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Published in:
arXiv

Published: 01/01/2017

Document Version
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

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Asymptotically Optimal Load Balancing Topologies

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July 20, 2017

Abstract

We consider a system of N servers inter-connected by some underlying graph topology G_N. Tasks arrive at the various servers as independent Poisson processes of rate λ. Each incoming task is irrevocably assigned to whichever server has the smallest number of tasks among the one where it appears and its neighbors in G_N. Tasks have independent unit-mean exponentially distributed service times and leave the system upon service completion.

The above model has been extensively investigated in the case G_N is a clique. Since the servers are exchangeable in that case, the queue length process is quite tractable, and in particular it has been proved that for any λ < 1, the fraction of servers with two or more tasks vanishes in the limit as N → ∞. For an arbitrary graph G_N, the lack of exchangeability severely complicates the analysis, and the queue length process tends to be worse than for a clique. Accordingly, a graph G_N is said to be N-optimal or √N-optimal when the occupancy process on G_N is equivalent to that on a clique on an N-scale or √N-scale, respectively.

We prove that if G_N is an Erdős-Rényi random graph with average degree d(N), then it is with high probability N-optimal and √N-optimal if d(N) → ∞ and d(N)/√N log(N) → ∞ as N → ∞, respectively. This demonstrates that optimality can be maintained at N-scale and √N-scale while reducing the number of connections by nearly a factor N and √N/ log(N) compared to a clique, provided the topology is suitably random. It is further shown that if G_N contains Θ(N) bounded-degree nodes, then it cannot be N-optimal. In addition, we establish that an arbitrary graph G_N is N-optimal when its minimum degree is N − o(N), and may not be N-optimal even when its minimum degree is cN + o(N) for any 0 < c < 1/2.

1 Introduction

Background and motivation. In the present paper we explore the impact of the network topology on the performance of load-balancing schemes in large-scale systems. Load balancing algorithms play a key role in distributing service requests or tasks (e.g. compute jobs, data base lookups, file transfers, transactions) among servers in parallel-processing systems. Well-designed load balancing schemes provide an effective mechanism for improving relevant performance metrics experienced by users while achieving high resource utilization levels. The analysis and design of load balancing algorithms has attracted strong renewed interest in recent years, mainly

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urged by huge scalability challenges in large-scale cloud networks and data centers with immense numbers of servers.

In order to examine the impact of the network topology, we focus on a system of \( N \) servers inter-connected by some underlying graph \( G_N \). Tasks arrive at the various servers as independent Poisson processes of rate \( \lambda \). Each incoming task is immediately assigned to whichever server has the smallest number of tasks among the one where it arrives and its neighbors in \( G_N \). Tasks have independent unit-mean exponentially distributed service times and leave the system upon service completion.

The above model has been extensively investigated in case \( G_N \) is a clique. In that case, each task is assigned to the server with the smallest number of tasks across the entire system, which is commonly referred to as the Join-the-Shortest Queue (JSQ) policy. Under the above Markovian assumptions, the JSQ policy has strong stochastic optimality properties [8, 24, 25, 32]. Specifically, the queue length process is better balanced and smaller in a majorization sense than under any alternative non-anticipating task assignment strategy that does not have advance knowledge of the service times. By implication, the JSQ policy minimizes the mean overall queue length, and hence the mean waiting time as well. Since the servers are exchangeable in a clique topology, the queue length process is in fact quite tractable. In particular, for any \( \lambda < 1 \), the stationary fraction of servers with two or more tasks as well as the mean waiting time vanish in the limit as \( N \to \infty \).

Unfortunately, however, implementation of the JSQ policy in a clique topology raises two fundamental scalability concerns. First of all, for each incoming task the queue lengths need to be checked at all servers, giving rise to a prohibitive communication overhead in large-scale systems with massive numbers of servers. Second, executing a task commonly involves the use of some data, and storing such data for all possible tasks on all servers will typically require an excessive amount of storage capacity [31, 33]. These two burdens can be effectively mitigated in sparser graph topologies where tasks that arrive at a specific server \( i \) are only allowed to be forwarded to a subset of the servers \( N_i \). For the tasks that arrive at server \( i \), queue length information then only needs to be obtained from servers in \( N_i \), and it suffices to store replicas of the required data on the servers in \( N_i \). The subset \( N_i \) containing the peers of server \( i \) can be naturally viewed as its neighbors in some graph topology \( G_N \). In the present paper we consider the case of undirected graphs, but most of the analysis can be extended to directed graphs.

While sparser graph topologies relieve the scalability issues associated with a clique, the queue length process will be worse (in the majorization sense) because of the limited connectivity. Surprisingly, however, even quite sparse graphs can asymptotically match the optimal performance of a clique, provided they are suitably random, as we will further describe below.

**Related work.** The above model has been studied in [11, 28], focusing on certain fixed-degree graphs and in particular ring topologies. The results demonstrate that the flexibility to forward tasks to a few neighbors, or even just one, with possibly shorter queues significantly improves the performance in terms of the waiting time and tail distribution of the queue length. This resembles the so-called ‘power-of-two’ effect in the classical case of a complete graph where tasks are assigned to the shortest queue among \( d \) servers selected uniformly at random. As shown by Mitzenmacher [17, 18] and Vvedenskaya et al. [30], such a ‘power-of-\( d \)’ scheme provides a huge performance improvement over purely random assignment, even when \( d = 2 \), in particular super-exponential tail decay, translating into far better waiting-time performance. However, the results in [11, 28] also establish that the performance sensitively depends on the underlying graph topology, and that selecting from a fixed set of \( d - 1 \) neighbors typically does not match the performance of re-sampling \( d - 1 \) alternate servers for each incoming task from the entire
population, as in the power-of-d scheme in a complete graph. Further related problems have been investigated in [2, 15, 16, 19].

If tasks do not get served and never depart but simply accumulate, then our model as described above amounts to a so-called balls-and-bins problem on a graph. Viewed from that angle, a close counterpart of our problem is studied in Kenthapadi and Panigrahy [14], where in our terminology each arriving task is routed to the shortest of \(d \geq 2\) randomly selected neighboring queues. In this setup they show that if the underlying graph is almost regular with degree \(N^{\varepsilon}\), where \(\varepsilon\) is not too small, the maximum number of balls in a bin scales as \(\log(\log(N))/\log(d) + O(1)\). This scaling is the same as in the case when the underlying graph is a clique [3]. The classical balls-and-bins problem with a power-of-d scheme (often referred to as ‘multiple-choice’ algorithm), without any graph topology, has also been studied extensively, [3, 5]. Just like in the queueing scenario mentioned above, the power-of-d scheme provides a major improvement over purely random assignment (\(d = 1\)) where the maximum number of balls in a bin scales as \(\log(N)/\log(\log(N))\) [12]. Several further variations and extensions have been considered subsequently [1, 4, 6, 7, 10, 22, 23, 29].

As alluded to above, there are natural parallels between the balls-and-bins setup and the queueing scenario as considered in the present paper. These commonalities are for example reflected in the fact that the power-of-d scheme yields a similar dramatic performance improvement over purely random assignment in both settings.

However, there are also quite fundamental differences between the balls-and-bins setup and the queueing scenario, even in a clique topology, besides the obvious contrasts in the performance metrics. The distinction is for example evidenced by the fact that a simple round-robin strategy produces a perfectly balanced allocation in a balls-and-bins setup but is far from optimal in a queueing scenario. In particular, the stationary fraction of servers with two or more tasks under a round-robin strategy remains positive in the limit as \(N \to \infty\), whereas it vanishes under the JSQ policy. On a related account, since tasks get served and eventually depart in a queueing scenario, less balanced allocations with a large portion of vacant servers will generate fewer service completions and result in a larger total number of tasks. Thus different schemes yield not only various degrees of balance, but also variations in the aggregate number of tasks in the system. These differences arise not only in case of a clique, but also in arbitrary graph topologies, and hence our problem requires a fundamentally different approach than developed in [14] for the balls-and-bins setup. Moreover, [14] considers only the scaling of the maximum queue length, whereas we analyze a more detailed time-varying evolution of the entire system along with its stationary behavior.

**Approach and key contributions.** As mentioned above, the queue length process in a clique will be better balanced and smaller (in a majorization sense) than in an arbitrary graph \(G_N\). Accordingly, a graph \(G_N\) is said to be \(N\)-optimal or \(\sqrt{N}\)-optimal when the queue length process on \(G_N\) is equivalent to that on a clique on an \(N\)-scale or \(\sqrt{N}\)-scale, respectively. Roughly speaking, a graph is \(N\)-optimal if the fraction of nodes with \(i\) tasks, for \(i = 0, 1, \ldots\), behaves as in a clique as \(N \to \infty\). Since the latter fraction is zero in the limit for all \(i \geq 2\) in a clique in stationarity, the fraction of servers with two or more tasks vanishes in any graph that is \(N\)-optimal, implying that the mean waiting time vanishes as well. Furthermore, recent results for the JSQ policy [9] imply that in a clique of \(N\) nodes in the heavy-traffic regime the number of nodes with zero tasks and that with two tasks both scale as \(\sqrt{N}\) as \(N \to \infty\). Again loosely speaking, a graph is \(\sqrt{N}\)-optimal if in the heavy-traffic regime the number of nodes with zero tasks and that with two tasks when scaled by \(\sqrt{N}\) both evolve as in a clique as \(N \to \infty\). Formal definitions of asymptotic optimality
on an $N$-scale or $\sqrt{N}$-scale will be introduced in Section 2.

As one of the main results, we will demonstrate that, remarkably, asymptotic optimality can be achieved in quite sparse Erdős-Rényi random graphs (ERRGs). We prove that a sequence of ERRGs indexed by the number of vertices $N$ with $d(N) \to \infty$ as $N \to \infty$, is $N$-optimal. We further establish that the latter growth condition for the average degree is in fact necessary in the sense that any graph sequence that contains $\Theta(N)$ bounded-degree vertices cannot be $N$-optimal. This implies that a sequence of ERRGs with finite average degree cannot be $N$-optimal. The growth rate condition is more stringent for optimality on $\sqrt{N}$-scale in the heavy-traffic regime. Specifically, we prove that a sequence of ERRGs indexed by the number of vertices $N$ with $d(N)/(\sqrt{N} \log(N)) \to \infty$ as $N \to \infty$, is $\sqrt{N}$-optimal.

The above results demonstrate that the asymptotic optimality of cliques on an $N$-scale and $\sqrt{N}$-scale can be achieved in far sparser graphs, where the number of connections is reduced by nearly a factor $N$ and $\sqrt{N}/\log(N)$, respectively, provided the topologies are suitably random in the ERRG sense. This translates into equally significant reductions in communication overhead and storage capacity, since both are roughly proportional to the number of connections.

While quite sparse graphs can achieve asymptotic optimality in the presence of randomness, the worst-case graph instance may even in very dense regimes (high average degree) not be optimal. In particular, we prove that any graph sequence with minimum degree $N - o(N)$ is $N$-optimal, but that for any $0 < c < 1/2$ one can construct graphs with minimum degree $cN + o(N)$ which are not $N$-optimal for some $\lambda < 1$.

The key challenge in the analysis of load balancing on arbitrary graph topologies is that one needs to keep track of the evolution of number of tasks at each vertex along with their corresponding neighborhood relationship. This creates a major problem in constructing a tractable Markovian state descriptor, and renders a direct analysis of such processes highly intractable. Consequently, even asymptotic results for load balancing processes on an arbitrary graph have remained scarce so far. We take a radically different approach and aim to compare the load balancing process on an arbitrary graph with that on a clique. Specifically, rather than analyze the behavior for a given class of graphs or degree value, we explore for what types of topologies and degree properties the performance is asymptotically similar to that in a clique.

Our proof methodology builds on some recent advances in the analysis on the power-of-$d$ algorithm where $d = d(N)$ grows with $N$ [20, 21]. Specifically, we view the load balancing process on an arbitrary graph as a ‘sloppy’ version of that on a clique, and thus construct several other intermediate sloppy versions. By constructing novel couplings, we develop a method of comparing the load balancing process on an arbitrary graph with that on a clique. In particular, we bound the difference between the fraction of vertices with $i$ or more tasks in the two systems for $i = 1, 2, \ldots$, to obtain asymptotic optimality results. From a high level, conceptually related graph conditions for asymptotic optimality were examined using quite different techniques by Tsitsiklis and Xu [26, 27] in a dynamic scheduling framework (as opposed to load balancing context).

**Organization of the paper.** The remainder of the paper is organized as follows. In Section 2 we present a detailed model description and introduce some useful notation and preliminaries. In Section 3 we state the main results, along with a high-level outline of the proof arguments. Proofs of statements marked (⋆) have been provided in the appendix. For a sequence of probability measures $(P_N)_{N \geq 1}$, the sequence of events $(E_N)_{N \geq 1}$ is said to hold with high probability if $P_N(E_N) \to 1$ as $N \to \infty$. The symbols ‘$\xrightarrow{L}$’ and ‘$\xrightarrow{P}$’ will denote convergences in distribution and in probability, respectively.
2 Model description and preliminaries

Let \( \{G_N\}_{N \geq 1} \) be a sequence of simple graphs indexed by the number of vertices \( N \). For the \( N \)-th system with \( N \) servers, we assume that the servers are inter-connected by the underlying graph topology \( G_N \), where server \( i \) is identified with vertex \( i \) in \( G_N \), \( i = 1, 2, \ldots, N \). Tasks arrive at the various servers as independent Poisson processes of rate \( \lambda \). Each server has its own queue with a fixed buffer capacity \( b \) (possibly infinite). When a task appears at a server \( i \), it is immediately assigned to the server with the shortest queue among server \( i \) and its neighborhood in \( G_N \). If there are multiple such servers, one of them is chosen uniformly at random. If \( b < \infty \), and server \( i \) and all its neighbors have \( b \) tasks (including the ones in service), then the newly arrived task is discarded. The tasks have independent unit-mean exponentially distributed service times. The service order at each of the queues is assumed to be oblivious to the actual service times, e.g. First-Come-First-Served (FCFS).

For \( k = 1, \ldots, N \), denote by \( X_k(G_N, t) \) the queue length at the \( k \)-th server at time \( t \) (including the one possibly in service), and by \( X_{[k]}(G_N, t) \) the queue length at the \( k \)-th ordered server at time \( t \) when the servers are arranged in nondecreasing order of their queue lengths (ties can be broken in some way that will be evident from the context). Let \( Q_i(G_N, t) \) denote the number of servers with queue length at least \( i \) at time \( t \), \( i = 1, 2, \ldots, b \). It is important to note that \( \{Q_i(G_N, t)\}_{i \geq 1}^{t \geq 0} \) is itself not a Markov process, but the joint process \( \{Q_i(G_N, t)\}_{i \geq 1}^{N \geq 1} \) is Markov.

**Proposition 1 (★).** For \( \lambda < 1 \), the joint process \( \{Q_i(G_N, t)\}_{i \geq 1}^{N \geq 1}, \{X_k(G_N, t)\}_{k \geq 1}^{t \geq 0} \) has a unique steady state \( \{(Q_i(G_N, \infty))_{i \geq 1}, (X_k(G_N, \infty))_{k \geq 1}\}^{N \geq 1} \). Also, the sequence of marginals \( \{(Q_i(G_N, \infty))_{i \geq 1}\}^{N \geq 1} \) is tight with respect to the \( \ell_1 \)-topology.

### Asymptotic behavior of occupancy processes in cliques.

We now describe the behavior of the occupancy processes on a clique as the number of servers \( N \) grows large. Rigorous descriptions of the limiting processes are provided in the appendix.

The behavior on \( N \)-scale is observed in terms of the fractions \( Q_i(G_N, t) = Q_i(G_N, t)/N \) of servers with queue length at least \( i \) at time \( t \). When \( \lambda < 1 \), on any finite time interval,

\[
\{(Q_1(K_N, t), Q_2(K_N, t), \ldots)\}_{t \geq 0} \xrightarrow{\mathcal{L}} \{(Q_1(t), Q_2(t), \ldots)\}_{t \geq 0},
\]

as \( N \to \infty \), where \((q_1(\cdot), q_2(\cdot), \ldots)\) is some deterministic process. Furthermore, in steady state

\[
q_1(K_N, \infty) \to \lambda \quad \text{and} \quad q_1(K_N, \infty) \to 0 \quad \text{for all} \ i = 2, \ldots, b \quad \text{as} \ N \to \infty.
\]

Note that \( q_1(K_N, \cdot) \) is the fraction of non-empty servers. Thus \( q_1(K_N, \infty) \) is the steady-state scaled departure rate which should be equal to the scaled arrival rate \( \lambda \). Surprisingly, however, we observe that the steady-state fraction of servers with a queue length of two or larger is asymptotically negligible.

To analyze the behavior on \( \sqrt{N} \)-scale, we consider a heavy-traffic scenario (a.k.a. Halfin-Whitt regime) where the arrival rate at each server is given by \( \lambda(N)/N \) with

\[
(N - \lambda(N))/\sqrt{N} \to \beta > 0 \quad \text{as} \ N \to \infty.
\]

In order to describe the behavior in the limit, let \( Q(G_N, t) = (Q_1(G_N, t), Q_2(G_N, t), \ldots, Q_b(G_N, t)) \) be a properly centered and scaled version of the occupancy process \( Q(G_N, t) \), with

\[
\tilde{Q}_1(G_N, t) = -\frac{N - Q_1(G_N, t)}{\sqrt{N}}, \quad \tilde{Q}_i(G_N, t) = \frac{Q_i(G_N, t)}{\sqrt{N}}, \quad i = 2, \ldots, b.
\]
The reason why $Q_1(\cdot, \cdot)$ is centered around $N$ while $Q_i(\cdot, \cdot)$, $i = 2, \ldots, b$, are not, is because for $G_N = K_N$, the fraction of servers with a queue length of exactly one tends to one, whereas the fraction of servers with a queue length of two or larger tends to zero as $N \to \infty$, as mentioned above. Recent results for $Q(K_N, t)$ [9] show that from a suitable starting state,

$$\{Q_1(K_N, t), Q_2(K_N, t), \ldots\} \quad \overset{\mathbb{L}}{\longrightarrow} \quad \{Q_1(t), Q_2(t), 0, 0, \ldots\}$$

as $N \to \infty$, where $(Q_1(\cdot), Q_2(\cdot))$ is some diffusion process. This implies that over any finite time interval, there will be $O(\sqrt{N})$ servers with queue length zero and $O(\sqrt{N})$ servers with a queue length of two or larger, and hence all but $O(\sqrt{N})$ servers have a queue length of exactly one.

**Asymptotic optimality.** As stated in the introduction, a clique is an optimal load balancing topology, as the occupancy process is better balanced and smaller (in a majorization sense) than in any other graph topology. In general the optimality is strict, but it turns out that near-optimality can be achieved asymptotically in a broad class of other graph topologies. Therefore, we now introduce two notions of asymptotic optimality, which will be useful to characterize the performance in large-scale systems.

**Definition 1** (Asymptotic optimality). A graph sequence $G = \{G_N\}_{N \geq 1}$ is called ‘asymptotically optimal on $N$-scale’ or ‘$N$-optimal’, if for any $\lambda < 1$, the scaled occupancy process $(q_1(G_N, \cdot), q_2(G_N, \cdot), \ldots)$ converges weakly, on any finite time interval, to the process $(q_1(\cdot), q_2(\cdot), \ldots)$ given by (1).

Moreover, a graph sequence $G = \{G_N\}_{N \geq 1}$ is called ‘asymptotically optimal on $\sqrt{N}$-scale’ or ‘$\sqrt{N}$-optimal’, if for any $\lambda(N)$ satisfying (3), on any finite time interval, the process $(\bar{Q}_1(G_N, \cdot), \bar{Q}_2(G_N, \cdot), \ldots)$ as in (4) converges weakly to the process $(\bar{Q}_1(\cdot), \bar{Q}_2(\cdot), \ldots)$ given by (5).

Intuitively speaking, if a graph sequence is $N$-optimal or $\sqrt{N}$-optimal, then in some sense, the associated occupancy processes are indistinguishable from those of the sequence of cliques on $N$-scale or $\sqrt{N}$-scale. In other words, on any finite time interval their occupancy processes can differ from those in cliques by at most $o(N)$ or $o(\sqrt{N})$, respectively. For brevity, $N$-scale and $\sqrt{N}$-scale are often referred to as fluid scale and diffusion scale, respectively. In particular, due to the $\ell_1$-tightness of the scaled occupancy processes as stated in Proposition 1, we obtain that for any $N$-optimal graph sequence $\{G_N\}_{N \geq 1}$,

$$q_1(G_N, \infty) \to \lambda \quad\text{and}\quad q_i(G_N, \infty) \to 0 \quad\text{for all } i = 2, \ldots, b \quad\text{as } N \to \infty,$$

implying that the stationary fraction of servers with queue length two or larger and the mean waiting time vanish.

In large-scale systems it is often enough to achieve optimal performance on fluid and (in case of heavy traffic) diffusion scale. Thus, in the rest of the paper we will investigate under what conditions a graph sequence is $N$-optimal or $\sqrt{N}$-optimal.

### 3 Sufficient criteria for asymptotic optimality

In this section we first develop a criterion for asymptotic optimality of an arbitrary deterministic graph sequence on different scales. Next this criterion will be leveraged to establish optimality of a sequence of random graphs.

We start by introducing some useful notation, and two measures of well-connectedness. Let $G = (V, E)$ be any graph. For a subset $U \subseteq V$, define $\com(U) := |V \setminus N[U]|$ to be the set of all
vertices that are disjoint from \(U\), where \(N[U] := U \cup \{v \in V : \exists u \in U \text{ with } (u,v) \in E\}\). For any fixed \(\varepsilon > 0\) define

\[
\text{dis}_1(G, \varepsilon) := \sup_{u \subseteq V, |u| \geq \varepsilon |V|} \text{com}(u), \quad \text{dis}_2(G, \varepsilon) := \sup_{u \subseteq V, |u| \geq \varepsilon \sqrt{|V|}} \text{com}(u). \tag{7}
\]

The next theorem provides sufficient conditions for asymptotic optimality on \(N\)-scale and \(\sqrt{N}\)-scale in terms of the above two well-connectedness measures.

**Theorem 2 (\(\star\)).** For any graph sequence \(G = \{G_N\}_{N \geq 1}\),

(i) \(G\) is \(N\)-optimal if for any \(\varepsilon > 0\), \(\text{dis}_1(G_N, \varepsilon)/N \to 0\), as \(N \to \infty\).

(ii) \(G\) is \(\sqrt{N}\)-optimal if for any \(\varepsilon > 0\), \(\text{dis}_2(G_N, \varepsilon)/\sqrt{N} \to 0\), as \(N \to \infty\).

The next corollary is an immediate consequence of Theorem 2.

**Corollary 3.** Let \(G = \{G_N\}_{N \geq 1}\) be any graph sequence and \(d_{\text{min}}(G_N)\) be the minimum degree of \(G_N\). Then (i) If \(d_{\text{min}}(G_N) = N - o(N)\), then \(G\) is \(N\)-optimal, and (ii) If \(d_{\text{min}}(G_N) = N - o(\sqrt{N})\), then \(G\) is \(\sqrt{N}\)-optimal.

The rest of the section is devoted to a discussion of the main proof arguments for Theorem 2, focusing on the proof of \(N\)-optimality. The proof of \(\sqrt{N}\)-optimality follows along similar lines. We establish in Proposition 4 that if a system is able to assign each task to a server in the set \(\mathcal{S}^N(\pi(N))\) of the \(\pi(N)\) nodes with shortest queues, where \(\pi(N) = o(N)\), then it is \(N\)-optimal. Since the underlying graph is not a clique however (otherwise there is nothing to prove), for any \(\pi(N)\) not every arriving task can be assigned to a server in \(\mathcal{S}^N(\pi(N))\). Hence we further prove in Proposition 5 a stochastic comparison property implying that if on any finite time interval of length \(t\), the number of tasks \(\Delta^N(t)\) that are not assigned to a server in \(\mathcal{S}^N(\pi(N))\) is \(o_P(N)\), then the system is \(N\)-optimal as well. The \(N\)-optimality can then be concluded when \(\Delta^N(t)\) is \(o_P(N)\), which we establish in Proposition 6 under the condition that \(\text{dis}_1(G_N, \varepsilon)/N \to 0\) as \(N \to \infty\) as stated in Theorem 2.

To further explain the idea described in the above proof outline, it is useful to adopt a slightly different point of view towards load balancing processes on graphs. From a high level, a load balancing process can be thought of as follows: there are \(N\) servers, which are assigned incoming tasks by some scheme. The assignment scheme can arise from some topological structure as considered in this paper, in which case we will call it *topological load balancing*, or it can arise from some other property of the occupancy process, in which case we will call it *non-topological load balancing*. As mentioned earlier, under Markovian assumptions, the JSQ policy or the clique is optimal among the set of all non-anticipating schemes, irrespective of being topological or non-topological. Also, load balancing on graph topologies other than a clique can be thought of as a ‘sloppy’ version of that on a clique, when each server only has access to partial information on the occupancy state. Below we first introduce a different type of sloppiness in the task assignment scheme, and show that under a limited amount of sloppiness optimality is retained on a suitable scale. Next we will construct a scheme which is a hybrid of topological and non-topological schemes, whose behavior is simultaneously close to both the load balancing process on a suitable graph and that on a clique.

**A class of sloppy load balancing schemes.** Fix some function \(\pi : N \to N\), and recall the set \(\mathcal{S}^N(\pi(N))\) as before. Consider the class \(\text{CJSQ}(\pi(N))\) where each arriving task is assigned to one of
the servers in $\mathcal{S}^N(n(N))$. It should be emphasized that for any scheme in $\text{CJSQ}(n(N))$, we are not imposing any restrictions on how the incoming task should be assigned to a server in $\mathcal{S}^N(n(N))$. The scheme only needs to ensure that the arriving task is assigned to some server in $\mathcal{S}^N(n(N))$ with respect to some tie breaking mechanism. The next proposition provides a sufficient criterion for asymptotic optimality of any scheme in $\text{CJSQ}(n(N))$.

**Proposition 4 (★).** For $0 \leq n(N) < N$, let $\Pi \in \text{CJSQ}(n(N))$ be any scheme. (i) If $n(N)/N \to 0$ as $N \to \infty$, then $\Pi$ is $N$-optimal, and (ii) If $n(N)/\sqrt{N} \to 0$ as $N \to \infty$, then $\Pi$ is $\sqrt{N}$-optimal.

A bridge between topological and non-topological load balancing. For any graph $G_N$ and $n \leq N$, we first construct a scheme called $\text{I}(G_N, n)$, which is an intermediate blend between the topological load balancing process on $G_N$ and some kind of non-topological load balancing on $N$ servers. The choice of $n = n(N)$ will be clear from the context.

To describe the scheme $\text{I}(G_N, n)$, first synchronize the arrival epochs at server $v$ in both systems, $v = 1, 2, \ldots, N$. Further, synchronize the departure epochs at the $k$-th ordered server with the $k$-the smallest number of tasks in the two systems, $k = 1, 2, \ldots, N$. When a task arrives at server $v$ at time $t$, say, it is assigned in the graph $G_N$ to a server $v' \in N[v]$ according to its own statistical law. For the assignment under the scheme $\text{I}(G_N, n)$, first observe that if

$$\min_{u \in N[v]} X_u(G_N, t) \leq \max_{u \in \mathcal{S}(n)} X_u(G_N, t), \quad (8)$$

then there exists some tie-breaking mechanism for which $v' \in N[v]$ belongs to $\mathcal{S}(n)$ under $G_N$. Pick such an ordering of the servers, and assume that $v'$ is the $k$-th ordered server in that ordering, for some $k \leq n + 1$. Under $\text{I}(G_N, n)$ assign the arriving task to the $k$-th ordered server (breaking ties arbitrarily in this case). Otherwise, if (8) does not hold, then the task is assigned to one of the $n + 1$ servers with minimum queue lengths under $G_N$ uniformly at random.

Denote by $\Delta^N(\text{I}(G_N, n), T)$ the cumulative number of arriving tasks up to time $T \geq 0$ for which Equation (8) is violated under the above coupling. The next proposition shows that the load balancing process under the scheme $\text{I}(G_N, n)$ is close to that on the graph $G_N$ in terms of the random variable $\Delta^N(\text{I}(G_N, n), T)$.

**Proposition 5 (★).** The following inequality is preserved almost surely

$$\sum_{t=1}^{b} |Q_1(G_N, t) - Q_1(\text{I}(G_N, n), t)| \leq 2\Delta^N(\text{I}(G_N, n), t) \quad \forall t \geq 0, \quad (9)$$

provided the two systems start from the same occupancy state at $t = 0$.

In order to conclude optimality on $N$-scale or $\sqrt{N}$-scale, it remains to be shown that for any $T > 0$, $\Delta^N(\text{I}(G_N, n), T)$ is sufficiently small. The next proposition provides suitable asymptotic bounds for $\Delta^N(\text{I}(G_N, n), T)$ under the conditions on $\text{dis}_1(G_N, \varepsilon)$ and $\text{dis}_2(G_N, \varepsilon)$ stated in Theorem 2.

**Proposition 6.** For any $\varepsilon, T > 0$ the following holds.

(i) There exists $\varepsilon' > 0$ and $n_{\varepsilon'}(N)$ with $n_{\varepsilon'}(N)/N \to 0$ as $N \to \infty$, such that if $\text{dis}_1(G_N, \varepsilon')/N \to 0$ as $N \to \infty$, then $\mathbb{P}(\Delta^N(\text{I}(G_N, n_{\varepsilon'}), T)/N > \varepsilon) \to 0$.

(ii) There exists $\varepsilon > 0$ and $m_{\varepsilon'}(N)$ with $m_{\varepsilon'}(N)/\sqrt{N} \to 0$ as $N \to \infty$, such that if $\text{dis}_2(G_N, \varepsilon')/\sqrt{N} \to 0$ as $N \to \infty$, then $\mathbb{P}(\Delta^N(\text{I}(G_N, m_{\varepsilon'}), T)/\sqrt{N} > \varepsilon) \to 0$. 

The proof of Theorem 2 then readily follows by combining Propositions 4-6 and observing that the scheme \( I(G_N, n) \) belongs to the class CJJSQ\( (n) \) by construction.

**Proof of Proposition 6.** Fix any \( \varepsilon, T > 0 \) and choose \( \varepsilon' = \varepsilon/(2\lambda T) \). With the coupling described above, when a task arrives at some vertex \( v \) say, Equation (8) is violated only if none of the vertices in \( S(n_{\varepsilon'}(N)) \) is a neighbor of \( v \). Thus, the total instantaneous rate at which this happens is

\[
\lambda \text{com}(S(n_{\varepsilon'}(N), t)) \leq \lambda \sup_{U \subseteq V_N, |U| \geq n_{\varepsilon'}(N)} \text{com}(U),
\]

irrespective of what this set \( S^N(n(N)) \) actually is. Therefore, for any fixed \( T > 0 \),

\[
\Delta^N(I(G_N, n_{\varepsilon'}), T) \leq A \left( \lambda \sup_{U \subseteq V_N, |U| \geq n_{\varepsilon'}(N)} \text{com}(U) \right),
\]

where \( A(\cdot) \) represents a unit-rate Poisson process. This can then be leveraged to show that \( \Delta^N(I(G_N, n_{\varepsilon'}), T) \) is small on an N-scale and \( \sqrt{N} \)-scale, respectively, under the conditions stated in the proposition, by choosing a suitable \( n_{\varepsilon'} \).

Specifically, if \( d_{\text{dis}1}(G_N, \varepsilon')/N \to 0 \), then there exists \( n_{\varepsilon'}(N) \) with \( n_{\varepsilon'}(N)/N \to 0 \) such that \( d_{\text{dis}1}(G_N, \varepsilon') \leq n_{\varepsilon'}(N) \) for all \( N \geq 1 \), and hence \( \sup_{U \subseteq V_N, |U| \geq n_{\varepsilon'}(N)} \text{com}(U) \leq \varepsilon'N \). It then follows that with high probability,

\[
\limsup_{N \to \infty} \frac{1}{N} \Delta^N(I(G_N, n_{\varepsilon'}), T) \leq \limsup_{N \to \infty} \frac{1}{N} A \left( \lambda T \varepsilon' N \right) \leq 2\lambda T \varepsilon' = \varepsilon.
\]

Likewise, if \( d_{\text{dis}2}(G_N, \varepsilon')/\sqrt{N} \to 0 \), then there exists \( m_{\varepsilon'}(N) \) with \( m_{\varepsilon'}(N)/\sqrt{N} \to 0 \) such that \( d_{\text{dis}2}(G_N, \varepsilon') \leq m_{\varepsilon'}(N) \) for all \( N \geq 1 \), and hence \( \sup_{U \subseteq V_N, |U| \geq m_{\varepsilon'}(N)} \text{com}(U) \leq \varepsilon' \sqrt{N} \). It then follows that with high probability,

\[
\limsup_{N \to \infty} \frac{1}{\sqrt{N}} \Delta^N(I(G_N, m_{\varepsilon'}), T) \leq \limsup_{N \to \infty} \frac{1}{\sqrt{N}} A \left( \lambda T \varepsilon' \sqrt{N} \right) \leq 2\lambda T \varepsilon' = \varepsilon.
\]

\( \square \)

From the conditions of Theorem 2 it follows that if for all \( \varepsilon > 0 \), \( d_{\text{dis}1}(G_N, \varepsilon) \) and \( d_{\text{dis}2}(G_N, \varepsilon) \) are \( o(N) \) and \( o(\sqrt{N}) \), respectively, then the total number of edges in \( G_N \) must be \( \omega(N) \) and \( \omega(N \sqrt{N}) \), respectively. Theorem 7 below states that the super-linear growth rate of the total number of edges is not only sufficient, but also necessary in the sense that any graph with \( O(N) \) edges is asymptotically sub-optimal on N-scale.

**Theorem 7 (\( \star \)).** Let \( G = \{G_N\}_{N \geq 1} \) be any graph sequence, such that there exists a fixed integer \( M < \infty \) with

\[
\limsup_{N \to \infty} \frac{\# \{ v \in V_N : d_v \leq M \}}{N} > 0,
\]

where \( d_v \) is the degree of the vertex \( v \). Then \( G \) is sub-optimal on N-scale.

To prove Theorem 7, we show that starting from an all-empty state, in finite time, a positive fraction of servers in \( G_N \) will have at least two tasks. This will prove that the occupancy processes when scaled by \( N \) cannot agree with those in the sequence of cliques, and hence \( \{G_N\}_{N \geq 1} \) cannot be N-optimal. The idea of the proof can be explained as follows: If a system contains \( \Theta(N) \) bounded-degree vertices, then starting from all empty servers, in any finite time interval there
will be $\Theta(N)$ servers $u$ say, for which all the servers in $N[u]$ have at least one task. For all such servers an arrival at $u$ must produce a server with queue length two. Thus, it shows that the instantaneous rate at which servers of queue length two are formed is bounded away from zero, and hence $\Theta(N)$ servers of queue length two are produced in finite time.

Worst-case scenario. Next we consider the worst-case scenario. Theorem 8 below asserts that a graph sequence can be sub-optimal for some $\lambda < 1$ even when the minimum degree $d_{\min}(G_N)$ is $\Theta(N)$.

Theorem 8 (★). For any \( \{d(N)\}_{N \geq 1} \), such that $d(N)/N \to c$ with $0 < c < 1/2$, there exists $\lambda < 1$, and a graph sequence $G = \{G_N\}_{N \geq 1}$ with $d_{\min}(G_N) = d(N)$, such that $G$ is sub-optimal on $N$-scale.

To construct such a sub-optimal graph sequence, consider a sequence of complete bipartite graphs $G_N = (V_N, E_N)$, with $V_N = A_N \cup B_N$ and $|A_N|/N \to c \in (0,1/2)$ as $N \to \infty$. If this sequence were $N$-optimal, then starting from all empty servers, asymptotically the fraction of servers with queue length one would converge to $\lambda$, and the fraction of servers with queue length two or larger should remain zero throughout. Now note that for large $N$ the rate at which tasks join the empty servers in $A_N$ is given by $(1-c)\lambda$, whereas the rate of empty server generation in $A_N$ is at most $c$. Choosing $\lambda > c/(1-c)$, one can see that in finite time each server in $A_N$ will have at least one task. From that time onward with at least instantaneous rate $\lambda(\lambda-c)-c$, servers with queue length two start forming. The range for $c$ stated in Theorem 8 is only to ensure that there exists $\lambda < 1$ with $\lambda(\lambda-c)-c > 0$.

Asymptotic optimality of random graph sequence. Next we investigate how the load balancing process behaves on random graph topologies. Specifically, we aim to understand what types of graphs are asymptotically optimal in the presence of randomness (i.e., in the average case scenario). Theorem 9 below establishes sufficient conditions for asymptotic optimality of a sequence of inhomogeneous random graphs. Recall that a graph $G' = (V', E')$ is called a supergraph of $G = (V, E)$ if $V = V'$ and $E \subseteq E'$.

Theorem 9 (★). Let $G = \{G_N\}_{N \geq 1}$ be a graph sequence such that for each $N$, $G_N = (V_N, E_N)$ is a super-graph of the inhomogeneous random graph $G'_N$ where any two vertices $u, v \in V_N$ share an edge with probability $p_{uv}^N$.

(i) If for each $\epsilon > 0$, there exists subsets of vertices $V_N^c \subseteq V_N$ with $|V_N^c| < \epsilon N$, such that $\inf \{p_{uv}^N : u, v \in V_N^c\} = \omega(1/N)$, then $G$ is $N$-optimal.

(ii) If for each $\epsilon > 0$, there exists subsets of vertices $V_N^c \subseteq V_N$ with $|V_N^c| < \epsilon \sqrt{N}$, such that $\inf \{p_{uv}^N : u, v \in V_N^c\} = \omega(\log(N)/\sqrt{N})$, then $G$ is $\sqrt{N}$-optimal.

The proof of Theorem 9 relies on Theorem 2. Specifically, if $G_N$ satisfies conditions (i) and (ii) in Theorem 9, then the corresponding conditions (i) and (ii) in Theorem 2 hold.

As an immediate corollary to Theorem 9 we obtain an optimality result for the sequence of Erdős-Rényi random graphs.

Corollary 10. Let $G = \{G_N\}_{N \geq 1}$ be a graph sequence such that for each $N$, $G_N$ is a super-graph of $ER_N(p(N))$, and $d(N) = (N-1)p(N)$. Then (i) If $d(N) \to \infty$ as $N \to \infty$, then $G$ is $N$-optimal. (ii) If $d(N)/(\sqrt{N} \log(N)) \to \infty$ as $N \to \infty$, then $G$ is $\sqrt{N}$-optimal.
Theorem 2 can be further leveraged to establish the optimality of the following sequence of random graphs. For any \( N \geq 1 \) and \( d(N) \leq N - 1 \) such that \( Nd(N) \) is even, construct the *erased random regular* graph on \( N \) vertices as follows: Initially, attach \( d(N) \) half-edges to each vertex. Call all such half-edges *unpaired*. At each step, pick one half-edge arbitrarily, and pair it to another half-edge uniformly at random among all unpaired half-edges to form an edge, until all the half-edges have been paired. This results in a uniform random regular multi-graph with degree \( d(N) \) \([13, \text{Proposition 7.7}]\). Now the erased random regular graph is formed by erasing all the self-loops and multiple edges, which then produces a simple graph.

**Theorem 11 (★).** Let \( G = \{G_N\}_{N \geq 1} \) be a sequence of erased random regular graphs with degree \( d(N) \). Then (i) If \( d(N) \to \infty \) as \( N \to \infty \), then \( G \) is \( N \)-optimal. (ii) If \( d(N)/\left(\sqrt{N} \log(N)\right) \to \infty \) as \( N \to \infty \), then \( G \) is \( \sqrt{N} \)-optimal.

Note that due to Theorem 7, we can conclude that the growth rate condition for \( N \)-optimality in Corollary 10 (i) and Theorem 11 (i) is not only sufficient, but necessary as well. Thus informally speaking, \( N \)-optimality is achieved under the minimum condition required as long as the underlying topology is suitably random.

## 4 Conclusion

We considered load balancing processes in large-scale systems where the servers are interconnected by some graph topology. The load balancing performance is optimal when the topology is a clique. For arbitrary topologies we established sufficient criteria for which the performance of the load balancing process is asymptotically optimal on suitable scales. Leveraging these criteria we showed that optimality can be achieved in quite sparse topologies, provided the connections are suitably random in the Erdős-Rényi sense. In terms of worst-case instances though, a graph is guaranteed to be optimal on fluid scale when the minimum degree is \( N - o(N) \), but can be sub-optimal when the minimum degree is \( cN + o(N) \) with \( 0 < c < 1/2 \). What happens for \( 1/2 < c < 1 \) is an open question. Our proof technique relies heavily on a connectivity property entailing that any two sufficiently large portions of vertices share a lot of edges. This property does not hold however in many networks with connectivity governed by spatial attributes, such as geometric graphs. In future research we aim to examine asymptotic optimality properties of such spatial network models.

## Acknowledgment

The authors thank Nikhil Bansal for helpful discussions in the early stage of this work, and also for pointing out several relevant references. The work was financially supported by The Netherlands Organization for Scientific Research (NWO) through Gravitation Networks grant 024.002.003 and TOP-GO grant 613.001.012.

## References


Proof of Proposition 1. Note that if $b < \infty$, the the process $\{(q_i(G_N, t))_{i \geq 1}, (X_k(G_N, t))_{k=1}^N\}_{t \geq 0}$ is clearly ergodic for all $N \geq 1$. When $b = \infty$, to prove the ergodicity of the process, first fix any $N \geq 1$ and observe that the ergodicity of the queue length processes at various vertices amounts to proving the ergodicity of the total number of tasks in the system. Using the S-coupling and Proposition 13 in Appendix B.2 we obtain for all $t > 0$,

$$\sum_{i=m}^{\infty} Q_i(G_N, t) \leq \sum_{i=m}^{\infty} Q_i(G'_N, t), \quad \text{for all } m = 1, 2, \ldots,$$

provided the inequality holds at time $t = 0$, where $G'_N$ is the collection of $N$ isolated vertices. Thus in particular, the total number of tasks in the system with $G_N$ is upper bounded by that
with $G'_N$. Now the queue length process on $G'_N$ is clearly ergodic since it is the collection of independent subcritical $M/M/1$ queues. Next, for the $\ell_1$-tightness of $\{(q_i(G_N,\infty))_{i\geq 1}\}_{N\geq 1}$, we will use the following tightness criteria: Define

$$\mathcal{X} = \left\{ q \in [0,1]^b : q_i \leq q_{i-1} \text{ for all } i = 2, \ldots, b, \text{ and } \sum_{i=1}^b q_i < \infty \right\} \quad (12)$$

as the set of all possible fluid-scaled occupancy states equipped with $\ell_1$ topology.

**Lemma 12** ([21, Lemma 4.7]). Let $\{X_N^N\}_{N\geq 1}$ be a sequence of random variables in $\mathcal{X}$. Then the following are equivalent:

(i) $\{X_N^N\}_{N\geq 1}$ is tight with respect to product topology, and for all $\varepsilon > 0$,

$$\lim_{k \to \infty} \lim_{N \to \infty} \mathbb{P}\left( \sum_{i \geq k} X_i^N > \varepsilon \right) = 0. \quad (13)$$

(ii) $\{X_N^N\}_{N\geq 1}$ is tight with respect to $\ell_1$ topology.

Note that since $(q_i(G_N,\infty))_{i\geq 1}$ takes value in $[0,1]^\infty$, which is compact with respect to the product topology, Prohorov’s theorem implies that $\{(q_i(G_N,\infty))_{i\geq 1}\}_{N\geq 1}$ is tight with respect to the product topology. To verify the condition in (13), note that for each $m \geq 1$, Equation (11) yields

$$\lim_{N \to \infty} \mathbb{P}\left( \sum_{i \geq m} q_i(G_N,\infty) > \varepsilon \right) \leq \lim_{N \to \infty} \mathbb{P}\left( \sum_{i \geq m} q_i(G'_N,\infty) > \varepsilon \right) = (1-\lambda) \sum_{i \geq m} \lambda^i.$$

Since $\lambda < 1$, taking the limit $k \to \infty$, the right side of the above inequality tends to zero, and hence, the condition in (13) is verified. \hfill \Box

**Proof of Theorem 2.** (i) In order to prove the fluid-level optimality of $G_N$, fix any $\varepsilon > 0$. Observe from Proposition 5 and Proposition 6 (i) that there exists $\varepsilon' > 0$ such that with high probability

$$\sup_{t \in [0,T]} \frac{1}{N} \sum_{i=1}^b |Q_i(G_N,t) - Q_i(I(G_N,n_{\varepsilon'}(N)),t)| \leq \frac{2A_N(T)}{N} \leq \varepsilon.$$

Furthermore, since $I(G_N,n_{\varepsilon'}(N)) \in CJSQ(n_{\varepsilon'}(N))$ and $n_{\varepsilon'}(N)/N \to 0$, Proposition 4 yields

$$\sup_{t \in [0,T]} \sum_{i=1}^b |q_i(I(G_N,n_{\varepsilon'}(N)),t) - q_i(t)| \xrightarrow{P} 0 \quad \text{as } N \to \infty.$$

Thus since $\varepsilon > 0$ is arbitrary, we obtain with high probability as $N \to \infty$,

$$\sup_{t \in [0,T]} \sum_{i=1}^b |q_i(G_N,t) - q_i(t)| \leq \varepsilon'',$$

for all $\varepsilon'' > 0$, which completes the proof of Part (i).
(ii) To prove the diffusion-level optimality of $G_N$, again fix any $\varepsilon > 0$. As in Part (i), using Proposition 5 and Proposition 6 (ii), there exists $\varepsilon' > 0$

$$\sup_{t \in [0,T]} \frac{1}{\sqrt{N}} \sum_{i=1}^{b} |Q_t(I(G_N, t) - Q_t(I(G_N, m_{\varepsilon'}(N)), t))| \leq \frac{\Delta_{I}(T)}{\sqrt{N}} \leq \varepsilon.$$ 

Furthermore, since $I(G_N, m_{\varepsilon'}(N)) \in CJSQ(m_{\varepsilon'}(N))$ and $m_{\varepsilon'}(N)/\sqrt{N} \to 0$, Proposition 4 yields

$$\{(Q_t(I(G_N, m_{\varepsilon'}(N)), t), Q_t(I(G_N, m_{\varepsilon'}(N)), t), \ldots)\}_{t \geq 0} \overset{\mathcal{L}}{\to} \{(Q_t(t), Q_2(t), \ldots)\}_{t \geq 0'},$$

as $N \to \infty$, where the process $(Q_1(\cdot), Q_2(\cdot), \ldots)$ given by (5). Since $\varepsilon > 0$ is arbitrary, we thus obtain

$$\{(Q_t(G_N, t), Q_t(G_N, t), \ldots)\}_{t \geq 0} \overset{\mathcal{L}}{\to} \{(Q_t(t), Q_2(t), \ldots)\}_{t \geq 0'},$$

as $N \to \infty$, which completes the proof of Part (ii). \hfill \Box

Proof of Theorem 9. In this proof we will verify the conditions stated in Theorem 2 for fluid and diffusion level optimality. Fix any $\varepsilon' = 2\varepsilon > 0$.

(i) Observe that for $G_N = (V_N, E_N)$ as described in Theorem 9 (i), we can get $V_N^c \subseteq V_N$ with $|V_N^c| < \varepsilon N$, such that $p(N) := \inf \{p^N_{uv} : u, v \in V_N^c\}$ with $Np(N) \to \infty$ as $N \to \infty$. Now, for any function $n : N \to \mathbb{N}$,

$$\begin{align*}
\mathbb{P} & (\exists V_1, V_2 \subseteq V_N, \text{ with } |V_1| \geq \varepsilon' N, |V_2| \geq n(N), \text{ and } E_N(V_1, V_2) = 0) \\
& \leq \mathbb{P} (\exists V'_1, V'_2 \subseteq V_N \setminus V_N^c, \text{ with } |V'_1| \geq \varepsilon N, |V'_2| \geq n(N), \text{ and } E_N(V'_1, V'_2) = 0) \\
& \leq \mathbb{P} (\exists V'_1, V'_2 \subseteq V_N \setminus V_N^c, \text{ with } |V'_1| = \varepsilon N, |V'_2| = n(N), \text{ and } E_N(V'_1, V'_2) = 0) \\
& \leq \binom{N(1-\varepsilon)}{\varepsilon N} \binom{N-2\varepsilon N}{n(N)} (1-p(N))^{\varepsilon N n(N)} \\
& \\leq \frac{1}{[\varepsilon^\varepsilon(1-\varepsilon)^{1-\varepsilon}]^N} \times \frac{\binom{N}{n(N)}}{(1-p(N))^{N(1-\varepsilon)}} \times \exp(-\varepsilon Np(N)n(N)) \\
& \lesssim \frac{\exp(-\varepsilon Np(N)n(N)) \times \exp(n(N) \ln(N))}{\exp\left(N \ln[\varepsilon^\varepsilon(1-\varepsilon)^{1-\varepsilon}]\right) \exp(-n(N))}.
\end{align*}$$

(14)

Choosing $n(N) = N/\sqrt{Np(N)}$ say, it can be seen that for any $p(N)$ such that $Np(N) \to \infty$ as $N \to \infty$, $n(N)/N \to 0$ and the above probability goes to 0. Therefore for any $\varepsilon', \delta > 0$, (14) yields

$$\mathbb{P} (\text{DIS}_1(G_N, \varepsilon') > \delta N) \leq \mathbb{P} (\exists U \subseteq V_N : |U| \geq \varepsilon' N \text{ and COM}(U) \geq \delta N) \to 0, \text{ as } N \to \infty.$$

(ii) Again, for $G_N = (V_N, E_N)$ as described in Theorem 9 (i), we can get $V_N^c \subseteq V_N$ with $|V_N^c| < \varepsilon \sqrt{N}$, such that $p(N) := \inf \{p^N_{uv} : u, v \in V_N^c\}$ with $Np(N)/\sqrt{N \log(N)} \to \infty$ as $N \to \infty$. 

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Now as in Part (i), for any function \( n : \mathbb{N} \to \mathbb{N} \),
\[
P \left( \exists V_1, V_2 \subseteq V_N, \text{ with } |V_1| \geq \epsilon' \sqrt{N}, |V_2| \geq n(N), \text{ and } E_N(V_1, V_2) = 0 \right)
\]
\[
\leq P \left( \exists V_1', V_2' \subseteq V_N \setminus V_N', \text{ with } |V_1'| = \epsilon \sqrt{N}, |V_2'| = n(N), \text{ and } E_N(V_1', V_2') = 0 \right)
\]
\[
\leq \left( \frac{N - \epsilon \sqrt{N}}{\epsilon \sqrt{N}} \right) \left( \frac{N - 2 \epsilon \sqrt{N}}{n(N)} \right) (1 - p(N)) \epsilon \sqrt{N} n(N) \]
\[
\leq N^{\epsilon \sqrt{N}/2} \exp(\epsilon \sqrt{N}) \times n(N) \exp \left( \frac{-\epsilon n(N)}{\sqrt{N}} + n(N) \left( 1 - \frac{n(N)}{N - \epsilon \sqrt{N}} \right) \right) \times \exp(-\epsilon \sqrt{N} p(N)n(N)).
\]
\[
(15)
\]
Choosing \( n(N) = \sqrt{N \ln(N)/p(N)} \), it can be seen that as \( N \to \infty \), \( n(N)/\sqrt{N} \to 0 \) and the above probability converges to 0. Therefore for any \( \epsilon', \delta > 0 \), (15) yields
\[
P \left( \text{dis}_2(G_N, \epsilon') > \delta \sqrt{N} \right) \leq P \left( \exists U \subseteq V_N : |U| \geq \epsilon' \sqrt{N} \text{ and } \text{com}(U) \geq \delta \sqrt{N} \right) \to 0, \text{ as } N \to \infty.
\]
This completes the proof of Theorem 9.

\[\square\]

**Proof of Theorem 11.** We will again verify the conditions stated in Theorem 2 for fluid and diffusion level optimality. For \( k \geq 1 \), denote \([2k - 1]!! = (2k - 1)(2k - 3)\ldots 3.1\).

(i) For any function \( n : \mathbb{N} \to \mathbb{N} \),
\[
P \left( \exists V_1, V_2 \subseteq V_N, \text{ with } |V_1| \geq \epsilon N, |V_2| \geq n(N), \text{ and } E_N(V_1, V_2) = 0 \right)
\]
\[
\leq P \left( \exists V_1', V_2' \subseteq V_N \setminus V_N', \text{ with } |V_1'| \geq \epsilon N, |V_2'| \geq n(N), \text{ and } E_N(V_1', V_2') = 0 \right)
\]
\[
\leq P \left( \exists V_1', V_2' \subseteq V_N \setminus V_N', \text{ with } |V_1'| = \epsilon N, |V_2'| = n(N), \text{ and } E_N(V_1', V_2') = 0 \right)
\]
\[
\leq \left( \frac{N}{\epsilon N} \right) \left( \frac{N - \epsilon N}{n(N)} \right) \frac{(Nd(N)(1 - \epsilon) - 1)!!}{(Nd(N) - 1)!!} \frac{(Nd(N) - n(N)d(N) - 1)!!}{(Nd(N)(1 - \epsilon) - n(N)d(N) - 1)!!}
\]
\[
\leq \frac{1}{(\epsilon^2(1 - \epsilon)^{1-\epsilon})N} \times \left( \frac{N}{n(N)} \right)^{n(N)} \frac{n(N)}{N^{1-\epsilon}} \times \exp(-\epsilon d(N)n(N))
\]
\[
\leq \frac{\exp(-\epsilon d(N)n(N)) \times \exp(n(N) \ln(N))}{\exp(N \ln[\epsilon^2(1 - \epsilon)^{1-\epsilon}]) \exp(-n(N))}.
\]
Choosing \( n(N) = N/\sqrt{d(N)} \) say, it can be seen that for any \( p(N) \) such that \( d(N) \to \infty \) as \( N \to \infty \), \( n(N)/N \to 0 \) and the above probability goes to 0. Therefore for any \( \epsilon', \delta > 0 \), (16) yields
\[
P \left( \text{dis}_1(G_N, \epsilon') > \delta N \right) \leq P \left( \exists U \subseteq V_N : |U| \geq \epsilon' N \text{ and } \text{com}(U) \geq \delta N \right) \to 0, \text{ as } N \to \infty.
\]
(ii) Again, as in Part (i), for any function \( n : \mathbb{N} \to \mathbb{N} \),
\[
\mathbb{P} \left( \exists V_1, V_2 \subseteq V_N, \text{ with } |V_1| \geq \epsilon' \sqrt{N}, |V_2| \geq n(N), \text{ and } E_N(V_1, V_2) = 0 \right)
\leq \mathbb{P} \left( \exists V'_1, V'_2 \subseteq V_N \setminus V_N^0, \text{ with } |V'_1| = \epsilon \sqrt{N}, |V'_2| = n(N), \text{ and } E_N(V_1, V_2) = 0 \right)
\leq \exp \left( \frac{\epsilon \sqrt{N} \log(N)}{2} - \frac{n(N) d(N)}{\sqrt{N}} \right).
\]

Now, choosing \( n(N) = \sqrt{N}/\sqrt{d(N)/(\sqrt{N} \log(N))} \), it can be seen that as \( N \to \infty \), \( n(N)/\sqrt{N} \to 0 \) and the above probability converges to 0. Therefore for any \( \epsilon', \delta > 0 \), (17) yields
\[
\mathbb{P} \left( \text{DIS}_2(G_N, \epsilon') > \delta \sqrt{N} \right) \leq \mathbb{P} \left( \exists U \subseteq V_N : |U| \geq \epsilon' \sqrt{N} \text{ and } \text{com}(U) \geq \delta \sqrt{N} \right) \to 0, \quad \text{as } N \to \infty.
\]

\[\square\]

**Proof of Theorem 7.** For brevity, denote by \( \Xi_N(M) \subseteq V_N \) the set of all vertices with degree at most \( M \). Since \( |\Xi_N(M)|/N \leq 1 \), from (10) we have a convergent subsequence \( \{\Xi_{n_n}(M)\}_{n_n \geq 1} \) with \( \{n_n\}_{n_n \geq 1} \subseteq \mathbb{N} \), such that \( |\Xi_{n_n}(M)|/N \to \xi > 0 \), as \( N \to \infty \). For the rest of the proof we will considered the asymptotic statements along this subsequence, and hence omit the subscript \( n \).

Let the system start from an occupancy state where all the vertices in \( \Xi_N(M) \) are empty. We will show that in finite time, a positive fraction of vertices in \( \Xi_N(M) \) will have at least two tasks. This will prove the fluid limit sample path cannot agree with that of the sequence of cliques, and hence \( \{G_N\}_{n \geq 1} \) cannot be N-optimal. The idea of the proof is as follows: If a graph contains \( \Theta(N) \) bounded degree vertices, then starting from all empty servers, in any finite time interval there will be \( \Theta(N) \) servers \( u \) say, for which all the servers in \( N[u] \) have at least one task. For all such servers an arrival at \( u \) must produce a server of queue length two. Thus, it shows that the instantaneous rate at which servers of queue length two are formed is bounded away from zero, and hence \( \Theta(N) \) servers of queue length two are produced.

Let \( u \) be a vertex with degree \( M \) or less in \( G_N \). Consider the event \( \xi_N(u, t) \) that at time \( t \) all vertices in \( N[u] \) have at least one job. Note that since \( M < \infty \) is fixed, for any \( \epsilon > 0 \),
\[
\mathbb{P} \left( \xi_N(u, t) \right) \geq \delta(t) \quad \text{for some } \delta(t) > 0, \text{ for all } N \geq 1.
\]

To see this, note that \( \delta(t) \) is the probability that before time \( t \) there are \( M + 1 \) arrivals at vertex \( u \) and no departure has taken place. Also observe that for \( u, v \in V_N \),
\[
\mathbb{P} \left( \xi_N(u, t) \cap \xi_N(v, t) \right) \geq \delta(t)^2.
\]

Indeed since the probability of the event \( \xi_N(u, t) \cap \xi_N(v, t) \) can be lower bounded by the event that before time \( t \) there are \( M + 1 \) arrivals at vertex \( u \), \( M + 1 \) arrivals at vertex \( v \), and no departure has taken place from \( N[u] \cup N[v] \). Thus, at time \( t \), the fraction of vertices in \( \Xi_N(M) \) for which all the neighboring vertices have at least one task, is lower bounded by \( \delta(t) \). Now the proof is completed by considering the following: let \( u \) be a vertex of degree \( M < \infty \) for which all the neighbors have at least one task. Then at such an instance if a task arrives at server \( u \), it must be assigned to a server with queue length one, and hence a server with queue length two will
be formed. Therefore the total scaled instantaneous rate at which the number of queue length 2 is being formed at time $t$ is at least $\delta(t) > 0$, which also gives the total rate of increase of the fraction of vertices with at least 2 tasks.

**Proof of Theorem 8.** Fix a $c > 0$. Construct the graph sequence $\{G_N\}_{N \geq 1}$ as a sequence of complete bipartite graphs with size of one partite set of the $N$-th graph to be $\lceil cN \rceil$, i.e., $V_N = A_N \cup B_N$, such that $|A_N| = \lceil cN \rceil$ and $B_N = V_N \setminus A_N$, and the edge set is given by $E_N = \{(u, v) : u \in A_N, v \in B_N\}$. Note that $d_{\text{min}}(G_N)/N \to c$, as $N \to \infty$. We will show that for any $0 < c < 1/2$, there exists $\lambda$, such that $G$ is sub-optimal on $N$-scale.

Assume on the contrary that $G$ is $N$-optimal. Denote by $q_{1,A}^N(t)$ and $q_{1,B}^N(t)$ the fractions of vertices with at least $i$ tasks in partite sets $A_N$ and $B_N$, respectively. Assume $q_{2,A}^N(0) = 0$, for all $N$. Observe that as long as $c - q_{1,A}^N > 0$, any external arrival to servers $B_N$ will be assigned to an empty server in $A_N$. Similarly, as long as $1 - c - q_{1,B}^N > 0$, any external arrival to servers $A_N$ will be assigned to an empty server in $B_N$. Thus one can show that as $N \to \infty$, until $q_{1,A}^N$ hits $c$, the processes $\{q_{1,A}^N(t)\}$ and $\{q_{2,B}^N(t)\}$ converges weakly to a deterministic process described by the following set of ODE’s:

$$
q_{1,A}^N(t) = \lambda(1 - c) - q_{1,A}(t) \\
q_{1,B}^N(t) = \lambda c - q_{1,B}(t).
$$

(19)

When $\lambda > c/(1 - c)$, in finite time $q_{1,A}(t)$ hits $\epsilon$. Since the total scaled arrival rate into the system of $N$ servers is $\lambda$, $q_{1,B}(t)$ must approach $\lambda - c$ as $t \to \infty$. Now we claim that when the fraction of servers in $A_N$ with queue length one is $c$, if a task appears at a server $v$ in $B_N$ that has queue length one, then with high probability it will be assigned to a server in $A_N$. To see this, note that at such an arrival if there is an empty server in $A_N$, then the arriving task is clearly assigned to the idle server, otherwise, when there is no empty server in $A_N$, the arriving task is assigned uniformly at random among the vertices in $N[v]$ having queue length one. Since there are $\Theta(N)$ vertices in $A_N$ with queue length one, the arriving task with probability $1 - O(1/N)$ joins a server in $A_N$. Therefore, the total scaled rate of tasks arriving into the servers in $A_N$ is at least $\lambda(\lambda - c)$, whereas the total scaled rate at which tasks can leave from $A_N$ is at most $c$. Thus if $\lambda(\lambda - c) > c$, then in finite time, a positive fraction of servers in $A_N$ will have queue length two or larger.

$$
\lambda(\lambda - c) > c \implies \lambda > \frac{c + \sqrt{c^2 + 4c}}{2}.
$$

It can be seen that $(c + \sqrt{c^2 + 4c})/2 < 1$ for any $c \in (0, 1/2)$. This completes the proof of Theorem 8.

**B Coupling and stochastic ordering**

**B.1 Stack formation and deterministic ordering**

In order to prove the stochastic comparisons among the various schemes (topological of non-topological), as in [21], we describe the many-server system as an ensemble of stacks, in a way that two different ensembles can be ordered. In this formulation, at each step, items are added or removed according to some rule. From a high level, we then show that if two systems follow some specific rules, then at any step, the two ensembles maintain some kind of deterministic
ordering. We will see that this deterministic ordering turns into an almost sure ordering when the systems are S-coupled.

Each server along with its queue is thought of as a stack of items, and we always consider the stacks to be arranged in nondecreasing order of their heights. The ensemble of stacks then represents the empirical CDF of the queue length distribution, and the $i^{th}$ horizontal bar corresponds to $Q_i^\Pi$ (for some task assignment scheme $\Pi$), as depicted in Figure 1a. If an arriving item happens to land on a stack which already contains $b$ items, then the item is discarded, and is added to a special stack $L^\Pi$ of discarded items, where it stays forever.

Any two ensembles $A$ and $B$, each having $N$ stacks and a maximum height $b$ per stack, are said to follow Rule($n_A, n_B, k$) at some step, if either an item is removed from the $k^{th}$ stack in both ensembles (if nonempty), or an item is added to the $n_A^{th}$ stack in ensemble $A$ and to the $n_B^{th}$ stack in ensemble $B$.

**Proposition 13.** For any two ensembles of stacks $A$ and $B$, as described above, if at any step Rule($n_A, n_B, k$) is followed for some value of $n_A$, $n_B$, and $k$, with $n_A \leq n_B$, then the following ordering is always preserved: for all $m \leq b$,

$$
\sum_{i=m}^{b} Q_i^A + L^A \leq \sum_{i=m}^{b} Q_i^B + L^B. \tag{20}
$$

This proposition says that, while adding the items to the ordered stacks, if we ensure that in ensemble $A$ the item is always placed to the left of that in ensemble $B$, and if the items are removed from the same ordered stack in both ensembles, then the aggregate size of the $b - m + 1$ highest horizontal bars as depicted in Figure 1a plus the cumulative number of discarded items is no larger in $A$ than in $B$ throughout.

**Proof of Proposition 13.** We prove the ordering by forward induction on the time-steps, i.e., we assume that at some step the ordering holds, and show that in the next step it will be preserved. In ensemble $\Pi$, where $\Pi = A, B$, after applying Rule($n_A, n_B, k$), the updated lengths of the horizontal bars are denoted by $Q_i^\Pi$, $i \geq 1$. Also, define $I^\Pi(c) := \max \{i \geq 0 : Q_i^\Pi \geq N - c + 1\}$, $c = 1, \ldots, N$, with the convention that $Q_0^\Pi \equiv N$. 

Figure 1: (a) The occupancy state of the system; When the servers are arranged in nondecreasing order of their queue lengths, $Q_i$ represents the width of the $i^{th}$ row. (b) High-level view of the CJSQ($n(N)$) class of schemes, where the servers are arranged in the nondecreasing order of their queue lengths, and the arrival must be assigned through the left tunnel.
Now if the rule prescribes removal of an item from the $k^{th}$ stack, then the updated ensemble will have the values
\[ \tilde{Q}_i^\Pi = \begin{cases} Q_i^\Pi - 1, & \text{for } i = I^\Pi(k), \\ Q_j^\Pi, & \text{otherwise,} \end{cases} \quad (21) \]
if $I^\Pi(k) \geq 1$; otherwise all the $Q_i^\Pi$-values remain unchanged. On the other hand, if the rule produces the addition of an item to stack $n^\Pi$, then the values will be updated as
\[ \tilde{Q}_i^\Pi = \begin{cases} Q_i^\Pi + 1, & \text{for } i = I^\Pi(n^\Pi) + 1, \\ Q_j^\Pi, & \text{otherwise,} \end{cases} \quad (22) \]
if $I^\Pi(n^\Pi) < b^\Pi$, otherwise all values remain unchanged.

Fix any $m \leq b$. Observe that in any event the $Q_i$-values change by at most one at any step, and hence it suffices to prove the preservation of the ordering in the case when (20) holds with equality:
\[ \sum_{i=m}^{b} Q_i^A + L^A = \sum_{i=m}^{b} Q_i^B + L^B. \quad (23) \]

We distinguish between two cases depending on whether an item is removed or added. First suppose that the rule prescribes removal of an item from the $k^{th}$ stack from both ensembles. Observe from (21) that the value of $\sum_{i=m}^{b} Q_i^\Pi + L^\Pi$ changes if and only if $I^\Pi(k) \geq m$. Also, since removal of an item can only decrease the sum, without loss of generality we may assume that $I_B(k) \geq m$, otherwise the right side of (23) remains unchanged, and the ordering is trivially preserved. From our initial hypothesis,
\[ \sum_{i=m+1}^{b} Q_i^A + L^A \leq \sum_{i=m+1}^{b} Q_i^B + L^B. \quad (24) \]
This implies
\[ Q_m^A = \sum_{i=m}^{b} Q_i^A - \sum_{i=m+1}^{b} Q_i^A \geq \sum_{i=m}^{b} Q_i^B - \sum_{i=m+1}^{b} Q_i^B = Q_m^B. \quad (25) \]
Also,
\[ I_B(k) \geq m \iff Q_m^B \geq N - k + 1 \iff Q_m^A \geq N - k + 1 \iff I_A(k) \geq m. \quad (26) \]
Therefore the sum $\sum_{i=m}^{b} Q_i^A + L^A$ also decreases, and the ordering is preserved.

Now suppose that the rule prescribes addition of an item to the respective stacks in both ensembles. From (22) we get that after adding an item, the value of $\sum_{i=m}^{b} Q_i^\Pi + L^\Pi$ increases only if $I^\Pi(n^\Pi) \geq m$. As in the previous case, we assume (23), and since adding an item can only increase the concerned sums, we assume that $I_A(n_A) \geq m$, because otherwise the left side of (23) remains unchanged, and the ordering is trivially preserved. Now from our initial hypothesis we have
\[ \sum_{i=1}^{b} Q_i^A + L^A \leq \sum_{i=1}^{b} Q_i^B + L^B. \quad (27) \]
Combining (23) with (27) gives
\[\sum_{i=1}^{m-1} Q_i^A = \left( \sum_{i=1}^{b} Q_i^A + L^A \right) - \left( \sum_{i=m}^{b} Q_i^A + L^A \right) \leq \left( \sum_{i=1}^{b} Q_i^B + L^B \right) - \left( \sum_{i=m}^{b} Q_i^B + L^B \right) = \sum_{i=1}^{m-1} Q_i^B.\] (28)

Observe that
\[I_A(n_A) \geq m \iff Q_m^A \geq N - n_A + 1 \iff Q_m^A \geq N - n_B + 1 \iff I_B(n_B) \geq m.\] (29)

Hence, the value of \(\sum_{i=m}^{b} Q_i^B + L^B\) also increases, and the ordering is preserved. \(\square\)

### B.2 Stochastic ordering

We now use the deterministic ordering established in Proposition 13 in conjunction with the S-coupling construction to prove a stochastic comparison between any specific scheme from the class \(\text{CJSQ}(n(N))\) and the ordinary JSQ policy. Call any two systems \(S\)-coupled, if they have synchronized arrival clocks and departure clocks of the \(k\)-th longest queue, for \(1 \leq k \leq N\) (‘S’ in the name of the coupling stands for ‘Server’). As described earlier, the class \(\text{CJSQ}(n(N))\) contains all schemes that assign incoming tasks by some rule to any of the \(n\) ordered servers. Observe that when \(n(N) = 0\), the class contains only the ordinary JSQ policy. Also, if \(n^{[1]}(N) < n^{[2]}(N)\), then \(\text{CJSQ}(n^{[1]}(N)) \subset \text{CJSQ}(n^{[2]}(N))\). Let \(\text{MJSQ}(n(N))\) be a particular scheme that always assigns incoming tasks to precisely the \((n(N) + 1)\)th ordered server. Notice that this scheme is effectively the JSQ policy when the system always maintains \(n(N)\) idle servers, or equivalently, uses only \(N - n(N)\) servers, and \(\text{MJSQ}(n(N)) \in \text{CJSQ}(n(N))\). For brevity, we suppress \(n(N)\) in the notation for the remainder of this subsection.

Consider three \(S\)-coupled systems following respectively the JSQ policy, any scheme from the class \(\text{CJSQ}\), and the \(\text{MJSQ}\) scheme. Recall that \(Q_i^{[\Pi]}(t)\) is the number of servers with at least \(i\) tasks at time \(t\) and \(L_i^{[\Pi]}(t)\) is the total number of lost tasks up to time \(t\), for the schemes \(\Pi = \text{JSQ}, \text{CJSQ}, \text{MJSQ}\). The following proposition provides a stochastic ordering for any scheme in the class \(\text{CJSQ}\) with respect to the ordinary JSQ policy and the \(\text{MJSQ}\) scheme.

**Proposition 14.** For any fixed \(m \geq 1\),

(i) \[\left\{ \sum_{i=m}^{b} Q_i^{\text{JSQ}}(t) + L^{\text{JSQ}}(t) \right\}_{t \geq 0} \preceq_{st} \left\{ \sum_{i=m}^{b} Q_i^{\text{CJSQ}}(t) + L^{\text{CJSQ}}(t) \right\}_{t \geq 0},\]

(ii) \[\left\{ \sum_{i=m}^{b} Q_i^{\text{MJSQ}}(t) + L^{\text{MJSQ}}(t) \right\}_{t \geq 0} \preceq_{st} \left\{ \sum_{i=m}^{b} Q_i^{\text{MJSQ}}(t) + L^{\text{MJSQ}}(t) \right\}_{t \geq 0},\]

provided the inequalities hold at time \(t = 0\).

The above proposition has the following immediate corollary, which will be used to prove bounds on the fluid and the diffusion scale.

**Corollary 15.** In the joint probability space constructed by the \(S\)-coupling of the three systems under respectively \(\text{JSQ}, \text{MJSQ}, \text{and any scheme from the class} \text{CJSQ}\), the following ordering is preserved almost surely throughout the sample path: for any fixed \(m \geq 1\)
and the tail bound in Proposition \( \overline{4} \) as given
\[ \sum_{b=m}^t Q_m^{\text{JSQ}}(t) = \sum_{i=m+1}^t Q_i^{\text{MJSQ}}(t) + L^{\text{JSQ}}(t) - L^{\text{MJSQ}}(t), \]
\[ \sum_{b=m}^t Q_m^{\text{CJSQ}}(t) \leq \sum_{i=m+1}^t Q_i^{\text{MJSQ}}(t) + L^{\text{JSQ}}(t) - L^{\text{MJSQ}}(t), \]

provided the inequalities hold at time \( t = 0 \).

**Proof of Proposition 14.** We first S-couple the concerned systems. Let us say that an incoming task is assigned to the \( \eta^\Pi \)-th ordered server under scheme \( \Pi \), \( \Pi = \text{JSQ, CJSQ, MJSQ} \). Then observe that, under the S-coupling, almost surely, \( \eta^{\text{JSQ}} \leq \eta^{\text{CJSQ}} \leq \eta^{\text{MJSQ}} \). Therefore, Proposition 13 ensures that in the probability space constructed through the S-coupling, the ordering is preserved almost surely throughout the sample path. \( \square \)

**Proof of Proposition 4.** We show that if \( n(N)/N \to 0 \) and \( n(N)/\sqrt{N} \to 0 \), then any scheme in the class \( \text{CJSQ}(n(N)) \) has the same process-level limits on \( N \)-scale and \( \sqrt{N} \)-scale, respectively. This establishes the asymptotic optimality on respective scales.

(i) Define \( \bar{N} = N - n(N) \) and \( \bar{\lambda}(\bar{N}) = \lambda(N) \). Observe that the MJSQ(\( n(N) \)) scheme with \( N \) servers can be thought of as the clique with \( \bar{N} \) servers and arrival rate \( \bar{\lambda}(\bar{N})/\bar{N} \) per server. Also, since \( n(N)/N \to 0 \),

\[ \frac{\bar{\lambda}(\bar{N})}{\bar{N}} = \frac{\lambda(N)}{N - n(N)} \to \lambda \quad \text{as} \quad \bar{N} \to \infty. \]

Furthermore, observe that the limit of the scaled occupancy processes in Theorem 17 as given by (31) is characterized by the parameter \( \lambda \) only, and hence the fluid limit of the MJSQ(\( n(N) \)) scheme is the same as that of the clique.

Now, observe from the fluid limit of the occupancy processes of cliques that if \( \lambda < 1 \), then for any buffer capacity \( b \geq 1 \), and any starting state, the fluid-scaled cumulative overflow is negligible, i.e., for any \( t \geq 0 \), \( L^N(t)/N \overset{P}{\to} 0 \). Since the above fact is induced by the fluid limit only, the same holds for the MJSQ(\( n(N) \)) scheme. Therefore, using the lower and upper bounds in Corollary 15 and the tail bound in Proposition 14, we complete the proof of (i).

(ii) To show that if \( n(N)/\sqrt{N} \to 0 \) as \( N \to \infty \), then the MJSQ(\( n(N) \)) scheme has the same diffusion limit as the occupancy processes of cliques, define \( \bar{N} = N - n(N) \) and \( \bar{\lambda}(\bar{N}) = \lambda(N) \). As mentioned earlier, the MJSQ(\( n(N) \)) scheme with \( N \) servers can be thought of as the clique with \( \bar{N} \) servers and arrival rate \( \bar{\lambda}(\bar{N})/\bar{N} \) per server. Also, since \( n(N)/\sqrt{N} \to 0 \),

\[ \frac{N - \bar{\lambda}(\sqrt{N})}{\sqrt{N}} = \frac{N - n(N) - \lambda(N)}{\sqrt{N - n(N)}} \to \beta > 0 \quad \text{as} \quad \bar{N} \to \infty. \]

Furthermore, observe that the diffusion limit of the occupancy processes of cliques in [9, Theorem 2] as given in (33) is characterized by the parameter \( \beta > 0 \), and hence the diffusion limit of the MJSQ(\( n(N) \)) scheme is the same as that of the occupancy processes of cliques.

Observe from the diffusion limit of the cliques that if \( \beta > 0 \), then for any buffer capacity \( b \geq 2 \), and suitable initial state as described in Theorem 19, the cumulative overflow is negligible, i.e., for any \( t \geq 0 \), \( L^N(t) \overset{P}{\to} 0 \). Indeed observe that if \( b \geq 2 \), and \( \{Q_2^N(t)\}_{t \geq 0} \) is a tight sequence, then the sequence of processes \( \{Q_2^N(t)\}_{t \geq 0} \) is stochastically bounded. Therefore, on any finite time interval, there will be only \( O_P(\sqrt{N}) \) servers with queue length more than one, whereas, for an overflow event to occur all the \( N \) servers must have at least two pending tasks. Therefore, for
any \( t \geq 0, \)

\[
\limsup_{N \to \infty} \mathbb{P} \left( L^N(t) > 0 \right) \leq \limsup_{N \to \infty} \mathbb{P} \left( \sup_{s \in [0,t]} Q_2^N(s) = N \right) \\
\leq \limsup_{N \to \infty} \mathbb{P} \left( \sup_{s \in [0,t]} \bar{Q}_2^N(s) = \sqrt{N} \right) = 0.
\]

Since the above fact is implied by the diffusion limit only, the same holds for the MJSQ\(\{n(N)\}\) scheme. Therefore, using the lower and upper bounds in Corollary 15, we complete the proof of (ii).

We now present one further key tool for stochastic comparison of two different (topological or non-topological) systems. Consider two S-coupled systems following schemes \( \Pi_1 \) and \( \Pi_2 \). Fix a specific arrival epoch, and let the arriving task join the \( n_{\Pi_1} \)-th ordered server in the \( i \)-th system following scheme \( \Pi_i \), \( i = 1, 2 \) (ties can be broken arbitrarily in both systems). We say that at a specific arrival epoch the two systems differ in decision, if \( n_{\Pi_1} \neq n_{\Pi_2} \), and denote by \( \Delta_{\Pi_1,\Pi_2}(t) \) the cumulative number of times the two systems differ in decision up to time \( t \).

**Proposition 16.** For two S-coupled systems under schemes \( \Pi_1 \) and \( \Pi_2 \) the following inequality is preserved almost surely

\[
\sum_{i=1}^{b} \left| Q_1^{\Pi_i}(t) - Q_2^{\Pi_i}(t) \right| \leq 2\Delta_{\Pi_1,\Pi_2}(t) \quad \forall \ t \geq 0,
\]

provided the two systems start from the same occupancy state at \( t = 0 \), i.e., \( Q_1^{\Pi_i}(0) = Q_2^{\Pi_i}(0) \) for all \( i = 1, 2, \ldots, b \).

**Proof of Proposition 16.** We will again use forward induction on the event times of arrivals and departures. Let the inequality (30) hold at time epoch \( t_0 \), and let \( t_1 \) be the next event time. We distinguish between two cases, depending on whether \( t_1 \) is an arrival epoch or a departure epoch.

If \( t_1 \) is an arrival epoch and the systems differ in decision, then observe that the left side of (30) can only increase by two. In this case, the right side also increases by two, and the inequality is preserved. Therefore, it is enough to prove that the left side of (30) remains unchanged if the two systems do not differ in decision. In that case, assume that both \( \Pi_1 \) and \( \Pi_2 \) assign the arriving task to the \( k \)-th ordered server. Recall from the proof of Proposition 13 the definition of \( I_{\Pi} \) for some scheme \( \Pi \). If \( I_{\Pi_1}(k) = I_{\Pi_2}(k) \), then the left side of (30) clearly remains unchanged. Now, without loss of generality, assume \( I_{\Pi_1}(k) < I_{\Pi_2}(k) \). Therefore,

\[
Q_{I_{\Pi_1}(k)+1}^{\Pi_1}(t_0) < Q_{I_{\Pi_1}(k)+1}^{\Pi_2}(t_0) \quad \text{and} \quad Q_{I_{\Pi_2}(k)+1}^{\Pi_1}(t_0) < Q_{I_{\Pi_2}(k)+1}^{\Pi_2}(t_0).
\]

After an arrival, the \((I_{\Pi_1}(k)+1)\)-th term in the left side of (30) decreases by one, and the \((I_{\Pi_2}(k)+1)\)-th term increases by one. Thus the inequality is preserved.

If \( t_1 \) is a departure epoch, then due to the S-coupling, without loss of generality, assume that a potential departure occurs from the \( k \)-th ordered server. Also note that a departure in either of the two systems can change at most one of the \( Q_i \)-values. If at time epoch \( t_0 \), \( I_{\Pi_1}(k) = I_{\Pi_2}(k) \) = \( i \), then both \( Q_i^{\Pi_1} \) and \( Q_i^{\Pi_2} \) decrease by one, and hence the left side of (30) does not change. Otherwise, without loss of generality assume \( I_{\Pi_1}(k) < I_{\Pi_2}(k) \). Then observe that

\[
Q_{I_{\Pi_1}(k)}^{\Pi_1}(t_0) < Q_{I_{\Pi_1}(k)}^{\Pi_2}(t_0) \quad \text{and} \quad Q_{I_{\Pi_2}(k)}^{\Pi_1}(t_0) < Q_{I_{\Pi_2}(k)}^{\Pi_2}(t_0).
\]
Furthermore, after the departure, $Q_{I_{i1}}^{1}(k)$ decreases by one. Therefore $|Q_{I_{i1}}^{1}(k) - Q_{I_{i1}}^{2}(k)|$ increases by one, and $Q_{I_{i2}}^{1}(k)$ decreases by one, thus $|Q_{I_{i1}}^{1}(k) - Q_{I_{i2}}^{1}(k)|$ decreases by one. Hence, in total, the left side of (30) remains the same.

Proof of Proposition 5. With the construction of the scheme $I(G_N, n)$, note that when a task arrives at some vertex $v$ say, the load balancing process on $G_N$ and the scheme $I(G_N, n)$ can differ in decision only if none of the vertices in $S(n)$ is a neighbor of $v$, i.e., when Equation (8) is not satisfied. Thus Proposition 16 completes the proof.

C Limiting behavior of sequence of cliques

In this section we will present the known facts about the behavior of the queue lengths process when the underlying topology is a clique, i.e. the JSQ policy.

C.1 Behavior on N-scale

In the fluid-level analysis, we consider the subcritical regime where $\lambda(N)/N \to \lambda < 1$ as $N \to \infty$. Recall $\chi$ from Equation (12). For any $q \in \chi$, denote $m(q) = \text{min}(i: q_i + 1 < 1)$, with the convention that $q_{b+1} = 0$ if $b < \infty$. Note that $m(q) < \infty$, since $q \in \ell_1$. If $m(q) = 0$, then define $p_0(q) = 1$ and $p_i(q) = 0$ for all $i \geq 1$. If $m(q) > 0$, distinguish two cases, depending on whether the normalized arrival rate $\lambda$ is larger than $1 - q_{m(q)} + 1$ or not. If $\lambda < 1 - q_{m(q)} + 1$, then define $p_{m(q) - 1}(q) = 1$ and $p_i(q) = 0$ for all $i \neq m(q) - 1$. On the other hand, if $\lambda > 1 - q_{m(q)} + 1$, then $p_{m(q) - 1}(q) = (1 - q_{m(q)} + 1)/\lambda$, $p_{m(q)}(q) = 1 - p_{m(q) - 1}(q)$, and $p_i(q) = 0$ for all $i \neq m(q) - 1, m(q)$. Note that the assumption $\lambda < 1$ ensures that the latter case cannot occur when $m(q) = b < \infty$.

Theorem 17. [21, Theorem 2.1] Assume $q(K_N, 0) \to q^\infty$ in $\chi$ and $\lambda(N)/N \to \lambda < 1$ as $N \to \infty$. Then the sequence of processes $\{q(K_N, t)\}_{t \geq 0}$ converges weakly with respect to the $\ell_1$ topology, to the limit $\{q(t)\}_{t \geq 0}$ that satisfies the system of integral equations

$$q_i(t) = q_i^\infty + \lambda \int_0^t p_{i-1}(q(s))ds - \int_0^t (q_i(s) - q_{i+1}(s))ds, \quad i = 1, \ldots, b,$$

where the coefficients $p_i(\cdot)$ are as defined earlier.

The coefficient $p_i(q)$ represents the instantaneous fraction of incoming tasks assigned to servers with a queue length of exactly $i$ in the fluid-level state $q \in \chi$. Assuming $m(q) < b$, a strictly positive fraction $1 - q_{m(q)} + 1$ of the servers have a queue length of exactly $m(q)$. Since $K_N$ always assigns to the shortest queue, the fraction of incoming tasks that get assigned to servers with a queue length of $m(q) + 1$ or larger is zero: $p_i(q) = 0$ for all $i = m(q) + 1, \ldots, b - 1$. Also, tasks at servers with a queue length of exactly $i$ are completed at (normalized) rate $q_i - q_{i+1}$, which is zero for all $i = 0, \ldots, m(q) - 1$, and hence the fraction of incoming tasks that get assigned to servers with a queue length of $m(q) - 2$ or less is zero as well: $p_i(q) = 0$ for all $i = 0, \ldots, m(q) - 2$. This only leaves the fractions $p_{m(q) - 1}(q)$ and $p_{m(q)}(q)$ to be determined. Now observe that the fraction of servers with a queue length of exactly $m(q) - 1$ is zero. If $m(q) = 0$, then clearly the incoming tasks will join the empty queue, and thus, $p_{m(q)} = 1$, and $p_i(q) = 0$ for all $i \neq m(q)$. Furthermore, if $m(q) \geq 1$, since tasks at servers with a queue length of exactly $m(q)$ are completed at (normalized) rate $1 - q_{m(q)} + 1 > 0$, incoming tasks can be assigned
to servers with a queue length of exactly \( m(q) - 1 \) at that rate. We thus need to distinguish between two cases, depending on whether the normalized arrival rate \( \lambda \) is larger than \( 1 - q_{m(q)+1} \) or not. If \( \lambda < 1 - q_{m(q)+1} \), then all the incoming tasks can be assigned to a server with a queue length of exactly \( m(q) - 1 \), so that \( p_{m(q)-1}(q) = 1 \) and \( p_{m(q)}(q) = 0 \). On the other hand, if \( \lambda > 1 - q_{m(q)+1} \), then not all incoming tasks can be assigned to servers with a queue length of exactly \( m(q) - 1 \) active tasks, and a positive fraction will be assigned to servers with a queue length of exactly \( m(q) \): \( p_{m(q)-1}(q) = (1 - q_{m(q)+1})/\lambda \) and \( p_{m(q)}(q) = 1 - p_{m(q)-1}(q) \).

It is easily verified that the unique fixed point \( \bar{q}^* = (q^*_1, q^*_2, \ldots, q^*_b) \) of the system of differential equations in (31) is given by

\[
q^*_i = \begin{cases} 
\lambda, & i = 1, \\
0, & i = 2, \ldots, b.
\end{cases}
\] (32)

The fixed point in (32) in conjunction with the interchange of limits result in Proposition 18 below indicates that in stationarity the fraction of servers with a queue length of two or larger is negligible. Let

\[
\pi^N(\cdot) = \lim_{t \to \infty} \mathbb{P}(q(K_N, t) = \cdot)
\]

be the stationary measure of the occupancy states of the \( N \)th system.

**Proposition 18.** [21, Proposition 2.2] Let \( \pi^N \) be the stationary measure of the occupancy states of the occupancy state process of \( K_N \). Then \( \pi^N \xrightarrow{\mathcal{L}} \pi^* \) as \( N \to \infty \) with \( d(N) \to \infty \), where \( \pi^* = \delta_{\bar{q}^*} \) with \( \delta_x \) being the Dirac measure concentrated upon \( x \), and \( \bar{q}^* \) as in (32).

### C.2 Behavior on \( \sqrt{N} \)-scale

In the diffusion-limit analysis, assume that \( \lambda(N) \) satisfies (3), and \( \hat{Q}(K_N, t) \) is as in (4).

**Theorem 19.** [9, Theorem 2] Assume \( \hat{Q}_1(K_N, 0) \xrightarrow{\mathcal{L}} \hat{Q}_1(0) \) in \( \mathbb{R} \) as \( N \to \infty \), buffer capacity \( b \geq 2 \), and there exists some \( k \geq 2 \) such that \( \hat{Q}_{k+1}(K_N, 0) = 0 \) for all sufficiently large \( N \). For \( d(N)/(\sqrt{N} \log N) \to \infty \), the sequence of processes \( \{\hat{Q}(K_N, t)\}_{t \geq 0} \) converges weakly to the limit \( \{\hat{Q}(t)\}_{t \geq 0} \), where \( \hat{Q}_1(t) \equiv 0 \) for \( i \geq k + 1 \) and \( (\hat{Q}_1(t), \hat{Q}_2(t), \ldots, \hat{Q}_k(t)) \) are the unique solutions in \( D_{\mathbb{R}^k}[0, \infty) \) of the stochastic integral equations

\[
\begin{align*}
\hat{Q}_1(t) &= \hat{Q}_1(0) + \sqrt{2}W(t) - \beta t + \int_0^t (-\hat{Q}_1(s) + \hat{Q}_2(s))ds - U_1(t), \\
\hat{Q}_2(t) &= \hat{Q}_2(0) + U_1(t) - \int_0^t (\hat{Q}_2(s) - \hat{Q}_3(s))ds, \\
\hat{Q}_i(t) &= \hat{Q}_i(0) - \int_0^t (\hat{Q}_i(s) - \hat{Q}_{i-1}(s))ds, \quad i = 3, \ldots, k
\end{align*}
\] (33)

for \( t \geq 0 \), where \( W \) is the standard Brownian motion and \( U_1(t) \) is the unique nondecreasing nonnegative process in \( D_{\mathbb{R}}[0, \infty) \) satisfying \( \int_0^\infty 1_{\hat{Q}_1(t)<0}dU_1(t) = 0 \).

Observe that \( -\hat{Q}_1^N \) is the scaled number of vacant servers. Thus, Theorem 19 shows that over any finite time interval, there will be \( O_P(\sqrt{N}) \) servers with queue length zero and \( O_P(\sqrt{N}) \) servers with a queue length larger than two, and hence all but \( O_P(\sqrt{N}) \) servers have a queue length of exactly one.