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A new approach to constrained state estimation for discrete-time linear systems with unknown inputs

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Summary
This paper addresses the problem of estimating the state for a class of uncertain discrete-time linear systems with constraints by using an optimization-based approach. The proposed scheme uses the moving horizon estimation philosophy together with the game theoretical approach to the $\mathcal{H}_\infty$ filtering to obtain a robust filter with constraint handling. The used approach is constructive since the proposed moving horizon estimator (MHE) results from an approximation of a type of full information estimator for uncertain discrete-time linear systems, named in short $\mathcal{H}_\infty$-MHE and $\mathcal{H}_\infty$–full information estimator, respectively. Sufficient conditions for the stability of the $\mathcal{H}_\infty$-MHE are discussed for a class of uncertain discrete-time linear systems with constraints. Finally, since the $\mathcal{H}_\infty$-MHE needs the solution of a complex minimax optimization problem at each sampling time, we propose an approximation to relax the optimization problem and hence to obtain a feasible numerical solution of the proposed filter. Simulation results show the effectiveness of the robust filter proposed.

KEYWORDS
constrained estimation, moving horizon estimation, optimization, robust estimation, uncertain linear systems

1 | INTRODUCTION

State estimation methods are used in a broad range of application areas including, but not restricted to, aerospace, robotics, communication, control, signal processing, system biology, and process engineering, see for instance, several studies,[1-5] where the well-known estimators like the Kalman filter and its linear and nonlinear variations are mainly used. However, when the model considered in the filtering process is uncertain, the central design premise of most classic and Kalman-based estimators is violated and hence the filter performance is compromised.

In this sense, robust estimation has attracted the interest of researchers to cope with different uncertainty sources and to limit the performance degradation of nonrobust observers and filters like the Luenberger observer, the Kalman filter, and many of their variations.[6,7] The problem of estimating the state of uncertain linear systems is well-known from the beginning of the 1980s.[7] Since then, countless publications have been provided from many authors by using different approaches. Regarding only the linear framework, the contributions have been focused to tackle two main sources of uncertainty, i.e., uncertainty on the parameters of the system[8,9] and uncertainty on the statistics of the disturbing noises.[7,8,10,11] Contributions addressing both uncertainty sources in the same statement of the problem are also available.[8,12,13] A literature review on state estimation for uncertain linear systems is found elsewhere.[14]

The moving horizon estimator (MHE) has shown to be an effective estimation scheme able to handle constraints even in the nonlinear framework.[15,16] Typically, the MHE solves the constrained estimation problem when parameter uncertainties are somehow negligible and the additive noises have known statistics as in the Kalman filter. The MHE rewrites the optimal estimation problem in an optimization-based procedure, allowing the natural addition of useful process insight in the form of constraints.[15-18] Some outstanding applications of MHE are found in previous studies[19-22] in topics ranging from actuator fault detection, observer design for linear-parameter-varying systems with uncertain measurements, large-scale nonlinear processes with delayed lab measurements, and model-based output feedback. The basic strategy of MHE, regarding only to the linear framework, reformulates the estimation problem as a quadratic program using a moving, fixed-size estimation window. The fixed-size window is needed to bound the computational burden derived of the solution of the optimization problem. This is the principal difference of MHE with the full information estimator, FIE in short.[15,17,18]

However, when models are uncertain, either because there exists uncertainty in the model parameters or in the statistics of the inputs, the classic MHE exhibits poor performance.[23,24] Main efforts to provide an estimation strategy for linear systems with uncertainty under a moving horizon approach are mainly due to Alessandri and coworkers.[25-30] In these contributions, the authors tackle the above problem through the formulation of a minimax optimization problem using the classic cost function provided in Rao et al.[18] A general issue in these contributions is the difficulty to provide a numerical solution to the stated problem even in an unconstrained scenario; the given solutions are the result of successive reformulations of the original statement under complicated and unrealistic assumptions. This is explained by the fact that typically minimax formulations are still complex to solve numerically.[29,31-33] An alternative approach was found in Sui and Johansen,[34] where the use of a pre-estimating linear observer in the forward prediction equations of the MHE cost function allowed to reduce the effects of uncertainty.

The present work provides an understandable and clear path to achieve a constrained estimation strategy for discrete-time linear systems with unknown inputs using a moving horizon approach. Our contribution differs from the ones presented by Alessandri et al because we consider the uncertainty in the statistics of the inputs instead of the uncertainty in the model parameters. Moreover, our formulation allows a direct addressing of constraints. Finally, we provide a way to circumvent the resulting minimax problem by means of the solution of two successive quadratic programs.

This work follows the ideas presented in Garcia-Tirado et al.[35] Robustness against unknown inputs is guaranteed by using the game theoretical approach of the $\mathcal{H}_\infty$ filtering.[10] For the sake of completeness, we present first a robust FIE in a $\mathcal{H}_\infty$-FIE, and then its moving horizon approximation, denoted in short as $\mathcal{H}_\infty$-MHE. As this approximation is made by using the concept of window shifting, the filter stability for both the $\mathcal{H}_\infty$-FIE and the $\mathcal{H}_\infty$-MHE is investigated following some results presented in Keerthi and Gilbert[36] and Rao.[15] Finally, a numerical approximation of the minimax optimization problem generated at each sampling time in the $\mathcal{H}_\infty$-MHE is provided to avoid the computational burden and complexity of such optimization problem. The filter with the numerical approximation is denoted as the $\mathcal{H}_\infty$-aMHE.

The paper is organized as follows. In Section 2, the problem of estimating the state of a constrained discrete-time linear system with unknown inputs using an optimization-based framework is stated. Then, the moving horizon approximation of the stated problem is provided in Section 3 along with a suitable definition of the *arrival cost*, the matrix recursion weighting the estimation error, and a suitable numerical approximation of the stated problem, denoted as $\mathcal{H}_\infty$-aMHE. A stability analysis for both the $\mathcal{H}_\infty$-FIE and the $\mathcal{H}_\infty$-MHE schemes is provided in Section 4. Then, a numerical example showing the benefits of the proposed scheme and its ability to handle constraints is given in Section 5. Finally, conclusions and future work are given in Section 6.

2 PROBLEM STATEMENT

Consider the uncertain discrete-time linear system

$$\begin{aligned} x_{k+1} &= Ax_k + Gw_k, \\ y_k &= Cx_k + v_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^p$ is the system output, and $w_k \in \mathbb{R}^m$ and $v_k \in \mathbb{R}^p$ are the model and measurement uncertainties, respectively, and A, G, and C are time-invariant matrices of corresponding order. w_k and v_k are assumed to be unknown. It is also known that the states and disturbances satisfy the following constraints:

$$x_k \in \mathbb{X}, w_k \in \mathbb{W}, \text{and} v_k \in \mathbb{V}, \tag{2}$$

where the sets $\mathbb{X}$, $\mathbb{W}$, and $\mathbb{V}$ are assumed to be polyhedral and convex.

Consider the constrained estimation problem for (1). The estimate of x_t given the measurement sequence $\{y_0, \cdots, y_{t-1}\}$, denoted as $\hat{x}_{t|t-1}$, is obtained by the solution of the state equation (1) if the a priori estimates of the initial state $x_{0|t-1}$ and the disturbance sequence $\{w_0, \cdots, w_{t-1}\}$ are known, ie, $\hat{x}_{0|t-1}$ and $\{\hat{w}_0, \cdots, \hat{w}_{t-1}\}$, respectively,

$$\hat{x}_{t|t-1}\left(t; \hat{x}_{0|t-1}, \left\{w_j\right\}_{j=0}^{t-1}\right) = A^t \hat{x}_{0|t-1} + \sum_{j=0}^{t-1} A^{t-j-1} G \hat{w}_j. \tag{3}$$

The constrained estimation problem for the uncertain discrete-time linear system (1) can be formulated as the solution of the following minimax problem:

$$\begin{gathered} \bar{\psi}_t^* = \min_{x_0} \max_{\{w_k\},\{v_k\}} \bar{\psi}_t\left(x_0, \{w_k\}, \{v_k\}\right) \\ \text{s.t} x_k \in \mathbb{X}, w_k \in \mathbb{W}, \text{ and } v_k \in \mathbb{V} \end{gathered} \tag{4}$$

with $\bar{\psi}_t$ the objective function defined by

$$\bar{\psi}_t\left(x_0, \{w_k\}, \{v_k\}\right) = \frac{1}{2}\|x_0 - \bar{x}_0\|^2_{\Pi_0^{-1}} - \frac{1}{2\gamma}\sum_{k=0}^{t-1}\left(\|w_k\|^2_{Q^{-1}} + \|v_k\|^2_{R^{-1}}\right), \tag{5}$$

where $\bar{x}_0$ is the a priori guess of the initial state, $\{w_k\} = \{w_k\}_{k=0}^{t-1}$, $\{v_k\} = \{v_k\}_{k=0}^{t-1}$, $\gamma > 0$ the performance bound, $\|y\|_S^2 = y^T S y$, and Π_0^{-1}, Q^{-1}, and R^{-1} are assumed to be symmetric positive definite matrices. Π_0^{-1} and $\bar{x}_0$ give the a priori information of the filter at time $t = 0$.

Since v_k is a function of the measurements and the system state, (5) can be finally rewritten as

$$\bar{\psi}_t\left(x_0, \{w_k\}\right) = \frac{1}{2}\|x_0 - \bar{x}_0\|^2_{\Pi_0^{-1}} - \frac{1}{2\gamma}\sum_{k=0}^{t-1}\left(\|w_k\|^2_{Q^{-1}} + \|y_k - Cx_k\|^2_{R^{-1}}\right). \tag{6}$$

This problem is referred as $\mathcal{H}_\infty$-FIE, because all the available measurements are considered from $k = 0$ to $k = t - 1$. Some classical papers had used the term *full information* for control systems where an enlarged output is made up from the state and disturbance.[37] However, in this paper, the term *full information* is used in the sense of estimation as used in Rao et al[15,18] and Jazwinski.[38] An analytic solution to the $\mathcal{H}_\infty$-FIE can be obtained in absence of constraints.[39]

Remark 1. The cost function 5 comes from the definition of the following disturbance attenuation function [7,10]

$$\psi_t\left(x_0, \{w_k\}, \{v_k\}\right) = \frac{\|x_0 - \bar{x}_0\|^2_{\Pi_0^{-1}}}{\sum_{k=0}^{t-1}\left(\|w_k\|^2_{Q^{-1}} + \|v_k\|^2_{R^{-1}}\right)}, \tag{7}$$

which is set to obtain an optimization-based estimation scheme in a $\mathcal{H}_\infty$ setting. Since the direct minimization of the above disturbance attenuation function is not tractable, a common practice is to force (7) to fulfil a performance bound; that is, the estimation strategy will try to find the best initial state under the worst-case disturbances such that

$$\psi_t\left(x_0, \{w_k\}, \{v_k\}\right) \le \frac{1}{\gamma}. \tag{8}$$

Remark 2. An objective function in the form of a disturbance attenuation function provides the elements to formulate the estimation problem in a worst-case framework.[7,10] This type of costs functions is common in the $\mathcal{H}_\infty$ filtering.[7,10,40,41]

3 | THE H∞-MHE

In the previous section, the constrained estimation problem for discrete-time linear systems with unknown inputs was stated. Although there exist strategies to solve (4), the problem size grows with time as the estimator processes more data. To make the estimation problem feasible and computationally tractable, the size of the problem needs to be bounded.

In this section, the $\mathcal{H}_\infty$-MHE is presented as an approximation to the $\mathcal{H}_\infty$-FIE. This moving horizon approximation helps to rewrite the estimation problem presented in the last section as a fixed dimension min-max optimization problem. To this end, the estimation problem is split in a moving estimation window plus a term summarizing the effect of old data. One way to summarize the old data is by means of the concept of arrival cost, which is discussed later for the problem of interest.

A moving horizon approximation of the $\mathcal{H}_\infty$-FIE will allow to solve the estimation problem for linear systems with uncertain inputs even in the constrained case. This is the main advantage of this approach with respect to other contributions. Moreover, full information approaches make sense in a theoretical frame since these are not feasible to be implemented in practice. Finally, despite of the need to solve a complex min-max problem at each sampling time, we propose a way to circumvent the computational burden derived of the numerical solution of such optimization problems.

3.1 | The constrained $\mathcal{H}_\infty$-MHE

Consider a system that is dynamically described by (1) subject to (2), where w_k and v_k are unknown. Given the measurement sequence $\{y_{t-N}, \cdots, y_{t-1}\}$, the estimate of x_t, denoted as $\hat{x}_{t|t-1}$, is computed by means of (3) but modified as

$$\hat{x}_{t|t-1} = A^{t-N}\hat{x}_{t-N|t-1} + \sum_{j=t-N}^{t-1} A^{t-j-1} G\hat{w}_{j|t-1}, \tag{9}$$

where $\hat{x}_{t-N|t-1} = x^*_{t-N}$ and $\{\hat{w}_{k|t-1}\}_{k=t-N}^{t-1} = \{w^*_{k|t-1}\}_{k=t-N}^{t-1}$ are obtained from the solution of the following minimax problem:

$$\begin{gathered}\hat{\psi}_t^* = \min_{x_{t-N}} \max_{\{w_k\}} \hat{\psi}_t\left(x_{t-N}, \{w_k\}\right) \\ \text{s.t.} x_k \in \mathbb{X}, w_k \in \mathbb{W}, \text{and} v_k \in \mathbb{V},\end{gathered} \tag{10}$$

where $\hat{\psi}_t$ is

$$\begin{aligned}\hat{\psi}_t\left(x_{t-N}, \{w_k\}\right) = {} & \frac{1}{2}\|x_{t-N} - \bar{x}_{t-N}\|^2_{\Pi^{-1}_{t-N}} \\ & - \frac{1}{2\gamma}\sum_{k=t-N}^{t-1}\left(\|w_k\|^2_{Q^{-1}} + \|y_k - C_k x_k\|^2_{R^{-1}}\right) + \frac{1}{2}\hat{\psi}^*_{t-N}.\end{aligned} \tag{11}$$

From the last formulation, there are two issues that need to be discussed further, the definition of the arrival cost and the weighting matrix Π_k. These will be explained below.

3.2 | The arrival cost in the $\mathcal{H}_\infty$-MHE and the weighting matrix Π_k

The arrival cost is an important concept in estimation theory since it helps to approximate the effect of the old data on the state at the beginning of the estimation window.[15] The arrival cost is an analogue concept of the *cost to go*, which is widely used in most model predictive control formulations. In the case of the $\mathcal{H}_\infty$-MHE, the classic arrival cost must be redefined to fit in the robust statement of the estimation problem. Consider again the $\mathcal{H}_\infty$-FIE. The associated objective function (6) can be rearranged by dividing it into two parts

$$\bar{\psi}_t\left(x_0, \{w_k\}_{k=0}^{t-1}\right) = \bar{\psi}_{t-N}\left(x_0, \{w_k\}_{k=0}^{t-N-1}\right) - \frac{1}{2\gamma}\sum_{k=t-N}^{t-1}\left(\|w_k\|^2_{Q^{-1}} + \|y_k - Cx_k\|^2_{R^{-1}}\right), \tag{12}$$

where N is the so-called estimation horizon. This parameter helps the estimation procedure to bound the size of the optimization problem to be solved at each sample time. Note that the first term in the right-hand side of (12) is the cost from $k = 0$ to $k = t - N - 1$ and the second term acts from $k = t - N$ to $k = t - 1$. The cost $\bar{\psi}_{t-N}$ must be approximated by the arrival cost to avoid the use of the complete set of data. Therefore, the redefinition of the optimization problem (4) with cost (6) using the arrival cost becomes

$$\psi_t^* = \min_{x_{t-N}} \max_{\{w_k\}_{k=t-N}^{t-1}} \Theta_{t-N}(x_{t-N}) - \frac{1}{2\gamma}\sum_{k=t-N}^{t-1}\left(\|w_k\|^2_{Q^{-1}} + \|y_k - Cx_k\|^2_{R^{-1}}\right), \tag{13}$$

where for $z \in \mathcal{R}_t$, $\Theta_t(z)$ is the arrival cost defined as

$$\Theta_t(z) := \min_{x_0} \max_{\{w_k\}_{k=0}^{t-1}} \left\{ \bar{\psi}_t \left(x_0, \{w_k\}_{k=0}^{t-1}\right) : x\left(t; x_0, \{w_k\}_{k=0}^{t-1}\right) = z \right\} \tag{14}$$

with $\mathcal{R}_t$ the reachable set of states as defined in Rao[15]

$$\mathcal{R}_t = \begin{cases} x_0 \in \mathbb{X}, \\ x\left(t; x_0, \{w_k\}_{k=0}^{t-1}\right) = x(k; x_0, \{w_k\}_{k=0}^{t-1}) \in \mathbb{X} \forall k = 0, \cdots, t \\ w_k \in \mathbb{W}, \forall k = 0, \cdots, t-1, \end{cases} \tag{15}$$

where the optimization is also subject to the constraints in (10). The indexes on w_k are recovered, for the sake of clarity. Following a similar reasoning as in Rao,[15] we approximate (14) as

$$\Theta_t(z) = (z - \bar{x}_t)^T \Pi_t^{-1} (z - \bar{x}_t) + \bar{\psi}_t^*, \tag{16}$$

where Π_k with $k = 0, \ldots, t$ is a matrix weighting the confidence on the a priori estimation of the state at time k. This approximation also follows in this formulation since optimizing the arrival cost will give back $\bar{x}_{t|t-1}$, our previous best estimate. The above recursion matrix is important for two reasons. First, it provides a quantification of how good is the estimate.[38] It also helps to prove convergence and stability of the filter. Moreover, as this weighting matrix is part of the arrival cost, then it helps to summarize the old data to obtain the moving horizon approximation.

At time t, the arrival cost can be used to rewrite (13) as

$$\begin{aligned} \hat{\psi}_t^*(x_{t-N}, \{w_k\}) := \min_{x_{t-N}} \max_{\{w_k\}_{k=t-N}^{t-1}} & -\frac{1}{2\gamma} \sum_{k=t-N}^{t-1} \left(\|w_k\|_{Q^{-1}}^2 + \|y_k - Cx_k\|_{R^{-1}}^2 \right) \\ & + \frac{1}{2}(x_{t-N} - \bar{x}_{t-N})^T \Pi_{t-N}^{-1} (x_{t-N} - \bar{x}_{t-N}) \\ & + \frac{1}{2}\bar{\psi}_{t-N}^*, \end{aligned} \tag{17}$$

where $\hat{\psi}_t^*$ is used instead of $\bar{\psi}_t^*$ to make a difference between the costs used in the $\mathcal{H}_\infty$-MHE and $\mathcal{H}_\infty$-FIE, respectively. Now, $\bar{x}_{t-N}$ turns out to be the moving horizon estimate of the state at time $t - N$ instead of an a priori guess. Therefore, the pair $(\bar{x}_{t-N}, \Pi_{t-N})$ summarizes the prior information at time $t - N$. For $t \le N$, the $\mathcal{H}_\infty$-MHE is equivalent to the $\mathcal{H}_\infty$-FIE.

Recursive computation of the estimation error weighting Π_k.

The following lemma gives a recursion for Π_k.

Lemma 1. *Let $\gamma > 0$ be a prescribed level of noise attenuation. The following recursion describes the dynamic evolution of the error weighting matrix of the $\mathcal{H}_\infty$-MHE:*

$$\Pi_{k+1} = A_k \Pi_k \left[I + \frac{1}{\gamma} C_k^T R_k^{-1} C_k \Pi_k \right]^{-1} A_k^T + \gamma G_k Q_k G_k^T, \tag{18}$$

where $k = t - N, \ldots, t - 1$ and Π_0, R_k, and Q_k are user-defined positive definite matrices.

□

Proof. the proof is in Garcia-Tirado et al.[35]

3.3 | Approximate numerical solution to the constrained $\mathcal{H}_\infty$-MHE problem

Constrained minimax optimization problems are, in general, hard to solve numerically.[33] To circumvent the computational burden derived from the above problem for the posed estimation strategy, a numerical approximation is provided. The filter derived from this approximation is referred in short as the $\mathcal{H}_\infty$-aMHE. With the approximation, the range of applications to which the $\mathcal{H}_\infty$-MHE can be applied is considerably broadened.

Consider the cost function (11) in matrix form as

$$\begin{aligned} \hat{\psi}_t (x_{t-N}, \{w_k\}) = & \frac{1}{2}(x_{t-N} - \bar{x}_{t-N})^T \Pi_{t-N}^{-1} (x_{t-N} - \bar{x}_{t-N}) \\ & - \frac{1}{2\gamma} \left[\tilde{w}^T \bar{Q} \tilde{w} + (Y - \Gamma x_{t-N} - \Xi \tilde{w})^T \bar{R} (Y - \Gamma x_{t-N} - \Xi \tilde{w}) \right] \\ & + \frac{1}{2}\hat{\psi}_{t-N}^*, \end{aligned} \tag{19}$$

where $\tilde{w} = \{w_k\}_{k=t-N}^{t-1}$, $\bar{x}_{t-N}$ is an a priori guess of the state provided by the $\mathcal{H}_\infty$-aMHE estimate of the state at time $t - N$, and $\bar{Q}$ and $\bar{R}$ are block diagonal matrices defined as

$$\bar{Q} = \oplus_{j=t-N}^{t-1} Q_j^{-1} = \begin{bmatrix} Q^{-1} & & & \\ & Q^{-1} & & \\ & & \ddots & \\ & & & Q^{-1} \end{bmatrix}, \bar{R} = \oplus_{j=t-N}^{t-1} R_j^{-1} = \begin{bmatrix} R^{-1} & & & \\ & R^{-1} & & \\ & & \ddots & \\ & & & R^{-1} \end{bmatrix}.$$

Making the products, ordering, and neglecting terms not involved in the optimization problem, (19) is rewritten as follows:

$$\begin{aligned} \bar{\psi}_t (x_{t-N}, \{w_k\}) = & \frac{1}{2} x_{t-N}^T \left(\Pi_{t-N}^{-1} - \gamma^{-1} \Gamma^T \bar{R} \Gamma\right) x_{t-N} - \frac{1}{2\gamma} \tilde{w}^T \left(\Xi^T \bar{R} \Xi + \bar{Q}\right) \tilde{w} \\ & + \frac{1}{\gamma} \left[\left(Y^T \bar{R} \Gamma - \gamma \bar{x}_{t-N}^T \Pi_{t-N}^{-1}\right) x_{t-N} + Y^T \bar{R} \Xi \tilde{w} - \tilde{w}^T \Xi^T \bar{R} \Gamma x_{t-N}\right]. \end{aligned} \tag{20}$$

It is worth to note that the two first terms of the previous equation are quadratic with respect to the optimization parameters. The pure linear terms associated with these parameters are also given. However, there is only one term involving both the estimate of the state at the beginning of the window and the disturbance sequence, ie, $\tilde{w}^T \Xi^T \bar{R} \Gamma x_{t-N}$. Therefore, the minimax optimization is not separable, and hence the application of an specialized algorithm is needed. To avoid the application of such a complex minimax optimization algorithm, we provide an approximation of the mixed term $\tilde{w}^T \Xi^T \bar{R} \Gamma x_{t-N}$ from a reasoning supported by the following remark:

Remark 3. At time t, the estimate of the state at the beginning of the time window $\hat{x}_{t-N|t-1}$ is sought together with the disturbance sequence $\{w_k\}_{k=t-N}^{t-1}$ from the solution of the minimax problem (10). However, from the past data, there exist a $\mathcal{H}_\infty$-aMHE estimate of x_{t-N}, which was found by using the measurement sequence $\{y_{t-2N}, \cdots, y_{t-N-1}\}$, ie, $\hat{x}_{t-N|t-N-1}$. Therefore, $\hat{x}_{t-N|t-N-1}$ is a good guess for $\hat{x}_{t-N|t-1}$. Then, the term $\tilde{w}^T \Xi^T \bar{R} \Gamma \hat{x}_{t-N|t-N-1}$ becomes a good approximation of $\tilde{w}^T \Xi^T \bar{R} \Gamma \hat{x}_{t-N|t-1}$.

By using the last reasoning, the minimax optimization becomes separable and allows the computation of the disturbance sequence maximizing the cost function. Then, the disturbance sequence is computed as follows:

$$\begin{aligned} \tilde{w}^* = & \arg \max_{\{w\}} -\frac{1}{2\gamma} \tilde{w}^T \left(\Xi^T \bar{R} \Xi + \bar{Q}\right) \tilde{w} + \frac{1}{\gamma} \left(Y^T \bar{R} \Xi - \bar{x}_{t-N}^T \Gamma^T \bar{R} \Xi\right) \tilde{w} \\ & \text{s.t} w_k \in \mathbb{W}, \end{aligned} \tag{21}$$

which can be solved as a constrained quadratic program. Once the disturbance sequence is found, we no longer need to approximate $\tilde{w}^T \Xi^T \bar{R} \Gamma \hat{x}_{t-N|t-N-1}$ with $\tilde{w}^T \Xi^T \bar{R} \Gamma \hat{x}_{t-N|t-1}$, and the problem is set as it was posed originally. Therefore, the initial condition minimizing the original problem is computed as

$$\begin{aligned} x_{t-N}^* = & \arg \min_{x_{t-N}} \frac{1}{2} x_{t-N}^T \tilde{M} x_{t-N} + \frac{1}{\gamma} \left[Y^T \bar{R} \Gamma - \gamma \bar{x}_{t-N}^T \Pi_{t-N}^{-1} - (\tilde{w}^*)^T \Xi^T \bar{R} \Gamma\right] x_{t-N} \\ & \text{s.t. } x_k \in \mathbb{X} \text{ and } v_k \in \mathbb{V}, \text{ with } \tilde{M} = \Pi_{t-N}^{-1} - \gamma^{-1} \Gamma^T \bar{R} \Gamma. \end{aligned} \tag{22}$$

4 | STABILITY ANALYSIS OF THE H∞-MHE

As it was stated before, the stability of the $\mathcal{H}_\infty$-MHE must be investigated since, at every sampling time, only a subset of the entire data is used due to the approximation. The main motivation of the moving horizon setting is the possibility of including constraints on the estimation problem to improve the quality of the estimates. As Rao and coworkers pointed out,[18] the implications of constraints for the estimator are more subtle than for the regulator, i.e., the model predictive controller. The difference is that the estimator has no control over the behavior of the state of the system. A poor choice of constraints may prevent the convergence to the true state of the system.[18] Examples of how a poor choice of constraints may prevent the convergence of the classic MHE are given in Rao.[15]

The discussion about the stability of the filter is based on the work on stability of optimization-based systems of Keerthi and Gilbert,[36] which was substantially adapted by Rawlings and coworkers to guarantee convergence and stability of the classic MHE.[18] Our contribution follows a similar derivation. To present the stability results, some definitions about zero-sum dynamic games and some auxiliary results are needed. In the following stability proof, Theorems 1-3 were taken from Başar and Bernhard[42] and de Souza et al,[43] respectively, and Lemmas 4 and 6 from Rao.[15]

Consider the cost function (6) defined over $\mathbb{X} \times \mathbb{W}$. Equation (6), $x_0 \in \mathbb{X}$, and $w_0 \in \mathbb{W}$ define a static zero-sum game to estimate x_1 in a $\mathcal{H}_\infty$-FIE setting.[42] If there exists a pair $\left(x_0^* \in \mathbb{X}, w_0^* \in \mathbb{W}\right)$ such that

$$\min_{x_0 \in \mathbb{X}} \bar{\psi}(x_0, w_0^*) = \max_{w_0 \in \mathbb{W}} \bar{\psi}(x_0^*, w_0) = \bar{\psi}(x_0^*, w_0^*) = \bar{\psi}^*, \tag{23}$$

then the pair $\left(x_0^*, w_0^*\right)$ is called a (pure-strategy) saddle-point solution. The saddle-point solution will also satisfy the following inequality:

$$\bar{\psi}(x_0^*, w_0) \leq \bar{\psi}(x_0^*, w_0^*) \leq \bar{\psi}(x_0, w_0^*), \forall x_0, w_0 \in \mathbb{X} \times \mathbb{W}. \tag{24}$$

The following theorems from Başar and Bernhard[42] guarantee the existence of a saddle-point solution of the $\mathcal{H}_\infty$-FIE to estimate x_1 from x_0 .

Theorem 1. *Let $\mathbb{X}$, $\mathbb{W}$ be compact, and $\bar{\psi}$ be continuous in the pair (x_0, w_0). Then, there exists a saddle-point solution in mixed policies.*

Proof. See Başar and Bernhard.[42] □

Theorem 2. *In addition to the hypothesis of Theorem 1 above, let $\mathbb{X}$ and $\mathbb{W}$ be convex, $\bar{\psi}$ is convex in $x_0 \in \mathbb{X}$ for every $w_0 \in \mathbb{W}$ and concave for $w_0 \in \mathbb{W}$ for every $x_0 \in \mathbb{X}$. Then, there exists a saddle-point in pure policies. If, furthermore, $\bar{\psi}$ is strictly convex-concave, the saddle-point solution is unique.*

Proof. See Başar and Bernhard.[42] □

Remark 4. By definition, the concept of mixed strategy covers pure strategy.[42]

As the time goes forward, every zero-sum dynamic game needs to be solved at each sample time. Unlike the zero-sum static games, the zero-sum dynamic games are subject to the system dynamics. We refer to x_0 and $\{w_k\}_{k=0}^{t-1}$ as the policies minimizing and maximizing $\bar{\psi}_t$, respectively. Let $X = \mathbb{X}$ and W be the spaces of the policies x_0 and $\{w_k\}_{k=0}^{t-1}$, respectively, where W is the cartesian product of $t-1$ times $\mathbb{W}$. A pair of policies $\left(x_0^*, \{w_k\}^*\right)$ constitutes a saddle-point solution for the game $\{\bar{\psi}; X, W\}$ if for all $\left(x_0, \{w_k\}_{k=0}^{t-1}\right) \in X \times W$

$$\bar{\psi}_t\left(x_0^*, \{w_k\}_{k=0}^{t-1}\right) \leq \bar{\psi}_t * \left(x_0^*, \{w_k^*\}_{k=0}^{t-1}\right) \leq \bar{\psi}_t\left(x_0, \{w_k^*\}_{k=0}^{t-1}\right), \forall x_0, \{w_k\}_{k=0}^{t-1}, \tag{25}$$

where $\bar{\psi}^*$ denotes the value of the game.[42]

Now, the monotonicity of the saddle-point value of the $\mathcal{H}_\infty$-FIE is discussed by using the following result.

Lemma 2. *Consider Γ_t, Ξ_t, $\bar{Q}_t$, and $\bar{R}_t$ defined as*

$$\Gamma_t = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{t-1} \end{bmatrix}, \Xi_t = \begin{bmatrix} CG & 0 & \cdots & 0 \\ CAG & CG & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ CA^{t-1}G & CA^{t-2}G & & CG \end{bmatrix}, \tag{26}$$

$$\bar{Q}_t = \bigoplus_{j=1}^{t} Q_j^{-1}, \bar{R}_t = \bigoplus_{j=1}^{t} R_j^{-1}, \tag{27}$$

where $\bigoplus_{j=1}^{t} D = diag(D, \ldots, D)$ with D repeated t times.

Assume $\Pi_0, \bar{Q}_t$, and $\bar{R}_t$ are definite positive. Omitting the time dependence of the above matrices, $\bar{\psi}_t(x_0, \{w_k\})$ in (6) is strictly convex in x_0 and strictly concave in $\{w_k\}_{k=0}^{t-1}$ iff

$$\Pi_0^{-1} - \gamma^{-1}\left(\Gamma^T \bar{R} \Gamma\right) \succ 0, \tag{28a}$$

$$-\gamma^{-1}\left(\Xi^T \bar{R} \Xi + \bar{Q}\right) \prec 0. \tag{28b}$$

Proof. Inequalities (28a) and (28b) are easily obtained by differentiating twice (6) with respect to x_0 and $\{w_k\}$, respectively, and then making these second-order derivatives greater than and lower than zero, respectively. □

From Theorem 2 and Lemma 2, it follows that the $\mathcal{H}_\infty$-FIE admits a unique (pure-strategy) saddle-point solution at each sampling time.

Now, let us switch to the $\mathcal{H}_\infty$-MHE. To show the saddle-point feasibility of the solution for the $\mathcal{H}_\infty$-MHE, let the cost function (11) be considered

$$\hat{\psi}_t\left(x_{t-N},\{w_k\}\right) = \frac{1}{2}\|x_{t-N}-\bar{x}_{t-N}\|^2_{\Pi^{-1}_{t-N}} - \frac{1}{2\gamma}\sum_{k=t-N}^{t-1}\left(\|w_k\|^2_{Q^{-1}}+\|y_k-Cx_k\|^2_{R^{-1}}\right)+\hat{\psi}^*_{t-N}.$$

The following lemma provides a necessary and sufficient condition for (11) to be strictly concave in $\{w_k\}_{k=t-N}^{t-1}$ and strictly convex in x_{t-N}.

Lemma 3. *Consider $\Gamma_N, \Xi_N, \bar{Q}_N$, and $\bar{R}_N$ defined in the moving horizon approximation of $\mathcal{H}_\infty$-FIE as*

$$\Gamma_N = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^{N-1}\end{bmatrix}, \Xi_N = \begin{bmatrix} CG & 0 & \cdots & 0 \\ CAG & CG & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ CA^{N-1}G & CA^{N-2}G & & CG \end{bmatrix}, \tag{29}$$

$$\bar{Q}_N = \bigoplus_{j=1}^{N} Q_j^{-1}, \bar{R}_N = \bigoplus_{j=1}^{N} R_j^{-1}. \tag{30}$$

Assume $\Pi_0, \bar{Q}_N$, and $\bar{R}_N$ are positive definite. For the quadratic two-person zero-sum dynamic game described by (1) and (10), the functional $\hat{\psi}_t$ in (11) is strictly concave in $\{w_k\}_{k=t-N}^{t-1}$ for all $x_{t-N} \in X$ and is strictly convex in x_{t-N}, for all $\{w_k\}_{k=t-N}^{t-1} \in W$ iff

$$\Pi_k^{-1} - \gamma^{-1}\left(\Gamma_N^T\bar{R}_N\Gamma_N\right) \succ 0, \tag{31}$$

where Π_k is given by (18) in Lemma 1:

$$\Pi_{k+1} = A_k\Pi_k\left[I+\frac{1}{\gamma}C_k^TR_k^{-1}C_k\Pi_k\right]^{-1}A_k^T+\gamma G_kQ_kG_k^T.$$

Proof. For the sake of simplicity, the indexes of Γ_N and Ξ_N are omitted. Concavity of $\hat{\psi}_t$ is guaranteed by (28b)

$$\gamma^{-1}(\Xi^T\bar{R}\Xi+\bar{Q}) \succ 0$$

provided $\bar{R}$ and $\bar{Q}$ positive definite. Since $\bar{\psi}_t$ is a quadratic functional of x_0, the requirement of strict convexity is equivalent to the existence of a unique solution to the optimal control problem

$$\min_{x_{t-N}\in X} \hat{\psi}_t(x_{t-N},\{w_k^*\}) \tag{32}$$

subject to the dynamics and for each sequence $\{w_k\}_{k=t-N}^{t-1} \in W$. Furthermore, since the Hessian matrix of $\hat{\psi}_t$ with respect to x_{t-N} is independent of $\{w_k\}_{k=t-N}^{t-1}$, the positive definiteness of (31) guarantees what is claimed. □

We showed that both the $\mathcal{H}_\infty$-FIE and $\mathcal{H}_\infty$-MHE problems guarantee saddle-point solutions at each sample time if conditions (28a), (28b) and (31), (28b) are fulfiled, respectively. Then, a sequence of saddle-points is expected as time grows up.

In the case under study, as constraints are dealt with, we use the modified Lyapunov stability theory as presented in Keerthi and Gilbert[36] and further adapted by Rawlings and coworkers.[18] The following definition gives the guidelines to state the stability of both the $\mathcal{H}_\infty$-FIE and the $\mathcal{H}_\infty$-MHE.[15]

Definition 1. The estimator is an *asymptotically stable observer* for the system

$$x_{k+1} = Ax_k, y_k = Cx_k \tag{33}$$

if for any ϵ there correspond a number $\delta > 0$ and a positive integer $\bar{t}$ such that $\|x_0-\bar{x}_0\| \le \delta$ and $\bar{x}_0 \in \mathbb{X}$, then $\|\bar{x}_t - A^tx_0\| \le \epsilon$ for all $t \ge \bar{t}$ and $\bar{x}_t \to A^tx_0$ as $t \to \infty$.

The following assumption gives a way to handle the constraints.[18]

Assumption 1. Suppose the system (33) with initial condition x_0 generates the data $y_k = CA^k x_0$. It is assumed the existence of $x_{0|\infty}$, $\{w_k\}_{k=0}^{\infty}$, and $\rho > 0$ such that

$$\begin{aligned}&\left(x_{0|\infty} - \bar{x}_0\right)^T \Pi_0^{-1} \left(x_{0|\infty} - \bar{x}_0\right) - \frac{1}{\gamma}\left(\sum_{k=0}^{\infty} w_{k|\infty}^T Q^{-1} w_{k|\infty} + v_{k|\infty}^T R^{-1} v_{k|\infty}\right),\\ &\geq -\rho \|x_0 - \bar{x}_0\|^2\end{aligned}$$

where $x_{k|\infty} \in \mathbb{X}$, $w_{k|\infty} \in \mathbb{W}$, $v_{k|\infty} \in \mathbb{V}$, $x_{k|\infty} := x(k, x_{0|\infty}, \{w_{j|\infty}\})$, and $v_{k|\infty} := y_k - Cx(k, x_{0|\infty}, \{w_{j|\infty}\})$.

Assumption 1 states the existence of a feasible state and disturbance sequence with bounded cost, if an infinite data sequence is considered. To establish asymptotic stability for both the $\mathcal{H}_\infty$-FIE and the $\mathcal{H}_\infty$-MHE, we require the following technical lemma.

Lemma 4. *Suppose (C, A) is observable and $N \geq n$. If*

$$\sum_{k=t-N}^{t-1} \|w_{k|t-1}\|_{Q^{-1}}^2 + \|v_{k|t-1}\|_{Q^{-1}}^2 \to 0,$$

then $\|\hat{x}_t - x_t\| \to 0$

Proof. The proof is available in Rao.[15] □

Now, the first result is presented, ie, the stability of the $\mathcal{H}_\infty$-FIE.

Proposition 1. *Assume Q, R, and Π_0 are positive definite, (C, A) is observable, and Assumption 1 holds. Then, the $\mathcal{H}_\infty$-FIE is an asymptotically stable observer for the system (33).*

Proof. Assume throughout the proof $t > n$, where n is the order of the system. Convergence of the cost sequence $\{\bar{\psi}_t^*\}$ is demonstrated first. The existence of the solution to the $\mathcal{H}_\infty$-FIE problem was already shown. Moreover, Assumption 1 guarantees a feasible trajectory given an infinite set of data. By optimality, $\bar{\psi}_k^* \geq -\rho\|x_0 - \bar{x}_0\|^2$ for all k, $\rho > 0$. Now, consider the cost function at times t and $t+1$

$$\begin{aligned}\bar{\psi}_t^* &= \|\hat{x}_{0|t-1} - \bar{x}_0\|_{\Pi_0^{-1}}^2 - \frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t-1}\|_{Q^{-1}}^2 + \|\hat{v}_{k|t-1}\|_{R^{-1}}^2\right)\\ \bar{\psi}_{t+1}^* &= \|\hat{x}_{0|t} - \bar{x}_0\|_{\Pi_0^{-1}}^2 - \frac{1}{\gamma}\left(\sum_{k=0}^{t} \|\hat{w}_{k|t}\|_{Q^{-1}}^2 + \|\hat{v}_{k|t}\|_{R^{-1}}^2\right)\\ &= \|\hat{x}_{0|t} - \bar{x}_0\|_{\Pi_0^{-1}}^2 - \frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t}\|_{Q^{-1}}^2 + \|\hat{v}_{k|t}\|_{R^{-1}}^2\right)\\ &\quad - \frac{1}{\gamma}\left(\|\hat{w}_{t|t}\|_{Q^{-1}}^2 + \|\hat{v}_{t|t}\|_{R^{-1}}^2\right),\end{aligned}$$

where $\hat{v}_{k|t} := y_k - C\hat{x}_{k|t}$ and $\hat{x}_{k|t} = x(k; \hat{x}_{0|t}, \{\hat{w}_{k|t}\})$. Let the difference between the optimal values at times $t+1$ and t be written as

$$\begin{aligned}\bar{\psi}_{t+1}^* - \bar{\psi}_t^* &= \|\hat{x}_{0|t} - \bar{x}_0\|_{\Pi_0^{-1}}^2 - \frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t}\|_{Q^{-1}}^2 + \|\hat{v}_{k|t}\|_{R^{-1}}^2\right)\\ &\quad - \frac{1}{\gamma}\left(\|\hat{w}_{t|t}\|_{Q^{-1}}^2 + \|\hat{v}_{t|t}\|_{R^{-1}}^2\right)\\ &\quad - \left[\|\hat{x}_{0|t-1} - \bar{x}_0\|_{\Pi_0^{-1}}^2 - \frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t-1}\|_{Q^{-1}}^2 + \|\hat{v}_{k|t-1}\|_{R^{-1}}^2\right)\right].\end{aligned} \tag{34}$$

Note that

$$\left(\hat{x}_{0|t}, \{\hat{w}_{k|t-1}\}_{k=0}^{t-1}\right) \quad \text{and} \quad \left(\hat{x}_{0|t}, \{\hat{w}_{0|t-1}, \cdots, \hat{w}_{t-1|t-1}, w_{t|t}\}\right)$$

are feasible at times $t-1$ and t, respectively, though not optimum (in the saddle-point sense). Also note that the last disturbance component to be applied at time t is not optimized, ie, it is a free parameter. Using these solutions, (34) becomes an inequality as it is verified by means of (24):

$$\begin{aligned}
\bar{\Psi}^*_{t+1} - \bar{\Psi}^*_t \leq & \|\hat{x}_{0|t} - \bar{x}_0\|^2_{\Pi_0^{-1}} - \frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right) \\
& - \frac{1}{\gamma}\left(\|w_{t|t}\|^2_{Q^{-1}} + \|v_{t|t}\|^2_{R^{-1}}\right) \\
& - \|\hat{x}_{0|t} + \bar{x}_0\|^2_{\Pi_0^{-1}} + \frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right) \\
= & -\frac{1}{\gamma}\left(\|w_{t|t}\|^2_{Q^{-1}} + \|v_{t|t}\|^2_{R^{-1}}\right)
\end{aligned} \tag{35}$$

Then, from the last inequality, the sequence $\left\{\bar{\Psi}^*_t\right\}$ is nonincreasing and bounded below by $-\rho\|x_0 - \bar{x}_0\|^2$, which in turns implies the convergence of the sequence

$$\{\bar{\Psi}^*_\infty\} \geq -\rho\|x_0 - \bar{x}_0\|^2 > -\infty.$$

Convergence implies for some fixed $N \geq n$,

$$\bar{\Psi}^*_t - \bar{\Psi}^*_{t-N} \leq -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right) \to 0 \tag{36}$$

as $t \to \infty$. Then, by Lemma 4, it follows that the estimation error $\|\hat{x}_t - A^t x_0\| \to 0$ as $t \to \infty$.

Stability is proved in the sense of the Definition 1. Let $\epsilon > 0$ and choose $\zeta \geq 0$, a sufficiently small upper bound of the cost in Lemma 4. If $\delta > 0$ is chosen such that $-\rho\delta^2 > -\zeta$, the following inequality is obtained for all $t \geq n$

$$\begin{aligned}
-\rho\delta^2 \leq & \|x^*_{0|t-1} - \bar{x}_0\|^2_{\Pi_0^{-1}} - \frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right) \\
\leq & \|\hat{x}_{0|t-1} - \bar{x}_0\|^2_{\Pi_0^{-1}} - \frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right).
\end{aligned}$$

The term $\|\hat{x}_{0|t-1} - \bar{x}_0\|^2_{\Pi_0^{-1}} \geq 0$ is dependent of the initial guess $\bar{x}_0$. If the initial guess is a good approximation of the real initial state, then the above cost is close to zero. As it is expected, this cost tends to a constant as time increases since the more data are available, the better is the estimation of the initial state. On the other hand, the term

$$-\frac{1}{\gamma}\left(\sum_{k=0}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right) \leq 0$$

is always decreasing as more data are processed. Therefore, the following inequality is also true:

$$-\zeta \leq -\rho\delta^2 \leq -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right).$$

By Lemma 4, if the above inequality holds, the $\mathcal{H}_\infty$-FIE is an asymptotically stable observer for the system (33) in the sense of Definition 1. Therefore, if the initial estimation error $\|x_0 - \hat{x}_0\| \leq \delta$, then the estimation error $\|\hat{x}_t - A^t x_0\| \leq \epsilon$ for all $t \geq n$ as claimed. □

To prove stability of the $\mathcal{H}_\infty$-MHE, we need to guarantee the positive definiteness of the estimation error weighting matrix Π_t. As Π_t is computed from a Riccati recursion, the positive definiteness of the unique solution is well established by the following technical theorem.

Theorem 3. *Subject to $\Pi_0 > 0$, then the detectability of (C, A) and the nonexistence of unreachable modes of $(A, GQ^{1/2})$ on the unique circle are necessary and sufficient conditions for*

$$\lim_{t\to\infty} \Pi_t = \Pi_\infty,$$

where Π_∞, with initial condition Π_0, is the unique stabilizing solution of the Riccati Equation (18).

Proof. The proof is available in de Souza et al.[43] □

From Theorem 3, if Π_0 is chosen such that $\Pi_0 \geq \Pi_\infty$, then Π_k is positive definite $\forall k \geq 0$. An alternative scenario is when G is nonsingular that implies GQG^T positive definite. Then, Π_k is also positive definite $\forall k \geq 0$. Before proceeding with the stability of the $\mathcal{H}_\infty$-MHE, the following Lemma is posed.

Lemma 5. *Consider the reachable set of states at time t generated by a feasible initial condition x_0 and disturbance sequence $\{w\}_{k=0}^{t-1}$, as shown in (15). The error weighting matrix Π_t, defined by (18), satisfies the following inequality for all $p \in \mathcal{R}_t$:*

$$(p-\hat{x}_t)^T\Pi_t^{-1}(p-\hat{x}_t) + \hat{\psi}_t^* \geq \min_{x_{t-N}}\max_{\{w_k\}} \{\hat{\psi}_t(x_{t-N}, \{w_k\}) : x(N; x_{t-N}, \{w_k\}) = p\} := \hat{\Theta}_t(p),$$

where the minimization and the maximization are subject to the constraint (2).

We proceed in a similar way as in Rao.[15] Before proving Lemma 5, we use the following technical lemma for general quadratic programs.

Lemma 6. *Let $\theta z = z^TQz$ where the matrix Q is symmetric definite positive and the sets Γ and Ω are closed and convex with $\Gamma \subseteq \Omega$. If a solution exists to the following quadratic programs $\theta(\hat{z}) = \min_{z\in\Omega}\theta(z)$, and $\theta(\bar{z}) = \min_{z\in\Gamma}\theta(z)$, then $\theta(\bar{z}) \geq \theta\hat{z} + \theta(\Delta z)$ where $\Delta z = \bar{z} - \hat{z}$.*

Proof. The proof is in Rao.[15] □

It is important to state that the above lemma applies for the posed min-max programs as long as the disturbance sequence $\{w_k\}$ is fixed with its optimum value.

Proof. [Lemma 5]. Without loss of generality, let us consider $\hat{x}_{t-N} = 0$. Consider an arbitrary $p \in \mathcal{R}_T$. Let

$$\left(\bar{x}_{t-N|t-1}, \left\{\bar{w}_{k|t-1}\right\}_{t-N}^{t-1}\right) = \min_{z}\max_{\{w_k\}} \{\hat{\psi}_t(z, \{w_k\}) : x(N; z; \{w_k\}) = p\},$$

where the optimization is subject to constraints in (4). If

$$\Delta x_{t-N|t-1} := \bar{x}_{t-N|t-1} - \hat{x}_{t-N|t-1}, \Delta\bar{w}_{k|t-1} := \bar{w}_{k|t-1} - \hat{w}_{k|t-1},$$

then, by Lemma 6,

$$\hat{\Theta}_t(p) \leq \hat{\psi}_t^* + \hat{\psi}_t\left(\Delta x_{t-N|t-1}, \left\{w_{k|t-1}\right\}\right)$$

since all the costs are lower than zero. Now, if we choose $p = \hat{x}_t$, then both $x_{t-N|t-1} = 0$ and $\left\{w_{k|t-1}\right\} = 0 \forall k \in t-N, \cdots, t-1$, as a matter of fact.

Let $\Delta p := p - \hat{x}_t$. Then, the following inequality can be verified

$$\hat{\psi}_t\left(\Delta x_{t-N|t-1}, \left\{w_{k|t-1}\right\}\right) \leq \min_{z}\max_{w_k}\left\{\hat{\psi}_t(\Delta z, \{w_k|t-1\} : x(N; \Delta z, \left\{\Delta w_{k|t-1}\right\})\right\}$$
$$= (p-\hat{x}_t)^T\Pi_t^{-1}(p-\hat{x}_t)$$

and the Lemma follows as claimed. □

We present the second important result of this contribution, ie, the proposition stating the stability of the $\mathcal{H}_\infty$-MHE.

Proposition 2. *Suppose the matrices Q, R, and Π_0 are positive definite, (C, A) is observable, Assumption 1 holds, $N \geq 0$, and either*

1. *The matrix G is nonsingular, or*
2. *$(A, GQ^{1/2})$ is controllable and $\Pi_0 \geq \Pi_\infty$.*

Then, the constrained $\mathcal{H}_\infty$-MHE is an asymptotically stable observer for the system (33).

Proof. As in the stability proof of the $\mathcal{H}_\infty$-FIE, convergence of the cost sequence $\left\{\hat{\psi}_t^*\right\}$ is demonstrated first. Then, the stability of the filter is shown in the sense of Definition 1. An optimal solution to the constrained $\mathcal{H}_\infty$-MHE problem exists as stated by Lemma 3 and Assumption 1. By definition, and stated by (36),

$$\bar{\psi}_t^* - \bar{\psi}_{t-N}^* \leq -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right) \to 0.$$

As the aim is to demonstrate that $-\rho\|x_0 - \bar{x}_0\|^2$ is a uniform bound, let an induction argument be considered. The case when $t \leq N$ is equivalent to the $\mathcal{H}_\infty$-FIE. It was already shown that $-\rho\|x_0 - \bar{x}_0\|^2$ is indeed a lower bound to the objective function (6). By using Lemma 5,

$$\hat{\Theta}_t(x_{t|\infty}) \leq (x_{t|\infty} - \hat{x}_t)^T \Pi_t^{-1}(x_{t|\infty} - \hat{x}_t) + \hat{\psi}_t^*.$$

For the induction argument, assume that

$$\hat{\Theta}_{t-N}(x_{t-N|\infty}) \leq (x_{t-N|\infty} - \hat{x}_{t-N})^T \Pi_{t-N}^{-1}(x_{t-N|\infty} - \hat{x}_{t-N}) + \hat{\psi}_{t-N}^*.$$

By optimality, the induction assumption, and the properties related to the arrival cost, for all $t \geq N$,

$$\min_{x_{t-N}} \max_{\{w_k\}} \left\{ -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|w_k\|^2_{Q^{-1}} + \|v_k\|^2_{R^{-1}}\right) + \hat{\Theta}_{t-N}(x_{t-N}) : \right.$$
$$\left. x(N; x_{t-N}, \{w_k\}) = x_{t|\infty} \right\} \geq -\rho\|x_0 - \hat{x}_0\|^2,$$

where the optimization is subject to constraint 4. The above is also true by Assumption1, because the solution to the estimation problem by using an infinite set of data is feasible. Using the induction argument the following inequality also holds

$$\begin{aligned}
&\min_{x_{t-N}} \max_{\{w_k\}} \left\{ -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|w_k\|^2_{Q^{-1}} + \|v_k\|^2_{R^{-1}}\right) \right. \\
&\left. + \ (x_{t-N} - \hat{x}_{t-N})^T \Pi_{t-N}^{-1}(x_{t-N} - \hat{x}_{t-N}) + \hat{\psi}_{t-N}^* \ : x(N; x_{t-N}, \{w_k\}) = x_{t|\infty} \right\} \geq \\
&\min_{x_{t-N}} \max_{\{w_k\}} \left\{ -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|w_k\|^2_{Q^{-1}} + \|v_k\|^2_{R^{-1}}\right) + \hat{\Theta}_{t-N}(x_{t-N}) : \right. \\
&\left. x(N; x_{t-N}, \{w_k\}) = x_{t|\infty} \right\} \geq -\rho\|x_0 - \hat{x}_0\|^2.
\end{aligned}$$

Finally, by Lemma 5,

$$\begin{aligned}
&(x_t - \hat{x}_t)^T \Pi_t^{-1}(x_t - \hat{x}_t) + \hat{\psi}_t^* \\
&\geq \min_{x_{t-N}} \max_{\{w_k\}} \left\{ -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|w_k\|^2_{Q^{-1}} + \|v_k\|^2_{R^{-1}}\right) + \hat{\Theta}_{t-N}(x_{t-N}) : \right. \\
&\left. x(N; x_{t-N}, \{w_k\}) = x_{t|\infty} \right\} \geq -\rho\|x_0 - \hat{x}_0\|^2,
\end{aligned}$$

where is verified that

$$\hat{\psi}_t^* \geq -\rho\|x_0 - \hat{x}_0\|^2$$

with every optimization procedure fulfilling the constraint (2). Hence, the sequence $\{\hat{\psi}_t^*\}$ is monotone nonincreasing and bounded below by $-\rho\|x_0 - \hat{x}_0\|^2$. As verified before, convergence implies (36) as $t \to \infty$. By Lemma 4, the estimation error $\|\hat{x}_t - A^t x_0\| \to 0$ as $t \to \infty$. Now, the stability proof follows a similar procedure as for the $\mathcal{H}_\infty$-FIE. Let $\epsilon > 0$ and choose $\zeta > 0$ sufficiently small for $t = N$ such that it is an upper bound of the cost in Lemma 4. Choose $\delta > 0$ such that $-\rho\delta^2 > -\zeta$, then the following inequality holds for all $t \geq N$

$$\begin{aligned}
-\rho\delta^2 &\leq \|\hat{x}_{t-N|t-1} - \hat{x}_{t-N}\|^2_{\Pi_{t-N}^{-1}} - \frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right) \\
+ \hat{\psi}_{t-N}^* &\leq \|\hat{x}_{t-N|t-1} - \hat{x}_{t-N}\|^2_{\Pi_{t-N}^{-1}} - \frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right)
\end{aligned}$$

since $\hat{\psi}_{t-N}^* < 0$. Then, using a similar argument that for the $\mathcal{H}_\infty$-FIE

$$-\zeta < -\rho\delta^2 \leq -\frac{1}{\gamma}\left(\sum_{k=t-N}^{t-1} \|\hat{w}_{k|t-1}\|^2_{Q^{-1}} + \|\hat{v}_{k|t-1}\|^2_{R^{-1}}\right).$$

Therefore, using Lemma 4, if the initial estimation error $\|x_0 - \hat{x}_0\| \leq \delta$, then the estimation error $\|\hat{x}_{t-N} - A^t x_0\| \leq \epsilon$ for all $t \geq N$ as claimed. □

5 | NUMERICAL EXAMPLE

In this example, the approximate numerical solution provided by the $\mathcal{H}_\infty$-aMHE is tested into the spring-mass-damper system from Appleby.[44] The dynamics of the system are modeled by means of the following uncertain continuous-time linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{-k_0}{m_0} & \frac{-b_0}{m_0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_0} \end{bmatrix} w, \tag{37a}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v, \tag{37b}$$

where $x = [\, x_1 \;\; x_2 \,]^T$ is the state of the system with x_1 the position of the mass and x_2 its velocity, w is the disturbance force, v is the measurement uncertainty, m_0 is the nominal mass, b_0 is the nominal viscous damping coefficient, and k_0 is the nominal spring constant. The nominal parameters are known to be $1/m_0 = 1.25, b_0 = 0.15$, and $k_0 = 5$. The objective is to estimate the state by measuring x_1, taking into account the unknown inputs.

Because we are considering discrete-time designs, the system is discretized using a zero-order hold with a sampling time of $T_s = 0.1$ second. The matrices of the discrete-time model are

$$A_d = \begin{bmatrix} 0.9691 & 0.0980 \\ -0.6127 & 0.9507 \end{bmatrix}, B_d = \begin{bmatrix} 0.0062 \\ 0.1225 \end{bmatrix},$$

$$C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_d = 0.$$

In this example, the measurement uncertainty v_k is assumed to be zero-mean white-noise with an unknown covariance. The modeling uncertainty w_k is assumed to be a random variable of unknown type. It is a priori known that $w_k > 1$. This information is added to the filter to improve the estimates. Two filters are tested in this scenario, the analytic and hence unconstrained $\mathcal{H}_\infty$-MHE, from now on u$\mathcal{H}_\infty$-MHE[35,39] and the constrained $\mathcal{H}_\infty$-aMHE. The u$\mathcal{H}_\infty$-MHE is the filter resulting from the solution of the problem 10 without constraints. Therefore, analytical expressions for both $\hat{x}_{t-N}$ and $\{\hat{w}_k\}_{k=t-N}^{t-1}$ are available. The initial condition for the plant and filters are $(x_{1,0}, x_{2,0}) = (2, 4)$ and $(x^f_{1,0}, x^f_{2,0}) = (0, 0)$, respectively. The tuning parameters for both filters were found by a trial and error procedure and are given in Table 1.

Figure 1 shows the time response of the u$\mathcal{H}_\infty$-MHE and the $\mathcal{H}_\infty$-aMHE schemes in the presented scenario. A zoom on the first 5s of Figure 1 is shown in Figure 2 in order to show that initial conditions for the plant and filters are indeed different. Although an approximation was made in the $\mathcal{H}_\infty$-aMHE, the time response of this filter is considerably better than the u$\mathcal{H}_\infty$-MHE. The latter fact is appreciated by the steady-state error in the estimate of the state x_2 of the $u\mathcal{H}_\infty$-MHE, since the filter has not included the a priori information about the modeling noise. The above result is also supported by Table 2 where the mean square error (MSE) is calculated for both filters using 3 sets of different initial conditions. Note that the MSEs for x_1 are small for both filters while the MSE for x_2 is considerably better in the $\mathcal{H}_\infty$-aMHE. This is an expected result since the modeling uncertainty affects directly the state x_2 in the presented example.

TABLE 1 Tuning parameters for u$\mathcal{H}_\infty$-MHE and constrained $\mathcal{H}_\infty$-aMHE

Filter	N	Q_k	R_k	P_o	γ
u$\mathcal{H}_\infty$-MHE	8	0.1	10	$diag(10, 10)$	100
$\mathcal{H}_\infty$-aMHE	8	0.1	10	$diag(0.01, 0.01)$	100

Abbreviation: MHE, moving horizon estimator.

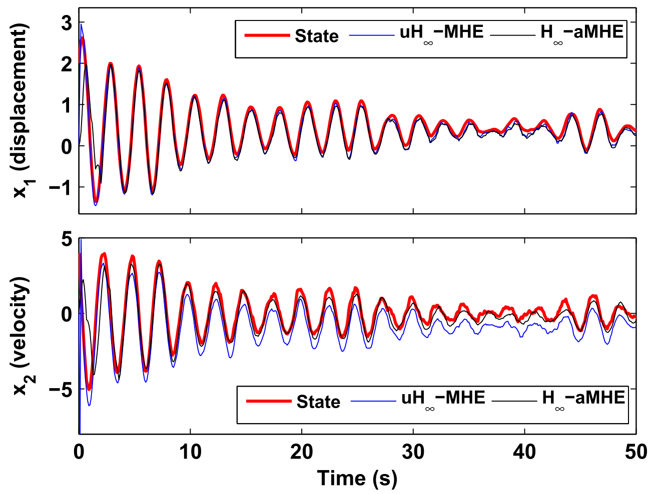

FIGURE 1 Time response of the analytic $\mathcal{H}_\infty$-MHE vs the $\mathcal{H}_\infty$-aMHE [Colour figure can be viewed at wileyonlinelibrary.com]

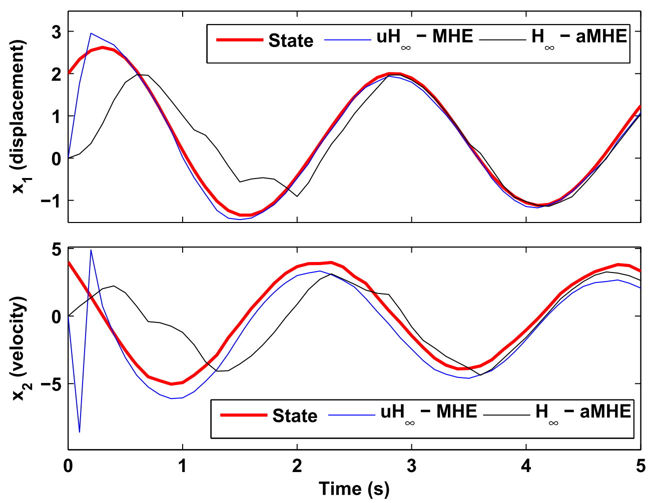

FIGURE 2 Zoom of Figure 1 from 0-5s [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2 Mean square error (MSE) of the filters

IC	State	u$\mathcal{H}_\infty$-MHE	$\mathcal{H}_\infty$-aMHE
0	MSE x_1	0.0092	0.0412
0	MSE x_2	0.9485	0.3359
1.5	MSE x_1	0.0087	0.0133
3.5	MSE x_2	0.8066	0.1144
3	MSE x_1	0.0091	0.0202
5	MSE x_2	0.7862	0.2202

Abbreviations: IC, initial conditions of the filters; MHE, moving horizon estimator.

6 | CONCLUSION

In this paper, we provided a theoretical basis to solve the constrained estimation problem for uncertain linear systems based on the MHE and the game theoretical approach to the $\mathcal{H}_\infty$ filtering.

As the key result, we demonstrated stability for both the $\mathcal{H}_\infty$-FIE and $\mathcal{H}_\infty$-MHE. This theoretical construction was mainly based on a modified Lyapunov theory for optimization-based systems, which had been used previously for the classic MHE.

The $\mathcal{H}_\infty$-MHE was derived for uncertain linear systems whose main uncertainty source is in form of additive noises with unknown statistics. As the main difference with previous contributions for uncertain linear systems, the filter was endowed with the ability of adding a priori information in form of equality and inequality constraints to improve the estimation quality.

A reliable approximation of the $\mathcal{H}_\infty$-MHE, named as the $\mathcal{H}_\infty$-aMHE, was provided to circumvent the direct numerical solution of a complex minimax optimization problem. Instead, a separable min-max optimization problem was posed, which produces a feasible numerical method to be solved even in real life applications.

Finally, as main open problems to be tackled in the near future are the formulation of new numeric strategies to solve minimax optimization problems, the filter formulation for linear systems with uncertain parameters, and the formulation of the nonlinear version of the $\mathcal{H}_\infty$-MHE.

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