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A Kleinman-Newton construction of the maximal solution of the infinite-dimensional control Riccati equation

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Abstract

Assuming only strong stabilizability, we construct the maximal solution of the algebraic Riccati equation as the strong limit of a Kleinman-Newton sequence of bounded nonnegative operators. As a corollary we obtain a comparison of the solutions of two algebraic Riccati equations associated with different cost functions. We show that the weaker strong stabilizability assumptions are satisfied by partial differential systems with collocated actuators and sensors, so the results have potential applications to numerical approximations of such systems. By means of a counterexample, we illustrate that even if one assumes exponential stabilizability, the Kleinman-Newton construction may provide a solution to the Riccati equation that is not strongly stabilizing.

Key words: Riccati equations, maximal solution, infinite-dimensional systems, Kleinman-Newton method, strong stabilizability.

1 Introduction and motivation

Let $\Sigma(A,B,C,D)$ be a state linear system on the separable Hilbert spaces Z, U and Y. This means that A, with domain $D(A) \subset Z$, is the infinitesimal generator of the C_0-semigroup $T(t)$ on Z and the other operators are bounded: $B \in \mathcal{L}(U,Z)$, $C \in \mathcal{L}(Z,Y)$, and $D \in \mathcal{L}(U,Y)$. In this paper we consider the bounded nonnegative solutions ($X \in \mathcal{L}(X)$, $X = X^* \geq 0$) of the operator Riccati equation

$$\langle \Pi z_1, A z_2 \rangle + \langle A z_1, \Pi z_2 \rangle + \langle C z_1, C z_2 \rangle = \langle (B^*\Pi + D^*C) z_1, R^{-1}(B^*\Pi + D^*C) z_2 \rangle, \tag{1}$$

where $z_1, z_2 \in D(A)$ and $R = I + D^*D$. It is well-known (see [2, Lemma 4.1.24]) that (1) is equivalent to the following version:

$$\Pi A z + A^*\Pi z + C^*Cz = (B^*\Pi + D^*C)^* R^{-1}(B^*\Pi + D^*C)z, \tag{2}$$

for $z \in D(A)$. In addition, it is well-known that this Riccati equation is directly related to the minimization of the cost criterium

$$J(z_0,u) = \int_0^\infty \left(\langle y(t), y(t) \rangle + \langle u(t), u(t) \rangle \right) dt,$$

where u, z_0 and y are related via

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \quad z(0) = z_0, \\ y(t) &= Cz(t) + Du(t). \end{aligned}$$

The assumption that there exists an $F \in \mathcal{L}(Z,U)$ such that $J(z_0, Fz) < \infty$ is called *optimizability*.

We construct the maximal solution of (1) as the strong limit of a sequence of bounded nonnegative operators. Our result is a partial generalization to strongly stabilizable systems of the following result from Curtain and Rodman [3] that was obtained for exponentially stabilizable systems. The corresponding finite-dimensional case was presented in Ran and Vreugdenhil [16].

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Theorem 1.1 *Suppose that $\Sigma(A,B,-,-)$ is exponentially stabilizable and there exists a bounded nonnegative X satisfying the following inequality for all $z \in D(A)$:*

$$\mathfrak{Z}^X(z) := \langle Xz,(A-BR^{-1}C_0)z\rangle + \langle (A-BR^{-1}C_0)z, Xz\rangle - \langle B^*Xz, B^*Xz\rangle + \langle Qz,z\rangle \geq 0. \tag{3}$$

Then there exists a nonincreasing sequence $X_n, n \geq 0$ of bounded nonnegative operators such that for $z \in D(A)$

$$(A-BB^*X_n)^*X_{n+1}z + X_{n+1}(A-BB^*X_n)z = -C_0^*C_0z - X_nBB^*X_nz,$$

and $A_{n+1} = A - BR^{-1}C_0 - BB^*X_n, n \geq 0$ generate exponentially stable semigroups. Moreover, the sequence $X_n, n \geq 0$ has the strong limit $X_{\max} \geq X$, which is the maximal bounded nonnegative solution to the inequality (3) *and to the Riccati equation $\mathfrak{Z}^X(z) = 0$.*

Note that for the special case $C_0 = 0$, $Q = C^*C$ the inequality (3) is trivially satisfied by $X = 0$. If, in addition, $\Sigma(A,B,C,-)$ is exponentially detectable, then $A - BB^*X_{\max}$ generates an exponentially stable semigroup, see [2, Theorem 6.2.7]. So as a corollary of Theorem 1.1 we obtain the infinite-dimensional generalization of the convergence of the Kleinman-Newton algorithm of [7]. A special case of Theorem 1.1 was also proved in the later paper by Burns, Sachs and Zietsman [1] under stronger assumptions, see [1, Theorems 6.2, 6.3] and Curtain and Iftime [6]. As explained in [1], the Klein-Newton algorithm has applications to the numerical approximation of very large scale Riccati equations.

The key assumption in Theorem 1.1 is exponential stabilizability of the infinite-dimensional system. However, many partial differential systems are not exponentially stabilizable, but do have nice strong stabilizability properties. In particular, partial differential systems with collocated inputs and outputs often have nice stabilizability properties. They can usually be formulated as a state linear system of the form $\Sigma(A,B,B^*,0)$ on a suitable state space, see Oostveen [15].

Our new contribution is to weaken the assumption of exponential stabilizability in [3] and in [1, Section 6] to a strong stabilizability assumption. Similar assumptions were made in Iftime, Zwart and Curtain [10] to obtain a representation of all self-adjoint solutions to the Riccati equation when A generates C_0-group. The following theorem is a special case of our main result, Theorem 3.1 in Section 3.

Theorem 1.2 *Consider the state linear system $\Sigma(A,B,C,0)$ on the separable Hilbert spaces Z, U and Y. Suppose that $\Sigma(A,B,C,0)$ is optimizable by $F \in \mathcal{L}(Z,U)$ and $A+BF$ generates a strongly stable semigroup, $T_{BF}(t)$. Then the following holds:*

(a) There exists a nonincreasing sequence $X_n, n \geq 0$, of bounded nonnegative operators such that for $z \in D(A)$

$$(A-BB^*X_n)^*X_{n+1}z + X_{n+1}(A-BB^*X_n)^*z = -C^*Cz - X_nBB^*X_nz,$$

and $A_{n+1} = A - BB^*X_n$, $n \geq 0$, generates a strongly stable semigroup;

(b) The sequence $X_n, n \geq 0$, has the strong limit $X_{\max} \geq 0$ which satisfies the Riccati equation (1) *(with $D = 0$);*

(c) $X_{\max}$ is the maximal solution to the Riccati equation (1) and to the inequality
$\langle Xz, Az\rangle + \langle Az, Xz\rangle - \langle B^*Xz, B^*Xz\rangle + \langle Cz, Cz\rangle \geq 0.$

The sequence $X_n, n \geq 0$, in Theorem 1.2 is known as the *Kleinman-Newton iterates*. In Section 3 it is further shown that, under additional assumptions, the semigroup generated by $A - BB^*X_{\max}$ is strongly stable. The following corollary is a special case of Theorem 3.1(d) in Section 3.

Corollary 1.3 *Suppose that the state linear system $\Sigma(A,B,C,0)$ satisfies the conditions in Theorem 1.2. If, in addition, there exists $L \in \mathcal{L}(Y,Z)$ such that $A+LC$ generates a strongly stable semigroup, then there exists a unique nonnegative solution $X_{\max}$ to* (1) *(with $D = 0$) and $A - BB^*X_{\max}$ generates a strongly stable semigroup.*

Clearly, if $\Sigma(A,B,C,0)$ is exponentially stabilizable, then all the assumptions in Theorem 1.2 are satisfied. As noted above, collocated systems of the type $\Sigma(A,B,B^*,0)$ are typically not exponentially stabilizable. However, if $\Sigma(A,B,B^*,0)$ is approximately controllable, A has compact resolvent and A generates a contraction semigroup, then all the assumptions in Theorem 1.2 and Corollary 1.3 are satisfied, see Oostveen [15, Lemma 2.2.6]. In Chapter 9 he gives several examples of partial differential equations with boundary control that can be formulated as such collocated systems on some appropriate Hilbert space. So our new convergence results have potential application to the numerical approximation of Riccati equations for such systems.

In the preliminaries Section 2 we define several stability concepts and collect key results needed for our proofs. The extended formulation to the above results and the proofs are given in Section 3. In addition, we obtain a result comparing the solutions of two Riccati equations, see Corollary 3.3. Finally, it is shown that the weaker strong stabilizability assumptions are satisfied by a class of partial differential systems with collocated actuators and sensors the type $\Sigma(A,B,B^*,0)$. In Section 4 we illustrate by means of a counterexample that, even if one assumes exponential stabilizability, the Kleinman-Newton construction may provide a solution to the Riccati equation that is not strongly stabilizing. Hence one needs to assume both strong output and strong input stabilizability to guarantee that the maximal solution to the Riccati equation constructed using the Kleinman-Newton algorithm is strongly stabilizing.

2 Preliminaries

We use the notation $L_2((a,b);U)$ for the set of Lebesgue measurable U-valued functions $f:(a,b)\to U$ such that

$$\int_a^b \|f(t)\|^2 dt < \infty, \quad 0\le a<b\le\infty.$$

The notation $L_2^{loc}([0,\infty);U)$ is the class of functions that are in $L_2((a,b);U)$ for all $(a,b)\subset[0,\infty)$.

First we recall some definitions of stability.

Definition 2.1 *Let $\Sigma(A,B,C,D)$ denote a state linear system on the Hilbert spaces Z,U,Y, where $B\in\mathcal{L}(U,Z)$, $C\in\mathcal{L}(Z,Y)$, $D\in\mathcal{L}(U,Y)$ and A is the infinitesimal generator of the C_0-semigroup $T(t)$.*

(a) $\Sigma(A,-,C,-)$ *is* output stable *if there exists $\gamma>0$ such that*
$$\int_0^\infty \|CT(t)z\|^2 dt \le \gamma\|z\|^2, \quad z\in Z.$$
(b) $\Sigma(A,B,-,-)$ *is* input stable *if there exists $\beta>0$ such that*
$$\int_0^\infty \|B^*T^*(t)z\|^2 dt \le \beta\|z\|^2, \quad z\in Z.$$
(c) $\Sigma(A,B,C,-)$ *is* input-output stable *if the extended input-output map defined by*
$$(\mathcal{F}^\infty u)(t)=\int_0^t CT(t-s)Bu(s)ds,\ u\in L_2^{loc}([0,\infty);U)$$
defines a bounded map from $L_2([0,\infty);U)$ to $L_2([0,\infty);Y)$.

(d) $\Sigma(A,B,C,D)$ *is* system stable *if it is input, output and input-output stable.*

(e) $T(t)$ *is* strongly stable *if $T(t)z\to0$ as $t\to\infty$ for all $z\in Z$.*

Note that input-output stability is often equivalently defined by $G\in H_\infty(U)$ as in Oostveen [15].

We remark that output (input) stability is also called infinite admissibility of the observation operator C (control operator B) and strong stability is also called asymptotic stability (see e.g. [9]).

Definition 2.2 *Consider the state linear system $\Sigma(A,B,C,0)$.*

(a) $\Sigma(A,B,C,-)$ *is* output stabilizable *if there exists an $F\in\mathcal{L}(Z,U)$ such that $\Sigma(A+BF,B,\binom{C}{F},-)$ is output stable.*

(b) $\Sigma(A,B,C,-)$ *is* strongly output stabilizable *if there exists an $F\in\mathcal{L}(Z,U)$ such that $\Sigma(A+BF,B,\binom{C}{F},-)$ is output stable and $A+BF$ generates a strongly stable semigroup.*

(c) $\Sigma(A,B,C,-)$ *is* input stabilizable *if there exists an $L\in\mathcal{L}(Y,Z)$ such that $\Sigma(A+LC,\begin{pmatrix} B & L\end{pmatrix},C,-)$ is input stable.*

(d) $\Sigma(A,B,C,-)$ *is* strongly input stabilizable *if there exists an $L\in\mathcal{L}(Y,Z)$ such that $\Sigma(A+LC,\begin{pmatrix} B & L\end{pmatrix},C,-)$ is input stable and $A+LC$ generates a strongly stable semigroup.*

Remark 2.3 *Note that $\Sigma(A,B,C,-)$ is output stabilizable if and only if it is optimizable.*

The following lemma on Lyapunov equations was first proved in Grabowski [9, Theorems 3 and 4].

Lemma 2.4 *The Lyapunov equation*

$$\langle Xz,Az\rangle+\langle Az,Xz\rangle=-\langle Cz,Cz\rangle, \quad z\in D(A),$$

has a bounded nonnegative solution X if and only if $\Sigma(A,-,C,-)$ is output stable. If $T(t)$ is strongly stable, then X is the unique bounded nonnegative solution.

We also need some related results on Lyapunov equations.

Lemma 2.5 *Suppose that A generates the C_0-semigroup $T(t)$ on Z and $L\in\mathcal{L}(Z,U)$. Suppose that $X\in\mathcal{L}(Z)$ satisfies the following inequality*

$$\langle Az,Xz\rangle+\langle Xz,Az\rangle\le-\langle Lz,Lz\rangle, \quad z\in D(A). \tag{4}$$

(a) *If $X\in\mathcal{L}(X)$ is nonnegative, then*
$$\int_0^\infty \|LT(t)z\|^2 dt\le\langle Xz,z\rangle, \quad z\in Z;$$
(b) *If $T(t)$ is strongly stable, then $X\ge0$;*

Proof: Substitute $z=T(t)z_0$ in (4) for $t>0$ and an arbitrary $z_0\in D(A)$ to obtain

$$\frac{d}{dt}\langle T(t)z_0,XT(t)z_0\rangle\le-\langle LT(t)z_0,LT(t)z_0\rangle=-\|LT(t)z_0\|^2.$$

Integrating from 0 to t we obtain

$$\langle T(t)z_0,XT(t)z_0\rangle-\langle Xz_0,z_0\rangle\le-\int_0^t\|LT(s)z_0\|^2 ds.$$

Since $D(A)$ is dense in Z the above extends to all $z_0\in Z$, and we obtain

$$\langle T(t)z_0,XT(t)z_0\rangle+\int_0^t\|LT(s)z_0\|^2 ds\le\langle Xz_0,z_0\rangle.$$

a. If X is nonnegative, then for all $t>0$ and all $z_0\in Z$

$$\int_0^t\|LT(s)z_0\|^2 ds\le\langle Xz_0,z_0\rangle.$$

b. If $T(t)$ is strongly stable, then $\|T(t)z_0\|\to0$ as $t\to\infty$ and $\langle Xz_0,z_0\rangle\ge\int_0^\infty\|LT(s)z_0\|^2 ds\ge0$. ∎

Lemma 2.6 *Let $\Sigma(A,B,C,-)$ be a state linear system on the Hilbert space Z and $P,Q \in \mathcal{L}(Z,W)$, where W is a Hilbert space. Suppose that $X = X^* \in \mathcal{L}(Z)$ is a nonnegative solution of the Lyapunov equation*

$$A^*Xz + XAz = -C^*Cz + P^*Qz + Q^*Pz, \quad z \in D(A).$$

If $PT(t)z$ and $QT(t)z$ are in $L_2([0,\infty);W)$, then $CT(t)z$ is in $L_2([0,\infty);Y)$.

Proof: For $z \in D(A)$ we have that the Lyapunov equation is equivalent to

$$\frac{d}{dt}\langle T(t)z, XT(t)z\rangle = \\ -\|CT(t)z\|^2 + \langle PT(t)z, QT(t)z\rangle + \langle QT(t)z, PT(t)z\rangle.$$

Thus for $t > 0$ there holds

$$\langle T(t)z, XT(t)z\rangle = \langle z, Xz\rangle - \int_0^t \|CT(\tau)z\|^2 d\tau \\ + \int_0^t [\langle PT(\tau)z, QT(\tau)z\rangle + \langle QT(\tau)z, PT(\tau)z\rangle]\, d\tau.$$

Since $X \geq 0$ we find that

$$\int_0^t \|CT(\tau)z\|^2 d\tau \leq \langle z, Xz\rangle + \\ \int_0^t [\langle PT(\tau)z, QT(\tau)z\rangle + \langle QT(\tau)z, PT(\tau)z\rangle]\, d\tau.$$

Since the last two integrals can be estimated independently of t, it follows that $CT(t)z$ is in $L_2([0,\infty);Y)$. ■

The following result on strong stability is from Oostveen and Curtain [14, Lemma 12].

Lemma 2.7 *If $\Sigma(A,B,-,-)$ is input stable, $T(t)$ is a strongly stable C_0-semigroup and $u(\cdot) \in L_2([0,\infty);U)$, then*

$$\lim_{t\to\infty} \int_0^t T(t-s)Bu(s)ds = 0. \tag{5}$$

We finish this section with a result on Riccati equations.

Theorem 2.8 *If $\Sigma(A,B,C,D)$ is output stabilizable, then there exists a minimal bounded nonnegative solution Π of the Riccati equation (1). Moreover, the closed-loop system $\Sigma\left(A - BR^{-1}D^*C - BR^{-1}B^*\Pi, B, \begin{pmatrix} S^{-\frac{1}{2}}C \\ R^{-\frac{1}{2}}B^*\Pi \end{pmatrix}, \begin{pmatrix} DR^{-\frac{1}{2}} \\ R^{-\frac{1}{2}} \end{pmatrix}\right)$ is output stable and input-output stable. If, in addition, $\Sigma(A,B,C,D)$ is input stabilizable, then the closed-loop system is system stable. Moreover, if $\Sigma(A,B,C,D)$ is strongly input stabilizable, then the Riccati equation (1) has a unique nonnegative solution and the closed-loop system is strongly stable.*

Proof: The first part was shown in Curtain and Opmeer [5, Lemma 3.4], where it was also shown that it suffices to prove the result for $D = 0$. Suppose now that the system with $D = 0$ is strongly input stabilizable, i.e., there exists $L \in \mathcal{L}(Y,Z)$ such that $\Sigma(A + LC, \begin{pmatrix} B & L \end{pmatrix}, C, 0)$ is input stable and $A + LC$ generates a strongly stable semigroup. Using [2, Theorem 3.2.1], for all $z \in Z$ we have

$$T_{-BB^*\Pi}(t)z = \\ T_{LC}(t)z - \int_0^t T_{LC}(t-s)\,(B,L)\begin{pmatrix} B^*\Pi \\ C \end{pmatrix} T_{-BB^*\Pi}(s)zds.$$

Since $T_{LC}(t)$ is strongly stable and the closed loop system is input stable, from Lemma 2.7 we conclude that $T_{-BB^*\Pi}(t)$ is strongly stable. Note that Lemma 6.2.4 in [2] can be generalized to show that Π must be the maximal solution to the Riccati equation. Since it is both the maximal and the minimal solution, we conclude that it is the unique nonnegative solution. ■

3 Main results and proofs

We first reformulate Theorem 1.2 to include the $D \neq 0$ case. Here we use the term output stabilizable instead of optimizable, see Remark 2.3.

Theorem 3.1 *Let $\Sigma(A,B,C,D)$ be a state linear system with state space Z, and denote $R = I + D^*D$ and $S = I + DD^*$. Suppose that*

- *$\Sigma(A,B,C,D)$ is strongly output stabilizable with the feedback $F \in \mathcal{L}(Z,U)$, and that*
- *$\Sigma\left(A + BF, B, \begin{pmatrix} C \\ F \end{pmatrix}, D\right)$ is input stable.*

Under the above assumptions the following holds:

(a) There exists a sequence X_n, $n \geq 0$ of bounded nonnegative operators such that

$$X_0 \geq \ldots X_{n-1} \geq X_n \geq \ldots \geq 0, \text{ and}$$

$$A_{n+1} = A - BR^{-1}D^*C - BR^{-1}B^*X_n,\ n \geq 0,$$

generate a strongly stable semigroup;

(b) The sequence X_n, $n \geq 0$, has the strong limit $X_{\max} = X^*_{max} \geq 0$, which is the maximal (nonnegative) solution to the Riccati equation (1) and to the following Riccati inequality

$$\langle Xz, (A - BR^{-1}D^*C)z\rangle + \langle (A - BR^{-1}D^*C)z, Xz\rangle \\ -\langle B^*Xz, R^{-1}B^*Xz\rangle + \langle Cz, S^{-1}Cz\rangle \geq 0. \tag{6}$$

(c) If, in addition, there exists $L \in \mathcal{L}(Y,Z)$ such that $\Sigma(A + LC, \begin{pmatrix} B \\ L \end{pmatrix}, C, D)$ is input stable, then the closed-loop system

$$\Sigma\left(A - BR^{-1}D^*C - BR^{-1}B^*X_{\max}, B, \begin{pmatrix} S^{-\frac{1}{2}}C \\ R^{-\frac{1}{2}}B^*X_{\max} \end{pmatrix}, \begin{pmatrix} DR^{-\frac{1}{2}} \\ R^{-\frac{1}{2}} \end{pmatrix}\right)$$

is system stable, i.e. it is input, output and input-output stable;

(d) *If, in addition to the assumptions in part c., $A+LC$ generates a strongly stable semigroup, then the closed-loop generator $A-BR^{-1}D^*C-BR^{-1}B^*X_{\max}$ generates a strongly stable semigroup.*

It is readily verified that the Riccati equation associated with $\Sigma(A,B,C,D)$ is the same as that associated with $\Sigma(A-BR^{-1}D^*C, BR^{-\frac{1}{2}}, S^{-\frac{1}{2}}C, 0)$, where $R=I+D^*D$ and $S=I+DD^*$. Moreover, the assumptions in Theorem 3.1 on $\Sigma(A,B,C,D)$ are also equivalent to those on $\Sigma(A-BR^{-1}D^*C, BR^{-\frac{1}{2}}, S^{-\frac{1}{2}}C, 0)$. In particular, there exists $\tilde{F}\in\mathcal{L}(Z,U)$ such that $A-BR^{-1}D^*C+BR^{-\frac{1}{2}}\tilde{F}$ generates a strongly stable semigroup and the system $\Sigma\left(A-BR^{-1}D^*C+BR^{-\frac{1}{2}}\tilde{F}, BR^{-\frac{1}{2}}, \left(\begin{smallmatrix} S^{-\frac{1}{2}}C \\ \tilde{F}\end{smallmatrix}\right), D\right)$ is input and output stable if and only if there exists $F\in\mathcal{L}(Z,U)$ such that $A+BF$ generates a strongly stable semigroup and $\Sigma(A+BF, B, \left(\begin{smallmatrix} C \\ F\end{smallmatrix}\right), D)$ is input and output stable. The equivalence is via $\tilde{F}=R^{\frac{1}{2}}F+R^{-\frac{1}{2}}D^*C$ for then

$$\Sigma\left(A-BR^{-1}D^*C+BR^{-\frac{1}{2}}\tilde{F}, BR^{-\frac{1}{2}}, \left(\begin{smallmatrix} S^{-\frac{1}{2}}C \\ \tilde{F}\end{smallmatrix}\right), D\right) = \Sigma\left(A+BF, BR^{-\frac{1}{2}}, \left(\begin{smallmatrix} S^{-\frac{1}{2}}C \\ R^{\frac{1}{2}}F+R^{-\frac{1}{2}}D^*C\end{smallmatrix}\right), D\right).$$

A dual remark applies to the existence of L in part (c) of Theorem 3.1. So it is sufficient to prove the results for the case $D=0$.

Proof of Theorem 3.1

It suffices to prove this for the following Riccati equation, i.e. $D=0$ ([2, Lemma 4.1.24]):

$$A^*\Pi z+\Pi Az-\Pi BB^*\Pi z+C^*Cz=0, \quad z\in D(A). \quad (7)$$

We use the notation $T_G(t)$ for the semigroup generated by $A+G$ where $G\in\mathcal{L}(Z)$ and the notation for A_n, X_n as given in the theorem.

(a): (i). First we show the existence of X_0 and X_1. Under our assumptions $\Sigma(A+BF, B, \left(\begin{smallmatrix} C \\ F\end{smallmatrix}\right), -)$ is output stable and by Lemma 2.4, the following Lyapunov equation

$$(A+BF)^*X_0z+X_0(A+BF)z=-C^*Cz-F^*Fz,$$

$z\in D(A)$, has a unique bounded nonnegative solution X_0.

Now consider the following for $z\in D(A)$:

$$\begin{aligned} &A_1^*X_0z+X_0A_1z \\ &=-F^*BX_0z-X_0B^*Fz-F^*Fz-C^*Cz-2X_0BB^*X_0z \\ &=-C^*Cz-(B^*X_0+F)^*(B^*X_0+F)z-X_0BB^*X_0z. \quad (8)\end{aligned}$$

So from Lemma 2.5 we conclude that

$$\int_0^\infty \|B^*X_0T_{-BB^*X_0}(t)z\|^2dt+\int_0^\infty \|CT_{-BB^*X_0}(t)z\|^2dt + \int_0^\infty \|(B^*X_0+F)T_{-BB^*X_0}(t)z\|^2dt \leq \langle X_0z, z\rangle. \quad (9)$$

Definition 2.1 gives that $\Sigma\left(A-BB^*X_0, B, \left(\begin{smallmatrix} C \\ F \\ B^*X_0\end{smallmatrix}\right), -\right)$ is output stable and so, by Lemma 2.4, there exists an unique nonnegative solution X_1 to the Lyapunov equation

$$\begin{aligned}(A-BB^*X_0)^*X_1z+X_1(A-BB^*X_0)z \\ =-X_0BB^*X_0z-C^*Cz, \quad z\in D(A). \quad (10)\end{aligned}$$

To show the strong stability of $T_{-BB^*X_0}$ we use the perturbation result from [2, Theorem 3.2.1]

$$T_{-BB^*X_0}(t)z=T_{BF}(t)z-\int_0^t T_{BF}(t-s)B(F+B^*X_0)T_{-BB^*X_0}(s)zds.$$

By (9), $u(\cdot)=(B^*X_0+F)T_{-BB^*X_0}(\cdot)z\in L_2([0,\infty);U)$. By assumption, $\Sigma(A+BF, B, -, -)$ is input stable and $T_{BF}(t)$ is a strongly stable semigroup. So we can apply Lemma 2.7 to conclude that $T_{-BB^*X_0}(t)z\to 0$ as $t\to\infty$ and $T_{-BB^*X_0}(t)$ is strongly stable. Hence X_1 is the unique solution to (10) (see Lemma 2.4).
(ii). For the induction step we suppose that for $m=0,1,...,n-1$ there exists a sequence of bounded nonnegative operators $X_m\in\mathcal{L}(Z)$ satisfying for $z\in D(A)$

$$\begin{aligned}(A-BB^*X_m)^*X_{m+1}z+X_{m+1}(A-BB^*X_m)^*z \\ =-C^*Cz-X_mBB^*X_mz. \quad (11)\end{aligned}$$

In addition, we suppose that $A_{m+1}:=A-BB^*X_m$ generates a strongly stable semigroup for $m=0,1,...,n-1$.

We show that $A_{n+1}=A-BB^*X_n$ generates a strongly stable semigroup and hence there exists a bounded $X_{n+1}=X_{n+1}^*\geq 0$, the unique solution to (11) for $m=n$.

Note that in part (a): (i) we have already shown the existence of the bounded, self-adjoint, nonnegative operators X_0, X_1 and $A_1=A-BB^*X_0$ generates a strongly continuous semigroup.
Step 1: We show that

$$\begin{aligned} &CT_{-BB^*X_n}(\cdot)z\in L_2([0,\infty);Y), \\ &FT_{-BB^*X_n}(\cdot)z\in L_2([0,\infty);U), \quad (12)\\ &B^*X_kT_{-BB^*X_n}(\cdot)z\in L_2([0,\infty);U), \quad k=0,\cdots,n.\end{aligned}$$

For $k = n-1, ..., 0$ and $z \in D(A)$ consider

$$\begin{aligned}
&A_{n+1}^* X_{n-k} z + X_{n-k} A_{n+1} z \\
&\quad = A^* X_{n-k} z + X_{n-k} A z - X_{n-k} BB^* X_n z - X_n BB^* X_{n-k} z \\
&\quad = -C^* C z - X_{n-k-1} BB^* X_{n-k-1} z + X_{n-k} BB^* X_{n-k-1} z \\
&\qquad + X_{n-k-1} BB^* X_{n-k} z - X_{n-k} BB^* X_n z - X_n BB^* X_{n-k} z \\
&\quad = -C^* C z - (X_{n-k} - X_{n-k-1}) BB^* (X_{n-k} - X_{n-k-1}) z \\
&\qquad + (X_{n-k} - X_n) BB^* (X_{n-k} - X_n) z - X_n BB^* X_n z. \qquad (13)
\end{aligned}$$

Choosing in (13) $k = 0$ we obtain:

$$\begin{aligned}
&A_{n+1}^* X_n z + X_n A_{n+1} z = -C^* C z \\
&\quad -(X_n - X_{n-1}) BB^* (X_n - X_{n-1}) z - X_n BB^* X_n z.
\end{aligned}$$

So by Lemma 2.5 we conclude that

$$\begin{aligned}
&C T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); Y), \\
&B^* X_n T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U) \text{ and} \\
&B^* X_{n-1} T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U).
\end{aligned}$$

Choosing in (13) $k = 1$ we obtain for $z \in D(A)$

$$\begin{aligned}
&A_{n+1}^* X_{n-1} z + X_{n-1} A_{n+1} z \\
&\quad = -C^* C z - (X_{n-1} - X_{n-2}) BB^* (X_{n-1} - X_{n-2}) z \\
&\qquad + (X_{n-1} - X_n) BB^* (X_{n-1} - X_n) z - X_n BB^* X_n z.
\end{aligned}$$

From the above $B^*(X_n - X_{n-1}) T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$, and so Lemma 2.6 implies that $B^* X_{n-2} T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$. Continuing in this fashion until $k = n-2$ we see that $B^* X_1 T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$.

For $k = n-1$ in (13) we obtain

$$\begin{aligned}
&A_{n+1}^* X_1 z + X_1 A_{n+1} z \\
&\quad = -C^* C z - (X_1 - X_0)^* BB^* (X_1 - X_0) z \\
&\qquad + (X_1 - X_n) BB^* (X_1 - X_n) z - X_n BB^* X_n z.
\end{aligned}$$

From the above $B^*(X_n - X_1) T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$, and so Lemma 2.6 implies that $B^* X_0 T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$.

Finally, we consider the X_0 case:

$$\begin{aligned}
&A_{n+1}^* X_0 z + X_0 A_{n+1} z \\
&\quad = A^* X_0 z + X_0 A z - X_n BB^* X_0 z - X_0 BB^* X_n z \\
&\quad = -C^* C z - F^* F z - X_0 BF z - F^* B^* X_0 z \\
&\qquad - X_n BB^* X_0 z - X_0 BB^* X_n z \\
&\quad = -C^* C z - (F + B^* X_0)^* (F + B^* X_0) z \\
&\qquad + (X_0 - X_n) BB^* (X_0 - X_n) z - X_n BB^* X_n z.
\end{aligned}$$

Since $B^*(X_n - X_0) T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$, Lemma 2.6 implies that $(F + B^* X_0) T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$. Combining all the above estimates, we obtain (12).
Step 2: We show that A_{n+1} generates a strongly stable semigroup.

$$A_{n+1} = A + BF - B(F + B^* X_0) + BB^*(X_0 - X_n).$$

Since $\Sigma(A + BF, B, -, -)$ is input stable, $T_{BF}(t)$ is strongly stable and $(F + B^* X_0) T_{-BB^* X_n}(\cdot) z$, $B^*(X_0 - X_n) T_{-BB^* X_n}(\cdot) z \in L_2([0,\infty); U)$, applying Lemma 2.7, we conclude that A_{n+1} generates a strongly stable semigroup. Finally, Lemma 2.4 implies that X_{n+1} is a nonnegative solution to (11) for $m = n$ and the uniqueness follows since A_{n+1} generates a strongly stable semigroup.
(iii). To show that $X_{n-1} \geq X_n$, for $z \in D(A)$ consider the following sequence of equalities

$$\begin{aligned}
&(A - BB^* X_{n-1})^* (X_{n-1} - X_n) z + \\
&\quad (X_{n-1} - X_n)(A - BB^* X_{n-1}) z \\
&\quad = A^* X_{n-1} z + X_{n-1} A z - 2 X_{n-1} BB^* X_{n-1} z \\
&\qquad - A_n^* X_n z - X_n A_n^* z \\
&\quad = -C^* C z + X_{n-2} BB^* X_{n-1} z + X_{n-1} BB^* X_{n-2} z \\
&\qquad - X_{n-2} BB^* X_{n-2} z - 2 X_{n-1} BB^* X_{n-1} z \\
&\qquad + C^* C z + X_{n-1} BB^* X_{n-1} z \\
&\quad = -(X_{n-1} - X_{n-2}) BB^* (X_{n-1} - X_{n-2}) z.
\end{aligned}$$

From Lemma 2.5 we conclude that $X_{n-1} \geq X_n$.
(b): We have a nonincreasing sequence of nonnegative operators that is bounded below by X. So by Krezig [13, Theorem 9.3.-3] we conclude that X_n converges strongly to a nonnegative operator $X_{\max} \in \mathcal{L}(Z)$. Taking inner products in (11) gives

$$\begin{aligned}
&\langle (A - BB^* X_m) z, X_{m+1} z \rangle + \langle X_{m+1} z, (A - BB^* X_m) z \rangle \\
&\qquad = -\|B^* X_m z\|^2 - \|C X_m z\|^2.
\end{aligned}$$

It can be seen that as $m \to \infty$ the above equality converges to the following Riccati equation

$$\begin{aligned}
&\langle Az, X_{\max} z \rangle + \langle X_{\max} z, Az \rangle \\
&\qquad - \langle B^* X_{\max} z, B^* X_{\max} z \rangle + \langle Cz, Cz \rangle = 0.
\end{aligned}$$

Parts (c) and (d) follow from Theorem 2.8. ■

The following result follows from Theorem 3.1 and Theorem 2.8.

Corollary 3.2 *Let $\Sigma(A, B, C, D)$ be a state linear system with state space Z and denote $R = I + D^* D$ and $S = I + DD^*$. Suppose that $\Sigma(A, B, C, D)$ is output stabilizable and strongly input stabilizable, and let $X \in \mathcal{L}(Z)$ be a bounded nonnegative operator satisfying (3) for all $z \in D(A)$. Then the conclusions of Theorem 3.1 part a and b still hold. Furthermore, $X_{\max}$ is the unique bounded nonnegative solution to the Riccati equation (1) and the closed-loop system*

$$\Sigma\left(A - BR^{-1} D^* C - BR^{-1} B^* X_{\max}, B, \begin{pmatrix} S^{-\frac{1}{2}} C \\ R^{-\frac{1}{2}} B^* X_{\max} \end{pmatrix}, \begin{pmatrix} DR^{-\frac{1}{2}} \\ R^{-\frac{1}{2}} \end{pmatrix}\right)$$
is strongly system stable.

As a corollary of Theorem 3.1, we further obtain a comparison between the maximal solutions of two different Riccati equations.

Corollary 3.3 *Let $\Sigma(A,B,C_i,0)$, for $i=1,2$, be state linear systems with the state-space Z. Suppose that there exists $F_i \in \mathcal{L}(Z,U)$ such that $A+BF_i$ generates a strongly stable semigroup and $\Sigma(A+BF_i,B,\begin{pmatrix} C_i \\ F_i \end{pmatrix},0)$ is input and output stable.*
Suppose that $M_i \in \mathcal{L}(U)$, $i=1,2$, are coercive and the Riccati equations

$$A^*\Pi z + \Pi A z - \Pi B M_i^{-1} B^* \Pi z + C_i^* C_i z = 0, \quad z \in D(A)$$

have the nonnegative solutions Q_1, Q_2, respectively. If $C_1C_1^* \geq C_2C_2^*$ and $M_1 \geq M_2$, then Π^1_{max} and Π^2_{max}, the maximal solutions to the above Riccati equations, exist and satisfy

$$\Pi^1_{max} \geq \Pi^2_{max} \geq Q_2.$$

Proof: Note that for $i=1,2$ we can always write $\tilde{B}_i = BM_i^{-\frac{1}{2}}$ and $\tilde{F}_i = M_i^{\frac{1}{2}}F$. Then $A+\tilde{B}_i\tilde{F}_i = A+BF$. For $z \in D(A)$ and $i=1,2$ define
$\mathfrak{z}_i^X(z) = \langle Xz, Az\rangle + \langle Az, Xz\rangle - \langle z, XBM_i^{-1}B^*Xz\rangle + \langle C_iz, C_iz\rangle$.
By Theorem 3.1 the maximal solutions, $\Pi^1_{\max}$, $\Pi^2_{\max}$ to both Riccati equations are also the maximal solutions to the inequalities $\mathfrak{z}_i^X(z) \geq 0$, $i=1,2$. Now $\mathfrak{z}_1^X(z)$ can be rewritten as follows:

$$\begin{aligned}
\mathfrak{z}_1^X(z) &= \langle Xz, Az\rangle + \langle Az, Xz\rangle - \langle z, XBM_2^{-1}B^*Xz\rangle \\
&\quad + \langle C_2z, C_2z\rangle + \langle z, XB(M_2^{-1} - M_1^{-1})B^*Xz\rangle \\
&\quad + \langle z, (C_1^*C_1 - C_2^*C_2)z\rangle \\
&= \mathfrak{z}_2^X(z) + \langle z, XB(M_2^{-1} - M_1^{-1})B^*Xz\rangle + \\
&\quad \langle z, (C_1^*C_1 - C_2^*C_2)z\rangle + \langle z, (C_1^*C_1 - C_2^*C_2)z\rangle \\
&\geq \mathfrak{z}_2^X(z).
\end{aligned}$$

The maximal solution $\Pi^2_{\max}$ satisfies the second Riccati equation, $\mathfrak{z}_2^{\Pi^2_{\max}} = 0$, and so we have

$$\mathfrak{z}_1^{\Pi^2_{\max}}(z) \geq \mathfrak{z}_2^{\Pi^2_{\max}}(z) = 0, \quad z \in D(A).$$

By Theorem 3.1 we conclude that

$$\Pi^1_{\max} \geq \Pi^2_{\max} \geq Q_2.$$

■

A class of collocated system which satisfy the assumptions of the results presented in this section is described in the following example.

Example 3.4 *Suppose that Z and U are separable Hilbert spaces, $B \in \mathcal{L}(U)$, and A generates a contraction semigroup on Z. If, in addition, A has compact resolvent and the collocated system $\Sigma(A,B,B^*,0)$ is approximately controllable, then $\Sigma(A-BB^*,B,B^*,0)$ is a strongly stable system (see for example, [15, Lemma 2.2.5]). So $\Sigma(A,B,B^*,0)$ is strongly stabilizable and strongly detectable and it satisfies the conditions of Corollary 3.2. Hence the corresponding Riccati equation has a maximal nonnegative solution $\Pi_{\max}$. But Theorem 2.8 implies that the corresponding Riccati equation has a unique bounded nonnegative solution and this is Π_{max}.*

4 Strong stability of $A - BB^*X_{\max}$

It is tempting to conjecture that if $\Sigma(A,B,C,0)$ is strongly output stabilizable by a feedback F and $\Sigma(A+BF,B,\begin{pmatrix} C \\ F \end{pmatrix},0)$ is input stable (i.e. the assumptions from Theorem 3.1 are satisfied), then $A-BB^*X_{\max}$ will generate a strongly stable semigroup. However, the following example shows that this is not the case even under the stronger assumption that $\Sigma(A,B,C,0)$ is exponentially stabilizable.

Example 4.1 *Consider the infinite-dimensional system $\Sigma(A,B,0,0)$ on the state space $Z = \ell_2(\mathbb{C}^2)$ with $A = \operatorname{diag}(A_n)$, $B = \operatorname{diag}(B_1)$, where*

$$A_n = \begin{pmatrix} -\frac{1}{n} + jn & 1 \\ 0 & \frac{1}{n} + jn \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To show that A generates a C_0-semigroup on $Z = \ell_2(\mathbb{C}^2)$ consider $A_0 = \operatorname{diag}(a_n I_{2\times 2})$ and $S(t) = \operatorname{diag}(e^{a_n t} I_{2\times 2})$, where $a_n = jn$. It is clear that $S(0) = I$ and $S(t+s) = S(t)S(s)$ for $s,t > 0$. To show that it is strongly continuous at the origin consider the following for $z \in Z$:

$$\begin{aligned}
\|S(t)z - z\|^2 &= \sum_{n=1}^{\infty} \left|e^{a_n t} - 1\right|^2 |z_{n,1}|^2 + \left|e^{a_n t} - 1\right|^2 |z_{n,2}|^2 \\
&= \sum_{n=1}^{\infty} (2 - 2\cos(nt))(|z_{n,1}|^2 + |z_{n,2}|^2).
\end{aligned}$$

So the series is uniformly convergent and we can take the limit as $t \to 0$ inside the summation to obtain $\|S(t)z - z\| \to 0$ as $t \to 0$. Thus A_0 generates the C_0-semigroup $S(t)$ on Z. Since A is a bounded perturbation of A_0 it does too, see [2, Theorem 3.2.1]. To show that $\Sigma(A,B,0,0)$ is exponentially stabilizable we choose the feedback $F = \operatorname{diag}(F_n)$, with

$$F_n = \begin{pmatrix} -2 + \frac{3}{n} - \frac{1}{n^2}, & -3 \end{pmatrix}.$$

Then

$$A + BF = \operatorname{diag}\begin{pmatrix} -\frac{1}{n} + jn & 1 \\ -2 + \frac{3}{n} - \frac{1}{n^2} & \frac{1}{n} + jn - 3 \end{pmatrix}.$$

This has the eigenvalues $\lambda_{n,1} = -1 + jn$, $\lambda_{n,2} = -2 + jn$, $n \in \mathbb{N}$. So all the eigenvalues lie in $Re(s) \leq -1$.

Than $A+BF = \mathrm{diag}(L_n \mathrm{diag}(\lambda_{n,1}, \lambda_{n,2}) L_n^{-1})$*, with*

$$L_n = \begin{pmatrix} 1 & 1 \\ -1+\frac{1}{n} & -2+\frac{1}{n} \end{pmatrix} \text{ and}$$

$$L_n^{-1} = \begin{pmatrix} 2-\frac{1}{n} & 1 \\ -1+\frac{1}{n} & -1 \end{pmatrix}.$$

Since all the elements of L_n *and* L_n^{-1} *are bounded, there exist constants* M_1 *and* M_2 *such that* $\|L_n\| \leq M_1$ *and* $\|L_n^{-1}\| \leq M_2$ *for* $n \geq 1$*. Thus the semigroup* $T_{BF}(t)$ *generated by* $A+BF$ *is exponentially stable.*

A simple calculation shows that for $\Sigma(A,B,0,0)$ *the solutions to the corresponding Riccati equation are* $X=0$ *and* $X_{\max} = \mathrm{diag}(X_n)$*, where* $X_n = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{n} \end{pmatrix}$*. So*

$$A - BB^* X_{\max} = \mathrm{diag}(M_n),$$

where $M_n = \begin{pmatrix} -\frac{1}{n}+jn & 1 \\ 0 & -\frac{1}{n}+jn \end{pmatrix}$*. Hence* $A - BB^*X_{\max}$ *generates the semigroup*

$$T_{\max}(t) = \mathrm{diag}(T_n(t)),$$

where $T_n(t) = \begin{pmatrix} e^{(-\frac{1}{n}+jn)t} & te^{(-\frac{1}{n}+jn)t} \\ 0 & e^{(-\frac{1}{n}+jn)t} \end{pmatrix}$.
Choose $N \in \mathbb{N}$ *and let* $z = (z_n)$ *with* $z_n \in \mathbb{C}^2$ *equal to zero when* $n \neq N$ *and* z_N *equal to* $\binom{0}{1}$*. We have*

$$\|T_{\max}(t)z\|^2 = \left| te^{(-\frac{1}{N}+jN)t} \right|^2 + \left| e^{(-\frac{1}{N}+jN)t} \right|^2.$$

Now choose $t = N$*. Then*

$$\|T_{max}(N)z\|^2 = e^{-2}(1+N^2).$$

Since z *has norm one, we see that*

$$\|T_{\max}(N)\| \geq e^{-1}N,$$

and so the C_0*-semigroup* $T_{\max}(t)$ *is unbounded and hence it is not strongly stable.* ■

The above counterexample shows that to guarantee a stabilizing solution to the Riccati equation using the Kleinman-Newton algorithm one needs to assume both strong output and strong input stabilizability.

5 Conclusion

The main new contribution of this paper has been to generalize the Kleinman-Newton iteration scheme for the infinite-dimensional control Riccati equation to allow for systems that are not exponentially stabilizable. This was first done for the exponentially stabilizable infinite-dimensional systems in [3]. In addition, a generalization of a comparison result was obtained under these weaker strong stabilizability assumptions. The weaker strong stabilizability assumptions are satisfied by many partial differential systems with collocated actuators and sensors of the type $\Sigma(A,B,B^*,0)$. Such systems are typically not exponentially stabilizable.

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