Higher-order asymptotic homogenization of periodic linear elastic composite materials at low scale separation

HIGHER-ORDER ASYMPTOTIC HOMOGENIZATION OF PERIODIC MATERIALS AT LOW SCALE SEPARATION

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Summary In this work, we investigate the limits of classical homogenization theories pertaining to homogenization of periodic composite materials at low scale separations and demonstrate the effectiveness of higher-order periodic homogenization in alleviating this limitation. Classical homogenization techniques are very effective for materials with large scale separation between the scale of the heterogeneity and the macro-scale dimension, but inaccurate at low scale separations. Literature suggests that asymptotic homogenization is capable of pushing the limit to smaller scale separation by taking on board higher-order terms of the asymptotic expansion. We show that the classical homogenization deviates from the actual solution for scale ratios below 10. Beyond this limit, the higher-order asymptotic homogenization solution still gives a very good approximation which becomes better as more higher-order terms are included. This results in a size-dependent macroscopic model, which indeed allows one to push the limitations of homogenization in the direction of less scale separation.

INTRODUCTION

All matter is heterogeneous at some scale, but frequently it is convenient to treat it as homogeneous. Some of the well-known examples are metal alloys, concrete, porous structures and fibrous composites. The distinct features of their microstructures respond quite differently to mechanical loading and hence their deformation is heterogeneously distributed at the fine scale. It is the combination of the different microstructural features which governs the overall response of the material to the loading.

Homogenization is a mathematical technique for studying partial differential equations with rapidly oscillating coefficients, which are typical of the equations that govern the physics of heterogeneous materials. An important aspect in the analysis of multiphase materials is to deduce their effective behavior (e.g. mechanical stiffness, thermal expansion properties, etc.) from the corresponding single-phase behaviors and the geometrical arrangement of the phases. This concept of rendering "homogeneous" a heterogeneous material is what we call homogenization.

Conventional homogenization methods are based on *a separation of scales*, given by: $l << L$, where l is the size of the heterogeneity and L represents the macroscopic length scale. However, if the microstructural size is of the same order as the macroscopic length scale, then most of the classical homogenization schemes break down. Literature [1-4] suggests that the asymptotic homogenization method is capable of pushing the limit to smaller scale separation, by generating a hierarchy of problems which can be solved sequentially to generate a solution that asymptotically converges to the exact (homogenized) solution.

Problem Description

A qualitative and quantitative assessment of the scale separation limits of the classical and the higher order periodic homogenization methods is performed on two-dimensional elastic two-phase composites, consisting of stiff circular inclusions in a soft matrix material, subjected to anti-plane shear by means of a periodic body force. This anti-plane shear problem can be described by the following partial differential equation:

$$\frac{\partial}{\partial x_1}\Big(G\frac{\partial u_3}{\partial x_1}\Big) + \frac{\partial}{\partial x_2}\Big(G\frac{\partial u_3}{\partial x_2}\Big) + F = 0 \tag{1}$$

where $u_3 = u(x_1, x_2)$ is the resulting out-of-plane displacement, G is the Shear Moduli distribution function and F is a bisinusoidal bodyforce given by $F = F_0 \sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)$.

The period of the material's microstructure is l and that of the body force is L, and the ratio L/l hence characterises the scale separation. The homogenized properties are defined not for a specific microstructural configuration with respect to a period of the body force, but by taking an ensemble average for a family of all possible microstructures.

Methodology

On the one hand, reference solutions are created using direct numerical simulation of a family of microstructural configurations for a range of scale ratios. On the other hand, asymptotic homogenization is used to obtain homogenized properties for zeroth order and higher orders. Predictions made using these homogenized properties are then compared against the reference solutions. Fig.1 shows a schematic of the outline of the solution methodology.

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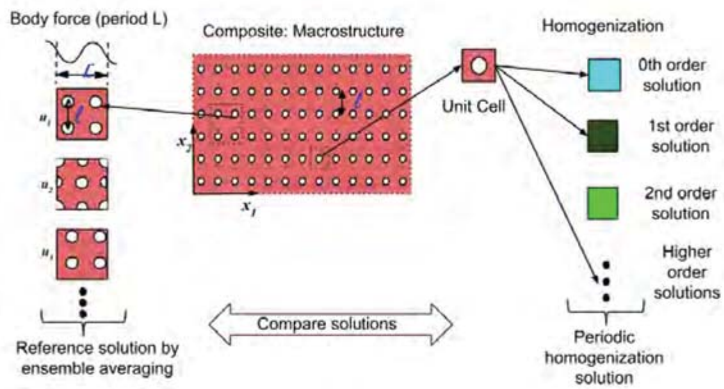

Figure 1: Schematic of the problem: Comparison of periodic homogenization solution with reference solution created by full-scale simulation for various scale ratios

RESULTS

Fig.2 shows the average peak displacement, normalized by that predicted by the classically homogenized solution, as a function of the scale ratio L/l. Shown are the reference solution, the classical homogenization solution and higher order solutions, for a phase contrast of 20. The zeroth order classical homogenization solution is independent of the scale ratio and hence is a straight line as shown in the plot. For scale ratios $L/l > 10$, the reference solution converges to this constant value, but at low scale separations it deviates from it significantly. The higher order periodic homogenization solutions closely match with the reference solution even for low scale ratios. The second order solution starts to deviate from the reference solution at $(L/l) = 4.5$, while the fourth order solution can still give a good approximation for even lower scale ratios.

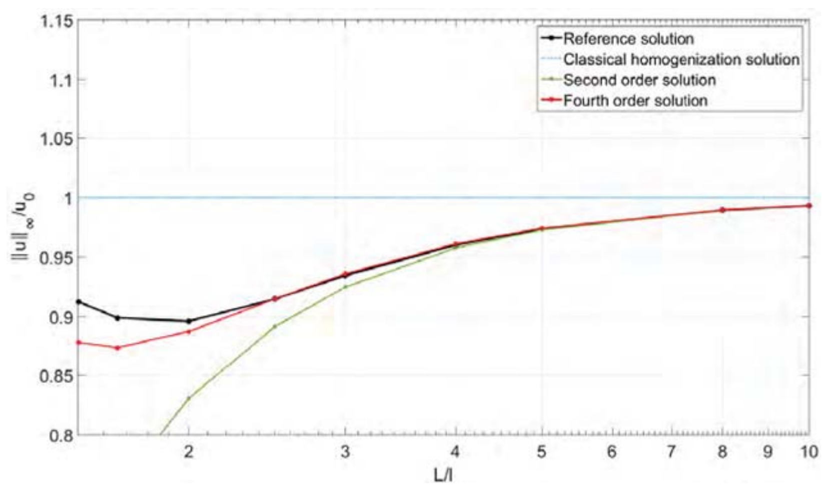

Figure 2: Scale separation plot: Norm of the displacement solution vs. scale separation, for various cases

CONCLUSIONS

We show that the zeroth order classical homogenization deviates from the actual solution for scale ratios *L/l* below 10. Beyond this limit, higher-order asymptotic homogenization solution gives a very good approximation. The approximation in the low scale separation regime becomes better as more higher-order terms are included. This shows that the higher-order theory indeed allows one to push the limitations of homogenization in the direction of less scale separation.

References

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