BACHELOR

Travelling-wave solutions for a linearized moving boundary problem

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Chapter 1

Abstract

This thesis discusses a 2-D moving boundary problem for the Laplacian in an exterior domain. These problems arise in the modelling of Hele-Shaw flow with either surface tension or kinetic undercooling as boundary condition. Here we study the Hele-Shaw problem with a combination of these two regularization effects. This thesis investigates the existence of travelling-wave solutions. After having identified the trivial solution on the unit circle, we derive the evolution equations for non-trivial domains that represent radial perturbations from the unit circle. By linearizing this operator problem and describing it in terms of Fourier series we find that solutions only exist when the surface tension coefficient is zero. For this case we give the form of the solution for a linearized problem explicitly in terms of Fourier coefficients.

Keywords: Hele-Shaw flow, moving boundary problem, kinetic undercooling, surface tension, Frechet derivative.

Contents

Chapter 2

Overview of content

In the first chapter we will derive and discuss a model for Hele-Shaw flow in an unbounded cell with an uniform velocity field at infinity. We will consider surface tension, kinetic undercooling as well as a combination of these two as dynamical boundary conditions. Furthermore, some background information for the applications and historical remarks of Hele-Shaw flow is given here. Chapter 3 and 4 describe the definitions and basic theorems of functional analysis preliminaries that will be used in the sequel of this thesis. These are mostly related to the mathematical concepts of Frechet differentiation and Sobolev spaces. Chapter 5 is about further discussion for each of the models, after having identified the "trivial" circle shape solutions we will look for non-trivial travelling-wave solutions. In Chapter 6 we rewrite our problem as an operator equation for an unknown function u which represents radial perturbations from the unit circle. We then calculate the linearization for these operator problems in Chapter 6 in terms of Frechet derivatives. Continuing this process in Chapter 8 we try to solve the linearized problem in terms of Fourier coefficients. In the final Chapter 9 we compare the results for the different boundary conditions.

2.1 Aim of the thesis

At the start of this project is was known that in the case of kinetic undercooling there exist solutions for the linearized Hele-Shaw problem [11]. These solutions were found in terms of Fourier coefficients. It was conjectured that this was not possible to do in the case of surface tension as regularization effect. The goal of the project was to check this fact and find out whether or not the problem with both kinetic undercooling and surface tension admits solutions in the linearized version.

Chapter 3

Introduction to problem

3.1 Introduction to Free boundary problems

A free boundary problem is a partial differential equation that needs to be solved for an unknown function ϕ defined on a domain that a priori is also unknown. In addition to the standard boundary conditions that are needed to solve PDEs, an additional kinetic condition must be set at the free boundary. Both the free boundary and the solution of the differential equations have to be determined [6]. Furthermore we call a free boundary problem in stationary when the domain Ω is moving over time, otherwise we call it in stationary.

3.2 Hele Shaw problem

This thesis is about several variations of an in stationary free boundary problems related to fluid dynamics, the so called Hele-Shaw flow. The Hele-Shaw model describes a fluid that is squeezed between two parallel surfaces, in such way that the situation can essentially be viewed as 2-dimensional. Here we take a fluid that is filling an unbounded domain exterior to an inviscid air bubble. The evolution of the boundary separating the outer fluid from the bubble is now of interest. Let $\Omega(t)$ be the evolving bubble domain and $\Gamma(t)$ it's boundary. The liquid is thus occupying the unbounded domain $\mathbb{R}^2 \setminus \Omega(t)$, see Figure 3.1 below.

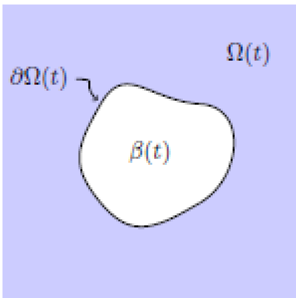


Figure 3.1: A picture of a bubble domain.

Further assumptions are that this liquid is incompressible and that the fluid velocity v is proportional to the gradient of the pressure p. This last assumption is known as Darcy's law, named after a 19th century hydraulic engineer [15]. In mathematical terms this can be written as

$$\begin{cases} v = -\nabla p \text{ in } \mathbb{R}^2 \setminus \Omega(t), \\ \nabla \cdot v = 0 \text{ in } \mathbb{R}^2 \setminus \Omega(t). \end{cases} \tag{3.1}$$

We assume that the motion is driven by a velocity field that is uniform like e_1 far away from the origin, this gives us our boundary condition at infinity.

$$v \approx e_1 \text{ for } |x| \to \infty. \tag{3.2}$$

On $\Gamma(t)$ we also place a kinematic condition on the boundary of the bubble. We define $V_n(t)$ to be the normal component of the velocity of the boundary. If we assume that the boundary only moves with the particles in the fluid, we then have the relation

$$V_n(t) = v(\cdot, t)|_{\Gamma(t)} \cdot n(t) \text{ on } \Gamma(t) \tag{3.3}$$

Taking this extra conditions into account and substituting $\phi = -p$ we get the following system of equations:

$$\begin{cases} \Delta\phi = 0 \text{ in } \mathbb{R}^2 \setminus \Omega(t), \\ V_n = \frac{\partial \phi}{\partial n} \text{ on } \Gamma(\text{t}), \\ \nabla\phi \approx e_1 \text{ for } |x| \to \infty. \end{cases} \tag{3.4}$$

We still need one boundary condition to complete our model, we consider both the homogeneous Robin condition and in homogeneous Dirichlet condition.

- **Kinetic undercooling**: At the moving boundary we demand

  $$\phi - l\partial_n \phi = 0 \text{ on } \Gamma(\text{t}). \tag{3.5}$$

  Here n is the normal in outward direction and $l \geq 0$ is the kinetic undercooling coefficient. The kinetic undercooling condition states that the potential at a point on the boundary is proportional to the normal velocity of on the boundary at that point. The name kinetic undercooling comes from the so called Stefan problem, which is a moving boundary problem related to the melting of ice. In this problem, certain thermodynamic effects on the interface between ice and water are modeled by equation (3.5). In our context Romero [13] proposed this term to relate to the interfacial curvature along the transverse direction to Hele-Shaw cell plate. [15].

- **Surface tension** At the moving boundary we assume

  $$\phi = -\gamma\kappa \text{ on } \Gamma(\text{t}). \tag{3.6}$$

  where κ stands for the curvature of the boundary, taken negative for convex domains. γ is a positive constant called the surface tension coefficient. This boundary condition is an approximation for the influences of surface tension forces on the free surface of the liquid hydraulic [15].

When we look at the combination of both, the boundary condition becomes:

$$\phi - l\partial_n \phi = -\gamma\kappa \text{ on } \Gamma(\text{t}). \tag{3.7}$$

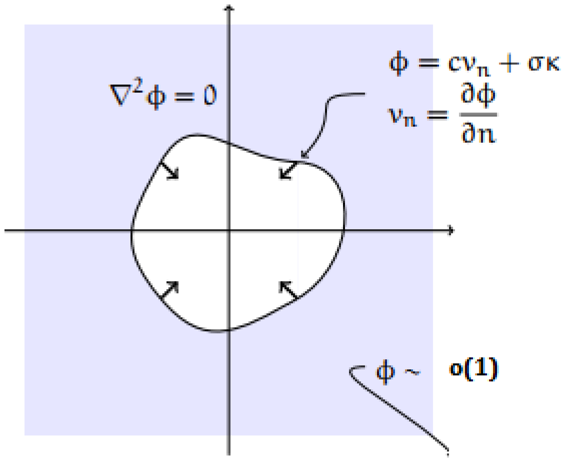


Figure 3.2: Bubble problem

3.3 Historical remarks and applications

The Hele-Shaw cell experiment was introduced in 1897 by a British engineer Henry Selby Hele-Shaw [14]. The experiments consisted of injecting one or more viscous fluids between two plates that are placed together. Although his initial purpose of the Hele-Shaw cell was to visualize streamlines of fluids around objects [9]. the Hele-Shaw cell provides a very useful tool to visualize moving boundary fluid problems as described above.

The very first analytical solutions were found almost 50 years later by Pelageya Yakovlevna Polubarinova-Kochina (1899-1999) and Lev Aleksandrovich Galin (1912-1981) in 1945 [8]. This was done for the case of a bounded domain with a source (or sink) and homogeneous Dirichlet conditions. They used Complex analysis and conformal mapping techniques to derive the "Polubarinova-Galin" for which many solutions have been found over the years.
The most famous example in free-boundary Hele-Shaw flow arises when the interaction of fluids with different viscosities is considered. If the fluid of lesser viscosity displaces the fluid with the greater viscosity the interface between them is unstable. This results in the developing of long-fingers of the fluid with lesser viscosity. The motion of travelling-waves were first considered by Taylor and Saffman [5].

Free-boundary Hele-Shaw flow remained up till now as a simple experimental and theoretical approximation of more complicated problems in physics and engineering that involve moving boundaries: for instance, groundwater flow, oil recovery, melting or freezing crystals (Stefan problems), streamers of charged particles. [14]

Chapter 4

Frechet derivative

4.1 Introduction

In this thesis we will make much use of Frechet derivatives of operators, this is a generalization of ordinary differentiation of real valued functions. In this section we give the elementary definitions and theorems that are related to this technique. The content of this section is mostly based on the book "Manifolds, Tensor Analysis, and Applications" by Ralph Abraham, J.E Marsden and Tudor Ratiu [1].
For a differentiable function $f : U \subseteq \mathbb{R} \to \mathbb{R}$, the interpretation of the derivative at a certain point $u_0 \in U$ is usually the slope of the tangent line to the graph of the function f at u_0. This slope $f'(u_0)$ can also be seen as a linear mapping, in the sense that $f(u) \approx f(u_0) + (u - u_0)f'(u_0)$. This can be generalized in the following way for functions in Banach spaces.

4.2 Definition of the Frechet derivative

Definition 1. *Let* $\mathbf{E}$ *and* $\mathbf{F}$ *be Banach spaces,* U *be an open set of* $\mathbf{E}$ *and let* $f : U \subseteq \mathbb{E} \to \mathbb{F}$ *be a function. We say that* f *is Frechet differentiable at the point* u_0 *if there exists a bounded linear mapping* $Df(u_0) : \mathbf{E} \to \mathbf{F}$ *such that for every* $\epsilon > 0$ *there exist an* $\delta > 0$ *such that whenever* $0 < ||u - u_0|| < \delta$ *we have*

$$\frac{||f(u) - f(u_0) - Df(u_0) \cdot (u - u_0)||}{||u - u_0||} < \epsilon. \tag{4.1}$$

4.3 Preliminaries

Here we collected some results that will be used later. We start with the proof of the uniqueness of the Frechet derivative.

Theorem 4.3.1 (Uniqueness)**.** *For a given function* $f : U \subseteq \mathbb{E} \to \mathbb{F}$ *there can be at most one such linear mapping at a given point* u_0*.*

Proof. Let L_1 and $L_2 \in L(\mathbf{E}, \mathbf{F})$ both satisfy the above definition and suppose that e is an unit vector in $\mathbf{E}$ so that $||e||= 1$. Let $u = u_0 + \lambda e$ then for $|\lambda| \neq 0$ small enough, we have

$$
\begin{aligned}
||L_1 e - L_2 e|| &= \frac{||L_1(u-u_0) - L_2(u-u_0)||}{||u-u_0||} \\
&\leq \frac{||f(u) - f(u_0) - L_1(u-u_0)||}{||u-u_0||} + \frac{||f(u) - f(u_0) - L_2(u-u_0)||}{||u-u_0||}.
\end{aligned}
\tag{4.2}
$$

Now letting λ approach zero we see that we have $||(L_1 - L_2)e||= 0$ for all unit vectors e. Therefore we must have $||L_1 - L_2||= 0$ and thus $L_1 = L_2$. □

This enables us to make the following definition.

Definition 2. *If f is differentiable at every $u \in \mathbf{E}$ then the map $Df : \mathbf{E} \to L(\mathbf{E}, \mathbf{F}); u \to Df(u)$ is called the derivative of f.*

Note that a function f is Frechet differentiable at u_0 if and only if there exists a linear map $Df(u_0)$ such that that the remainder term $r(h)$ defined by

$$
f(u_o + h) = f(u_0) + Df(u_0)h + r(h), \tag{4.3}
$$

has the property that $\frac{||r(h)||}{||h||} \to 0$ when $||h|| \to 0$.
We will use this notation in the following proofs.

Just as in the case of function in $\mathbb{R}^n$ we deduce that differentiability is stronger than continuity.

Theorem 4.3.2 (Differentiability implies continuity.)**.** *Suppose f is Frechet differentiable at $u \in \mathbb{E}$ then f must be continuous at u.*

Proof. Write

$$
||f(u+h) - f(u)||= ||Df(u)h + r(h)|| \leq ||Df(u)|| ||h|| + ||r(h)|| \to 0. \tag{4.4}
$$

Here we used the boundedness of the linear mapping $Df(u)$. □

All operators that are linear and bounded themselves must be Frechet differentiable.

Theorem 4.3.3 (Linear operators are differentiable)**.** *If f is a linear bounded operator, then f must be Frechet differentiable.*

Proof. It is easy to see that Definition 1 is satisfied, when we take $Df(u)$ to be f itself for each u. We then have

$$
\frac{||f(u+h) - f(u) - Df(u) \cdot (h)||}{||h||} = \frac{||f(u) + f(h) - f(u) - f(h)||}{||h||} = 0. \tag{4.5}
$$

□

The usual rules for calculating the derivative also hold true for Frechet derivatives, we start by proving linearity.

Theorem 4.3.4 (Linearity of derivative)**.** *Let $f, g : U \subset \mathbb{E} \to \mathbb{F}$ be differentiable mappings, α, β scalars then the mapping $\alpha f + \beta g$ is differentiable with derivative $\alpha Df + \beta Dg$.*

Proof. If u and $h \in \mathbb{E}$, we have

$$\begin{aligned} f(u+h) &= f(u) + Df(u)h + r_1(h), \\ g(u+h) &= g(u) + Dg(u)h + r_2(h), \end{aligned} \tag{4.6}$$

with

$$\frac{||r_i(h)||}{||h||} \to 0, i = 1, 2. \tag{4.7}$$

Adding these two gives

$$\alpha f(u+h) + \beta g(u+h) = \alpha f(u) + \beta g(u) + (\alpha Df(u) + \beta Dg(u))(h) + R(h), \tag{4.8}$$

where $R(h) = r_1(h) + r_2(h)$ satisfies

$$\frac{||R(h)||}{||h||} \to 0 \tag{4.9}$$

when $||h|| \to 0$. □

An useful property of the Frechet derivative is that we can differentiate operators that takes value in a Cartesian product of Banach spaces component wise.

Theorem 4.3.5 (Derivative of cartesian product)**.** *Let $f_i : U \subset \mathbb{E} \to \mathbb{F}_i$ be differentiable mappings for $1 \leq i \leq n$ then the Cartesian product function $f_1 \times ... \times f_n = f : U \to \mathbb{F}_i \times ... \times \mathbb{F}_n$ defined by $f(u) = (f_1(u), ..., f_n(u))$ is differentiable with derivative $Df_1 \times ..... \times Df_n$. Here we assume that the product norm on $\mathbb{F}_i \times ... \times \mathbb{F}_n$ is given by $||(u_1, ..., u_n)||_p = ||u_1||_1 + ... + ||u_n||_n$.*

Proof. For u and $h \in \mathbb{E}$, we have

$$\begin{aligned} f(u+h) &= (f_1(u+h), ..., f_n(u+h)) \\ &= (f_1(u) + Df_1(u)h + r_1(h)), ..., f_n(u) + Df_n(u)h + r_n(h)) \\ &= (f_1(u), ...f_n(u)) + (Df_1(u)h, ..., Df_n(u)h) + (r_1(h), ..., r_n(h)) \\ &= (f_1(u), ...f_n(u)) + (Df_1(u), ..., Df_n(u))h + R(h) \end{aligned} \tag{4.10}$$

with $||R(h)||_p = ||(r_1(h), ..., r_n(h))\,||_p = ||r_1(h)||_1 + ... + ||r_n(h)||_n$. We easily see that

$$\frac{||R(h)||}{||h||} \to 0 \tag{4.11}$$

when $||h|| \to 0$. □

The chain rule now asserts that the derivative of a composition of differentiable mappings is composition of the derivatives.

Theorem 4.3.6 (Chain rule)**.** *Suppose $f : U \subset \mathbf{E} \to V \subset \mathbf{F}$ and $g : V \subset \mathbf{F} \to \mathbf{G}$ are differentiable. Then the composition $f \circ g$ is differentiable and*

$$D(f \circ g) = Df \circ Dg \tag{4.12}$$

Proof. By assumption g is differentiable at $u \in U$ and f is differentiable at $g(u)$. We thus must have

$$g(u+h) = g(u) + Dg(u)h + r_1(h) \tag{4.13}$$

and for $v = g(u)$

$$f(v+k) = f(v) + Df(v)k + r_2(k) \tag{4.14}$$

where $\frac{||r_i(h)||}{||h||} \to 0$, for $i = 1, 2$.

This gives

$$\begin{aligned} f(g(u+h)) - f(g(u)) &= f(v + Dg(u)h + r_1(h)) - f(v) \\ &= Df(v)[Dg(u)h + r_1(h)] + r_2(Dg(u)h + r_1(h)). \end{aligned} \tag{4.15}$$

Now we define $r_3(h)$ by $f(g(u+h)) = f(g(u)) + Df(g(u)) \circ Dg(u)[h] + r_3(h)$,

we are done if we can show that

$$\lim_{||h|| \to 0} \frac{||r_3(h)||}{||h||} = 0. \tag{4.16}$$

From equation 4.15 we get

$$r_3(h) = Df(v)[r_1(h)] + r_2(Dg(u)h + r_1(h)). \tag{4.17}$$

Since $Df(v)$ is bounded linear mapping, it is continuous. Therefore

$$\lim_{h \to 0} \frac{D(f(v)[r_1(h)]}{||h||} = D(f(v)[\lim_{h \to 0} \frac{r_1(h)}{||h||}] = 0. \tag{4.18}$$

It remains to show that

$$\frac{||r_2(Dg(u)h + r_1(h))||}{||h||} \to 0. \tag{4.19}$$

Let $\epsilon \geq 0$, then we can find $\delta_{1,2,3}$ such that $||r_2(h)|| \leq ||h||\epsilon$ when $||h|| \leq \delta_1$, $||Dg(u)h + r_1(h))|| \leq \delta_1$ when $||h|| \leq \delta_2$ and $\frac{||r_1(h)||}{||h||} \leq 1$ when $||h|| \leq \delta_3$.
Thus for $||h|| \leq \min \delta_{1,2,3}$ we have

$$\begin{aligned} \frac{||r_2(Dg(u)h + r_1(h))||}{||h||} \leq \epsilon \left(\frac{||Dg(u)[h]|| + ||r_1(h)||}{||h||} \right) \leq \epsilon \left(\frac{||Dg(u)|| ||h||}{||h||} + \frac{||r_1(h)||}{||h||} \right) \\ \leq \epsilon(||Dg(u)|| + 1). \end{aligned} \tag{4.20}$$

This says the above expression can be made smaller than any positive number, which is what we wanted to prove. □

We now introduce the concept of a Banach algebra.

Definition 3 (Banach Algebra)**.** *An algebra is a vector space* $\mathbf{A}$ *such for each* $x, y \in \mathbf{A}$ *an unique product* $xy \in \mathbf{A}$ *is defined such that the following equation are satisfied*

$$(xy)z = x(yz), \tag{4.21}$$

$$x(y+z) = xy + xz, \tag{4.22}$$

$$(x+y)z = xz + yz, \tag{4.23}$$

$$\alpha(xy) = (\alpha)y = x(\alpha y). \tag{4.24}$$

$\mathbf{A}$ *is called an algebra with identity if* $\mathbf{A}$ *contains an element* e *such that for all* $x \in \mathbf{A}$,

$$ex = xe = x. \tag{4.25}$$

This element e *is called the identity of* $\mathbf{A}$.

A Banach algebra is a complete normed space $\mathbf{A}$*which is an algebra that there exists a positive constant* c *such that for all* $x, y \in \mathbf{A}$

$$||xy|| \leq c||x||||y|| \tag{4.26}$$

Theorem 4.3.7 (Product rule for derivative in Banach algebra)**.** *Suppose* $\mathbf{E}$ *is a Banach algebra and* $f, g : U \subset \mathbf{E} \to \mathbf{F}$ *are differentiable. Then the product* $f * g$ *is differentiable and*

$$D(f * g)(u)[h] = Df(u)[h] * g(u) + f(u) * Dg(u)[h]. \tag{4.27}$$

Proof. For $h \neq 0$ write

$$\begin{aligned}\Delta f * g &= f(u+h) * g(u+h) - f(u)g(u) \\ &= (f(u+h) - f(u)) * g(u+h) + f(u) * (g(u+h) - g(u)) \\ &= (f(u+h) - f(u)) * g(u) + f(t) * (g(u+h) - g(u)) \\ &+ (f(u+h) - f(u)) * (g(u+h) - g(u)).\end{aligned} \tag{4.28}$$

Therefore we have

$$\begin{aligned}&\frac{||f(u+h) * g(u+h) - f(u) * g(u) - (Df(u)[h] * g(u) + f(u) * Dg(u)[h])||}{||h||} \\ &\leq \frac{||(f(u+h) - f(u)) - Df(u)h||}{||h||}||g(u)|| + ||f(u)||\frac{||(g(u+h) - g(u)) - Dg(u)h||}{||h||} \\ &+ \frac{||(f(u+h) - f(u) - Df(u)h)||}{||h||}||(g(u+h) - g(u))|| + ||Df(u)||||(g(u+h) - g(u))||,\end{aligned} \tag{4.29}$$

which goes to zero if we take $||h|| \to 0$. □

Finally we have the following theorem which is an extension to the well known implicit function theorem for functions in $\mathbb{R}^n$.

Theorem 4.3.8. *Let* X, Y *and* Z *be Banach spaces, let* $f : X \times Y \to Z$ *be Frechet differentiable near* (x_o, y_0), *and suppose that* $f(x_0, y_0) = 0$. *Suppose that the Frechet derivative with respect to the second argument at* (x_0, y_0) *given by*

$$h \to Df(x_0, y_0)[0, h] \tag{4.30}$$

is bijective from Y *to* Z. *Then there exists a unique mapping* $y : U \to Y$ *with* U *an open ball around* x_0 *that is Frechet differentiable, that satisfies*

$$f(x, y(x)) = 0 \tag{4.31}$$

and $y(x_0) = y_o$

Proof. We will not prove this here, see [17]. □

Chapter 5

Sobolev spaces

5.1 Introduction

In this thesis we will make use of Sobolev function spaces, these are subspaces of L^2 consisting of functions which have Fourier coefficients that satisfy certain decay. Here we give an overview of the important definitions and theorems. This summary is based on the lecture notes "Introduction to Sobolev Spaces on the Circle" by Dave Gilliam [7].

A function $f \in L^2[0, 2\pi]$ has a Fourier representation

$$f = \sum_{k=0}^{\infty}(a_k \cos(k\theta)) + \sum_{k=1}^{\infty}(b_k \sin(k\theta)), \tag{5.1}$$

where the coefficients are given by

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta, \tag{5.2}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(k\theta)\, d\theta, \tag{5.3}$$

$$b_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\sin(k\theta)\, d\theta. \tag{5.4}$$

Here the convergence is in the L^2 norm, we have

$$\lim_{n\to\infty} ||a_0 + \sum_{k=1}^{n} a_k \cos(k\theta) + b_k \sin(k\theta) - f(\theta)||_{L_2} = 0, \tag{5.5}$$

with

$$||f||_{L_2} = \left(\int_0^{2\pi} |f(\theta)|^2\, d\theta\right)^{1/2}. \tag{5.6}$$

By Parseval's identity we have

$$\sum_{k=0}^{\infty}|a_k|^2 + \sum_{k=1}^{\infty}|b_k|^2 = \frac{1}{2\pi}||f||_{L_2}^2. \tag{5.7}$$

5.2 Definition Sobolev space

Now let us define the L^2-based Sobolev space of order s, for $s \in \mathbb{N}$.

Definition 4.

$$H^s := \left\{ \phi \in L^2[0, 2\pi] \,\middle|\, \sum_{k=0}^{\infty}(1+k^2)^s|a_k|^2 + \sum_{k=1}^{\infty}(1+k^2)^s|b_k|^2) < \infty \right\} \tag{5.8}$$

where a_k and b_k denote again the Fourier coefficients of ϕ.

Theorem 5.2.1. *H^s is a Hilbert space with inner product defined by*

$$\langle \phi, \psi \rangle_s = \sum_{k=0}^{\infty}(1+k^2)^s a_k \overline{\alpha_k} + \sum_{k=1}^{\infty}(1+k^2)^s b_k \overline{\beta_k} \tag{5.9}$$

where

$$\phi = \sum_{k=0}^{\infty}(a_k \cos(k\theta)) + \sum_{k=1}^{\infty}(b_k \sin(k\theta)) \tag{5.10}$$

and

$$\psi = \sum_{k=0}^{\infty}(\alpha_k \cos(k\theta)) + \sum_{k=1}^{\infty}(\beta_k \sin(k\theta)). \tag{5.11}$$

This implies the norm for H^s

$$||\phi||_s^2 = \sum_{k=0}^{\infty}(1+k^2)^s|a_k|^2 + \sum_{k=1}^{\infty}(1+k^2)^s|b_k|^2. \tag{5.12}$$

Furthermore this space is complete under this given norm.

Proof. It is not hard to see that H^s is closed under the operations of taking sums and products with scalars, and is thus indeed a linear space. We first show now that the inner product is well defined, in other words (5.9) is finite for all $\phi, \psi \in H^s$.
By the usual Cauchy-Schwarz inequality we have

$$\begin{aligned} &|\sum_{k=0}^{\infty}(1+k^2)^s a_k \overline{\alpha_k} + \sum_{k=1}^{\infty}(1+k^2)^s b_k \overline{\beta_k}| \leq |\sum_{k=0}^{\infty}(1+k^2)^s a_k \overline{\alpha_k}| + |\sum_{k=1}^{\infty}(1+k^2)^s b_k \overline{\beta_k}| \\ &\leq \sum_{k=0}^{\infty}(1+k^2)^s|a_k|^2 \sum_{k=0}^{\infty}(1+k^2)^s|\alpha_k|^2 + \sum_{k=1}^{\infty}(1+k^2)^s|b_k|^2 \sum_{k=0}^{\infty}(1+k^2)^s|\beta_k|^2 < \infty. \end{aligned} \tag{5.13}$$

Now we show completeness, let ϕ_n be a Cauchy sequence, we thus have that for every $\epsilon > 0$ we can find a N such that for every $n, k \geq N$ we have

$$||\phi_n - \phi_m|| \leq \epsilon, \tag{5.14}$$

which means that

$$\sum_{k=0}^{\infty}(1+k^2)^s|a_{k,n} - a_{k,m}|^2 + \sum_{k=1}^{\infty}(1+k^2)^s|b_{k,n} - b_{k,m}|^2 \leq \epsilon^2. \tag{5.15}$$

We have for $m, n \geq N$ and any k

$$|a_{k,n} - a_{k,m}| \leq \epsilon \tag{5.16}$$
$$|b_{k,n} - b_{k,m}| \leq \epsilon. \tag{5.17}$$

This shows that for any fixed k, the sequences (a_k) and (b_k) are Cauchy sequences. By completeness of the complex numbers, there are sequences a_k and b_k such that

$$a_{k,n} \to a_k \quad \text{for } n \to \infty, \tag{5.18}$$
$$b_{k,n} \to b_k \quad \text{for } n \to \infty. \tag{5.19}$$

From (5.15) we have that for every $N > 0$

$$\sum_{k=0}^{N}(1+k^2)^s|a_{k,n} - a_{k,m}|^2 + \sum_{k=1}^{N}(1+k^2)^s|b_{k,n} - b_{k,m}|^2 \leq \epsilon^2. \tag{5.20}$$

Passing to the limit in this finite sum gives

$$\sum_{k=0}^{N}(1+k^2)^s|a_k - a_{k,m}|^2 + \sum_{k=1}^{N}(1+k^2)^s|b_k - b_{k,m}|^2 \leq \epsilon^2. \tag{5.21}$$

Letting N now approach infinity, we finally get

$$\sum_{k=0}^{\infty}(1+k^2)^s|a_k - a_{k,m}|^2 + \sum_{k=1}^{\infty}(1+k^2)^s|b_k - b_{k,m}|^2 \leq \epsilon^2. \tag{5.22}$$

From this it follows that $\phi(\theta) = \sum_{k=0}^{\infty}(a_k \cos(k\theta)) + \sum_{k=1}^{\infty}(b_k \sin(k\theta))$ is actually in H^s, since

$$\begin{aligned}
\sum_{k=0}^{\infty}(1+k^2)^s|a_k|^2 + \sum_{k=1}^{\infty}(1+k^2)^s|b_k|^2 &= \sum_{k=0}^{\infty}(1+k^2)^s|a_k - a_{k,m} + a_{k,m}|^2 \\
&+ \sum_{k=1}^{\infty}(1+k^2)^s|b_k - b_{k,m} + b_{k,m}|^2 \\
&\leq 2\sum_{k=0}^{\infty}(1+k^2)^s(|a_k - a_{k,m}|^2 + |a_{k,m}|^2) \\
&+ 2\sum_{k=1}^{\infty}(1+k^2)^s(|b_k - b_{k,m}|^2 + |b_{k,m}|^2) < \infty.
\end{aligned} \tag{5.23}$$

We also have from (5.21) that $||\phi_n - \phi||_s \to 0$ and this completes the proof.

□

5.3 Preliminaries

We now show some results that will be needed later in the thesis.

Theorem 5.3.1. *Suppose $p \leq q$, then we have*

$$||\phi||_p \leq ||\phi||_q. \tag{5.24}$$

Proof. We easily see that (5.12) is monotonic in s. □

Theorem 5.3.2. *For a (weakly) differentiable $\phi \in H^{s+1}$ we have*

$$||\dot{\phi}||_s \leq ||\phi||_{s+1} \tag{5.25}$$

with $\dot{\phi}$ denoting the derivative of ϕ with respect to θ.

Proof. Let a_k and b_k be the Fourier coefficients of ϕ, we use the fact that differentiation is equivalent with multiplying the Fourier coefficient by k.

$$\begin{aligned}
||\dot{\phi}||_s^2 &= \sum_{k=0}^{\infty}(1+k^2)^s k^2 |a_k|^2 + \sum_{k=1}^{\infty}(1+k^2)^s k^2 |b_k|^2 \\
&\leq \sum_{k=0}^{\infty}(1+k^2)^s (1+k^2) |a_k|^2 + \sum_{k=1}^{\infty}(1+k^2)^s (1+k^2) |b_k|^2 \\
&= \sum_{k=0}^{\infty}(1+k^2)^{s+1} |a_k|^2 + \sum_{k=1}^{\infty}(1+k^2)^{s+1} |b_k|^2 \\
&= ||\phi||_{s+1}^2.
\end{aligned} \tag{5.26}$$

□

This shows that differentiating is a bounded linear operation from $H^s \to H^{s-1}$.

Theorem 5.3.3. *There exist a positive constant C such that for $s \geq 1, s \in \mathbb{N}$ we also have*

$$||\phi \cdot \psi||_s \leq C ||\phi||_s ||\psi||_s. \tag{5.27}$$

Here $\cdot$ denotes point wise multiplication of functions.

Proof. We will not prove this here, but refer to literature [7]. □

Notice that the identity element e in H^s is the function $u \equiv 1$, and that we can invert any function that is bounded away from zero. this makes H^s a Banach algebra with respect to the operation of point wise multiplication.
Now we are able to combine ideas of the Frechet derivative of the previous chapter with Sobolev spaces.
Let us define the operator $A_n : H^s \to H^s$ by

$$A_n[\psi](\theta) := (\psi(\theta))^n. \tag{5.28}$$

This operator is well defined by 5.3.3.

Theorem 5.3.4. *A_n is Frechet differentiable with*

$$DA_n(\psi)(h) = nA_{n-1}(\psi) \cdot h. \tag{5.29}$$

Proof. By the Binomial Theorem we can write

$$\begin{aligned}
\frac{||(u+h)^n - u^n - nA_{n-1}(u)h||}{||h||} &= \frac{||\sum_{k=0}^{n}\binom{n}{k}u^{n-k}h^k - u^n - nA_{n-1}(u)h||}{||h||} \\
&= \frac{||\sum_{k=2}^{n}\binom{n}{k}u^{n-k}h^k||}{||h||} \\
&\leq \frac{\sum_{k=2}^{n}\binom{n}{k}||u||^{n-k}||h||^k}{||h||} \\
&= \sum_{k=2}^{n}\binom{n}{k}||u||^{n-k}||h||^{k-1} \to 0. \qquad (5.30)
\end{aligned}$$

□

For a k times continuously differentiable, positive function u we write $u \in C_+^s \subset H_+^s$, and define inversion and square root operator $C_+^s \to C_+^s$

$$R[u](\theta) := \frac{1}{u(\theta)}, \qquad (5.31)$$

$$S[u](\theta) := \sqrt{\psi(\theta)}. \qquad (5.32)$$

These operators are again well defined by the elementary rules for differentation. We will now show that these operators are Frechet differentiable in a neighborhood of each element in C_+^s.

Theorem 5.3.5. *R and S are Frechet differentiable with u in H_+^s*

$$DR(u)h = \frac{-1}{u^2} \cdot h \qquad (5.33)$$

and

$$DS(u)h = \frac{1}{2\sqrt{u}} \cdot h. \qquad (5.34)$$

Proof. We apply the implicit function theorem 4.3.8 to the operators:

$$F(r, u) = r^2 - u = 0, \qquad (5.35)$$

$$G(s, u) = su - e = 0 \qquad (5.36)$$

It is easy to see that F and G are differentiable with

$$DF(r, u)(h, k) = 2rh - k, \qquad (5.37)$$

$$DG(s, u)(h, k) = sk + uh \qquad (5.38)$$

and that the partial derivatives with respect r and s are thus indeed invertible, when u is an invertible element. We conclude that I and S are indeed differentiable operators in a neighborhood around each $u_0 \in H_+^s$. Let W denote the operator

$$u \to u^2 \qquad (5.39)$$

then we have

$$[W \circ S]u = u. \tag{5.40}$$

Now differentiate both sides with respect to u by using the chain rule 4.3.6. This gives

$$DW(S(u)) \circ DS(u) = I, \tag{5.41}$$

here I now stands for the identity operator. From 5.3.4 we have that

$$DW(S(u))[k] = 2S[u]k. \tag{5.42}$$

This implies that we we must have

$$DS(u)[h] = \frac{1}{2\sqrt{u}} \cdot h. \tag{5.43}$$

Now for R we see that we have the operator equation

$$R * I = e \tag{5.44}$$

where $*$ denotes the point wise multiplication.
Differentiating this equation by using Theorem (5.3.3) gives

$$u * DR(u)[h] + R[u] * h = 0. \tag{5.45}$$

This gives the result

$$DR(u)[h] = -u^{-1}R[u] * h = \frac{-1}{u^2} \cdot h. \tag{5.46}$$

□

Chapter 6

Models

6.1 Known properties of problem

We here further describe the moving boundary problems introduced in Chapter 3. These consist of finding a $t \to \Omega(t) \subset \mathbb{R}^2$ that has boundary boundary $\Gamma(t)$ with exterior normal $n = n(t)$ and a potential function $\phi = \phi(\cdot, t)$ defined on $\mathbb{R}^2 \backslash \overline{\Omega(t)}$ such that

$$\left.\begin{array}{l} \Delta\phi = 0 \qquad \text{in } \mathbb{R}^2 \setminus \Omega(t) \\ \nabla\phi - \overrightarrow{e_1} = o(|x|^{-1}) \qquad \text{for } |x| \to \infty \\ \phi - l\partial_n\phi = -\gamma\kappa \text{ on } \Gamma(\text{t}) \\ V_n = \partial_n\phi \qquad \text{on } \Gamma(\text{t}) \end{array}\right\} \tag{6.1}$$

Here, $\overrightarrow{e_1}$ is the unit vector in the x_1 direction and V_n is the normal velocity of the moving boundary $\Gamma(t)$. The initial domain $\Omega(0)$ given and the parameter l, γ are non-negative constants. It can be shown that that the fixed time problem $(6.1)_{1,2,3}$ is uniquely solvable for a sufficiently smooth $\Omega(t)$ [16]. It is also known that for $l = 0$ and $\gamma = 0$ this problem is ill-posed [11]. We will not consider that case in this thesis, so we have either $l > 0$ or $\gamma > 0$.

We first show that $\Omega(t)$ is volume preserving through time with the help of the following lemma:

Lemma 6.1.1. *Let Γ_t be the position at time t of a smooth, closed curve in $\mathbb{R}^2$ that varies smoothly with t, and bounds at any time a domain Ω_t. Let $\mathbf{N}(\mathbf{x}, t)$ denote the unit outward normal on S_t and $\mathbf{V}(\mathbf{x}, t)$ is the velocity vector of the point $\mathbf{x}$ at time t. Suppose now that $f(\mathbf{x}, t)$ is a smooth function. Define $F = F(t) = \int_{\Omega(t)} f\, dx$, we then have*

$$\frac{dF}{dt} = \frac{d}{dt}\int_{\Omega(t)} f\, dx = \int_{\Omega(t)} \frac{\partial f}{\partial t}\, dx + \int_{\Gamma(t)} f V_n\, ds. \tag{6.2}$$

Proof. We transform the integral over a time dependent domain back to a fixed domain Ω_0, by the transformation $x = \psi(X, t)$. Here ψ solves the following system of ordinary differential equations

$$\frac{d\psi}{dt} = \mathbf{V}(\psi(X, t), t), \tag{6.3}$$

$$\psi(X, 0) = X. \tag{6.4}$$

This transformation has Jacobian determinant $\det\mathbf{J} = \det\mathbf{J}(t)$. Writing $g(X,t) = f(\psi(X,t),t)$ we have

$$\frac{dF}{dt} = \frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx = \frac{d}{dt}\int_{\Omega_0} g(X,t)\mathrm{Det}\mathbf{J}\,dX \tag{6.5}$$

$$= \int_{\Omega_0} \frac{\partial}{\partial t} g(X,t)\mathrm{Det}\mathbf{J}\,dX + \int_{\Omega_0} g(X,t)\frac{\partial}{\partial t}\mathrm{Det}\mathbf{J}\,dX. \tag{6.6}$$

Now we make use of Liouville's formula [3], which states that when the matrix valued function J satisfies

$$\frac{dJ}{dt} = AJ \tag{6.7}$$

where $A = A(t)$ is a real valued matrix, then the Jacobian determinant of J must satisfy

$$\frac{dDetJ}{dt} = Tr(A)DetJ. \tag{6.8}$$

From differentiating (6.3) with respect to X we get

$$\frac{\partial}{\partial t}\mathbf{J} = \mathrm{J}(\mathbf{V})\cdot\mathbf{J}. \tag{6.9}$$

By application of Liouville's formula we have

$$\frac{dDet\mathbf{J}}{dt} = \mathrm{Tr}(\mathrm{J}(\mathbf{V}))\mathrm{Det}\mathbf{J} = (\nabla\cdot\mathbf{V})\mathrm{Det}\mathbf{J}. \tag{6.10}$$

So we have

$$\frac{dF}{dt} = \int_{\Omega_0}\left[\frac{\partial g}{\partial t} + g\nabla\cdot\mathbf{V}\right]\mathrm{Det}\mathbf{J}\,dX. \tag{6.11}$$

From the chainrule we know that $\frac{\partial g}{\partial t} = \frac{\partial f}{\partial t} + \nabla f\cdot\mathbf{V}$ and after transforming the result back we get

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx = \int_{\Omega(t)} \frac{\partial f}{\partial t} + \nabla f\cdot\mathbf{V} + f\nabla\cdot\mathbf{V}\,dx. = \int_{\Omega(t)} \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{V})\,dx. \tag{6.12}$$

By using Divergence theorem we finally have

$$\int_{\Omega(t)} f\,dx = \int_{\Omega(t)} \frac{\partial f}{\partial t}\,dx + \int_{\Gamma(t)} fV_n\,ds. \tag{6.13}$$

This proves the lemma. □

By the previous lemma in the case that $f \equiv 1$ we can write due to 6.1:

Theorem 6.1.2.

$$\frac{\partial}{\partial t}\int_{\Omega(t)} dx = \int_{\Gamma(t)} V_n\,ds = \int_{\Gamma(t)} \partial_n\phi\,ds = 0. \tag{6.14}$$

The last step is consequence of the Divergence lemma and (6.1)˙2, in our case $\Delta\phi = 0$ outside $\Omega(t)$ and not inside. But in the following way we can work around that. For t fixed, consider a ball $B(R)$ with $R = R(t)$ large enough such that $\Gamma(t) \subset B(R)$. Define $W = B(R) \setminus \Gamma(t)$, then it is clear that

$$0 = \int_W \Delta\phi \, dx = \int_{\Gamma(t)} \partial_n \phi \, ds - \int_{\partial B(R)} \partial_n \phi \, ds \tag{6.15}$$

from the Gauss Divergence Theorem. So we find that

$$\int_{\Gamma(t)} \partial_n \phi \, ds = \int_{\partial B(R)} \partial_n \phi \, ds \tag{6.16}$$

and thus that the last integral does not depend on R. We can estimate:

$$\int_{\partial B(R)} \partial_n \phi \, ds = \int_{\partial B(R)} \nabla\phi \cdot \overrightarrow{n} \, ds \tag{6.17}$$

$$= \int_{\partial B(R)} e_1 \cdot \overrightarrow{n} \, ds + \int_{\partial B(R)} (\nabla\phi - e_1) \cdot \overrightarrow{n} \, ds \leq 2\pi R \text{ Max } |\nabla\phi - e_1| \to 0 \tag{6.18}$$

when R goes to infinity. This implies that

$$\int_{\partial B(R)} \partial_n \phi \, ds = 0, \tag{6.19}$$

So it can only be the case that $\int_{\Gamma(t)} \partial_n \phi = 0$ as well.

6.2 Coordinate transformations

To remove the in homogeneous term at infinity write $\phi = \psi + x_1$ or equivalently $\psi = \phi - x_1$. The straightforward calculations for this transformation are given below

$$\Delta\psi = \Delta\phi - \Delta x_1 = \Delta\phi = 0, \tag{6.20}$$

$$-\gamma\kappa = \phi - l\partial_n\phi = \psi + x_1 - l\partial_n(\psi + x_1) = \psi + x_1 - l\nabla(\psi + x_1) \cdot \overrightarrow{n} \tag{6.21}$$

$$= \psi + x_1 - l\nabla(\psi) \cdot \overrightarrow{n} - ln_1 = \psi + x_1 - l\partial_n(\psi) \; - ln_1, \tag{6.22}$$

$$\nabla\psi = \nabla\phi - \overrightarrow{e_1} = o(|x|)^{-1}, \tag{6.23}$$

$$V_n = \partial_n\phi = \nabla(\phi) \cdot \overrightarrow{n} = \nabla(\psi + x_1) \cdot \overrightarrow{n} = \nabla(\psi) \cdot \overrightarrow{n} + \overrightarrow{n_1} = \partial_n\psi + n_1. \tag{6.24}$$

We thus obtain the following system for ψ:

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathbb{R}^2 \setminus \Omega(t), \\ |\nabla\psi| = o(|x|^{-1}) & \text{for } |x| \to \infty, \\ V_n = \partial_n\psi + n_1 & \text{on } \Gamma(\text{t}), \end{cases} \tag{6.25}$$

where in the case of pure kinetic undercooling we have:

$$\psi - l\partial_n\phi = -x_1 + ln_1 \qquad \text{on } \Gamma(\text{t}). \tag{6.26}$$

In the case of pure surface tension we have :

$$\psi = -x_1 - \gamma\kappa \qquad \text{on } \Gamma_u. \tag{6.27}$$

The combination boundary condition is given by:

$$\psi - l\partial_n\psi = -x_1 + ln_1 - \gamma\kappa \qquad \text{on } \Gamma_u. \tag{6.28}$$

A final remark on (6.25) is that that equation $(6.1)_2$ implies boundedness of ψ. By estimating a line integral we get

$$|\psi(x)| = |\int_0^x \nabla\psi \cdot \overrightarrow{t}\, ds| \leq |x| \text{ Max } |\nabla\psi| \to 0 \tag{6.29}$$

when $|x| \to \infty$ Conversely we have that boundedness even implies that

$$\nabla\psi = O(|x|^{-2}) \tag{6.30}$$

see [16] for a verification of this fact. Therefore we can effectively replace the third condition with boundedness of ψ.

6.3 Traveling-wave solutions

We now make the ansatz that travelling-wave solutions exists, these are solutions of the form $\Omega(t) = \Omega + \overrightarrow{v}t$. These solutions thus not only conserve volume but also shape over time. Here $\overrightarrow{v} = (v_1, v_2) \in \mathbb{R}^2$ is the velocity vector of the moving volume and Ω being the shape of the bubble.

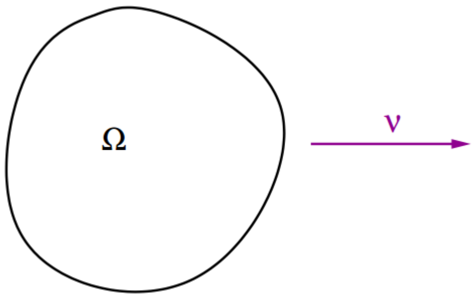


Figure 6.1: Traveling wave solution

By introducing a Cartesian coordinate system that is moving along with the bubble, Thus making a coordinate transformation $\eta = x - v_1t$ and $\xi = y - v_2t$, we end up with a stationary problem in the η-ξ plane.

We will now only write boundary condition (6.28) as it contains the other conditions as a trivial case. This them gives the stationary problem:

$$\begin{cases} \Delta\psi = 0 \qquad \text{in } \mathbb{R}^2 \setminus \Omega, \\ \psi - l\partial_n\psi = -x_1 + ln_1 - \gamma\kappa \qquad \text{on } \Gamma, \\ \psi = O(1) \qquad \text{for } |x| \to \infty, \\ V_n = \partial_n\psi + (e_1 - \overrightarrow{v}) \cdot n = 0 \qquad \text{on } \Gamma. \end{cases} \tag{6.31}$$

We now state a number of in variance properties that solutions of (6.31) satisfy. These are based on the article [11] by Günther & Prokert, where they were given in the case of only kinetic undercooling. Suppose now that $(\Omega, \psi, l, \gamma)$ is a solution of (6.31).

- Translation in variance
  for any $a \in \mathbb{R}^2$, $(\Omega + a, T_\alpha\psi, l, \gamma)$ is also a solution where

$$\Omega + a := \{x + a \,|\, x \in \Omega\}, \quad T_a\psi(x) := \psi(x - a) \tag{6.32}$$

- Scaling in variance
  for any $R \geq 0$, $(R\Omega, S_R\psi, Rl, R^2\gamma)$ is a solution with

$$R\Omega := \{Rx \,|\, x \in \Omega\}, \quad S_R\psi(x) := R\psi(x/R). \tag{6.33}$$

- Point reflection symmetry
  $(-\omega, \psi^-, l, \gamma)$ is also a solution where

$$-\Omega := \{-x \,|\, x \in \Omega\}, \quad \psi^-(x) := -\psi(-x). \tag{6.34}$$

There is a solution that is easily identifiable, namely the solution of a moving circle with radius 1, that is moving in the direction of e_1 with speed $\frac{2}{l+1}$. By separation of variables we find

$$\Omega = B(0,1), \quad \nu = \frac{2}{l+1}e_1, \quad \psi = \frac{l-1}{l+1}\frac{x_1}{|x|^2} + \gamma. \tag{6.35}$$

Together with the shifting and scaling properties we obtain a three-parameter family of solutions:

$$\Omega = B_R(a) := \{x \,|\, |x - a| < R\}, \overrightarrow{v} = \overrightarrow{v_0} = \frac{2R}{l+R}\overrightarrow{e_1}, \psi = R^2\frac{l-R}{l+R}\frac{x_1 - a_1}{|x-a|^2} + \frac{\gamma}{R}. \tag{6.36}$$

We demand three extra conditions to exclude the above degrees of freedom.

$$\int_\Omega dx = 2\pi, \int_\Omega x\, dx = 0. \tag{6.37}$$

So Ω has same area as the unit circle and geometric centroid at the origin.

6.4 Operator problem

We now restrict ourselves by only allowing star-shaped domains with respect to the origin, these are domains whose boundary is of the form $r = f(\theta)$ in polar coordinates, where f is a positive 2π-periodic function. Our goal is to reformulate the described problem as a non-linear operator equation on the unit circle $S := \partial B_1(0)$. We can identify a function $u(x_1, x_2)$ defined on S in a obvious way with a 2π periodic function on $\mathbb{R}$ by $u(\theta) = u(\cos\theta, \sin\theta)$.
Define for positive continuous function $u : S \to \mathbb{R}$

$$\Omega_u := \left\{ x \in \mathbb{R}^2 \,\middle|\, 0 < |x| < u(x/|x|) \right\} \cup \{(0,0)\}, \Gamma_u := \partial\Omega_u. \tag{6.38}$$

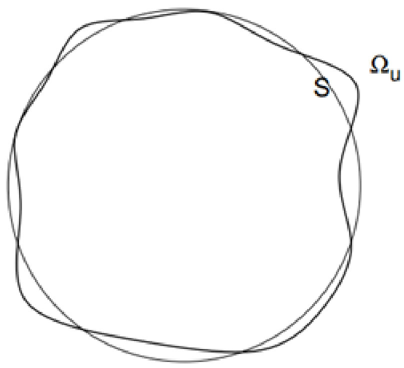


Figure 6.2: Perturbation of unit circle

In the case that $l > 0$ we can combine the two boundary conditions of (6.31). We see that we can rewrite our problem as

$$F(u, \overrightarrow{v}) = l\overrightarrow{v} \cdot n(u) - x_1(u) - \gamma\kappa(u) + A_l(u)[x_1 - ln_1(u) + \gamma\kappa] = 0. \tag{6.39}$$

Here

$$x(u)(\theta) = u(\theta)(\cos\theta, \sin\theta) \tag{6.40}$$

is a parameterization of Γ_u and

$$n(u)(\theta) = (u^2(\theta) + u'^2(\theta))^{-1/2}((u(\theta \sin\theta)', -(u(\theta)\cos\theta)') \tag{6.41}$$

is the exterior unit normal to Ω_u.
The curvature of Ω_u is mapped back to unit circle by

$$\kappa(u)(\theta) = \frac{u(\ddot{u} - u) - 2\dot{u}^2}{(u^2 + \dot{u}^2)^{3/2}}. \tag{6.42}$$

This expression for the curvature can be found in the literature, see for instance [2].
The operator $A_l(u)$ that is acting on a smooth function ϕ defined on S is given by $A_l(u)\phi = \psi \circ x(u)$ where ψ defined on $\mathbb{R}^2 \backslash \Omega_u$ solves the following exterior Robin problem

$$\begin{cases} \Delta\psi = 0 \qquad \text{in } \mathbb{R}^2 \setminus \Omega_u, \\ \psi - l\partial_n\phi = \phi \circ x(u)^{-1} \qquad \text{on } \Gamma_u, \\ \psi = O(1) \qquad \text{for } |x| \to \infty. \end{cases} \tag{6.43}$$

Finally by rewriting the scaling and shifting conditions in polar coordinates, we get

$$\int_0^{2\pi} u^2(\theta)\, d\theta = 2\pi, \int_0^{2\pi} u^3(\theta) \sin\theta\, d\theta = \int_0^{2\pi} u^3(\theta) \cos\theta\, d\theta = 0. \tag{6.44}$$

We can thus represent our problem as looking for a function $u \in H^s$ for sufficient high s and vector $v \in \mathbb{R}^2$ that solves

$$\begin{cases} F(u, \overrightarrow{v}) = 0 \quad \text{on S}, \\ \langle\ u - \mathbf{1}, \mathbf{1}\rangle = \langle\ u - \mathbf{1}, \cos\rangle = \langle\ u - \mathbf{1}, \sin\rangle = 0. \end{cases} \tag{6.45}$$

Note that in the case of $\gamma = 0$, the case of strictly kinetic undercooling, (6.39) simplifies to:

$$F(u, \overrightarrow{v}) = l\overrightarrow{v} \cdot n(u) - x_1(u) + A_l(u)[x_1 - ln_1(u)] = 0. \tag{6.46}$$

In the case of $l = 0$, we see that the equation 6.39 no longer holds true, in this case we work with a different operator namely $A(u)$. This operator is now defined on smooth functions ϕ defined on S by $A(u)\phi = (\partial_n\psi) \circ x(u)$ where ψ defined on $\mathbb{R}^2 \backslash \Omega_u$ solves the following exterior robin problem

$$\begin{cases} \Delta\psi = 0 \qquad \text{in } \mathbb{R}^2 \setminus \Omega_u, \\ \psi = \phi \circ x(u)^{-1} \qquad \text{on } \Gamma_u, \\ \psi = O(1) \qquad \text{for } |x| \to \infty. \end{cases} \tag{6.47}$$

The operator problem can now in this case be formulated as:

$$F(u, \overrightarrow{\nu}) = A(u)[\gamma\kappa(u) + x_1(u)] + (\nu - e_1) \cdot N(u) = 0. \tag{6.48}$$

Chapter 7

Linearization

In the previous chapter we rewrote each version of the boundary problem as an operator equation

$$F(u,v) = 0 \qquad \text{on S.} \tag{7.1}$$

Here we interpret the operator F in the case of pure kinetic undercooling as a mapping from

$$H^s[0,2\pi] \to H^{s-1}[0,2\pi]. \tag{7.2}$$

The image of F is in H^{s-1} because of the term related to the normal in (6.46), which involves first order derivatives. In the case of surface tension we see that F maps from

$$H^s[0,2\pi] \to H^{s-2}[0,2\pi] \tag{7.3}$$

because of the curvature term in (6.39). In this section we are going to calculate the Frechet derivative for F around $u \equiv 1$ (the trivial solution) in the sense of Chapter 3 for both cases.

7.1 Kinetic undercooling

Let us start simply with F from the problem without surface tension, we then have

$$F(u,\overrightarrow{v}) = l\overrightarrow{v} \cdot n(u) - x_1(u) + A_l(u)x_1 - ln_1(u). \tag{7.4}$$

By using the linearity property 4.3.4 and chain rule 4.3.6 we see that

$$\begin{aligned} D_uF(u,v_0)[h] = lv_0 \cdot n'(u)[h] - x_1'(u)[h] + A_l(u)[x_1'(u)[h] - ln_1'(u)[h] \\ +A_l'(u)[h][x_1(u) - ln_1(u)]. \end{aligned} \tag{7.5}$$

It is clear that $x(u)$ itself is a linear mapping in u, by theorem 7.26 and 4.3.5 we thus must have

$$x'(u)[h] = (\cos,\sin)h, \tag{7.6}$$

and thus

$$x_1'(u)[h] = \cos(\theta)h. \tag{7.7}$$

By using Theorems 4.3.5, 5.3.5 and 5.3.4 we calculate:

$$n'(u)[h] = \frac{\dot{h}e_\theta + he_r}{(u^2+\dot{u}^2)^{1/2}} + (-\dot{u}e_\theta + ue_r)(-1/2)\frac{1}{(u^2+\dot{u}^2)^{3/2}}(2uh + 2\dot{u}\dot{h}). \tag{7.8}$$

After substituting $u \equiv 1$ this reduces to cleans things up:

$$n'(1)[h] = -h^{\cdot} e_{\theta}. \tag{7.9}$$

Having done these calculation we see that

$$A_l(1)[x_1'(1)[h] - ln_1'(1)[h] = A_l(1)[\cos(\theta)h - l\dot{h}\sin\theta]. \tag{7.10}$$

It only remains to find $A_l'(u)[h][x_1(u) - ln_1(u)]$. For this we use the following lemma that is found in [12].

Lemma 7.1.1.

$$A_l'(u)h\phi = A_l(u)[(-\partial_r\psi\circ x(u))h + l((\partial_r\partial_n\psi\circ x(u)h + (\nabla\psi\circ x(u))\cdot n'(u)h)] + \partial_r\psi\circ x(u))h \tag{7.11}$$

where $\psi = A_l(u)\psi \circ x(u)^{-1}$

We will not prove this fact here, but we give a formal verification for this lemma in the appendix. Again this expression rather simplifies because $X(1)$ is the identity and $\partial_r = \partial_n$ on S. This gives

$$A_l'(1)[h][x_1(1) - ln_1(1)] = A_l(1)[(-\partial_n\psi h + l(\partial_n^2\psi h + \nabla\psi\cdot n'(1)h)] + \partial_n\psi)h, \tag{7.12}$$

with $\phi = (1-l)x_1$ and $\psi = \frac{1-l}{l+1}\frac{x_1}{|x|^2}$.
Calculating the normal derivatives for this particular ψ, we get

$$A_l'(u)[h][x_1(u) - ln_1(u)] = \frac{1-l}{l+1}A_l(1)[-\cos h + l(2\cos h + \dot{h}\sin)] - \cos h. \tag{7.13}$$

Combining everything we have we find that

$$L_1 h := D_u F(1, v_0)[h] = \frac{2}{l+1}(l\dot{h}\sin\theta - h\cos\theta + A_l[(1 + L - l^2)h\cos\theta - l^2\dot{h}\sin\theta]). \tag{7.14}$$

We now have linearized the equation in the case of pure kinetic undercooling.

7.2 Surface tension

We are now repeating the above calculations for the case of surface tension, where we have

$$F(u, \overrightarrow{\nu}) = A(u)[\gamma\kappa(u) + x_1(u)] + (\nu - e_1)\cdot N(u). \tag{7.15}$$

We again have that the derivative takes the following form;

$$D_u F(u, v_0)[h] = e_1\cdot n'(u)[h] + A(u)[\gamma\kappa'(u)[h] + x_1'(u)[h]] + A'(u)[h][-\gamma\kappa(u) + x_1(u)]. \tag{7.16}$$

As we already have calculated the Frechet derivatives for $n(u)$ and $x(u)$, we just have to calculate the derivative of $\kappa(u)$

$$\kappa'(u)[h] = \frac{h(\ddot{u}-u)+u(\ddot{h}-h)-4\dot{u}\dot{h}}{(u^2+\dot{u}^2)^{3/2}} + (u(\ddot{u}-u)-2\dot{u}^2)(-3/2)\frac{1}{(u^2+\dot{u}^2)^{5/2}}(2uh+2\dot{u}\dot{h}). \tag{7.17}$$

which simplifies to

$$\kappa'(u)[h] = \ddot{h} + h. \tag{7.18}$$

Because we use a different operator for this problem the derivative now takes another form:

$$A'(1)[h]) - \gamma + x_1) = \dot{h}\sin + 2h\cos + A(1)[h\cos]. \tag{7.19}$$

Putting this together gives

$$L_2 h := D_u F(1, 2e_1)[h] = 2\dot{h}\sin\theta + 2h\cos\theta + A(1)(\gamma(\ddot{h}+h) + 2h\cos\theta)). \tag{7.20}$$

7.3 Combination of both kinetic undercooling and surface tension

Finally, for the combination model we have

$$F(u, \overrightarrow{v}) = l\overrightarrow{v}\cdot n(u) - x_1(u) - \gamma\kappa(u) + A_l(u)[x_1 - ln_1(u) + \gamma\kappa(u)]. \tag{7.21}$$

This expression has a linearization of the form

$$D_u F(u, v_0)[h] = lv_0 \cdot n'(u)[h] - x_1'(u)[h] - \gamma\kappa'(u)[h] + A_l(u)[x_1'(u)[h] - ln_1'(u)[h] \tag{7.22}$$

$$+\gamma\kappa'(u)[h]] + A_l'(u)[h][x_1(u) - ln_1(u) + \gamma\kappa(u)]. \tag{7.23}$$

We already knew from the previous lemma what the derivative of the operator A_l looks like, because $X(1)$ is again an identity on S and $\partial_r = \partial_n$. We have

$$A_l'(1)[h][x_1(1) - ln_1(1) + \gamma\kappa(1)] = A_l(1)[(-\partial_n\psi h + l(\partial_n^2\psi h + \nabla\psi\cdot n'(1)h)] + \partial_n\psi h, \tag{7.24}$$

with $\phi = (1-l)x_1 + \gamma$ and $\psi = \frac{1-l}{l+1}\frac{x_1}{|x|^2} + \gamma$.
Combining all the above we find

$$L_3 h := D_u F(1, v_0)[h] = \frac{2}{l+1}(l\dot{h}\sin\theta - h\cos\theta) + A_l(1)[(1+l-l^2)h\cos\theta - l^2\dot{h}\sin\theta$$
$$+\frac{\gamma(\ddot{h}+h)(l+1)}{2}] - \gamma(\ddot{h}+h). \tag{7.25}$$

This completes the calculations for the linearizations. From now on we will only consider the linearization of (6.1) around $(\mathbf{1}, \overrightarrow{v_0})$. This the following system:

$$\begin{cases} l(\overrightarrow{v} - \overrightarrow{v_0}\cdot)(\cos\theta, \sin\theta) + L(u-\mathbf{1}) = 0, \\ \langle\, u-\mathbf{1}, \mathbf{1}\rangle = \langle\, u-\mathbf{1}, \cos\rangle = \langle\, u-\mathbf{1}, \sin\rangle = 0. \end{cases} \tag{7.26}$$

Chapter 8

Fourier representation

In this section we want to find out how the Fourier representation looks like for each linear operator L. Because the linearization involves the operator $A_l(1)$ and $A(1)$, we want to first find the Fourier representations for these operators. Let $\phi(\theta)$ be given in Fourier series as

$$\phi(\theta) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta. \tag{8.1}$$

Because $x(1)$ in (6.40) is now the identity we have to solve the Laplacian equation on the exterior of the unit circle

$$\begin{cases} \Delta\psi = 0 & \text{in } \mathbb{R}^2 \setminus B(0,1), \\ \psi - l\partial_n\phi = \phi & \text{on } S, \\ \psi = O(1) & \text{for } |x| \to \infty. \end{cases} \tag{8.2}$$

In terms of polar coordinates the Laplace operator becomes

$$\Delta\psi = \frac{\partial^2\psi}{\partial r^2} + 1/r\frac{\partial\psi}{\partial r} + 1/r^2\frac{\partial^2\psi}{\partial\theta^2}. \tag{8.3}$$

On the unit circle we know that $\partial_n = \partial_r$ and we thus have the system:

$$\begin{cases} \Delta\psi = \frac{\partial^2\psi}{\partial r^2} + 1/r\frac{\partial\psi}{\partial r} + 1/r^2\frac{\partial^2\psi}{\partial\theta^2} = 0 & \text{for } r \geq 1, \\ \psi - l\partial_r\phi = \phi & \text{on } r = 1, \\ \psi = O(1) & \text{for } r \to \infty\ . \end{cases} \tag{8.4}$$

After separation of variables $\psi(r,\theta) = R(r)\Theta(\theta)$, we get

$$r^2R"/R + rR'/R = \Theta"\theta = \lambda^2. \tag{8.5}$$

From this follows the eigenvalue problem

$$\begin{cases} \Theta" + \lambda^2\Theta = 0 \\ \Theta(0) = \Theta(2\pi), \end{cases} \tag{8.6}$$

and the equation

$$r^2R" + rR' - \lambda R = 0. \tag{8.7}$$

The eigenvalue problem (8.6) has solutions

$$\begin{cases} \lambda = \lambda_n = n \\ \Theta = \Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta) \end{cases}$$

for $n = 0, 1, 2....$ The equation (8.7) is an Euler equation with general solution

$$R(r) = Ar^{\lambda} + Br^{-\lambda}. \tag{8.8}$$

The boundedness of the solution now implies that A has to be zero. So the solution is of the form

$$\psi(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \sin(n\theta) + B_n \cos(n\theta)). \tag{8.9}$$

By (8.4) we see that on the circle it must be that

$$A_0 + \sum_{n=1}^{\infty}(A_n \sin(n\theta) + B_n \cos(n\theta)) + l\sum_{n=1}^{\infty}(nA_n \sin(n\theta) + nB_n \cos(n\theta))$$
$$= \phi(\theta) = a_0 + \sum_{k=0}^{\infty} a_k \cos k\theta + b_k \sin k\theta. \tag{8.10}$$

We thus have

$$(A_l(\mathbf{1})\phi)(\theta) = a_0 + (\sum_{k=0}^{\infty} a_k \cos k\theta + b_k \sin k\theta)/(1 + kl). \tag{8.11}$$

Using the same approach as above we find for the operator $A(1)$ the representation

$$(A(\mathbf{1})\phi)(\theta) = (\sum_{k=1}^{\infty} -ka_k \cos k\theta - kb_k \sin k\theta). \tag{8.12}$$

In the next sections we we will make much use of the following identities for cos and sin

$$\begin{aligned} 2\cos(m\theta)\cos(\theta) &= \cos((m+1)\theta) + \cos((m-1)\theta), \\ 2\sin(m\theta)\cos(\theta) &= \sin((m+1)\theta) + \sin((m-1)\theta), \\ 2\cos(m\theta)\sin(\theta) &= \sin((m+1)\theta) + \sin((m-1)\theta), \\ 2\sin(m\theta)\sin(\theta) &= -\cos((m+1)\theta) + \cos((m-1)\theta). \end{aligned} \tag{8.13}$$

8.1 Kinetic undercooling

For the first problem it is easy to find that $L_1[1] = -\frac{2l^2}{(l+1)^2}\cos\theta$, $L_1[\sin\theta] = L_1[\cos\theta] = 0$. In general for $k \geq 1$, we get

$$L_1[\sin k\theta] = \frac{l(k-1)}{(l+1)}(\left(1 - \frac{1}{1+(k+1)l}\right)\sin(k+1)\theta \tag{8.14}$$

$$-\left(1 + \frac{1}{1+(k-1)l}\right)\sin(k-1)\theta), \tag{8.15}$$

$$L1[\cos k\theta] = \frac{l(k-1)}{(l+1)}(\left(1 - \frac{1}{1+(k+1)l}\right)\cos(k+1)\theta \tag{8.16}$$

$$-\left(1 + \frac{1}{1+(k-1)l}\right)\cos(k-1)\theta). \tag{8.17}$$

Using the following Fourier expansion for h:

$$h(\theta) = \sum_{k=2}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta), \tag{8.18}$$

We find that

$$L_1[h](\theta) = \sum_{k=2}^{\infty}(\alpha_k \cos((k+1)\theta) - \beta_k \cos((k-1)\theta)$$
$$+(\alpha_k \sin((k+1)\theta) - \beta_k \sin((k-1)\theta),$$

with

$$\alpha_k = \frac{l(k-1)}{l+1}(1 - \frac{1}{1+(k+1)l}),$$
$$\beta_k = \frac{l(k-1)}{l+1}(1 + \frac{1}{1+(k-1)l}).$$

We can now solve the following equation

$$(L_1 h)\theta) = \sum_{k=1}^{\infty}(f_k \cos(k\theta + g_k \text{sin}(k\theta)), \tag{8.19}$$

where the coefficients f_k and g_k are related to a_k and b_k by

$$f_1 = -\beta_2 a_2, \qquad f_2 = -\beta_3 a_3, \qquad f_k = \alpha_{k-1}a_{k-1} - \beta_{k+1}a_{k+1}, \tag{8.20}$$

$$g_1 = -\beta_2 b_2, \qquad g_2 = -\beta_3 b_3, \qquad g_k = \alpha_{k-1}b_{k-1} - \beta_{k+1}b_{k+1}. \tag{8.21}$$

8.2 Surface tension

For the case of pure surface tension we have

$$L[1] = 2\cos\theta + A(1)(\gamma + 2\cos\theta)) = 2\cos\theta - 2\cos\theta = 0, \tag{8.22}$$

$$\begin{aligned} L[\cos\theta] &= 2\dot{\cos}\,\theta\sin\theta + 2\cos\theta\cos\theta + A(1)(\gamma(\ddot{\cos}\,\theta + \cos\theta) + 2\cos\theta\cos\theta)) \\ &= -2\sin\theta\sin\theta + 2\cos\theta\cos\theta + A(1)(\gamma(\ddot{\cos}\,\theta + \cos\theta) + 2\cos\theta\cos\theta)) \\ &= \cos 2\theta - 1 + \cos 2\theta + 1 + A(1)(\cos 2\theta + 1) \\ &= 2\cos 2\theta - 2\cos 2\theta = 0, \end{aligned} \tag{8.23}$$

and by similar calculation

$$L[\sin\theta] = 0. \tag{8.24}$$

This is corresponding to the translation and scaling invariances.

In general for $m \geq 1$, we get

$$\begin{aligned} L[\cos m\theta] &= 2\dot{\cos}\,m\theta\sin\theta + 2\cos m\theta\cos\theta + A(1)(\gamma(\ddot{\cos}\,m\theta + \cos m\theta) + 2\cos m\theta\cos\theta)) \\ &= -2m\sin m\theta\sin\theta + 2\cos m\theta\cos\theta + A(1)(\gamma(\ddot{\cos}\,m\theta + \cos m\theta) + 2\cos m\theta\cos\theta)) \\ &= -m(-\cos((m+1)\theta) + \cos((m-1)\theta)) + \cos((m+1)\theta) + \cos((m-1)\theta) \\ &+ A(1)(\gamma(-m^2\cos m\theta + \cos m\theta) + \cos((m+1)\theta) + \cos((m-1)\theta)) \\ &= (m+1)\cos((m+1)\theta) + (1-m)\cos((m-1)\theta)(m-1) + \gamma(m^3 - m)\cos m\theta \\ &- (m+1)\cos((m+1)\theta) \\ &+ (1-m)\cos((m-1)\theta) \\ &= \gamma(m^3 - m)\cos m\theta + 2(1-m)\cos((m-1)\theta). \end{aligned} \tag{8.25}$$

Again similarly for $\sin m\theta$,

$$L[\sin(m\theta)] = \gamma(m^3 - m)\sin m\theta + 2(1-m)\sin((m-1)\theta). \tag{8.26}$$

Note that we can restrict our attention to perturbation h of the trivial solution where the zeroth and first Fourier coefficients vanish, as the the trivial solution already satisfies the equations (6.44).

If we again set

$$h(\theta) = \sum_{k=2}^{\infty} a_k\cos(k\theta) + b_k\sin(k\theta) \tag{8.27}$$

as Fourier representation for h.

We then find

$$\begin{aligned} L[h](\theta) &= \sum_{k=2}^{\infty} a_k(\gamma(k^3 - k)\cos k\theta \\ &+ 2(1-k)\cos((k-1)\theta)) + b_k(\gamma(k^3 - k)\sin k\theta + 2(1-k)\sin((k-1)\theta)) \\ &= \sum_{k=2}^{\infty} a_k(\gamma(k^3 - k)\cos k\theta) + \sum_{k=2}^{\infty} a_k(2(1-k)\cos((k-1)\theta)) \\ &+ \sum_{k=2}^{\infty} b_k(\gamma(k^3 - k)\sin k\theta) + \sum_{k=2}^{\infty} b_k(2(1-k)\sin((k-1)\theta)). \end{aligned} \tag{8.28}$$

Now substitute $j = k - 1$,

$$\begin{aligned}
L[h](\theta) &= \sum_{k=2}^{\infty} a_k \gamma(k^3 - k)\cos k\theta \\
&+ \sum_{j=1}^{\infty} -2a_{j+1}(j\cos(j\theta)) + \sum_{k=2}^{\infty} b_k\gamma(k^3-k)\sin k\theta + \sum_{j=1}^{\infty} -2b_{j+1} j \sin(j\theta)) \\
&= -2a_2\cos(\theta) - 2b_2\sin(\theta) + \sum_{k=2}^{\infty}(a_k\gamma(k^3-k) - 2a_{k+1}k)\cos(k\theta) \\
&+ (b_k\gamma(k^3-k) - 2b_{k+1}k)\sin(k\theta).
\end{aligned} \tag{8.29}$$

This implies that the relation between the Fourier coefficients $Lh(\theta)$

$$Lh(\theta) = \sum_{k=2}^{\infty} f_k \cos(k\theta) + g_k \sin(k\theta) \tag{8.30}$$

and the Fourier coefficients of $h(\theta)$ are given by

$$f_1 = -2a_2, \quad f_k = a_k\gamma(k^3-k) - 2a_{k+1}k, \quad k \geq 2 \tag{8.31}$$

$$g_1 = -2b_2, \quad g_k = b_k\gamma(k^3-k) - 2b_{k+1}k. \quad k \geq 2. \tag{8.32}$$

8.3 Combination of kinetic undercooling and surface tension

For the third model we have in general for $m \geq 1$,

$$\begin{aligned}
L[\sin k\theta] = \gamma\frac{l(k^3-k)}{1+kl}\sin(k\theta) + \frac{l(k-1)}{(l+1)}(\left(1 - \frac{1}{1+(k+1)l}\right)\sin(k+1)\theta \\
- \left(1 + \frac{1}{1+(k-1)l}\right)\sin(k-1)\theta),
\end{aligned} \tag{8.33}$$

$$\begin{aligned}
L[\cos k\theta] = \gamma\frac{l(k^3-k)}{1+kl}\cos(k\theta) + \frac{l(k-1)}{(l+1)}(\left(1 - \frac{1}{1+(k+1)l}\right)\cos(k+1)\theta \\
- \left(1 + \frac{1}{1+(k-1)l}\right)\cos(k-1)\theta).
\end{aligned} \tag{8.34}$$

Using a Fourier series representation for h, as noted before we can restrict to the case where the first three coefficients vanish.

$$h(\theta) = \sum_{k=2}^{\infty}(a_k\cos(k\theta) + b_k\sin(k\theta)). \tag{8.35}$$

We thus have

$$\begin{aligned}
L[h](\theta) = \sum_{k=2}^{\infty}(\alpha_k\cos((k+1)\theta) - \beta_k\cos((k-1)\theta) + \zeta_k\cos k\theta)a_k \\
+(\alpha_k\sin((k+1)\theta) - \beta_k\sin((k-1)\theta) + \zeta_k\sin k\theta)b_k),
\end{aligned} \tag{8.36}$$

where

$$\alpha_k = \frac{l(k-1)}{l+1}(1 - \frac{1}{1+(k+1)l}), \tag{8.37}$$

$$\beta_k = \frac{l(k-1)}{l+1}(1 + \frac{1}{1+(k-1)l}), \tag{8.38}$$

$$\zeta_k = \gamma \frac{l(k^3-k)}{1+kl}. \tag{8.39}$$

With this we can calculate the Fourier coefficients of $(L[h])\theta$

$$(Lh)\theta) = \sum_{k=1}^{\infty}(f_k \cos(k\theta) + g_k \sin(k\theta)), \tag{8.40}$$

where the coefficients f_k and g_k are given by

$$f_1 = -\beta_2 a_2, \quad f_2 = -\beta_3 a_3 + \zeta_2 a_2, \qquad f_k = \alpha_{k-1}a_{k-1} - \beta_{k+1}a_{k+1} + \zeta_k a_k, \quad k \geq 3 \tag{8.41}$$

$$g_1 = -\beta_2 b_2, \quad g_2 = -\beta_3 b_3 + \zeta_2 b_2, \qquad g_k = \alpha_{k-1}b_{k-1} - \beta_{k+1}b_{k+1} + \zeta_k b_k \quad k \geq 3. \tag{8.42}$$

Chapter 9

Conclusion

In this thesis we analyzed a linearized model for Hele-Shaw flow with three kinds of boundary conditions. Here we will find out whether or not the equation

$$Lh = \alpha \cos\theta + \beta \sin\theta \tag{9.1}$$

has solutions in H^s, for some $s \geq 0$.

9.1 Kinetic undercooling

In the case of kinetic undercooling, we found the recursion (8.20). In this case the equation (9.1)

has a solution that is given by

$$h = \alpha\phi_1 + \beta\phi_2. \tag{9.2}$$

Here we have

$$\phi_1(\theta) := \frac{l+1}{l}\sum_{k=1}^{\infty} c_k \cos k\theta, \tag{9.3}$$

and

$$\phi_2(\theta) := \frac{l+1}{l}\sum_{k=1}^{\infty} c_k \sin k\theta \tag{9.4}$$

with the coefficients c_k given by

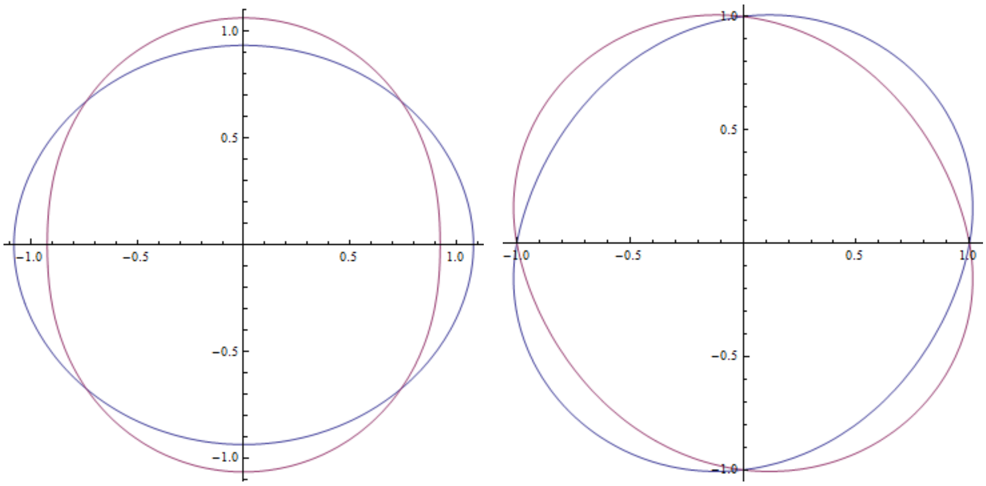
$$c_1 = -\frac{1}{\beta_2}, \quad c_k = \frac{-\alpha_2}{\beta_2}\cdot\frac{\alpha_4}{\beta_4}\cdots\frac{\alpha_{2k-2}}{\beta_{2k-2}}\cdot\frac{1}{\beta_{2k}} \tag{9.5}$$

where

$$\alpha_k = (k-1)(1-\frac{1}{1+(k+1)l}),$$
$$\beta_k = (k-1)(1+\frac{1}{1+(k-1)l}).$$

From 9.5 we see that

$$c_k = \frac{k-3/2}{k-1/2+1/l}c_{k-1}, \qquad k \geq 1 \tag{9.6}$$



(a) $l = 1/4$, $\vec{v} - \vec{v_0} = \pm 0.1\vec{e_1}$ (b) $l = 1/4$, $\vec{v} - \vec{v_0} = \pm 0.1\vec{e_2}$

Figure 9.1: Plots of bubble-domains

Relating the products of finite arithmetic progressions to the Gamma function, we have

$$c_k = C\frac{\Gamma(k-3/2)}{\Gamma(k-1/2+1/l)} \tag{9.7}$$

where the constant C is determined by

$$C = c_1\frac{\Gamma(1/2+1/l)}{\Gamma(-1/2)} = \frac{-1}{\beta_2}\frac{\Gamma(1/2+1/l)}{\Gamma(-1/2)}. \tag{9.8}$$

From the asymptotics for a quotient of Gamma functions we know

$$\frac{\Gamma(x+\beta)}{\Gamma(x+\alpha)} \sim x^{\beta-\alpha}, \qquad x \to \infty. \tag{9.9}$$

We thus have that $c_k \sim k^{-(1+1/l)}$, so that the functions ψ_1, ψ_2 are indeed H^s when $s < 1/l + 1/2$. This means that for any velocity vector near $\overrightarrow{v_0}$ we can can solve (7.26) for an unique function u. With this function u corresponds a travelling wave domain by (6.38).

In figure 9.1 we plotted some bubble-domains for several values for parameters $\overrightarrow{v}$ and l.

9.2 Surface tension

For the case of surface tension, we get the recurrent relation:

$$a_{k+1} = \gamma\frac{k^3-k}{2k}a_k = \frac{(k^2-1)}{2}a_k = \frac{(k+1)(k-1)}{2}a_k. \tag{9.10}$$

This gives

$$a_k = C\cdot\Gamma(k+1)\Gamma(k-1) = C\frac{1}{(k)(k-1)}\Gamma^2(k+1), \tag{9.11}$$

where again C is a constant determined by a_0.

By using Stirling's formula we find that

$$a_k \sim \frac{1}{k}(\frac{k}{e})^{2k}. \tag{9.12}$$

So that we find that (a_k) is not in l^2, and that the Fourier series corresponding with this sequence is definitely not in any H^s space.

9.3 Combination of surface tension and kinetic undercooling

The recusion 8.41 gives in this case

$$c_{k+1} = \frac{(k-1)(k+1)}{k+2/\ell}c_k + \frac{k-2}{k+2/\ell}c_{k-1}, \tag{9.13}$$

which implies that

$$c_{k-1}/c_k = O(1/k). \tag{9.14}$$

If we plug this in the recursion we have

$$c_{k+1} = \frac{(k-1)(k+1)}{k+2/\ell}(1+O(1/k^2))c_k \ , \tag{9.15}$$

and thus

$$c_k = A\frac{\Gamma(k+1)\Gamma(k-1)}{\Gamma(k+2/\ell)}(1+o(1)) \ , \tag{9.16}$$

because the infinite product of $1+O(1/k^2)$ is convergent. By the Stirling formula again we have

$$c_k \sim k^{k-1/2-2/\ell}e^{-k}. \tag{9.17}$$

Thus also in this case we see that the sequence (c_n) is not in l^2.

9.4 Final conclusion

We conclude that although kinetic undercooling and surface tension both regularize the Hele-shaw problem, only in the case of strict kinetic undercooling we find that there exist solutions to the linearized problem. In the case of surface tension the found Fourier coefficients diverge very fast and we can not identify any periodic function with them. In the combination case we see that something similar happens and so that it must be that the surface tension still determines the (linear) behaviour of the problem.

Chapter 10

Appendix

10.1 Formal derivative of the Boundary-value problem

Here we verify the structure of the linearization, Lemma 7.11 by formal variation of the domain.

We remind the reader that for a positive continuous function function u on S we defined

$$\Omega_u := \left\{ x \in \mathbb{R}^2 \,\middle|\, 0 < |x| < u(x/|x|) \right\} \cup \{(0,0)\}, \Gamma_u := \partial\Omega_u. \tag{10.1}$$

Let u_0, h, ψ be fixed smooth functions on S and consider the parameterization of exterior domains $\Omega_t^* := \mathbb{R}^2 \setminus \Omega_{u0+th}$ where $t \in I := (-\epsilon, \epsilon)$, $\Omega^* := \mathbb{R}^2 \setminus \Omega_0 = \mathbb{R}^2 \setminus \Omega_{u_0}$ and let $t \to \phi(\cdot, t) \in \mathrm{Diff}(\Omega^*, \Omega_t^*)$. $\phi_t : x \to x + f(x)th(x/|x|)x/|x|$, such that f is a smooth function with $f(x) = 0$ for $|x|$ large and f(x)=1 on the unit circle, which makes $\phi(\cdot, t)$ the identity for points far away from the origin. This will be convenient for later. Finally extend ψ by the function $g : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}$ with $g(x) = \psi(x/|x|)$
and define $w : \Omega^* \times I \to \mathbb{R}$ by

$$\begin{cases} \Delta\phi(\cdot,t)_* w = 0 & \text{in } \Omega_t^*, \\ \phi(\cdot,t)_* w - l\partial_{n_t}\phi(\cdot,t)_* w = g & \text{on } \Gamma_t^*, \\ \phi(\cdot,t)_* w = O(1) & \text{for } |x| \to \infty. \end{cases} \tag{10.2}$$

If we write $\dot{w} := \partial_t w(\cdot, t)|_{t=0}$, $\partial_r z(\zeta) := \nabla z(\zeta) \cdot \zeta/|\zeta|$ and set $\dot{w} = w' + h\partial_r w$,

we have the following lemma:

Lemma 10.1.1. *The above defined w' solves the following equations:*

$$\begin{cases} \Delta w' = 0 & \textit{in } \Omega^*, \\ w' - l\partial_n w' = -\partial_r w h + l(\partial_r\partial_n w h + \nabla w \cdot n'(u)[h]) & \textit{on } \Gamma^*, \\ w' = O(1) & \textit{for } |x| \to \infty. \end{cases} \tag{10.3}$$

Proof. By pulling the boundary condition back to our fixed domain Ω we have:

$$w - l\phi^*\partial_{n_{\Omega_t}}\phi_* w = w - l(n_{\Omega_t} \circ \phi) \cdot \nabla[(w \circ \phi^{-1})] \circ \phi = g \circ \phi \qquad \text{on } \partial\Omega. \tag{10.4}$$

Using differential rules for composition and inverse functions

$$\nabla[(w \circ \phi^{-1})](x) = D[\phi^{-1}]^T(x)\nabla[w](\phi^{-1}(x)) = D[\phi]^{-T}(\phi^{-1}(x))\nabla[w](\phi^{-1}(x)), \tag{10.5}$$

which gives us

$$w - l(n_{\Omega_t} \circ \phi) \cdot D\phi^{-T} \nabla w = g \circ \phi \qquad \text{on } \partial\Omega. \tag{10.6}$$

Differentiation with respect to t at $t = 0$ and noting that ϕ and thus ϕ^{-1} are identities at $t = 0$, this gives us

$$(\partial_t w - l(\partial_t n \cdot \nabla w + n \cdot \partial_t(D\phi^{-T}\nabla w)))|_{t=0} \tag{10.7}$$

$$= \partial_t w - l(\partial_t n \cdot \nabla w + n \cdot (\partial_t(D\phi^{-T})\nabla w + \phi^{-T}\partial_t \nabla w))|_{t=0} = \partial_v g, \tag{10.8}$$

where $v(\zeta) := (\phi(\zeta, t))^{\cdot}$.

If we make use of the identity $\dot{A}^{-1} = -A^{-1}\dot{A}A^{-1}$, the fact that we can interchange time and place derivatives and again notice that $D\phi$ is the identity matrix at $t = 0$.
We then get

$$\dot{w} - l(\dot{n} \cdot \nabla w + n \cdot \nabla \dot{w} - n \cdot (Dv)^T \nabla w) = \partial_v g \tag{10.9}$$

This used identity is easily proven by writing $0 = \frac{d}{dt}I = \frac{d}{dt}(AA^{-1}) = (\frac{d}{dt}A)A^{-1} + A(\frac{d}{dt}A^{-1})$, which gives $-(\frac{d}{dt}A)A^{-1} = A(\frac{d}{dt}A^{-1})$ and now multiply from the left by A^{-1} and evaluate at $t = 0$ to obtain the result.

We have

$$\begin{aligned} n \cdot \nabla \dot{w} - n \cdot (Dv)^T \nabla w &= n \cdot (\nabla(w' + \partial_v w) - (Dv)^T \nabla w)) \\ &= n \cdot \nabla w' + n \cdot (\nabla(\nabla w \cdot v) - (Dv)^T \nabla w) \\ &= n \cdot \nabla w' + n \cdot (\nabla(\partial_x w v_1 + \partial_y w v_2) - (Dv)^T \nabla w) \\ &= n \cdot \nabla w' + n_1(\partial_x^2 w v_1 + \partial_x v_1 \partial_x w + \partial_{y,x}^2 w v_2 + \partial_y w \partial_x v_2) \\ &+ n_2((\partial_{x,y}^2 w v_1 + \partial_x v_1 \partial_y w + \partial_y^2 w v_2 + \partial_y w \partial_y v_2) \\ &- n_1(\partial_x v_1 \partial_x w - \partial_x v_2 \partial_y w) - n_2(\partial_x v_2 \partial_x w - \partial_y v_2 \partial_y w))] \\ &= n \cdot \nabla w' + n_1 \partial_x^2 w v_1 + n_1 \partial_{x,y}^2 w v_2 \\ &+ n_2 \partial_{x,y}^2 w v_1 + n_2 \partial_y^2 w v_2. \end{aligned} \tag{10.10}$$

This is equivalent to

$$w' + \partial_v w - l\dot{n} \cdot \nabla w + n \cdot \nabla w' + n[\nabla^2 w]v = w' + \partial_v w - l\dot{n} \cdot \nabla w + n \cdot \nabla w' + \partial_v \partial_n w = \partial_v g, \tag{10.11}$$

here $\nabla^2 w$ denotes the Hessian matrix.
From the given diffeomorphism we can explicitly calculate the vector field v, $v(x) = \partial_t[x + th(x/|x|)x/|x|]|_{t=0} = h(x/|x|)x/|x|$ and thus we have $\partial_v = h\partial_r$ which implies that $\partial_v g = 0$ because g is constant in radial direction.
Taking this into accoun, we get

$$w' + \partial_r w h - l(\dot{n} \cdot \nabla w + \partial_n w' + \partial_r \partial_n w h) = 0. \tag{10.12}$$

This yields the boundary condition.
Fix $x_0 \in \Omega^*$ and let B be a open ball around x_0 such that $B \subset \Omega^*$ There exists a smooth function $\eta = \phi^{-1}; B \times J \to \Omega^*$ such that

$$\phi(\eta(x, t), t) = x. \tag{10.13}$$

Differentiating this equation with respect to t with the chain rule gives

$$D(\phi)^T\partial_t\eta + v(\eta(x,t)) = 0 \tag{10.14}$$

as noted before at $t = 0$, ϕ and η are the identity mappings.
So we obtain: $\dot{\eta}(x) = -\dot{\phi}(x) = -v(x)$. For $(x,t) \in B \times J$ define a function f by:

$$f(x,t) = w(\eta(x,t),t) = \phi_* w, \tag{10.15}$$

and from the defining equality for w we have

$$\Delta f = 0 \tag{10.16}$$

and after differentiation with respect to t

$$\nabla\dot{f} = \nabla(-\partial_v w + \dot{w}) = \nabla(-h\partial_r w + \dot{w}) = 0. \tag{10.17}$$

This yields the first equation.

Finally we show

$$w' = O(1) \qquad \text{for } |x| \to \infty \tag{10.18}$$

We constructed ϕ such that

$$\phi(x,t) = x, \qquad \text{for } |x| \text{ large,} \tag{10.19}$$

So we have for $|x| \geq R$, for a certain large enough $R > 0$.

$$\Delta w(\cdot,t) = 0 \tag{10.20}$$

and

$$\Delta w(\cdot,t) = O(1) \tag{10.21}$$

Now we apply the Kelvin transform $K : x \to \frac{x}{R}|x|^2$, and let $w^* = K_* w(\cdot,t)$ be the pull-back of w to the unit circle. We must have that w^* is also harmonic on the unit circle except in the origin, where it is undefined. However can be extended to a harmonic function on the whole unit disk. From the fact that we constructed a smooth family of diffeomorphisms with respect to t we know that the time derivative ∂_t exists everywhere except possibly in the origin. We thus want to show this derivative exists there and is bounded. For this we use an argument from complex analysis, we know that we can find a complex holomorphic function F in the unit circle such that w^* is the real part of $F(z,t)$. Because of the the fact that that F is holomorphic we can expand it into a taylor expansion:

$$F(z,t) = \sum_{n=0}^{\infty} a_n(t) z^n, \tag{10.22}$$

where a_n are given by the Cauchy-formula

$$a_n(t) = \frac{1}{2\pi i}\int \frac{F(\psi,t)}{(z)^{n+1}} dz, \tag{10.23}$$

around some circle enclosed in the unit circle.

We can also expand $\partial_t F(z,t)$ in a Laurent expansion,

$$\partial_t F(z,t) = \sum_{n=-\infty}^{\infty} b_n(t) z^n, \tag{10.24}$$

with

$$b_n(t) = \frac{1}{2\pi i} \int \frac{\partial_t F(z,t)}{z^{n+1}} dz, \tag{10.25}$$

We want to show that all b_n are zero for $n < 0$, so $\partial_t F(z,t)$ is indeed bounded around the origin. Because $\partial_t F(z,t)$ is continuous with respect to t over the path of integration, we can interchange integrals and time derivatives. We find that

$$\partial_t a_n = \int \frac{\partial_t F(\psi,t)}{z^{n+1}} dz = b_n. \tag{10.26}$$

But we already knew that all the a_n are zero for $n < 0$, so we conclude that (10.24) is a Taylor expansion as well. This completes the proof.

□

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