parameters in a way such that the surplus at a predetermined time is always different from the initial surplus.

Consider the following model. Let ε be a number less than a and strictly greater than zero. As before, claims arrive according to a Poisson process with rate λ . If the previous inter-arrival time is less than the threshold a , then the claim size is equal to μ_1 . Otherwise, the claim size is equal to μ_2 . It can be shown that $e^{-RU(t)}$ is not a martingale anymore, which is formulated in the following theorem.

Theorem 5.6.

In the threshold model described above, starting with initial surplus $u_0 = u$, $e^{-RU(t)}$ is a martingale if and only if $\mu_1 = \mu_2$, where R is the adjustment coefficient.

Proof.

Case $\mu_1 = \mu_2$

If $\mu_1 = \mu_2$, we just have the classical model as if no dependence structure is present. Therefore, $e^{-RU(t)}$ is a martingale by Theorem 5.5.

Case $\mu_1 \neq \mu_2$

Before getting started with the main proof of the case $\mu_1 \neq \mu_2$, some properties of the adjustment coefficient are established:

$$\begin{aligned}
& \mathbb{E}[e^{R(X_1 - cT_1)}] = 1 \\
\iff & \int_0^a e^{R(\mu_1 - ct)} \lambda e^{-\lambda t} dt + \int_a^\infty e^{R(\mu_2 - ct)} \lambda e^{-\lambda t} dt = 1 \\
\iff & \lambda e^{R\mu_1} \int_0^a e^{-(Rc + \lambda)t} dt + \lambda e^{R\mu_2} \int_a^\infty e^{-(Rc + \lambda)t} dt = 1 \\
\iff & \lambda e^{R\mu_1} \frac{1 - e^{-(Rc + \lambda)a}}{Rc + \lambda} + \lambda e^{R\mu_2} \frac{e^{-(Rc + \lambda)a}}{Rc + \lambda} = 1 \\
\iff & \lambda e^{R\mu_1} = \left(1 - \lambda e^{R\mu_2} \frac{e^{-(Rc + \lambda)a}}{Rc + \lambda}\right) \frac{Rc + \lambda}{1 - e^{-(Rc + \lambda)a}}.
\end{aligned} \tag{5.11}$$

Now, it has to be shown that $\mathbb{E}[e^{-RU(\varepsilon)}] \neq e^{-Ru}$. Let us first rewrite $\mathbb{E}[e^{-RU(\varepsilon)}]$ by conditioning on the number of arrivals before ε :

$$\begin{aligned}
\mathbb{E}[e^{-RU(\varepsilon)}] &= \sum_{k=0}^{\infty} e^{-R(u + c\varepsilon - k\mu_1)} \frac{(\lambda\varepsilon)^k e^{-\lambda\varepsilon}}{k!} \\
&= e^{-R(u + c\varepsilon) - \lambda\varepsilon} \sum_{k=0}^{\infty} \frac{(\lambda\varepsilon e^{R\mu_1})^k}{k!} \\
&= e^{-R(u + c\varepsilon) - \lambda\varepsilon} e^{\lambda\varepsilon e^{R\mu_1}} \\
&= e^{-Ru} e^{(\lambda e^{R\mu_1} - Rc - \lambda)\varepsilon}.
\end{aligned} \tag{5.12}$$

The proof goes by contradiction. Suppose that $e^{-RU(t)}$ is a martingale, then $\lambda e^{R\mu_1} = Rc + \lambda$ must hold, otherwise $e^{(\lambda e^{R\mu_1} - Rc - \lambda)\varepsilon} \neq 1$. By the last line of Equation (5.11), if $\lambda e^{R\mu_1} = Rc + \lambda$, then:

$$1 - \lambda e^{R\mu_2} \frac{e^{-(Rc + \lambda)a}}{Rc + \lambda} = 1 - e^{-(Rc + \lambda)a}. \tag{5.13}$$

Subsequently, from Equation (5.13) the following equality can be obtained:

$$\begin{aligned}
\implies & \lambda e^{R\mu_2} \frac{e^{-(Rc + \lambda)a}}{Rc + \lambda} = e^{-(Rc + \lambda)a} \\
\implies & \lambda e^{R\mu_2} = Rc + \lambda.
\end{aligned} \tag{5.14}$$

The last line of Equation (5.14) is in contradiction with the fact that $\lambda e^{R\mu_1} = Rc + \lambda$ and $\mu_1 \neq \mu_2$. Hence, $e^{-RU(t)}$ is not a martingale. ■

5.4 Conclusion

Two generalizations of the single threshold models are presented in chapter for which we successfully proved Lundberg's Inequality. In the first generalized risk model, the claim size distribution is equal to one of a finite number of different distributions, depending on the interval in which the previous inter-arrival falls in. The second risk model extends the latter one continuously.

Although Lundberg's Inequality may be valid for a certain model with dependency, not all applications of it may still hold. For instance, in the threshold model with deterministic claim sizes, $e^{-RU(t)}$ is a martingale if and only if $\mu_1 = \mu_2$.

Chapter 6

A Rouché problem in the $M/\overleftarrow{G}_M/1$ queueing model

In the last section of the summary of [2], the $M/\overleftarrow{G}_M/1$ queueing model was considered. An expression for the Laplace-Stieltjes transform of the sojourn time S was found with 2 constants, of which only C_1 was determined:

$$\tilde{S}(s) = \mathbb{E}[e^{-sS}] = -\frac{C_1 s(\lambda + \tau - s)(\lambda + \tau)\widetilde{X}_{F_2}(s) - \lambda s(\lambda - s)(\widetilde{X}_{F_2}(s) - \widetilde{X}_{F_1}(s))C_2}{(\lambda + \tau)[(\lambda - \lambda\widetilde{X}_{F_2}(s) - s)(\lambda + \tau - s) + \lambda(\lambda - s)(\widetilde{X}_{F_2}(s) - \widetilde{X}_{F_1}(s))]}, \quad (6.1)$$

where $\widetilde{X}_{F_i}(s)$ is the Laplace-Stieltjes transform F_i for $i = 1, 2$.

It is satisfying and challenging to determine the second constant as well, because $\tilde{S}(s)$ will then be fully specified. One approach to obtain C_2 makes use of a positive zero of the denominator of $\tilde{S}(s)$, provided it actually exists. The numerator of $\tilde{S}(s)$ must vanish at this point, otherwise $\tilde{S}(s)$ would explode, contradicting its finiteness. Subsequently, filling in this zero in the numerator would yield an extra equation, leading us to C_2 .

Before actually solving for a positive zero of the denominator of $\tilde{S}(s)$, one can try to establish its existence using Rouché's Theorem, which is the focus of this chapter. In Section 5.1, the Rouché problem is specified. Section 5.2 then treats a simple case in which we try to solve Rouché problem.

6.1 Problem description

For the notational convenience, define $N(s)$ and $D(s)$ as follows:

$$N(s) = C_1 s(\lambda + \tau - s)(\lambda + \tau) \widetilde{X}_{F_2}(s) - \lambda s(\lambda - s)(\widetilde{X}_{F_2}(s) - \widetilde{X}_{F_1}(s)) C_2,$$

$$D(s) = (\lambda - \lambda \widetilde{X}_{F_2}(s) - s)(\lambda + \tau - s) + \lambda(\lambda - s)(\widetilde{X}_{F_2}(s) - \widetilde{X}_{F_1}(s)),$$

then $\widetilde{S}(s) = -\frac{N(s)}{(\lambda + \tau)D(s)}$.

Now, rewrite $D(s)$ as follows:

$$\begin{aligned} D(s) &= (\lambda - s)(\lambda + \tau - s) - \lambda \tau \widetilde{X}_{F_2}(s) - \lambda(\lambda - s) \widetilde{X}_{F_1}(s) \\ &= -s(\lambda + \tau - s) + \lambda \tau (1 - \widetilde{X}_{F_2}(s)) + \lambda(\lambda - s)(1 - \widetilde{X}_{F_1}(s)). \end{aligned} \quad (6.2)$$

$s_0 = 0$ is already found to be a zero of $D(s)$. It suggests to divide $D(s)$ by s to get:

$$\frac{D(s)}{s} = -(\lambda + \tau - s) + \lambda \tau \frac{(1 - \widetilde{X}_{F_2}(s))}{s} + \lambda(\lambda - s) \frac{(1 - \widetilde{X}_{F_1}(s))}{s}. \quad (6.3)$$

By construction, all zeros of the denominator of $\widetilde{S}(s)$ excluding s_0 are just the zeros of $D(s)/s$. The main goal of this chapter is to use Rouché's Theorem to prove that $D(s)/s$ has at least one positive zero. First thing to note is that $D(s)/s$ is analytic in the right-half plane. In order to apply Rouché's Theorem, one still has to decompose $D(s)/s$ into two parts: $f(s)$ and $g(s)$ and find a closed contour K on the right-half plane, such that:

- $D(s)/s = f(s) + g(s)$,
- $|g(s)| > |f(s)|$ for all s on K ,
- $g(s)$ has at least one zero inside K .

If the above conditions are satisfied, the existence of a positive zero for $D(s)/s$ is guaranteed by Rouché's Theorem. Thus the challenge of this Rouché problem lies in the search for the functions $f(s)$, $g(s)$ and a contour K such that the above conditions hold.

6.2 Exponentially distributed service times

In general, it is not straightforward to find a decomposition of $D(s)/s$ with the desired conditions. In this section, a simple case is treated in which we are able to derive the desired result just partially. The service times of the model under consideration are exponentially distributed with parameter β_1 or β_2 , depending on whether the previous inter-arrival time is greater or less than an exponentially distributed threshold. The Laplace-Stieltjes transforms of the service time then become:

$$\widetilde{X}_{F_1}(s) = \frac{\beta_1}{\beta_1 + s}, \quad \widetilde{X}_{F_2}(s) = \frac{\beta_2}{\beta_2 + s}. \quad (6.4)$$

A choice of decomposition would be:

$$f(s) = \lambda\tau \frac{(1 - \widetilde{X}_{F_2}(s))}{s} + \lambda(\lambda - s) \frac{(1 - \widetilde{X}_{F_1}(s))}{s}, \quad (6.5)$$

$$g(s) = -(\lambda + \tau - s). \quad (6.6)$$

Substituting Equation (6.4) in Equation (6.5) gives:

$$f(s) = \frac{\lambda\tau}{\beta_2 + s} + \frac{\lambda(\lambda - s)}{\beta_1 + s}. \quad (6.7)$$

It is obvious that $\lambda + \tau$ is the unique positive zero of $g(s)$. Let K be the half circle on the right-half plane, of which the straight side lies on the imaginary axis. The radius is chosen large enough such that $\lambda + \tau$ is included in the contour K .

It remains to show that $|g(s)| > |f(s)|$ holds both on the round side K_1 and the straight side K_2 of the contour K . The round side will be treated first, having results also valid for general service times distributions. On K_2 , we are only able to prove $|g(s)| > |f(s)|$ for the case $\lambda \geq \beta_1$, but not for $\lambda < \beta_1$.

The round side K_1

On the round side K_1 , the variable s can be written in the trigonometric form: $s = r(\cos \varphi + i \sin \varphi)$ for $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. Thus, the absolute value of s is simply r on the round side of an half circle. It will be proven that $|f(s)|$ is bounded and $|g(s)|$ increases to infinity as $r \rightarrow \infty$. Therefore, the desired inequality is obtained if the radius r is chosen large enough.

Let us first try to find an upper bound for $|f(s)|$:

$$\begin{aligned} |f(s)| &= \left| \lambda\tau \frac{(1 - \widetilde{X}_{F_2}(s))}{s} + \lambda(\lambda - s) \frac{(1 - \widetilde{X}_{F_1}(s))}{s} \right| \\ &\leq \left| \lambda\tau \frac{(1 - \widetilde{X}_{F_2}(s))}{s} \right| + \left| \lambda(\lambda - s) \frac{(1 - \widetilde{X}_{F_1}(s))}{s} \right| \\ &= \lambda\tau \frac{|1 - \widetilde{X}_{F_2}(s)|}{|s|} + \lambda \left| \lambda \frac{(1 - \widetilde{X}_{F_1}(s))}{s} - s \frac{(1 - \widetilde{X}_{F_1}(s))}{s} \right| \\ &\leq \frac{2\lambda\tau}{|s|} + \lambda^2 \frac{|1 - \widetilde{X}_{F_2}(s)|}{|s|} + \lambda |1 - \widetilde{X}_{F_1}(s)| \\ &\leq \frac{2\lambda\tau}{|s|} + \frac{2\lambda^2}{|s|} + 2\lambda \\ &= \frac{2\lambda\tau}{r} + \frac{2\lambda^2}{r} + 2\lambda. \end{aligned} \quad (6.8)$$

In this computation, we make extensive use of $|\widetilde{X_{F_i}}(s)| \leq 1$ for $i = 1, 2$. The proof of this is easy:

$$|\widetilde{X_{F_i}}(s)| = \left| \int e^{-sx} dF_i(x) \right| \leq \int |e^{-sx}| dF_i(x) \leq \int 1 dF_i(x) = 1. \quad (6.9)$$

To be able to conclude that $|g(s)| > |f(s)|$, note that:

$$|g(r(\cos \varphi + i \sin \varphi))| = |\tau + \lambda - r(\cos \varphi + i \sin \varphi)| \xrightarrow{r \rightarrow \infty} \infty, \quad (6.10)$$

for all $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, such that $|g(s)| > |f(s)|$ holds if the radius of K_1 is big enough.

The straight side K_2

The points on K_2 can be written as $s = ix$ for $|x| \leq r$ and r fixed. We distinguish two cases: $\lambda \geq \beta_1$ and $\lambda < \beta_1$.

Case $\lambda \geq \beta_1$

Due to the stability condition, $|g(0)| > |f(0)|$ holds. To see this, rewrite the stability condition as follows:

$$\frac{1}{\lambda} > \frac{\tau}{\lambda + \tau} \frac{1}{\beta_2} + \frac{\lambda}{\lambda + \tau} \frac{1}{\beta_1} \quad (6.11)$$

$$\iff \lambda + \tau > \frac{\lambda\tau}{\beta_2} + \frac{\lambda^2}{\beta_1} \quad (6.12)$$

$$\iff |g(0)| > |f(0)|. \quad (6.13)$$

Moreover, the following two inequalities can be proven to be true for all x :

$$|g(ix)| \geq |g(0)|, \quad (6.14)$$

$$|f(0)| \geq |f(ix)|. \quad (6.15)$$

As a result, $|g(ix)| > |f(ix)|$ is true for all $|x| \leq r$. Inequality (6.14) is plain to see:

$$|g(ix)|^2 = |-(\lambda + \tau - ix)|^2 = (\lambda + \tau)^2 + x^2 \geq (\lambda + \tau)^2 = |g(0)|^2. \quad (6.16)$$

Inequality (6.15) is slightly more complicated:

$$\begin{aligned} |f(ix)| &= \left| \frac{\lambda\tau}{\beta_2 + ix} + \frac{\lambda(\lambda - ix)}{\beta_1 + ix} \right| \\ &\leq \left| \frac{\lambda\tau}{\beta_2 + ix} \right| + \left| \frac{\lambda(\lambda - ix)}{\beta_1 + ix} \right| \\ &\leq \frac{\lambda\tau}{\beta_2} + \left| \frac{\lambda(\lambda - ix)}{\beta_1 + ix} \right| \\ &= \frac{\lambda\tau}{\beta_2} + \frac{\lambda\sqrt{\lambda^2 + x^2}}{\sqrt{\beta_1^2 + x^2}} \\ &= \frac{\lambda\tau}{\beta_2} + \frac{\lambda^2}{\beta_1} \sqrt{\frac{1 + x^2/\lambda^2}{1 + x^2/\beta_1^2}} \\ &\leq \frac{\lambda\tau}{\beta_2} + \frac{\lambda^2}{\beta_1} \\ &= |f(0)|. \end{aligned} \quad (6.17)$$

The inequality between the fifth and the sixth line holds whenever $\lambda \geq \beta_1$, because then $\sqrt{\frac{1+x^2/\lambda^2}{1+x^2/\beta_1^2}}$ is less than 1. To conclude, $|g(ix)| > |f(ix)|$ on K_2 for $\lambda \geq \beta_1$.

Case $\lambda < \beta_1$

$$\begin{aligned}
|f(ix)| &= \left| \frac{\lambda\tau}{\beta_2 + ix} + \frac{\lambda(\lambda - ix)}{\beta_1 + ix} \right| \\
&\leq \left| \frac{\lambda\tau}{\beta_2 + ix} \right| + \left| \frac{\lambda(\lambda - ix)}{\beta_1 + ix} \right| \\
&\leq \frac{\lambda\tau}{\beta_2} + \left| \frac{\lambda(\lambda - ix)}{\beta_1} \right|
\end{aligned} \tag{6.18}$$

Unfortunately, we have now reached an upper bound of $|f(ix)|$ which is not strict enough. For instance, if the parameters are chosen in such a way that this upper bound is greater than $|g(ix)|$. For example, take $\lambda = \tau = \beta_2 = x = 1$ and $\beta_1 = 0.1$, then:

$$|g(ix)| = \sqrt{(\lambda + \tau)^2 + x^2} \approx 2.23607, \quad \frac{\lambda\tau}{\beta_2} + \left| \frac{\lambda(\lambda - ix)}{\beta_1} \right| = 2.28565.$$

Either, the upper bounds in Equation (6.18) we chose are not strict enough or the decomposition is not right.

The roots of $D(s)/s$

In case it is possible to establish the existence of a positive root, we can obtain the zeros of $D(s)/s$ by using Mathematica. To this end, it is multiplied by $(\beta_1 + s)(\beta_2 + s)$, which results in the following cubic equation:

$$\frac{D(s)}{s}(\beta_1 + s)(\beta_2 + s) = -(\lambda + \tau - s)(\beta_1 + s)(\beta_2 + s) + \lambda\tau(\beta_1 + s) + \lambda(\lambda - s)(\beta_2 + s) = 0. \tag{6.19}$$

By construction, the zeros of $D(s)/s$ must also satisfy Equation (6.20). Either Equation (6.20) has three real zeros, or it has one real zero and two complex zeros. The built-in command *Solve* of Mathematica solves this cubic equation and returns the three closed-form solutions. Let s_1 be a positive zero. We now can compute the last missing constant C_1 as follows:

$$C_2 = \frac{C_1(\lambda + \tau)(\lambda + \tau - s_1)\beta_2(\beta_1 + s_1)}{\lambda(\lambda - s_1)[\beta_2(\beta_1 + s_1) - \beta_1(\beta_2 + s_1)]}. \tag{6.20}$$

6.3 Conclusion

In this chapter, we have consider a Rouché problem in a relatively easy setting. Still, it has not been successful to establish the existence of a positive zero for $D(s)/s$. It might need further investigation to establish the results.

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