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Hoeksema, J.

*Award date:*
2017

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August 31, 2017

## Abstract

In this thesis we will try to understand the macroscopic behaviour of a large number of singularly interacting particles.

We consider a system of Brownian motions under a singular potential, and study the large-deviation properties of the empirical measures of their invariant measures and their paths. Amongst others, the latter can be a tool to establish a rigorous convergence of the particle system to its hydrodynamic limit, a system of partial differential equations describing the macroscopic evolution.

This approach is relatively straightforward for a non-singular potential, and our main contribution is to make a first step in the extension to a quite general class of singular potentials, in particular those that contain an attraction term. To do so we revisit the concept of a *quasi-continuous* function with respect to a large-deviation principle, first introduced in [BG99], which allows us to establish a LDP of our original system via a singular change of measure. We significantly extend their concept to more general topological spaces, and introduce an intuitive framework to construct and combine these functions and establish quasi-continuity via local approximation by continuous functions.

In the case of particles in metric spaces that weakly interact under an additive potential this approximation method gives us quasi-continuity—and hence a large-deviation principle—under quite general assumptions. This representation is especially simple for sub-logarithmic interactions.

Moreover, we also study quasi-continuity for the invariant measures for a system with attractive logarithmic interactions. Inspired by [BG99], we prove this via an discretization method, and show how this might be extended to more general cases.

Finally, we briefly discuss how our techniques might apply to the space of the paths of the interacting particles. As mentioned, the latter has many applications, and we hope to contribute to this in further research.

# Contents

# 1 Introduction

A lot of interesting physical or biological systems are composed of particles where some of them can strongly repel or attract each other, sometimes even stick together to make up new systems. One can think of aggregating bacteria, polymers, self-gravitating gases, ionic liquids, or even dislocations in certain crystals.

One of the main problems in the study of these systems is to predict their macroscopic behaviour. Namely, because there are usually quite many particles one would hope to approximate their collective behaviour in terms of the dynamics of quantities like densities alone.

It is therefore helpful to know which models for the microscopic dynamics lead to which macroscopic models. Because the particles usually interact with each other and the environment in a complicated way it is hard to show this convergence rigorously.

In this thesis, we hope to lay the groundwork to do so, by introducing a general framework for studying their large deviations, which gives asymptotic probabilities of these densities.

## 1.1 Background

We model our particle systems as a set of reversible stochastic differential equations in $\mathbb{R}^d$, driven by a interaction potential with a sub-logarithmic or logarithmic singularity. Although most of our results only apply for their invariant measures, we generalized these to rather abstract metric spaces, which were originally intended and developed for the use in studying the paths of the SDEs. Therefore we will still include both the invariant measures and the system of SDEs in the following discussion.

In the case of non-singular interactions, such as twice continuously differentiable coefficients, there exist strong solutions to these SDE's and large-deviation principles for their empirical measures are well known, see for example Chapter X, [dH08]. However, it is the singular nature of the interaction between dislocations that complicates its analysis.

While there are several existence results for SDE's with singular, especially in certain Sobolev or generalized spaces, see for example [FLT10], [FIR14], there is comparatively little known for large-deviation principles for attracting particles in multiple dimensions. In 1D the logarithmic repelling case is known as Dyson's Brownian motion, which is closely related to eigenvalues of random matrices, and its properties are well understood — even for an infinite number of particles — see for example [Osa13], and in particular [LLX14] and [Fon04] for convergence and large deviations results, where the latter is proven via the uniqueness of the corresponding McKean-Vlasov equation. Moreover, convergence without the use of large-deviation principles, usually in the form of so called propagation of chaos, are established for a general class of energies in [BÖ16], and see [WJ17] for an overview of mean field limits from physics.

Finally, in [GQ15] and [FJ15] they even establish this property for sub-logarithmic and logarithmic *attracting* energies in 2D, in their study of the Kegel-Seller equation for chemotaxis, but do not study large deviations. In particular, they show that the singularity in the logarithmic case is *always visited*.

Now, the approach that we explore here consists of changing the underlying measure of our singular system to that of a non-interacting system. In particular, the former can formally be written as Gibbs measures over path spaces induced by a singular energy.

Again, in the case of a bounded and continuous energy, large-deviation principles for a sequence of Gibbs measures are well known. Moreover, for invariant measures there exist LDPs for several scalings of temperature, see for example [DE13], [LSZ17]. Also, in [DZ03] this approach is extended especially for empirical processes, yet only for processes with generalized but still continuous coefficients.

Hence, our main contribution therefore is to try systematically lift this approach to a quite general class of singular energies, using the framework of *quasi-continuous* functions, a concept first introduced by [BG99]. Intuitively speaking these functions are such that the probability of a singularity decreases super-exponentially as the number of particles increases, which allows us to circumvent the singularity entirely. Even more so, they are shown to even be *equivalent* to the class of functions for which this change of measure induces a large-deviation principle.

## 1.2 Outline

First, we briefly introduce our model in Section 2 including all the necessary definitions and known results for continuous functions, and then list our main results for the singular case.

In Section 3 we introduce our framework for quasi-continuity, which serves as the main toolbox for proving LDPs of induced Gibbs measures in this thesis. Amongst others, we significantly extend the underlying techniques used in [BG99] to functions over metric spaces, and introduce a systematic approach to construct and combine them. We generalize several properties of continuous functions to the linear space of quasi-continuous functions, and for example show that there exists a quasi-analogue of the statement that uniform convergence of continuous functions implies the limit is continuous.

The latter enables us to establish quasi-continuity via linearity and approximation by continuous functions. Finally, we introduce an asymmetric generalization of our methods that are useful for systems with repulsive interactions.

In particular, our results imply that an LDP of an induced Gibbs measure follows from a simple upper bound on the partition function and upper bounds on local partition functions of certain error energies. This framework is then applied to Gibbs measures induced by additive energies for weakly interacting systems in Section 4 , in which we show that a certain exponential integrability of the potential is sufficient.

As shown in Section 5, this directly applies to sub-logarithmic potentials for invariant measures. Moreover, inspired by [BG99], we show that via a discretization technique it also holds for logarithmic energies.

Finally, in Section 6, we sketch a possible way to extend this approach to the space of stochastic processes.

# 2 Model and main results

In this section we will first introduce the necessary definitions and concepts, in particular those for large deviations. Next, we will describe our model and our main results so far, and discuss them within the context of contemporary results found in the literature.

## 2.1 Basic definitions

**Large deviations** Suppose we are concerned about asymptotic probabilities of sequences of measures $P^N$ of variables $z^N$, and in particular wonder if they converge almost surely. For example, when all measures are over $\mathbb{R}$ and have a density $\rho^N$ we can think of a 'large-deviation principle' for $P^N$ if there exists some function $I$ such that asymptotically,

$$\rho^N(x) \sim e^{-NI(x)}, \tag{2.1.1}$$

and moreover, that the probability of $z^N$ being in an interval $[a,b]$ is similar to (2.1.1) but now with the *highest possible exponent*, namely

$$P^N\big([a,b]\big) \sim \sup_{x\in[a,b]} e^{-NI(x)}. \tag{2.1.2}$$

However, since we consider infinite dimensional spaces, there is not always a concept of density. Yet the concept of (2.1.2) can still be made precise in the theory of large deviations, using asymptotic probabilities over arbitrary open and closed sets. We largely follow [DZ10] and [dH08].

Let $\mathcal{X}$ be a Polish space, i.e. such that there exists a complete separable metric that induces the topology, $\mathcal{B}(\mathcal{X})$ the Borel $\sigma$-field on $\mathcal{X}$, and $\mathcal{P}(\mathcal{X})$ the space of probability measures on $\mathcal{X}$.

**Definition 2.1.** *A function $I : \mathcal{X} \to [0,\infty]$ is a rate function if*

1. $I \not\equiv \infty$,
2. *$I$ is lower semi-continuous,*
3. *$I$ has compact sublevel sets $\{x \,|\, I(x) \le a\}$.*

Moreover, $D(I)$ is defined as

$$D(I) := \{x \,|\, I(x) < \infty\}, \tag{2.1.3}$$

and usually referred to as the *domain* of $I$. Finally, for any rate function $I$ and measurable set $A$,

$$I(A) := \inf_{x\in A} I(x). \tag{2.1.4}$$

Now, let $z^N$ be a sequence of $\mathcal{X}$-valued random variables, with laws $P^N \in \mathcal{P}(\mathcal{X})$.

**Definition 2.2.** *The sequence of measures $P^N$ satisfies a* large-deviation principle *with rate function $I$ if for all open sets $O$ and closed sets $C$,*

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} \log P^N\big(\,C\,\big) &\le -I(C), \\ \liminf_{N\to\infty} \frac{1}{N} \log P^N\big(\,O\,\big) &\ge -I(O). \end{aligned} \tag{2.1.5}$$

Finally, a technical result that we will use repeatedly in several proofs, is the following:

**Lemma 2.3.** *Suppose $P^N$ satisfies a large-deviation principle with rate function $I$. Then for any $x \in \mathcal{X}$,*

$$\begin{aligned} \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log P^N\big(\,B_\epsilon(x)\,\big) &= -I(x), \\ \lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log P^N\big(\,B_\epsilon(x)\,\big) &= -I(x). \end{aligned} \tag{2.1.6}$$

It is a special case of Theorem 4.1.18 of [DZ10], but it will also follow from the proof of Lemma 3.16

**Remark 2.1.** Note that for any two positive sequences $a_n$, $b_n$, it is easy to see that (see for example Lemma 1.2.15, [DZ10]),

$$\limsup_{n\to\infty} \frac{1}{n}\log(a_n + b_n) = \max\left\{ \limsup_{n\to\infty}\frac{1}{n}\log a_n \,,\, \limsup_{n\to\infty}\frac{1}{n}\log b_n \right\}. \tag{2.1.7}$$

Thus, we have for any two sets $A_1$ and $A_2$,

$$\limsup_{N\to\infty}\frac{1}{N}\log P^N\big(\, A_1 \cup A_2 \,\big) \le \max\left\{ \limsup_{N\to\infty}\frac{1}{N}\log P^N\big(\, A_1\,\big) \,,\, \limsup_{N\to\infty}\frac{1}{N}\log P^N\big(\, A_2\,\big) \right\}. \tag{2.1.8}$$

And hence similarly for any finite union $\cup_i A_i$. With that in mind, a large-deviation principle simply states that you can do the same for certain *infinite* unions of sets.

**Varadhan's Lemma** As mentioned, one method of establishing LDPs is by relating them to another system, one that is usually better understood. In our case the former is actually a system of interacting particles, and latter is one with non-interacting particles. The transfer of one system to another is done via Varadhan's Lemma, when the transformation is continuous.

Namely, suppose we have a bounded and continuous function $\mathcal{E} : \mathcal{X} \to \mathbb{R}$, then we consider the family of sequences of measures $Q_\beta^N$ for $\beta \in \mathbb{R}$,

$$\frac{dQ_\beta^N}{dP^N} := \frac{e^{-\beta N \mathcal{E}(z^N)}}{Z_\beta^N}, \tag{2.1.9}$$

with $Z_\beta^N$ a normalization constant.

Because of convention and physical background, we usually we refer to $\mathcal{E}$ as an *energy* and $\beta$ as *inverse temperature*, and $Q_\beta^N$ as the measures induced by $\mathcal{E}$. Moreover, although we will not use the term throughout the rest of this thesis, the constants $Z_\beta^N$ as a function of $\beta$ are referred to as the *partition function* of $\mathcal{E}$.

Many systems, in particular the ones that we study in this section, can be written in this form, and $\mathcal{E}$ usually corresponds to a driving energy functional, and and for positive and low enough temperature the measures $Q_\beta^N$ coalesce around to the minimizer of $\mathcal{E}$.

However we also consider 'negative' temperatures — not that it corresponds to actual negative temperature, but merely that we switch the sign of $\beta\mathcal{E}$ — which can corresponds to switching between attractive and repulsive systems.

Now, we have the following theorem for $Q_\beta^N$, as an application of Varadhan's Lemma as noted in Theorem 4.3.1 of [DZ10].

**Theorem 2.4** (Varadhan)**.** *Suppose $P^N$ satisfies a large-deviation principle with rate function $I$, and suppose $\mathcal{E}$ is continuous and bounded.*

*Then for each $\beta \in \mathbb{R}$, it holds that,*

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[\, e^{-\beta N\mathcal{E}(z^N)}1_C \,\Big] &\le J_\beta(C),\\ \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[\, e^{-\beta N\mathcal{E}(z^N)}1_O \,\Big] &\ge J_\beta(O).\end{aligned} \tag{2.1.10}$$

*Here $J_\beta$ is defined as*

$$J_\beta(x) := \beta\mathcal{E}(x) + I(x). \tag{2.1.11}$$

*In particular, $Q_\beta^N$ satisfies a LDP for all $\beta$ with rate function $\mathcal{F}_\beta$,*

$$\mathcal{F}_\beta(x) := J_\beta(x) - \inf_{y\in\mathcal{X}} J_\beta(y). \tag{2.1.12}$$

In particular, the so called *Laplace principle* holds, namely

$$\lim_{N\to\infty} \frac{1}{N} \log Z_\beta^N = \inf_{y\in\mathcal{X}} J_\beta(y). \tag{2.1.13}$$

For a unbounded but continuous function this can be easily extended via an exponential moment condition, see [DZ10].

Moreover, because of the asymptotic nature of Varadhan's Lemma, note that we also can relax the assumption of continuity of a function $\mathcal{E}$ if, as $N$ goes to infinity, $\mathcal{E}$ uniformly approximates a continuous function.

**Theorem 2.5.**

*Let $\mathcal{E}_1$, $\mathcal{E}_2$ be two functions. For each $N$, we consider the essential supremum $\|\mathcal{E}_1-\mathcal{E}_2\|_\infty^N$, which is the smallest number such that*

$$|\mathcal{E}_1-\mathcal{E}_2|(z^N) \le \|\mathcal{E}_1-\mathcal{E}_2\|_\infty^N \qquad P^N\text{-almost everywhere.} \tag{2.1.14}$$

*Suppose that we have*

$$\lim_{N\to\infty} \|\mathcal{E}_1-\mathcal{E}_2\|_\infty^N = 0. \tag{2.1.15}$$

*Then if $\mathcal{E}_1$ induces a sequence of measures $Q_1^N$ that satisfies a LDP with rate function $\mathcal{F}$, then $\mathcal{E}_2$ induces a sequence of measures $Q_2^N$ that satisfies a LDP with the same rate function $\mathcal{F}$.*

*Proof.* Note that for any closed set $C$,

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-N\mathcal{E}_2(z^N)}\, 1_C \right] &= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-N(\mathcal{E}_2-\mathcal{E}_1)(z^N)}\, e^{-N\mathcal{E}_1(z^N)}\, 1_C \right] \\
&\le \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{N\|\mathcal{E}_2-\mathcal{E}_1\|_\infty^N}\, e^{-N\mathcal{E}_1(z^N)}\, 1_C \right] \\
&\le \limsup_{N\to\infty} \|\mathcal{E}_2-\mathcal{E}_1\|_\infty^N + \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-N\mathcal{E}_1(z^N)}\, 1_C \right] \\
&\le -\inf_{y\in C} \mathcal{E}_1(y) + I(y),
\end{aligned} \tag{2.1.16}$$

where we used (2.1.15) and the LDP for $\mathcal{E}_1$.

Similarly, for open sets,

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-N\mathcal{E}_2(z^N)}\, 1_O \right] &= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-N(\mathcal{E}_2-\mathcal{E}_1)(z^N)}\, e^{-N\mathcal{E}_1(z^N)}\, 1_O \right] \\
&\ge \liminf_{N\to\infty} \|\mathcal{E}-\mathcal{E}_\lambda\|_\infty^N + \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-N\mathcal{E}_1(z^N)}\, 1_O \right] \\
&\ge -\inf_{y\in O} \mathcal{E}_\lambda(y) + I(y).
\end{aligned} \tag{2.1.17}$$

□

Similarly, one could also consider sequences $\mathcal{E}_\lambda$ that converge with respect to $\lim_{N\to\infty} \|\cdot\|_\infty^N$.

Now, our approach is to systemically study what are some of the weakest conditions on $\mathcal{E}$ for Varadhan's Lemma to apply, and in which way these functions can be approximated by bounded and continuous functions $\mathcal{E}_\lambda$ in an appropriate way.

In particular, we will see that the class of quasi-bounded and quasi-continuous functions introduced in Section 3 consists of all the functions such that Varadhan's lemma holds for *all real temperatures*, and for merely quasi-continuous functions a large deviation principle will be found to apply only 'locally'.

Moreover, we will show that quasi-continuity can be established by approximation, where instead of the asymptotic uniform norm $\|\cdot\|_\infty^N$ we will use a local exponential estimate. See Section 3.2 for an immediate overview of both of these results.

**Empirical measures** The particle systems that we study are all *exchangeable*, which means that we only regard the configuration consisting of our particles, not the individual particles themselves. In particular, $\mathcal{X}$ will itself actually be a probability space and the random variables $z^N$ will be random *empirical measures.*

Thus, let $\mathcal{X} = \mathcal{P}(S)$, where $S$ is a Polish space. Moreover, $\mathcal{X}$ is equipped with the *weak topology*, which is the topology generated by the sets

$$\{\mu \in \mathcal{P}(S) : \Big| \int_S \phi \, d\mu - c \Big| < \delta\}, \tag{2.1.18}$$

for $c \in \mathbb{R}$, $\delta > 0$, and $\phi \in C_b(S)$, the space of bounded and continuous functions on $S$. Because of Theorem D.8, [DZ10], it follows that $\mathcal{X}$ is also Polish, hence satisfying our assumption on $\mathcal{X}$ of the beginning of this section.

Now consider for any $N$ the collection $X_i^N$, $1 \le i \le N$, where $X_i^N$ are $S$-valued random variables on some probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P})$, with joint law $\tilde{P}^N \in \mathcal{P}(S^N)$. To avoid confusion, note that $X_i^N$ and their realizations $x_i^N$ do not lie in $\mathcal{X}$, despite the notational similarity.

Now, for any fixed $N$ we define the empirical measure $z^N \in \mathcal{X}$ as follows:

$$z^N(\omega) \coloneqq \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N(\omega)}, \qquad \omega \in \Omega_N. \tag{2.1.19}$$

Denote by $\pi^N : S^N \to \mathcal{P}(S)$ the continuous projection of the random vector $X^N$ to the empirical measure,

$$\pi^N(X^N) = z^N, \tag{2.1.20}$$

and denote by $P^N \in \mathcal{P}(\mathcal{X})$ (hence $P^N \in \mathcal{P}(\mathcal{P}(S))$) the law of $z^N$ induced by $\tilde{P}^N$,

$$P^N \coloneqq \tilde{P}^N \circ \pi^{-1}. \tag{2.1.21}$$

Similarly, we will usually compare multiple systems, and whenever we refer to the laws $Q^N \in \mathcal{P}(\mathcal{P}(S))$ of the empirical measure $z^N$, induced by the laws $\tilde{Q}^N \in \mathcal{P}(S^N)$ of the particle system, we mean

$$Q^N \coloneqq \tilde{Q}^N \circ \pi^{-1}. \tag{2.1.22}$$

Moreover, define $R(\nu \| \mu)$, with $\nu, \mu \in \mathcal{X}$, the *relative entropy* of $\nu$ with respect to $\mu$,

$$R(\nu \| \mu) \coloneqq \begin{cases} \displaystyle\int_S \log \frac{d\nu}{d\mu} \, d\nu & \text{if } \mu \text{ absolutely continuous w.r.t. } \mu, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.1.23}$$

Then, in the case where for any $N$, all $X_i^N$ are independent and identically distributed with common law $\mu_0 \in \mathcal{X}$, it is known that the random variables $z^N$ satisfy a large-deviation principle, see Theorem 6.2.10, [DZ10]. Namely, let $P^N \in \mathcal{P}(\mathcal{X})$ be the laws of $z^N$ for this system.

**Theorem 2.6** (Sanov)**.** *The sequence $P^N$ satisfies a large-deviation principle with rate function $I(\mu) \coloneqq R(\mu \| \mu_0)$.*

## 2.2 Model

We consider a system of $N$ particles that are driven by a possibly singular interaction potential $V : \Lambda \to \bar{\mathbb{R}}$, and which are under the influence of some type of noise that models thermal effects. Moreover, we disregard any inertial effects.

To be precise, we model the particles by the following coupled set of stochastic differential equations,

$$\begin{aligned} dX_i^N(t) &= -\frac{1}{N}\sum_{i,j\neq i} \nabla V(X_i^N - X_j^N)\,dt + dB_i^N(t), \qquad t\in[0,T], \\ X_i(0) &\text{ satisfies law } (\rho_0)^N. \end{aligned} \tag{2.2.1}$$

Here $B_i^N(t)$, $t \in [0,T]$ are $N$ independent Brownian motions on some $\Lambda \subseteq \mathbb{R}^d$, with common initial law $\rho_0$ on $\Lambda$.

Alternatively, seen as a single stochastic differential equation over $\Lambda^N$, the system can be written in the following form:

$$dX^N(t) := -N\nabla E_V^N(X^N)\,dt + dB^N(t), \tag{2.2.2}$$

where

$$E_V^N(x) := \frac{1}{N^2}\sum_{i,j\neq i} V(x_i - x_j). \tag{2.2.3}$$

Note the inequality of $i \neq j$ because of the possible singularity in $V$.

The system (2.2.1) is sometimes referred to as a *first-order model*, which can be seen as a over-damped version of *second-order* models, which do incorporate the acceleration of the particles. See for example also [WJ17] and [Hol16]) for a discussion on contemporary results on mean field limits and propagation of chaos of both first and second order models and their differences.

In this section we assume the interaction potential $V$ to be possibly singular at the origin but otherwise continuous (see also Section 5 for more general assumptions). And for most part of this thesis, we consider $V$ to have at worst a *logarithmic singularity.* This is because in any dimension a first-order system of attracting particles under any stronger singularity is bound to stick together and eventually collapse for any temperature.

For this thesis however we are interested in the case where — at least for some high enough temperature — the first-order system will be asymptotically chaotic and converge to a nice hydrodynamic limit as the number of particles increases, and hence we restrict ourselves mostly to logarithmic potentials.

Yet, for a repulsive interaction potential this restriction can be somewhat relaxed, and by a generalization of our methods we will also consider *Riesz potentials*,

$$V(x) = |x|^{-s}, \tag{2.2.4}$$

with $0 < s < d$. Moreover, the case $s = d-2$ for $d > 2$, and $V(x) = -\log|x|$ when $d = 2$, is referred to as the *Coulomb potential*, which is a fundamental solution of the Laplace equation in $\mathbb{R}^d$.

For $s \geq d$, in the case of *hyper-singular* Riesz interaction, the system of (2.2.1) might still be well defined and convergence might still be established. However, this convergence might be so fast that the large-deviation principle in rate $N$ will be *trivial* even in the case of invariant measures, since the rate function can blow up because of the non-integrability of $V$.

See also Section 2.3 for a overview and discussion of our results, and [HLSS16], where they consider large deviations for the renormalized interaction potential $V := \frac{V}{N^{s/d}}$, when $s \geq d$.

**Invariant measures** Whenever the system (2.2.1) is well defined and reversible — which is the case when for example $V$ is twice bounded continuously differentiable, see [dH08, p. 112] — the density of the invariant measure can also be written in Gibbs form. Namely, let $\tilde{P}^N \in \mathcal{P}(\Lambda^N)$ be the law of independent particles in $\Lambda^N$, i.e. $P^N := (\mu_0)^N$ for some $\mu_0 \in \mathcal{P}(\Lambda)$ and $\tilde{Q}^N \in \mathcal{P}(\Lambda^N)$ the law of the interacting particles, then

$$\frac{d\tilde{Q}_V^N}{d\tilde{P}^N}(x_1, \ldots, x_N) := \frac{1}{Z_V^N} e^{-NE_V^N(x_1,\ldots,x_N)}, \tag{2.2.5}$$

with $Z_V^N$ a normalization constant, and $x_i \in \Lambda$ for all $1 \leq i \leq N$. Hence, with the formalism and notation for empirical measures, see Section 2.1, let $\mathcal{X} := \mathcal{P}(\Lambda)$ equipped with the weak topology, and define the empirical measure $z^N \in \mathcal{P}(\Lambda)$ as before, namely

$$z^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i^N}, \tag{2.2.6}$$

where the random variables $X_i^N \in \Lambda$ have either joint law $\tilde{Q}_V^N$ or $\tilde{P}^N$. Note that $z^N$, a random variable, is not related to the normalization constant $Z_V^N$, and the notation merely stems from convention. Additionally, note that $x_i \notin \mathcal{X}$.

Moreover, let $Q_V^N \in \mathcal{P}(\mathcal{X})$, i.e. $Q_V^N \in \mathcal{P}(\mathcal{P}(\Lambda))$, be the law of $z^N$ induced by $\tilde{Q}_V^N$ (see Section 2.1), and similarly, $P^N$ the law of the non-interacting system. Then $P^N$ and $Q_V^N$ are also absolutely continuous in the case of continuous $V$, and for $\mu \in \mathcal{X}$,

$$\frac{dQ_V^N}{dP^N}(\mu) := e^{-N\mathcal{E}_V(\mu)}. \tag{2.2.7}$$

Here $\mathcal{E}_V : \mathcal{X} \to \bar{\mathbb{R}}$, and

$$\mathcal{E}_V(\mu) := \int_{x \neq y} V(x-y) d\mu(x) d\mu(y). \tag{2.2.8}$$

The question of large-deviation principles for $Q_V^N$ for arbitrary $V$ is an important question in its own right with many applications in statistical physics — see also the discussion for our main results in Section 2.3 — even when not directly derived from a well defined and reversible system of SDE's in the form (2.2.1). Hence, in this thesis we will also study quite general conditions on $V$ for a LDP of $Q_V^N$ to hold.

Now, note that $\mathcal{E}_V(\mu)$ is not continuous on $\mathcal{X}$, even for continuous $V$. However, let $\bar{\mathcal{E}}_V(\mu)$ be defined as

$$\bar{\mathcal{E}}_V(\mu) := \int V(x-y) d\mu(x) d\mu(y). \tag{2.2.9}$$

Then for bounded continuous $V$, $\bar{\mathcal{E}}_V(\mu)$ is continuous and bounded, and $\bar{\mathcal{E}}_V(\mu) = \mathcal{E}_V(\mu)$ for every absolutely continuous $\mu$. Moreover, for every $N$,

$$\mathcal{E}_V(\mu) := \bar{\mathcal{E}}_V(z^N) - \frac{1}{N} V(0) \tag{2.2.10}$$

Hence, by the results of the Section 2.1, Theorem 2.5, it is easy to see that for bounded continuous $V$, $Q_V^N$ still satisfies a large-deviation principle with rate function $\mathcal{F}_V$, defined by

$$\begin{aligned} \mathcal{F}_V(\mu) &:= J_V(\mu) - \inf_{\nu \in \mathcal{X}} J_V(\nu), \\ J_V(\mu) &:= \mathcal{E}_V(\mu) + R(\mu \| \mu_0). \end{aligned} \tag{2.2.11}$$

**Empirical processes** Similarly, note $X_i^N \in S := C([0,T] \to \Lambda)$, the space of continuous paths in $\Lambda$. Moreover, let $\tilde{Q}^N \in \mathcal{P}(S^N)$ be the law of the system $X^N$ defined by (2.2.1), and $W^N \in \mathcal{P}(S^N)$ the law of the independent Brownian motions $B^N$ on $\Lambda$.

Then, for simplicity using the similar notation as before, consider the *empirical process* $z^N \in \mathcal{X} := \mathcal{P}(S)$,

$$z^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i^N}. \tag{2.2.12}$$

Again, to avoid confusion, note that $X_i^N \notin \mathcal{X}$ — $X_i^N \in S$ are random paths and $\mathcal{X} = \mathcal{P}(S)$ is the space of measures over paths, which is the space upon which we study large deviations, and the overlap in notation is merely to be consistent with Section 3.

Now, similar as before, let $\mathrm{Q}^N \in \mathcal{P}(\mathcal{X})$, i.e. $\mathrm{Q}^N \in \mathcal{P}(\mathcal{P}(S))$, be the laws of the random variable $z^N$ induced by $\tilde{Q}^N$ (see Section 2.1), and similarly, $\mathrm{P}^N \in \mathcal{P}(\mathcal{X})$ the law for the empirical process of the non-interacting system induced by $W^N$.

Whenever $V$ is twice continuously differentiable, it is known that strong solutions to (2.2.1) exist, see [dH08, p. 112]. Moreover, $\tilde{Q}^N$ and $\tilde{P}^N$, and hence $\mathrm{Q}^N$ and $\mathrm{P}^N$ are absolutely continuous and such that

$$\frac{d\mathrm{Q}^N}{d\mathrm{P}^N}(\mu) = e^{NF_V(\mu)}. \tag{2.2.13}$$

Here $F_V : \mathcal{X} \to \bar{\mathbb{R}}$, is a functional dependent on $V$, equivalent to zero whenever $V = 0$. Namely, $F_V = F_V^1 + F_V^2 + F_V^3$, with

$$\begin{aligned}
F_V^1(\mu) &:= -\int_0^T \int_S \left| \int_S \nabla V(x_t - y_t)\, d\mu(x) \right|^2 d\mu(y)\, dt, \\
F_V^2(\mu) &:= \int_S \int_0^T \Delta V(x_t - y_t)\, d\mu(x)\, d\mu(y), \\
F_V^3(\mu) &:= -\int_S \Big( V(x_T - y_T) - V(x_0 - y_0) \Big) d\mu(x)\, d\mu(y).
\end{aligned} \tag{2.2.14}$$

Hence, similarly as in the previous section, $\mathrm{Q}^N$ satisfies a large-deviation principle with rate function $\mathcal{F}_V$,

$$\begin{aligned}
\mathcal{F}_V(\mu) &:= J_V(\mu) - \inf_{\nu \in \mathcal{X}} J_V(\nu), \\
J_V(\mu) &:= R(\mu \| W) - F_V(\mu).
\end{aligned} \tag{2.2.15}$$

Recall, $R(\mu\|W)$ is the relative entropy of $\mu$ with respect to the law $W$ of a Brownian motion on $\Lambda$ for $t \in [0,T]$.

## 2.3 Main results

We briefly list our main results, which are mostly a particular case from results on invariant measures, as stated Section 5.3, where we also show LDPs for more general potentials. Moreover, we compare them with results from the literature. Finally, we pose a conjecture of how these results could be extended to empirical processes.

**Invariant measures**
For this section we assume that $V$ is possibly singular at the origin, but otherwise continuous. Then in Section 5 we will prove the following.

**Theorem 2.7.**
*Suppose there is some $\beta^* > 1$ such that:*

$$\sup_y \int_{\Lambda^2} e^{\beta |V(y-x)|}\, dx < \infty. \tag{2.3.1}$$

*Then the sequence of measures $Q_V^N$ induced by $V$ satisfies a LDP with rate function $\mathcal{F}_V$.*

In particular this encompasses the case where $V$ is attractive and has a logarithmic singularity, which was shown in two dimensions for dipole systems in [BG99], which was the origin of the concept of quasi-continuity and our method. Moreover, because of quasi-continuity we prove a slightly stronger consequence, namely that for *any* temperature $\beta \in \mathbb{R}$, the measures induced by $\beta V$ will satisfy a LDP conditioned on a certain subset $A_\beta \subset \mathcal{X}$. The set $A_\beta$ can be interpreted as a restriction on the clustering of particles, see Section5.4.

Moreover, we have the following asymmetric generalization,

**Theorem 2.8.** *Suppose that for all $\beta \geq 0$, it holds that*

$$\sup_y \int_{\Lambda} e^{-\beta V(y-x)}\, dx < \infty. \tag{2.3.2}$$

*And moreover,*

$$\int_{\Lambda^2} |V(x-y)|\, dx\, dy < \infty. \tag{2.3.3}$$

*Then the sequence of measures $Q^N$ induced by $V$ satisfies a LDP with rate function $\mathcal{F}_V$.*

This implies a large-deviation principle for empirical measures under repulsive Riesz interaction potentials (see Section 2.2), which is also established in more generality in [CGZ14] and [DLR15], using techniques from respectively potential and stochastic control theory, in which they also consider stronger topologies and dependence of the inverse temperature $\beta$ on $N$. We also refer to these for more information on the minimizers of $F_V$ and additional properties, which we do not consider here.

As mentioned, for any stronger singularity a LDP might no longer be meaningful, since because of the non-integrability of $V$ the rate function might blow up up for any measure that is continuous on a subset of $\Lambda$. In [HLSS16] the hyper-singular Riesz case is studied, where the potential is renormalized by a factor $N^{\frac{s}{d}}$. In this case they show that there does exists a LDP with rate $N$ with a rate function similar to $F_V$, but $\mathcal{E}_V$ is only defined by contraction over a term involving energies of *microscopic point configurations* and a relative entropy with respect to a Poisson point process.

The latter technique for large-deviation principles for microscopic configurations also arises in the usual logarithmic and Riesz case when $\beta_N = N$, see for example [Leb16] [LSZ17].

Finally, see Section 5 for more general assumptions on $V$ which are only defined by pointwise approximation.

**Empirical processes** Similar to the case for invariant measures, we can investigate LDPs for the empirical process of a system with attracting particles corresponding to (2.2.1).

Although formally they can fall within the framework of LDPs for Gibbs measures on metric spaces, we unfortunately did not yet fully establish this link to make definitive statements about large deviation principles for any given simple $V$. However, we will give an approximation result that requires a rather strong assumption about the representation of $\tilde{Q}_V^N$, as we will show in Lemma 6.1.

However, allow us to still make the following conjecture.

**Conjecture 2.9.** *Suppose that for each $\beta \in \mathbb{R}$,*

$$\sup_{y \in S} \mathbb{E} \exp\left[\beta \int_0^T \left|\nabla V(y(t) - B(t))\right|^2 dt\right] < \infty. \tag{2.3.4}$$

*Then for each $N$, the system of SDE's given by* (2.2.1) *has a weak solution and the sequence of measures $\mathrm{Q}^N$ induced by $V$ satisfy a LDP with rate function $\mathcal{F}_V$.*

# 3 Quasi-continuity

## 3.1 Introduction

In this section we will introduce a class of functions that are not necessarily continuous or bounded yet such that Varadhan's Lemma (as paraphrased in Theorem 2.4) apparently still applies. Even more so, under certain conditions they are even *equivalent* — a function $\mathcal{E}$ belongs to this class if and only if Varadhan's Lemma applies for *all real temperatures*.

These functions — called *quasi-continuous* and *quasi-bounded* functions, with respect to a sequence of probability measures $P^N$ — can be characterized in several ways. One is via a probabilistic interpretation, in which the probability of a discontinuity of $\mathcal{E}$ at any point will vanish sufficiently fast as $N$ goes to infinity. It is this concept that was coined in [BG99], and it was their results that provided the inspiration for this thesis.

However, we will also give a different interpretation, by introducing a quasi-analogue of the uniform norm of $\mathcal{E}$ on any given set $A$. Just as the uniform norm plays an important role in characterizing and studying continuous functions, our quasi-analogue will be an important tool in studying quasi-continuous functions. Moreover, it will provide us with an intuitive calculus that will greatly reduce the complexity of some our results and proofs.

In particular, we will show how quasi-continuous and quasi-bounded functions form a linear space and can be neatly transformed and manipulated. Finally, we will how these functions can arise as limits of bounded continuous functions, which will our main tool in Section 4 and 5 to establish LDPs.

**Outline**
We will first sketch the necessary concepts and list the main results in Section 3.2, such as the relation between quasi-continuity and Varadhan's Lemma, and how quasi-continuity can be established via approximation. Next, in Section 3.3, we will precisely define quasi-continuity, and list its various properties and transformations. Next, in Section 3.4 the relation with large-deviation principles will be studied, which will imply Theorems 3.1 and 3.4 of Section 3.2.

Moreover, we will provide several convergence results in Section 3.5, in which we show how quasi-continuity can be established by the approximation of $\mathcal{E}$ by continuous functions. As mentioned, this will be our main tool in proving large-deviation principles.

In Section 3.6 we will investigate what kind of restrictions quasi-continuity places on the rate functions of their induced LDPs. Since the properties of the rate functions are usually easier established or verified than large-deviation principles themselves — for which we will give explicit examples in Sections 4.5 for Gibbs measures — this provides us a convenient way for finding candidates for quasi-continuity.

Finally, in Section 3.7, we will give a generalization of our results tailored to asymmetric LDPs, in the sense that we only consider *positive temperatures*. This will be for example important in studying systems with very singular but purely repulsive potentials, such as the Riesz potentials listed in Section 2.2.

## 3.2 Main results

First, recall Varadhan's Lemma, as paraphrased in Theorem 2.4, and the notation and basic results as introduced in Section 2.1. We will consider a sequence of random variables $z^N \in \mathcal{X}$, with corresponding laws $P^N \in \mathcal{P}(\mathcal{X})$, and we assume that the sequence of measures $P^N$ satisfies a large-deviation principle with rate function $I$. Moreover, for a given measurable — but possibly singular — function $\mathcal{E} : \mathcal{X} \to \bar{\mathbb{R}}$, we consider the sequence of measures $Q^N_\beta$ defined by

$$\frac{dQ^N_\beta}{dP^N}(x) := \frac{e^{-\beta N \mathcal{E}(x)}}{Z^N_\beta}, \tag{3.2.1}$$

for any $\beta \in \mathbb{R}$ such that the normalization constants $Z^N_\beta$ are finite for all $N$.

Moreover, because of the possible singularity of $\mathcal{E}$ — it is possible that for example $\mathcal{E}(x) = \infty$ for some $x \in \mathcal{X}$ — we extend $J_\beta$ and $\mathcal{F}_\beta$ of Section 2.1 for each $\beta \in \mathbb{R}$ as follows,

$$\mathcal{F}_\beta(x) := J_\beta(x) - J_\beta(\mathcal{X}), \tag{3.2.2}$$

with

$$J_\beta(x) := \begin{cases} \beta\mathcal{E}(x) + I(x) & x \in D(I), \\ +\infty & x \notin D(I), \end{cases} \tag{3.2.3}$$

and as before, for any measurable set $A$,

$$J_\beta(A) := \inf_{x \in A} J_\beta(x). \tag{3.2.4}$$

By Theorem 2.4, whenever $\mathcal{E}$ is continuous and bounded, the sequence of measures $Q^N_\beta$ satisfies a LDP with rate function $\mathcal{F}_\beta$ for all $\beta \in \mathbb{R}$. As mentioned, this can be extended when we pose certain restrictions on $\mathcal{E}$, such as when $\mathcal{E}$ is quasi-continuous and quasi-bounded. Moreover, we will show that boundedness implies quasi-boundedness, and similarly, that continuity implies quasi-continuity - and hence our results are genuine extensions of the framework for large deviations as outlined in Section 2.1.

Now, one consequence of quasi-continuity of $\mathcal{E}$ with respect to some sequence of measures $P^N$, see Definition 3.5, will be shown to be that for all $x \in D(I)$ and all $\delta > 0$ asymptotically

$$P^N\Big(z^N \in B_\epsilon(x)\,,\, |\mathcal{E}(z^N) - \mathcal{E}(x)| > \delta\Big) \leq e^{-C_\epsilon N}, \tag{3.2.5}$$

where $B_\epsilon(x)$ are open balls around $x$ with radius $\epsilon$, and $C_\epsilon$ is a constant depending on $x$ such that

$$\lim_{\epsilon \to 0} C_\epsilon = \infty. \tag{3.2.6}$$

Similarly, we will show that quasi-boundedness, see Definition 3.8, will imply that asymptotically for large $N$ and large $\delta > 0$,

$$P^N\big(\,|\mathcal{E}(z^N)| > \delta\,\big) \leq e^{-\delta N}. \tag{3.2.7}$$

In other words, for a quasi-bounded and quasi-continuous function $\mathcal{E}$ the probability of a discontinuity around $x$ vanishes exponentially in $N$, and the rate becomes super-exponential as we take smaller and smaller neighbourhoods of $x$. Moreover, the probability of $\mathcal{E}$ being larger than any $\delta$ vanishes exponentially in $N$ and $\delta$. In the context of large deviations this allows us to disregard the discontinuity and unboundedness of $\mathcal{E}$ entirely, and treat the function as continuous and bounded for all our purposes. In particular, it allows us to extend Varadhan's Lemma, as shown below.

**Theorem 3.1.** ***Extended Varadhan's Lemma for quasi-continuous functions***
*Suppose $P^N$ satisfies a large-deviation principle with rate function $I$. Then the following statements are equivalent:*

1. *$\mathcal{E}$ is quasi-continuous and quasi-bounded.*
2. *For all $x \in D(I)$ and all $\beta \in \mathbb{R}$, we have $|\mathcal{E}(x)| < \infty$ and*
$$\lim_{\epsilon \to 0} [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} = 0, \tag{3.2.8}$$
*where*
$$[\![\mathcal{E}]\!]_A := \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{N|\mathcal{E}(z^N)|} \,\middle|\, A \right]. \tag{3.2.9}$$
*Moreover, for all $\beta \in \mathbb{R}$ it holds that*
$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} < \infty. \tag{3.2.10}$$
3. *The sequence of measures $Q_\beta^N$ satisfies a LDP for all $\beta \in \mathbb{R}$ with rate function $\mathcal{F}_\beta$.*

Here $\mathbb{E}^N$ are expectations with respect to $P^N$.

The non-negative and convex functional $[\![\cdot]\!]_A$ plays a similar role for quasi-continuous functions as the uniform norm for continuous functions. However, it is not homogeneous, and that is why the equivalence between the various statements only holds because we consider them for *all temperatures*. If this assumption is relaxed, the equivalences are no longer guaranteed.

Now, recall Theorem 2.5, in which we showed how Varadhan's Lemma can be extended via approximation in an asymptotic uniform norm. Using the functional $[\![\cdot]\!]_A$ defined above, we can generalize this significantly, via the route of quasi-continuity.

**Theorem 3.2.**
*Suppose $\mathcal{E}_\lambda$ are quasi-bounded and quasi-continuous for all $\lambda$, converge pointwise to $\mathcal{E}$ on $D(I)$, and that for all $\beta \in \mathbb{R}$*
$$\lim_{\lambda\to 0} [\![\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{\mathcal{X}} = 0. \tag{3.2.11}$$
*Then $\mathcal{E}$ is quasi-bounded and quasi-continuous.*

However, without quasi-boundedness, which is equivalent to (3.2.10) for all $\beta \in \mathbb{B}$, Theorems 3.1 and Theorem 3.2 do not apply. We will give an explicit and important example in Section 5.4, where $\mathcal{E}$ is induced by particle systems with logarithmic potentials.

For such a function a different approach is necessary. However, we will show there still exists a *localized* approximation theorem, and that there is still an equivalence between quasi-continuity and *localized* and *unnormalized* large-deviation principles with 'rate function' $J_\beta$ (see Theorem 3.13). Moreover, we will see that in this case we can still establish a *full* LDP, but only on some finite temperature range. Namely,

**Theorem 3.3.**
*Suppose $\mathcal{E}_\lambda$ are quasi-continuous for all $\lambda$, converge pointwise to $\mathcal{E}$ on $D(I)$, and that for all $x \in D(I)$ and all $\beta \in \mathbb{R}$*
$$\lim_{\epsilon\to 0} \lim_{\lambda\to 0} [\![\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{B_\epsilon(x)} = 0. \tag{3.2.12}$$
*Then $\mathcal{E}$ is quasi-continuous.*

**Theorem 3.4.**
*Let $\beta^* > 0$. Suppose $\mathcal{E}$ is quasi-continuous and such that for all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$,*
$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} < \infty. \tag{3.2.13}$$
*Then for all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$ it holds that $Q_\beta^N$ satisfies a LDP with rate function $\mathcal{F}_\beta$.*

## 3.3 Quasi-continuity and properties

In this section we will investigate quasi-continuity and list its various properties. We will give two formulations: one in terms of probability, and one in terms of non-negative and convex functionals $[\![\cdot]\!]_A$, where the latter play the same role for quasi-boundedness and quasi-continuity as the uniform norm for bounded and continuous functions. As mentioned, this provides us with an intuitive form of calculus on quasi-continuous functions which will greatly simplify the analysis for the remaining sections.

We will first give a straightforward extension of the definition of quasi-continuity as coined in [BG99, p. 218], which we will henceforth denote as *weak quasi-continuity.*

**Definition 3.5.** *A function $\mathcal{E}$ is* weakly quasi-continuous *(w.q.c.) with respect to $P^N$ if for all $x \in D(I)$ and all $\delta > 0$ it holds that $|\mathcal{E}(x)| < \infty$, and*

$$\lim_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log P^N \left( |\mathcal{E}(z^N) - \mathcal{E}(x)| > \delta \,\middle|\, B_\epsilon(x) \right) = -\infty. \tag{3.3.1}$$

Again, the idea behind the definition is that for a weakly quasi-continuous function the probability of the size of a discontinuity at a point $x \in D(I)$ being bigger then some $\delta$ is very small, even super-exponentially in $N$.

Now, for our purposes we will need a slightly stronger statement then weak quasi-continuity, formulated in terms of $[\![\cdot]\!]_A$. Here the latter was already mentioned in Section 3.2, but because of its importance we will repeat it here.

**Definition 3.6.** *For any set $A \subset \mathcal{X}$ and any function $\mathcal{E} : \mathcal{X} \to \bar{\mathbb{R}}$,*

$$[\![\mathcal{E}]\!]_A := \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{N|\mathcal{E}(z^N)|} \,\middle|\, A \right]. \tag{3.3.2}$$

Using $[\![\cdot]\!]_A$, we can precisely define quasi-continuity as follows.

**Definition 3.7.** *$\mathcal{E}$ is* quasi-continuous *(q.c.) if for all $x \in D(I)$ and all $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}(x)| < \infty$, and*

$$\lim_{\epsilon \to 0} [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} = 0. \tag{3.3.3}$$

That this is truly a stronger statement than weak quasi-continuity is not that trivial, but we will verify it in Lemma 3.12.

Similarly, we can straightforwardly define a quasi-analogue of boundedness.

**Definition 3.8.** *$\mathcal{E}$ is* locally quasi-bounded *(l.q.b.) if for all $x \in D(I)$ and all $\beta \in \mathbb{R}$ we have*

$$\limsup_{\epsilon \to 0} [\![\beta\mathcal{E}]\!]_{B_\epsilon(x)} < \infty. \tag{3.3.4}$$

*Moreover, $\mathcal{E}$ is* quasi-bounded *(q.c.) if if for every $\beta \in \mathbb{R}$,*

$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} < \infty. \tag{3.3.5}$$

Note the similarities to the definition of continuity and local boundedness at a point, where $[\![\cdot]\!]_A$ is simply replaced by $\|\cdot\|_{A,\infty}$, the uniform norm on $A$. Moreover, as the terms would suggest, quasi-boundedness implies local quasi-boundedness (Lemma 3.10).

Now, while $[\![\cdot]\!]_A$ is non-negative and convex, as shown below, it is not a (semi)norm because it's not homogeneous. However, it still satisfies the following nice properties.

**Lemma 3.9.**

$$[\![0]\!]_A = 0, \tag{3.3.6a}$$

$$[\![\mathcal{E}]\!]_A \geq 0, \tag{3.3.6b}$$

$$[\![\mathcal{E}]\!]_A = [\![-\mathcal{E}]\!]_A = [\![\,|\mathcal{E}|\,]\!]_A . \tag{3.3.6c}$$

*For any constant $c \in \mathbb{R}$,*

$$[\![c]\!]_A = |c|. \tag{3.3.7}$$

*Moreover,*

$$[\![\mathcal{E}]\!]_A \leq \|\mathcal{E}\|_{A,\infty}, \tag{3.3.8}$$

*and whenever $|\mathcal{E}_1| \leq |\mathcal{E}_2|$ on $A$, it holds that*

$$[\![\mathcal{E}_1]\!]_A \leq [\![\mathcal{E}_2]\!]_A . \tag{3.3.9}$$

*Additionally, for any $\alpha \in [0,1]$,*

$$[\![\alpha\mathcal{E}_1 + (1-\alpha)\mathcal{E}_2]\!]_A \leq \alpha\, [\![\mathcal{E}_1]\!]_A + (1-\alpha)\, [\![\mathcal{E}_2]\!]_A . \tag{3.3.10}$$

*Hence,*

$$\begin{aligned} [\![\beta\mathcal{E}]\!]_A &\geq |\beta|\, [\![\mathcal{E}]\!]_A , \qquad \textit{if } |\beta| \geq 1, \\ [\![\beta\mathcal{E}]\!]_A &\leq |\beta|\, [\![\mathcal{E}]\!]_A , \qquad \textit{if } |\beta| \leq 1. \end{aligned} \tag{3.3.11}$$

*Finally, in particular, for any fixed $\mathcal{E}$ and $A$, $[\![\beta\mathcal{E}]\!]_A$ is convex (hence continuous) in $\beta$ whenever $[\![\beta\mathcal{E}]\!]_A < \infty$, and non-decreasing in $\beta$ for $\beta \geq 0$.*

*Proof.* The first six properties follow directly from Definition 3.6. For the convexity, we use Hölder's inequality with exponent $\alpha^{-1}$ with $\alpha \in (0,1)$, namely,

$$\begin{aligned} [\![\alpha\mathcal{E}_1 + (1-\alpha)\mathcal{E}_2]\!]_A &= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ \exp\Big(\alpha N\mathcal{E}_1 + (1-\alpha)N\mathcal{E}_2\Big) \Big|\, A \Big] \\ &\leq \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\big[ e^{N\mathcal{E}_1} \big| A\big]^{\alpha}\, \mathbb{E}^N\big[ e^{N\mathcal{E}_2} \big| A\big]^{1-\alpha} \\ &= \alpha\, [\![\mathcal{E}_1]\!]_A + (1-\alpha)\, [\![\mathcal{E}_2]\!]_A . \end{aligned} \tag{3.3.12}$$

Moreover, note that $[\![\beta\mathcal{E}]\!]_A = [\![|\beta|\mathcal{E}]\!]_A$ by (3.3.6c), and using convexity with $\mathcal{E}_1 = \mathcal{E}$, $\mathcal{E}_2$ and $\alpha = |\beta|$, it follows that for any $\beta$ with $|\beta| \leq 1$, that

$$[\![\beta\mathcal{E}]\!]_A \leq |\beta|\, [\![\mathcal{E}]\!]_A . \tag{3.3.13}$$

Now replacing $\mathcal{E}$ by $\frac{\mathcal{E}}{\beta}$, and $\beta$ by $1/\beta^*$ in the last expression, we see that for any $\beta^*$ with $|\beta^*| \geq 1$

$$[\![\mathcal{E}]\!]_A \leq \frac{1}{|\beta^*|}\, [\![\beta^*\mathcal{E}]\!]_A . \tag{3.3.14}$$

and the result follows. Finally, note that for any $\beta \geq 1$ it follows that

$$\begin{aligned} [\![\beta\mathcal{E}]\!]_A &\geq |\beta|\, [\![\mathcal{E}]\!]_A \\ &\geq [\![\mathcal{E}]\!]_A . \end{aligned} \tag{3.3.15}$$

□

In contrast to the uniform norm, it does in general *not* hold that for a $O \subset A$,

$$[\![\mathcal{E}]\!]_O \leq [\![\mathcal{E}]\!]_A . \tag{3.3.16}$$

However, we can still compare rates over different sets, as seen below.

**Lemma 3.10.**
*Denote by $O^o$ the interior of a set $O$. Then we have the following:*

1. *For any $\beta \geq 1$, and $O \subset A$ with $I(O^o) < \infty$,*
$$[\![\mathcal{E}]\!]_O \leq \frac{1}{\beta}\left([\![\beta\mathcal{E}]\!]_A + \limsup_{N\to\infty}\frac{1}{N}\log\frac{P^N(A)}{P^N(O)}\right). \tag{3.3.17}$$
2. *Quasi-boundedness of $\mathcal{E}$ implies local quasi-boundedness of $\mathcal{E}$.*
3. *Suppose that for all $\beta \in \mathbb{R}$,*
$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} = 0. \tag{3.3.18}$$
*Then for all $\beta \in \mathbb{R}$, and all sets $A$ with $I(A^o) < \infty$,*
$$[\![\beta\mathcal{E}]\!]_A = 0. \tag{3.3.19}$$

*Proof.* 1. Because of the large-deviation principle for $P^N$, it holds that for any $A$, $O \subset A$, with $I(O^o) < \infty$,
$$\begin{aligned}0 &\geq \limsup_{N\to\infty}\frac{1}{N}\log P^N\big(A\big)\\ &\geq \liminf_{N\to\infty}\frac{1}{N}\log P^N\big(O\big)\\ &\geq -I(O^o)\\ &> -\infty.\end{aligned} \tag{3.3.20}$$
In particular, for any large enough $N$, both $P^N(O)$ and $P^N(A)$ are non-zero, and the exponential rates are bounded.

Hence, using conditional Hölder's inequality with exponent $\beta \geq 1$ and afterwards writing out the conditional expectations,
$$\begin{aligned}[\![\mathcal{E}]\!]_O &= \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N|\mathcal{E}(z^N)|}\Big|1_O\Big]\\ &\leq \limsup_{N\to\infty}\frac{1}{\beta N}\log\mathbb{E}^N\Big[e^{\beta N|\mathcal{E}(z^N)|}\Big|1_O\Big]\\ &\leq \limsup_{N\to\infty}\frac{1}{\beta N}\log\mathbb{E}^N\Big[e^{\beta N|\mathcal{E}(z^N)|}1_A\Big] - \beta^{-1}\liminf_{N\to\infty}\frac{1}{N}\log P^N\big(O\big)\\ &\leq \beta^{-1}\,[\![\mathcal{E}]\!]_{\mathcal{A}} + \beta^{-1}\limsup_{N\to\infty}\frac{1}{N}\log P^N\big(A\big) - \beta^{-1}\liminf_{N\to\infty}\frac{1}{N}\log P^N\big(O\big),\end{aligned} \tag{3.3.21}$$
where the third inequality follows from the fact that $O \subset A$.

2. Note that for any $x \in D(I)$ and all $\epsilon > 0$, it holds that
$$\begin{aligned}I(B_\epsilon(x)) &\leq I(x)\\ &< \infty.\end{aligned} \tag{3.3.22}$$
Hence, using the previous result of (3.3.17) with $A = \mathcal{X}$, $O = B_\epsilon(x)$ and $\beta = 1$,
$$\begin{aligned}[\![\mathcal{E}]\!]_{B_\epsilon(x)} &\leq [\![\mathcal{E}]\!]_{\mathcal{X}} - \liminf_{N\to\infty}\frac{1}{N}\log P^N(B_\epsilon(x))\\ &\leq [\![\mathcal{E}]\!]_{\mathcal{X}} + I(x)\\ &\leq \infty,\end{aligned} \tag{3.3.23}$$
where the latter inequality follows from quasi-boundedness of $\mathcal{E}$. Similarly, substituting $\mathcal{E}$ by $\beta\mathcal{E}$, local quasi-boundedness follows.

3. Similarly, using (3.3.17), we have for all $\beta \geq 1$,

$$\begin{aligned}
[\![\mathcal{E}]\!]_A &\leq \frac{1}{\beta}\Big( [\![\beta\mathcal{E}]\!]_{\mathcal{X}} - \liminf_{N\to\infty} \frac{1}{N} \log P(A)\Big) \\
&= -\frac{1}{\beta} \liminf_{N\to\infty} \frac{1}{N} \log P(A) \\
&\leq \frac{1}{\beta} I(A^o).
\end{aligned} \tag{3.3.24}$$

Letting $\beta \to \infty$, the result follows.

□

Finally, we list a few properties that enable us to easily transform and combine quasi-continuous functions.

**Lemma 3.11.** ***Transformations of quasi-continuous functions***

1. *Continuity at $x$ implies quasi-continuity at $x$.*
2. *For any continuous function $f : \mathbb{R}^d \to \mathbb{R}$ and $d$ weakly quasi-continuous functions $\mathcal{E}_i$ on $\mathcal{X}$, the function $\mathcal{E}(x) := f(\mathcal{E}_1(x), \ldots, \mathcal{E}_d(x))$ is weakly quasi-continuous. In particular, the space of w.q.c. functions is an algebra.*
3. *For any Lipschitz continuous function $f : \mathbb{R}^d \to \mathbb{R}$ and $d$ quasi-continuous functions $\mathcal{E}_i$ on $\mathcal{X}$ the function $\mathcal{E}(x) := f(\mathcal{E}_1(x), \ldots, \mathcal{E}_d(x))$ is quasi-continuous. In particular, the space of q.c. functions is a linear space.*

   *Similarly, for any Lipschitz continuous function $f : \mathbb{R}^d \to \mathbb{R}$ and $d$ either quasi-bounded or locally q.b. functions $\mathcal{E}_i$ on $\mathcal{X}$, the function $\mathcal{E}(x) := f(\mathcal{E}_1(x), \ldots, \mathcal{E}_d(x))$ is quasi-bounded or locally q.b. respectively. In particular, the space of quasi-bounded and the space of locally quasi-bounded functions are linear spaces.*

*Proof.* 1. It is easy to see that continuity implies quasi-continuity, since for a continuous function $\mathcal{E}$ we have for all $x$

$$\lim_{\epsilon\to 0} \|\mathcal{E} - \mathcal{E}(x)\|_{B_\epsilon(x),\infty} = 0. \tag{3.3.25}$$

Since the uniform norm dominates $[\![\cdot]\!]_{B_\epsilon(x)}$, see (3.3.8), quasi-continuity follows.

2. Recall that in $\mathbb{R}^d$ all $p$-norms, with $p \geq 1$ and including the maximum norm $\|\cdot\|_\infty$, are all equivalent. Hence, for simplicity, we will use the maximum norm when discussing continuity, and the $\|\cdot\|_1$-norm for Lipschitz continuity.

   Now, note that by definition of the weak quasi-continuity for $\mathcal{E}_i$, for any $x \in D(I)$ it holds that $|\mathcal{E}_i(x)| < \infty$ for all $i$. Moreover, since $f$ is continuous, it follows that for every $a \in \mathbb{R}^d$ and every $\delta > 0$ there exists a $\delta' > 0$ such that for every $b \in \mathbb{R}^d$ it holds that

$$\|a - b\|_\infty \leq \delta' \implies |f(a) - f(b)| \leq \delta. \tag{3.3.26}$$

   Hence, for any fixed $x \in D(I)$ and $\delta > 0$, there exists a $\delta' > 0$ such that for every $y \in \mathcal{X}$,

$$\begin{aligned}
|\mathcal{E}(x) - \mathcal{E}(y)| > \delta &\implies \max_i |\mathcal{E}_i(x) - \mathcal{E}_i(y))| > \delta' \\
&\implies \exists i : |\mathcal{E}_i(x) - \mathcal{E}_i(y)| > \delta.
\end{aligned} \tag{3.3.27}$$

   Recall that $P(\cup A_i) \leq \sum_i P(A_i)$, hence for any fixed $x \in D(I)$, $\delta > 0$, there exists a $\delta' > 0$ such that for all open balls $B_\epsilon(x)$:

$$\begin{aligned}
P^N\Big( |\mathcal{E}(z^N) - \mathcal{E}(x)| > \delta \,\Big|\, B_\epsilon(x)\Big) &\leq P^N\Big( \exists i : |\mathcal{E}_i(z^N) - \mathcal{E}_i(x)| > \delta' \,\Big|\, B_\epsilon(x)\Big) \\
&\leq \sum_{i=1}^{d} P^N\Big( |\mathcal{E}_i(z^N) - \mathcal{E}_i(x)| > \delta' \,\Big|\, B_\epsilon(x)\Big).
\end{aligned} \tag{3.3.28}$$

Hence, using (2.1.7), for any $\epsilon > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log P^N \left( |\mathcal{E}(z^N) - \mathcal{E}(x)| > \delta \,\middle|\, B_\epsilon(x) \right) \le \max_i \limsup_{N\to\infty} \frac{1}{N} \log P^N \left( |\mathcal{E}_i(z^N) - \mathcal{E}_i(x)| > \delta' \,\middle|\, B_\epsilon(x) \right). \tag{3.3.29}$$

The latter is the maximum of a finite number of exponential rates, and using weak quasi-continuity of $\mathcal{E}_i$ at $x$ for all $i$, the result follows when we take the limit $\epsilon \to 0$. Moreover, note that (pointwise) addition and multiplication are continuous operations.

3. Similarly, for any Lipschitz continuous function $f$ with Lipschitz constant $L$, where for simplicity we use the $\|\cdot\|_1$ on $\mathbb{R}^d$ — which is equivalent to all $p$-norms on $\mathbb{R}^d$ — it holds that for any two points $a, b \in \mathbb{R}^d$,

$$|f(a) - f(b)| \le L\|a - b\|_1. \tag{3.3.30}$$

Hence for any $x, y \in \mathcal{X}$,

$$|\mathcal{E}(x) - \mathcal{E}(y)| \le L \sum_{i=1}^d |\mathcal{E}_i(x) - \mathcal{E}_i(y)|. \tag{3.3.31}$$

Thus, using (3.3.6c), (3.3.9) and convexity, it follows that for any open set $A$,

$$\begin{aligned} [\![\mathcal{E} - \mathcal{E}(x)]\!]_A &\le \left[\!\!\left[ L \sum_{i=1}^d |\mathcal{E}_i - \mathcal{E}_i(x)| \right]\!\!\right]_A \\ &\le \sum_{i=1}^d \frac{1}{d} [\![\, dL\, |\mathcal{E}_i - \mathcal{E}_i(x)| \,]\!]_A \\ &= \sum_{i=1}^d \frac{1}{d} [\![\, dL\, (\mathcal{E}_i - \mathcal{E}_i(x)) \,]\!]_A . \end{aligned} \tag{3.3.32}$$

However, since by definition of q.c. of $\mathcal{E}_i$ for each $i$, it holds that

$$\lim_{\epsilon\to 0} [\![\, dL\, (\mathcal{E}_i - \mathcal{E}_i(x)) \,]\!]_{B_\epsilon(x)} = 0, \tag{3.3.33}$$

and quasi-continuity of $\mathcal{E}$ follows. Again, note that addition is Lipschitz continuous.

Finally, similarly, for quasi-bounded (and locally quasi-bounded) one can repeat the same trick. Namely note that for any any Lipschitz continuous function $f$ with Lipschitz constant $L$, it holds that for any point $a \in \mathbb{R}^d$ and the zero vector $0 \in \mathbb{R}^d$,

$$|f(a) - f(0)| \le L\|a\|_1. \tag{3.3.34}$$

hence for all points $x \in \mathcal{X}$,

$$|\mathcal{E}(x)| \le \|f(0)\|_1 + L \sum_{i=1}^d |\mathcal{E}_i(x)|, \tag{3.3.35}$$

and hence for any quasi-bounded $\mathcal{E}_i$,

$$\begin{aligned} [\![\mathcal{E}]\!]_{\mathcal{X}} &\le \|f(0)\| + \left[\!\!\left[ L \sum_{i=1}^d |\mathcal{E}_i - \mathcal{E}_i(x)| \right]\!\!\right]_{\mathcal{X}} \\ &\le \|f(0)\| + \sum_{i=1}^d \frac{1}{d} [\![\, dL\, (\mathcal{E}_i - \mathcal{E}_i(x)) \,]\!]_{\mathcal{X}} \\ &< \infty, \end{aligned} \tag{3.3.36}$$

and similarly, for locally q.b. $\mathcal{E}_i$ and any $x \in D(I)$,

$$\begin{aligned} \lim_{\epsilon\to 0} [\![\mathcal{E}]\!]_{B_\epsilon(x)} &\le \lim_{\epsilon\to 0} \left( \|f(0)\| + \sum_{i=1}^d \frac{1}{d} [\![\, dL\, (\mathcal{E}_i - \mathcal{E}_i(x)) \,]\!]_{B_\epsilon(x)} \right) \\ &< \infty. \end{aligned} \tag{3.3.37}$$

□

**Comments**

**Remark 3.1.** While we assumed $\mathcal{E}$ to be measurable, this is actually not necessary. Namely, for any sequence of measurable functions $\mathcal{E}^N : \mathcal{X} \to \bar{\mathbb{R}}$ and a function $\mathcal{E}_0 : \mathcal{X} \to \bar{\mathbb{R}}$, we can for example define quasi-continuity of the collection

$$\{\, \{\mathcal{E}^N\}_N \,,\, \mathcal{E}_0 \,\}, \tag{3.3.38}$$

by the statement that for any $x \in D(I)$ and any $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}_0(x)| < \infty$ and

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ \, e^{N|\mathcal{E}^N(z^N) - \mathcal{E}_0(x)|} \, \Big| \, B_\epsilon(x) \, \Big] = 0. \tag{3.3.39}$$

In other words, we simply replace in every definition $\mathcal{E}(z^N)$ by $\mathcal{E}^N(z^N)$, and $\mathcal{E}(x)$ for any $x \in D(I)$ by $\mathcal{E}_0(x)$.

In particular, one can dispense with measurability questions of $\mathcal{E}$ itself, and for example consider particle systems where the potential $V$ is dependent on the number of particles.

**Remark 3.2.** While it might not be immediately clear why in the definition of (weak) quasi-continuity, Definition 3.5, we restrict ourselves to $D(I)$, this comes from the fact that for any $x \notin D(I)$,

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log P^N \big( \, B_\epsilon(x) \, \big) = -\infty, \tag{3.3.40}$$

which follows from Lemma 2.3. Note that by definition $I(x) = \infty$ for any $x \notin D(I)$.

Hence the probability of $z^N$ being near $x \notin D(I)$ is already so small that for the purpose of large deviations we apparently do not even *need* any continuity properties of $\mathcal{E}$ at points outside $D(I)$, only some boundedness properties. For details, see the proof of Theorem 3.17.

**Remark 3.3.** Note that in the proof of Lemma 3.11, concerning Lipschitz continuous transformations, it appears to be again essential that the Definition 3.7 for quasi-continuity holds for *all* temperatures.

**Remark 3.4.** For a locally quasi-bounded function $\mathcal{E}$ — see Definition 3.8 — while for every $x \in D(I)$ and $\beta^* > 0$ there exists by definition a small enough ball such that $[\![\beta^*\mathcal{E}]\!]_{B_\epsilon(x)} < \infty$, this does not imply that $[\![\beta\mathcal{E}]\!]_{B_\epsilon(x)} < \infty$ for all $\beta > \beta^*$. Even more counter-intuitive, it does not even imply that $[\![\beta^*\mathcal{E}]\!]_O < \infty$ for all sets $O \subset A$.

For functions $\mathcal{E}$ that are derived from particle systems with logarithmic interaction potentials these properties can fail, as mentioned at the end of Section 3.2. Namely, for most sets $A$ there is a critical $\beta^*$ such that $[\![\beta\mathcal{E}]\!]_A = \infty$ for all $\beta \geq \beta^*$. We will give explicit examples of this phenomenon in Section 5.

Note that this critical behaviour only occurs for *strictly* local quasi-bounded functions, since for a quasi-bounded function $\mathcal{E}$ it follows from Lemma 3.10 that for any $A$ with $I(A^o) < \infty$ and *all* $\beta \in \mathbb{R}$ it holds that $[\![\beta\mathcal{E}]\!]_A < \infty$.

**Remark 3.5.** Finally, although the definition of quasi-continuity and quasi-boundedness in Definitions 3.7 and 3.8 and the probabilistic interpretations given by (3.2.5) and (3.2.7) might seem very different, it can be shown that they are actually asymptotically equivalent. We will give one implication below.

**Lemma 3.12.** *Suppose $\mathcal{E}$ is quasi-bounded and quasi-continuous. Then it is weakly quasi-continuous, and moreover,*

$$\lim_{\epsilon\to 0} \limsup_{(\delta,N)\to(\infty,\infty)} \frac{1}{\delta N} \log P^N \Big( \, \big|\mathcal{E}(z^N)\big| > \delta \, \Big) = -\infty. \tag{3.3.41}$$

*Proof.* Note that we can bound the probabilities via an exponential Chebyshev inequality, namely for any $\beta > 0$ we have

$$\begin{aligned} P^N\Big( |\mathcal{E}(z^N) - \mathcal{E}(x)| > \delta \,\Big|\, A \Big) &\le P^N\Big( e^{\beta N |\mathcal{E}(z^N) - \mathcal{E}(x)|} > e^{\beta N \delta} \,\Big|\, A \Big) \\ &\le e^{-\beta N \delta} \mathbb{E}^N\Big[ e^{\beta N |\mathcal{E}(z^N) - \mathcal{E}(x)|} \,\Big|\, A \Big]. \end{aligned} \tag{3.3.42}$$

Hence, for any $\beta > 0$ and $\epsilon > 0$,

$$\limsup_{N\to\infty} \frac{1}{N} \log P^N\Big( \, |\mathcal{E}(z^N) - \mathcal{E}(x)| > \delta \,\Big|\, B_\epsilon(x) \, \Big) \le -\beta\delta + [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} \,. \tag{3.3.43}$$

Thus, because of quasi-continuity,

$$\begin{aligned} \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log P^N\Big( \, |\mathcal{E}(z^N) - \mathcal{E}(x)| > \delta \,\Big|\, B_\epsilon(x) \, \Big) &\le -\beta\delta + \lim_{\epsilon\to 0} [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} \\ &= -\beta\delta. \end{aligned} \tag{3.3.44}$$

Since the latter holds for all $\beta$, we let $\beta \to \infty$ and quasi-continuity follows.

Similarly, it follows that for any $N$, $\delta > 0$, and $\beta > 0$,

$$\frac{1}{N} \log P^N\Big( |\mathcal{E}(z^N)| > \delta \Big) \le \frac{1}{N} \log \mathbb{E}^N\Big[ e^{\beta N |\mathcal{E}(z^N)|} \Big] - \delta\beta. \tag{3.3.45}$$

Now fix $\beta > 0$. Since $\mathcal{E}$ is quasi-bounded, it follows that there exists a constant $C$ and large enough $N^*$ such that for all $N > N^*$

$$\frac{1}{N} \log \mathbb{E}^N\Big[ e^{\beta N |\mathcal{E}(z^N)|} \Big] < C, \tag{3.3.46}$$

and hence,

$$\limsup_{(\delta,N)\to(\infty,\infty)} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{\beta N |\mathcal{E}(z^N)|} \Big] = 0. \tag{3.3.47}$$

Thus,

$$\limsup_{(\delta,N)\to(\infty,\infty)} \frac{1}{N} \log P^N\Big( |\mathcal{E}(z^N)| > \delta \Big) \le -\beta. \tag{3.3.48}$$

Since this holds for every $\beta > 0$, we have proven (3.3.41). □

## 3.4 Relation to large deviations

The power of quasi-continuity lies in its connection with large deviations. Apparently, quasi-continuity is as 'continuous' as you need for Varadhan's Lemma to work. The main results of this section are quite technical, but they will allow us to switch between the different notions of quasi-continuity, pointwise exponential estimates, and *localized* and *unnormalized* large-deviation principles, see below.

First, recall the definitions of $J_\beta$ and $[\![\cdot]\!]_A$, see (3.2.3) and (3.3.2),

$$J_\beta(x) := \begin{cases} \beta\mathcal{E}(x) + I(x) & x \in D(I), \\ +\infty & x \notin D(I). \end{cases} \tag{3.4.1}$$

$$[\![\mathcal{E}]\!]_A := \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{N|\mathcal{E}(z^N)|} \,\Big|\, A\Big]. \tag{3.4.2}$$

Finally, recall that $D(I) := \{x\,|\,I(x) < \infty\}$, see (2.1.3), and that for any set $A$ (see (3.2.4)),

$$J_\beta(A) := \inf_{x\in A} J_\beta(x). \tag{3.4.3}$$

**Theorem 3.13.** ***Equivalence quasi-continuity and localized LDPs***
*Suppose $P^N$ satisfies a large-deviation principle with rate function $I$. Then the following statements are equivalent*

*(1). $\mathcal{E}$ is quasi-continuous.*

*(2). For all $x \in D(I)$ and all $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}(x)| < \infty$ and*

$$\lim_{\epsilon\to 0} [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} = 0. \tag{3.4.4}$$

*(3). For all $x \in D(I)$ and all $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}(x)| < \infty$ and*

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta N\mathcal{E}(z^N)}\, 1_{B_\epsilon(x)}\Big] = -J_\beta(x), \tag{3.4.5a}$$

$$\lim_{\epsilon\to 0}\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta N\mathcal{E}(z^N)}\, 1_{B_\epsilon(x)}\Big] = -J_\beta(x). \tag{3.4.5b}$$

*(4). For all $x' \in D(I)$ and all $\beta' \in \mathbb{R}$ there exists an open neighbourhood $A'$ of $x'$ such that*

- *(a) for $J_{\beta'}$ it holds that on $A'$*
  - *i. $J_{\beta'} \not\equiv \infty$,*
  - *ii. $J_{\beta'}$ is lower semi-continuous,*
  - *iii. $J_{\beta'}$ has pre-compact sublevel sets $\{x\,|\,x \in A',\, J_{\beta'}(x) \le a\}$.*
- *(b)*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta' N\mathcal{E}(z^N)}\, 1_{A'}\Big] < \infty. \tag{3.4.6}$$

- *(c) For all open sets $O$ and closed sets $C$, with $O, C \subset A'$, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta' N\mathcal{E}(z^N)}\, 1_{C}\Big] \le -J_\beta(C), \tag{3.4.7a}$$

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta' N\mathcal{E}(z^N)}\, 1_{O}\Big] \ge -J_\beta(O). \tag{3.4.7b}$$

The adjectives localized and unnormalized refer in this case to the fact that we consider large-deviation principles on sets $A' \subset \mathcal{X}$ instead of $\mathcal{X}$ itself, and the fact that the exponential integrals are not divided by some normalization constants $Z_\beta^N$ such as for $Q_\beta^N$ in Theorem 3.1. See also Remark 3.7 for a discussion on why we can not simply use (re)normalized versions of the above statements, which even for $A' = \mathcal{X}$ is an essential distinction that was apparently missed in [BG99].

Now, the proof of Theorem 3.13 will be split into several parts. First, we will show in Lemma 3.14 how to go from localized LDPs to pointwise estimates such as (3.4.5), and by Lemma's 3.15 and 3.16 it will follow that these are equivalent to quasi-continuity.

Next, in Theorem 3.17, we will prove how pointwise estimates and an additional exponential moment condition imply a unnormalized large-deviation principle, which will complete the proof of Theorem 3.13.

Finally, we will show how these results imply Theorem 3.1 — which stated the equivalence between large-deviation principles and quasi-bounded and quasi-continuous functions, and Theorem 3.4 — which showed how to go from quasi-continuity to large-deviation principles for a certain finite temperature range.

**Lemma 3.14.** *Let $\beta \in \mathbb{R}$ and $A$ be a open set. Suppose that $J_\beta$ is lower semi-continuous on $A$, and such that for all open sets $O$ and closed sets $C$, with $O, C \subset A$, it holds that*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_C \Big] \le -J_\beta(C), \tag{3.4.8a}$$

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_O \Big] \ge -J_\beta(O). \tag{3.4.8b}$$

*Then we have for all $x \in A \cap D(I)$*

$$\lim_{\epsilon\to0}\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] = -J_\beta(x), \tag{3.4.9a}$$

$$\lim_{\epsilon\to0}\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] = -J_\beta(x). \tag{3.4.9b}$$

*Proof.* Consider any fixed $x \in A \cap D(I)$. Note that for small enough $\epsilon$ it holds that $B_\epsilon(x) \subset A$ and hence by (3.4.8b) for small enough $\epsilon$,

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \right] &\ge -J_\beta(B_\epsilon(x)) \\ &\ge -J_\beta(x),\end{aligned} \tag{3.4.10}$$

where the latter inequality follows from the fact that $x \in B_\epsilon(x)$ for all $\epsilon > 0$ and hence

$$\begin{aligned}J_\beta(B_\epsilon(x)) &= \inf_{y\in B_\epsilon(x)} J_\beta(y) \\ &\le J_\beta(x).\end{aligned} \tag{3.4.11}$$

Thus, it follows that

$$\lim_{\epsilon\to0}\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \right] \ge -J_\beta(x). \tag{3.4.12}$$

Similarly, by (3.4.8b) for small enough $\epsilon$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \right] \le -J_\beta(\bar{B}_\epsilon(x)). \tag{3.4.13}$$

However, by lower semi-continuity of $J_\beta$,

$$\lim_{\epsilon\to0} J_\beta(\bar{B}_\epsilon(x)) \ge J_\beta(x), \tag{3.4.14}$$

and hence

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\beta N\mathcal{E}(z^N)}\,1_{B_\epsilon(x)}\right]\le -J_\beta(x). \tag{3.4.15}$$

Thus, the pointwise estimates of (3.4.9) follow, namely,

$$\begin{aligned}\lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\beta N\mathcal{E}(z^N)}\,1_{B_\epsilon(x)}\right]&= -J_\beta(x),\\ \lim_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\beta N\mathcal{E}(z^N)}\,1_{B_\epsilon(x)}\right]&= -J_\beta(x).\end{aligned} \tag{3.4.16}$$

□

In particular, note from the proof of Lemma 3.14 that one can similarly derive a pointwise estimate for $P^N$ itself — also previously shown in Lemma 2.3. Since we will employ it numerous times throughout the proofs of the rest of this section, we will repeat it here. Namely,

$$\begin{aligned}\lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P^N\big(B_\epsilon(x)\big)&= -I(x),\\ \lim_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log P^N\big(B_\epsilon(x)\big)&= -I(x).\end{aligned} \tag{3.4.17}$$

Now, we will first need some technical estimates, which will allow us to switch between quasi-continuity and exponential rates of $\pm\mathcal{E}$, as is clear from Lemma 3.16.

**Lemma 3.15.**

$$[\![\mathcal{E}]\!]_A = \max\left\{\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N\mathcal{E}(z^N)}\,\Big|\,A\Big],\ \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-N\mathcal{E}(z^N)}\,\Big|\,A\Big]\right\}, \tag{3.4.18}$$

$$\begin{aligned}[\![\mathcal{E}]\!]_A &\ge \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N\mathcal{E}(z^N)}\,\Big|\,A\Big]\\ &\ge \liminf_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N\mathcal{E}(z^N)}\,\Big|\,A\Big]\ge -[\![\mathcal{E}]\!]_A\,.\end{aligned} \tag{3.4.19}$$

*Proof.* 1. We compute $[\![\mathcal{E}]\!]_A$ by splitting $A$ up into two parts depending on the sign of $\mathcal{E}(z^N)$ and using Remark 2.1,

$$\begin{aligned}[\![\mathcal{E}]\!]_A &= \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N|\mathcal{E}(z^N)|}\,\Big|\,A\Big]\\ &= \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N\mathcal{E}(z^N)}\,1_{\mathcal{E}(z^N)\ge 0}+e^{-N\mathcal{E}(z^N)}\,1_{\mathcal{E}(z^N)<0}\,\Big|\,A\Big]\\ &= \max\Bigg\{\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N\mathcal{E}(z^N)}\,1_{\mathcal{E}(z^N)\ge 0}\,\Big|\,A\Big],\\ &\qquad\qquad \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-N\mathcal{E}(z^N)}\,1_{\mathcal{E}(z^N)<0}\,\Big|\,A\Big]\Bigg\}\\ &\le \max\left\{\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N\mathcal{E}(z^N)}\,\Big|\,A\Big],\ \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-N\mathcal{E}(z^N)}\,\Big|\,A\Big]\right\}.\end{aligned} \tag{3.4.20}$$

The reverse inequality trivially follows since $\pm\mathcal{E}(x)\le|\mathcal{E}(x)|$ for all $x$.

2. Note that by applying Jensen's inequality to the transformation $x\to\frac{1}{x}$ on the positive variable $\exp\{\mathcal{E}(z^N)\}$, it follows that for fixed $N$

$$
\begin{aligned}
\Big|\log \mathbb{E}^N\Big[ e^{N\mathcal{E}(z^N)} \Big| A\Big]\Big| &= \max\Big\{ -\log \mathbb{E}^N\Big[ e^{N\mathcal{E}(z^N)} \Big| A\Big], \log \mathbb{E}^N\Big[ e^{N\mathcal{E}(z^N)} \Big| A\Big]\Big\} \\
&\leq \max\Big\{ \log \mathbb{E}^N\Big[ e^{-N\mathcal{E}(z^N)} \Big| A\Big], \log \mathbb{E}^N\Big[ e^{N\mathcal{E}(z^N)} \Big| A\Big]\Big\}.
\end{aligned}
\tag{3.4.21}
$$

Dividing by $N$ and taking the limit supremum, the latter is equal to $[\![\mathcal{E}]\!]_A$ according to the previous part, and the result follows.

□

**Lemma 3.16.**

*Let $\beta^* > 0$, and let $A$ be a open set. Then the following are equivalent.*

*(1). For all $x \in A \cap D(I)$ and all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$ we have $|\mathcal{E}(x)| < \infty$ and*

$$
\lim_{\epsilon\to 0} [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} = 0.
\tag{3.4.22}
$$

*(2). For all $x \in A \cap D(I)$ and all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$ we have $|\mathcal{E}(x)| < \infty$ and*

$$
\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N\mathcal{E}(z^N)}\, 1_{B_\epsilon(x)}\Big] = -J_\beta(x),
\tag{3.4.23a}
$$

$$
\lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N\mathcal{E}(z^N)}\, 1_{B_\epsilon(x)}\Big] = -J_\beta(x).
\tag{3.4.23b}
$$

*Proof.*

Fix $x \in D(I) \cap A$ with $|\mathcal{E}(x)| < \infty$. Note that for any fixed $N$, $\beta$, and a set $O$,

$$
\begin{aligned}
\frac{1}{N}\log \mathbb{E}^N\Big[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))} \Big| O\Big] &= \frac{1}{N}\log \mathbb{E}^N\Big[ e^{-\beta N\mathcal{E}(z^N)} \Big| O\Big] + \beta\mathcal{E}(x) \\
&= \frac{1}{N}\log \mathbb{E}^N\Big[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\, 1_O\Big] - \frac{1}{N}\log P^N\big(O\big) + \beta\mathcal{E}(x).
\end{aligned}
\tag{3.4.24}
$$

Setting $O = B_\epsilon(x)$, using the pointwise LDP for $P^N$ of (3.4.17), and taking limits, it follows that

$$
\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))} \Big| 1_{B_\epsilon(x)}\right] = \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N\mathcal{E}(z^N)}\, 1_{B_\epsilon(x)}\right] + J_\beta(x).
\tag{3.4.25}
$$

Similarly,

$$
\lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))} \Big| 1_{B_\epsilon(x)}\right] = \lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N\mathcal{E}(z^N)}\, 1_{B_\epsilon(x)}\right] + J_\beta(x).
\tag{3.4.26}
$$

Hence (2), which concerns the pointwise estimates of (3.4.23), is equivalent to the statement that for any $x \in D(I) \cap A$ and any $\beta$ with $|\beta| < \beta^*$ it holds that $|\mathcal{E}(x)| < \infty$ and

$$
\begin{aligned}
\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))} \Big| 1_{B_\epsilon(x)}\right] &= 0, \\
\lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\left[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))} \Big| 1_{B_\epsilon(x)}\right] &= 0.
\end{aligned}
\tag{3.4.27}
$$

However, because of (3.4.19) and (3.4.18) we have the following (in)equalities for any set $O$,

$$\begin{aligned}
[\![\beta(\mathcal{E}-\mathcal{E}(x))]\!]_O &= \max\left\{ \limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\,1_O \right], \limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\,1_O \right] \right\} \\
&\geq \min\left\{ \liminf_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\,1_O \right], \liminf_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\,1_O \right] \right\} \\
&\geq -[\![\beta(\mathcal{E}-\mathcal{E}(x))]\!]_O
\end{aligned} \tag{3.4.28}$$

In particular, it is now easy to see that (1) implies (2). Moreover, since if (2) holds than (3.4.27) holds for *both* $+\beta$ and $-\beta$, for any $\beta$ with $|\beta| < \beta^*$, it also follows that (2) implies (1), since then

$$\begin{aligned}
&\lim_{\epsilon\to 0} [\![\beta(\mathcal{E}-\mathcal{E}(x))]\!]_{B_\epsilon(x)} \\
&= \max\left\{ \lim_{\epsilon\to 0}\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\,B_\epsilon(x) \right], \lim_{\epsilon\to 0}\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\,B_\epsilon(x) \right] \right\} \\
&= \max\left\{0, 0\right\}.
\end{aligned} \tag{3.4.29}$$

□

**Theorem 3.17.** *Let $\beta' \in \mathbb{R}$ and $A'$ be an open set. Suppose that for every $x \in D(I) \cap A'$ it holds that*

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{-\beta' N\mathcal{E}(z^N)}\,1_{B_\epsilon(x)} \right] = -J_{\beta'}(x), \tag{3.4.30a}$$

$$\lim_{\epsilon\to 0}\liminf_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\left[ e^{-\beta' N\mathcal{E}(z^N)}\,1_{B_\epsilon(x)} \right] = -J_{\beta'}(x). \tag{3.4.30b}$$

*, with $J_\beta(x)$ possibly infinite. Additionally, suppose there exists a $\gamma > 1$ such that*

$$\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\Big[ e^{-\gamma\beta' N\mathcal{E}(z^N)}\,1_{A'} \Big] < \infty. \tag{3.4.31}$$

*Then on $A'$, $J_{\beta'}$ is lower semi-continuous with pre-compact sublevel sets, and for any open set $O$ and closed $C$ with $O, C \subset A'$ it holds that*

$$\limsup_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\Big[ e^{-\beta' N\mathcal{E}(z^N)}\,1_C \Big] \leq -J_{\beta'}(C), \tag{3.4.32a}$$

$$\liminf_{N\to\infty} \frac{1}{N}\log \mathbb{E}^N\Big[ e^{-\beta' N\mathcal{E}(z^N)}\,1_O \Big] \geq -J_{\beta'}(O). \tag{3.4.32b}$$

*In particular, $\inf_{x\in A'} J_{\beta'}(x) > -\infty$, and hence $-\beta'\mathcal{E} < \infty$ on $D(I)$.*

*Proof.* The proof is as follows. First, we will show in part $(a)$ of the proof that (3.4.31) can be used to extend the pointwise estimates of (3.4.30) to all $x \in A'$, including those outside $D(I)$ — i.e. with $I(x) = \infty$. Moreover, it will be used to prove a unnormalized version of exponential tightness.

The rest of the proof is adapted from the proofs of Theorems 4.1.11 and 4.1.18 and Lemma 1.2.18 of [DZ10], see also Remark 3.7. In part $(b)$ we use the extended pointwise estimates to subsequently establish on $A'$ the lower bound for open sets, lower semi-continuity of $J_{\beta'}$, and the upper bound for compact sets. The unnormalized exponential tightness is then used in part $(c)$ to lift the latter to closed sets and to show that the sublevel sets of $J_{\beta'}$ are pre-compact, and hence the proof is concluded.

**(a) Extension of $J_{\beta'}$ and pointwise estimates outside $D(I)$ and exponential tightness**

Now, take any $x \in A' \backslash D(I)$, i.e. with $I(x) = \infty$. Then for any small enough $\epsilon$ such that $B_\epsilon(x) \subset A'$, we can show by Hölder's inequality with exponent $\gamma$,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \right] \le \frac{1}{\gamma} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\gamma\beta' N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \right] \\ + (1-\gamma^{-1}) \limsup_{N\to\infty} \frac{1}{N} \log P^N\big( B_\epsilon(x) \big).\end{aligned} \tag{3.4.33}$$

The first term is finite by (3.4.31), and since $\gamma > 1$ and hence $(1-\gamma^{-1}) > 0$, it follows from the pointwise estimate of $P^N$, see (3.4.17), that

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \right] = -\infty. \tag{3.4.34}$$

Recall that by definition $J_{\beta'}(x) = \infty$ since $x \notin D(I)$, and hence we have shown that (3.4.30) holds for *all* $x \in A'$.

Moreover, since $P^N$ is exponentially tight there exists a sequence of compact sets $K_M$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log P^N\big( K_M^c \big) \le -M. \tag{3.4.35}$$

Now, similarly as above, and using the inclusion of $K_M^c \cap A'$ in both $K_M^c$ and $A'$,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)}\, 1_{K_M^c \cap A'} \Big] \le \gamma^{-1} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\gamma\beta' N \mathcal{E}(z^N)}\, 1_{A'} \Big] \\ + (1-\gamma^{-1}) \limsup_{N\to\infty} \frac{1}{N} \log P^N\big( K_M^c \big).\end{aligned} \tag{3.4.36}$$

Since the first term is finite by (3.4.31) and independent of $M$, and the second term goes to $-\infty$ as $M \to \infty$, we can renumber the sets $K_M$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)}\, 1_{K_M^c \cap A'} \Big] \le -M. \tag{3.4.37}$$

**(b) Lower bounds for open sets, lower semi-continuity of $J_\beta$, and upper bound for compact sets**

Fix any open set $O$ with $O \subset A'$. Note that for any $x \in O$ there exists a small enough $\epsilon > 0$ such the ball $B_\epsilon(x)$ is contained in $O$, and hence from the pointwise estimate of (3.4.30b) it follows that,

$$\begin{aligned}\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_O \right] &\ge \lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \right] \\ &= -J_{\beta'}(x).\end{aligned} \tag{3.4.38}$$

Therefore, by taking the supremum over all $x \in O$, we have shown the lower bound (3.4.32b).

In particular, note that by taking $O = A'$ and using (3.4.31) that $J_\beta(x) > -\infty$, and hence $-\beta' \mathcal{E} < \infty$ on $A'$, since

$$\begin{aligned}-J_{\beta'}(x) &\le \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{A'} \right] \\ &\le \gamma^{-1} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' \gamma N \mathcal{E}(z^N)} 1_{A'} \right] \\ &< \infty.\end{aligned} \tag{3.4.39}$$

Moreover, fix any $x \in A'$ and $a \in \mathbb{R}$, then if $J_{\beta'}(x) > a$ it follows from the pointwise estimate (3.4.30a) that there exists a small enough $\tilde{\epsilon} > 0$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{B_{\tilde{\epsilon}}(x)} \right] < -a. \tag{3.4.40}$$

Hence for all $y \in B_{\tilde{\epsilon}}(x)$,

$$\begin{aligned} -a &> \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{B_{\tilde{\epsilon}}(x)} \right] \\ &\geq \limsup_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{B_{\epsilon}(y)} \right] \\ &= -J_{\beta'}(y), \end{aligned} \tag{3.4.41}$$

where the last equality follows from the pointwise estimate (3.4.30b).

Therefore the sets $\{x \,|\, x \in A',\, J_{\beta'}(x) > a\}$ are open, and since they are contained in the open set $A'$ they are also open in the sub-space topology on $A'$ — hence $J_{\beta'}$ is lower semi-continuous on $A'$ with respect to the sub-space topology. But since $A'$ is open in $\mathcal{X}$ it follows that $J_{\beta'}$ is also lower semi-continuous on $A'$ with respect to the original topology on $\mathcal{X}$.

We will now show the upper bound for compact sets. Fix some $\delta > 0$, and some compact set $K \subset A'$, and define $J^{\delta}_{\beta'}(x) := \min\{J_{\beta'}(x) - \delta, 1/\delta\}$. Because of the pointwise limit (3.4.30a) we can find for any $x \in K$ a small enough ball $B_x$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)} 1_{B_x} \right] \leq -J^{\delta}_{\beta'}(x). \tag{3.4.42}$$

Since $K$ is compact and can be covered by the collection $\cup_{x\in K} B_x$, there exists a finite subcover $\{B_{x_i}\}_i$, and hence by Remark 2.1,

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)}\, 1_K \right] &\leq \max_i \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta' N \mathcal{E}(z^N)}\, 1_{B_{x_i}} \right] \\ &\leq -J^{\delta}_{\beta'}(x). \end{aligned} \tag{3.4.43}$$

Letting $\delta \to 0$, we obtain the desired result.

### (c) Extension to upper bound for closed sets and pre-compact sublevel sets of $J_{\beta'}$

To go from an upper bound for compact sets to one for to closed sets, namely (3.4.32a), we will use the exponential tightness of the unnormalized measures as shown in (3.4.37).

Fix any closed set $C$ with $C \subset A'$, and let $K_M$ be the sequence of compact sets satisfying (3.4.37). Then, using Remark 2.1, we have for all $M$ the following

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)}\, 1_C \Big] \\ &\quad = \max\Big\{ \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)}\, 1_{C\cap K_M} \Big],\, \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)}\, 1_{C\cap K_M^c} \Big] \Big\} \\ &\quad \leq \max\Big\{ -J_{\beta'}(C\cap K_M)\,,\, \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)}\, 1_{A'\cap K_M^c} \Big] \Big\}. \end{aligned} \tag{3.4.44}$$

Where the final inequality follows from the fact that $C \subset A'$ and $C \cap K_M$ is compact, hence using the upper bound for compact sets established previously. Now, since the inequality holds for *all* $M$, letting $M \to \infty$ and exploiting (3.4.37), we have that

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)}\, 1_C \Big] &\leq \lim_{M\to\infty} \max\Big\{ -J_{\beta'}(C\cap K_M)\,,\, -M \Big\} \\ &\leq -J_{\beta'}(C). \end{aligned} \tag{3.4.45}$$

Additionally, using the lower bound (3.4.32b) for the open sets $A' \cap K_M^c$,

$$\begin{aligned} -J_{\beta'}(K_M^c \cap A') &\leq \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{\beta' N \mathcal{E}(z^N)} 1_{K_M^c \cap A'} \Big] \\ &\leq -M. \end{aligned} \tag{3.4.46}$$

Hence

$$M < \inf_{x \in K_M^c \cap A'} J_{\beta'}(x), \tag{3.4.47}$$

and thus

$$\begin{aligned} \{x \,|\, x \in A',\, J_{\beta'}(x) \leq M\} &\subset K_M \cap A' \\ &\subset K_M. \end{aligned} \tag{3.4.48}$$

Hence the sublevel sets of $J_{\beta'}$ are contained in the compact sets $K_M$, and thus they are pre-compact. See Remark 3.6 for why they are *not* necessarily compact instead of pre-compact.

Concluding, we have established both the upper bound for closed sets (3.4.32a) and lower bound for open sets (3.4.32b), and that $J_{\beta'}$ is lower semi-continuous with pre-compact sublevel sets, hence the proof is complete. □

We are now ready to prove Theorem 3.13, which stated the equivalence between quasi-continuity and localized large-deviation principles.

*Proof of Theorem 3.13.*

**(1)** $\Longleftrightarrow$ **(2)**

Recall, this stated that $\mathcal{E}$ is quasi-continuous if and only if for all $x \in D(I)$ and all $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}(x)| < \infty$ and

$$\lim_{\epsilon\to 0} [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} = 0. \tag{3.4.49}$$

However, this is true simply by the definition of quasi-continuity, see Definition 3.7.

**(2)** $\Longleftrightarrow$ **(3)**

Here it is implied that 3.4.49 holds if and only if for every $x \in D(I)$ and every $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}(x)| < \infty$ and the following pointwise estimates are satisfied,

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{-\beta N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \Big] = -J_\beta(x), \tag{3.4.50a}$$

$$\lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{-\beta N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \Big] = -J_\beta(x), \tag{3.4.50b}$$

which follows directly from Lemma 3.16 with $A' := \mathcal{X}$ and $\beta^* := \infty$.

**(3)** $\Longrightarrow$ **(4)**

To prove this, we need to show that by assumption of **(3)** for every $x' \in D(I)$ and all $\beta' \in \mathbb{R}$, there exists an open neighbourhood $A'$ of $x'$ such that $J_{\beta'}$ is unequivalent to $+\infty$, lower semi-continuous and with pre-compact sublevel sets, and moreover,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{-\beta' N \mathcal{E}(z^N)} 1_{A'} \Big] < \infty, \tag{3.4.51}$$

and such that for all open sets $O$ and closed sets $C$, with $O, C \subset A'$, it holds that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{-\beta' N \mathcal{E}(z^N)} 1_C \Big] \leq -J_\beta(C), \tag{3.4.52a}$$

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{-\beta' N \mathcal{E}(z^N)} 1_O \Big] \geq -J_\beta(O). \tag{3.4.52b}$$

Now, fix any $x' \in D(I)$ and $\beta' \in \mathbb{R}$. First, note that by the definition of $I$ and by the assumption of **(3)** that $0 \le I(x') < \infty$ and $|\mathcal{E}(x')| < \infty$, and hence $-\infty < J_\beta(x') < \infty$ for any $\beta \in \mathbb{R}$. In particular it follows that $J_{\beta'} \not\equiv \infty$ on any neighbourhood containing $x'$.

Similarly, for every $\gamma > 1$ it holds that $J_{\gamma\beta'}(x') > -\infty$. Hence, by assumption and the limit of (3.4.50a),

$$\begin{aligned}\lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\gamma\beta' N\mathcal{E}(z^N)}1_{B_\epsilon(x')}\right] &= -J_{\gamma\beta'}(x') \\ &< \infty.\end{aligned} \tag{3.4.53}$$

Thus we can find a small enough $\epsilon' > 0$ such that the corresponding exponential rate over the ball $B_{\epsilon'}(x')$ is finite, namely

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\gamma\beta' N\mathcal{E}(z^N)}1_{B_{\epsilon'}(x')}\right] < \infty. \tag{3.4.54}$$

Now set $A' := B_{\epsilon'}(x')$. Note that by assumption for all $x \in D(I)\cap A'$ the pointwise estimates 3.4.50 hold for $\beta = \beta'$. Thus, by Theorem 3.17, it follows that $J_{\beta'}$ satisfies all the necessary properties on $A'$, and that the upper bound for closed sets (3.4.52a) and the lower bound for open sets (3.4.52b) hold. Finally, note that (3.4.51) follows from (3.4.54) by Hölder's inequality with exponent $\gamma$.

**(4)** $\Longrightarrow$ **(3)**

We first note that one part of this direction — where by assumption of the localized LDPs of (3.4.52) around every $x' \in D(I)$ for every $\beta' \in \mathbb{R}$ should imply the pointwise estimates of (3.4.50) — follows directly from Lemma 3.14.

Finally, to show that $|\mathcal{E}(x')| < \infty$ for every $x' \in D(I)$ and conclude the proof, note that (3.4.52b) and (3.4.51) on an appropriate $A'$ for any fixed $x' \in D(I)$ and $\beta' \in \mathbb{R}$ imply that

$$\begin{aligned}-J_{\beta'}(x) &\le \lim_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-\beta' N\mathcal{E}(z^N)}\,1_{B_\epsilon(x')}\Big] \\ &\le \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-\beta' N\mathcal{E}(z^N)}\,1_{A'}\Big] \\ &< \infty.\end{aligned} \tag{3.4.55}$$

Hence $J_{\beta'}(x') > -\infty$, and since we can choose $\beta' \in \mathbb{R}$ arbitrarily, it follows that $|\mathcal{E}(x)| < \infty$. □

Recall Theorem 3.1 and Theorem 3.4, which respectively stated the equivalence between large-deviation principles and quasi-bounded and quasi-continuous functions, and showed how to go from pointwise estimates to LDPs on some finite temperature range. They broadly follow from Theorems 3.4 and 3.17, with the additional step of normalization, see below.

*Proof of Theorem 3.1.*
First, see and recall Theorems 3.1 and 3.13, and the proof of the latter, shown above.

**(1)** $\Longleftrightarrow$ **(2)**

Note that **(1)** and **(2)** of Theorem 3.1 are similar to those of Theorem 3.13 but with an additional boundedness condition, namely that the fact that quasi-boundedness is equivalent to the fact that for all $\beta \in \mathbb{R}$ it holds that

$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} < \infty. \tag{3.4.56}$$

However, this is simply the definition of quasi-boundedness, see Definition 3.8.

**(2) $\implies$ (3)**

Note that by Theorem 3.1, it follows that quasi-continuity implies **(3)** of Theorem 3.1, i.e. that for all $x \in D(I)$ and all $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}(x)| < \infty$ and

$$\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \Big] = -J_\beta(x), \tag{3.4.57a}$$

$$\lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} 1_{B_\epsilon(x)} \Big] = -J_\beta(x). \tag{3.4.57b}$$

Moreover, by the assumption of quasi-boundedness we have that for all $\beta \in \mathbb{R}$

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} \Big] < \infty. \tag{3.4.58}$$

Hence, by Theorem 3.17, it follows that for every $\beta \in \mathbb{R}$ the unnormalized measures satisfy a LDP on the entire space $\mathcal{X}$, i.e. $J_\beta$ is lower semi-continuous on $\mathcal{X}$ with pre-compact sublevel sets, and for any open set $O$ and closed set $C$, it holds that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} 1_C \Big] \le -J_\beta(C), \tag{3.4.59a}$$

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} 1_O \Big] \ge -J_\beta(O). \tag{3.4.59b}$$

In particular, since $\mathcal{X}$ is open and closed, it follows by combining (3.4.59a) and (3.4.59b) that we can replace the limit infimum/supremum of (3.4.59) by an actual limit, i.e. it holds that for every $\beta \in \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} \Big] = -J_\beta(\mathcal{X}). \tag{3.4.60}$$

Now, fix $\beta \in \mathbb{R}$, and recall the normalized measures $Q^N_\beta$, defined by

$$\frac{dQ^N_\beta}{dP^N}(x) := \frac{e^{-\beta N \mathcal{E}(x)}}{Z^N_\beta}, \tag{3.4.61}$$

with normalization constants $Z^N_\beta$ given by

$$Z^N_\beta = \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} \Big], \tag{3.4.62}$$

and recall the function $\mathcal{F}_\beta$,

$$\mathcal{F}_\beta := J_\beta(x) - J_\beta(\mathcal{X}). \tag{3.4.63}$$

Note that by (3.4.58) and (3.4.60) it follows that $J_\beta(\mathcal{X}) > -\infty$, and since by assumption $|\mathcal{E}(x)| < \infty$ for any $x \in D(I)$ it follows that $J_\beta \not\equiv +\infty$, and in particular $J_\beta(\mathcal{X}) < \infty$. Hence, the function $\mathcal{F}_\beta$ is well-defined, non-negative and with $\mathcal{F}_\beta \not\equiv \infty$. Moreover, recall that $J_\beta$ is lower semi-continuous on $\mathcal{X}$ with pre-compact sublevel sets. However, lower semi-continuity on $\mathcal{X}$ implies that the sublevel sets are closed, and hence they are compact. Thus, we have concluded that $\mathcal{F}_\beta$ is a rate function.

Now, it follows by (3.4.59a) and (3.4.60) that for every closed set $C$,

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log Q^N_\beta(C) &= \limsup_{N\to\infty} \left( \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} 1_C \Big] - \frac{1}{N} \log Z^N_\beta \right) \\
&= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} 1_C \Big] + J_\beta(\mathcal{X}) \\
&\le -(J_\beta(C) - J_\beta(\mathcal{X})) \\
&= -\mathcal{F}_\beta(C)
\end{aligned} \tag{3.4.64}$$

Similarly, by (3.4.59b) and (3.4.60) it follows that for every open set $O$,

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N\big(\, O \,\big) \geq -\mathcal{F}_\beta(O). \tag{3.4.65}$$

Hence, we have proven for every $\beta \in \mathbb{R}$ that $Q_\beta^N$ satisfies a large-deviation principle with rate function $\mathcal{F}_\beta$.

**(3) $\implies$ (2)**

Finally, to conclude the proof, note that by Lemma 3.14 a LDP for $Q_\beta^N$ with rate function $\mathcal{F}_\beta$ implies that for any $x \in \mathcal{X}$

$$\begin{aligned} \lim_{\epsilon\to 0}\limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N\big(\, 1_{B_\epsilon(x)} \,\big) &= -\mathcal{F}_\beta(x), \\ \lim_{\epsilon\to 0}\liminf_{N\to\infty} \frac{1}{N} \log Q_\beta^N\big(\, 1_{B_\epsilon(x)} \,\big) &= -\mathcal{F}_\beta(x). \end{aligned} \tag{3.4.66}$$

However, note that for any $\epsilon > 0$ it holds that

$$\begin{aligned} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] &= \limsup_{N\to\infty} \Big( \frac{1}{N} \log Q_\beta^N\big(\, 1_{B_\epsilon(x)} \,\big) + \frac{1}{N} \log Z_\beta^N \Big) \\ &\leq \limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N\big(\, 1_{B_\epsilon(x)} \,\big) + \limsup_{N\to\infty} \frac{1}{N} \log Z_\beta^N. \end{aligned} \tag{3.4.67}$$

And, similarly,

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] \geq \liminf_{N\to\infty} \frac{1}{N} \log Q_\beta^N\big(\, 1_{B_\epsilon(x)} \,\big) + \liminf_{N\to\infty} \frac{1}{N} \log Z_\beta^N. \tag{3.4.68}$$

Hence, since by the assumption of the Laplace principle it holds that

$$J_\beta(\mathcal{X}) = \lim_{N\to\infty} \frac{1}{N} \log Z_\beta^N, \tag{3.4.69}$$

it follows that by the definition of $\mathcal{F}_\beta$ that

$$\begin{aligned} \lim_{\epsilon\to 0}\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] &= -J_\beta(x) \\ \lim_{\epsilon\to 0}\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[\, e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] &= -J_\beta(x). \end{aligned} \tag{3.4.70}$$

Finally, similar as in the proof of Theorem 3.13, note that by assumptions $J_\beta(\mathcal{X} > -\infty$ and hence $J_\beta(x) > -\infty$ for all $x \in \mathcal{X}$ and all $\beta \in \mathbb{R}$. Hence, by the definition of $J_\beta$, $|\mathcal{E}(x)| < \infty$ for all $x \in D(I)$. □

*Proof of Theorem 3.4.*
Finally, recall Theorem 3.4, which states how to go from quasi-continuity and the fact that for all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$ with some $\beta^* > 0$ that

$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} < \infty, \tag{3.4.71}$$

to a LDP on $\mathcal{X}$ for $Q_\beta^N$ for all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$.

However, note that this is simply the direction (2) $\implies$ (3) of Theorem 3.1, but with $\beta \in \mathbb{R}$ now restricted with $|\beta| < \beta^*$. Namely, quasi-continuity implies pointwise estimates for all $x \in D(I)$, which by Theorem 3.17 is lifted to a unnormalized LDP for all with $|\beta| < \beta^*$. The step from a unnormalized LDP to one for $Q_\beta^N$ is exactly as done in the proof of Theorem 3.4, see above, and we will not repeat it here.

□

**Comments**

**Remark 3.6.** Note that in the formulation of localized LDPs on a open set $A'$, as stated in Theorem 3.17, we do not use the *sub-space topology* on $A'$. Namely, we only consider the subsets $O$ and $C$ of $A$, such that $O$ is open and $C$ is closed in the topology on $\mathcal{X}$. While any set $O$ that is open in the subspace topology of $A'$ is also open in $\mathcal{X}$, and hence satisfies

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{-\beta' N \mathcal{E}(z^N)} \, 1_O \Big] \geq -J_{\beta'}(O), \tag{3.4.72}$$

the upper bound for closed sets,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ e^{-\beta' N \mathcal{E}(z^N)} \, 1_C \Big] \leq -J_{\beta'}(C), \tag{3.4.73}$$

does *not* necessarily hold for any $C$ that is closed in the sub-space topology. The reason for this is that in the proof of Theorem 3.17 we lift an upper bound on compact sets to sets $C \subset A'$ such that $C \cap K_M$ is compact, but where the compact sets $K_M$ — induced by the exponential tightness of $P^N$ on the entire space $\mathcal{X}$ — do *not necessarily lie in* $A'$, and hence $K_M \cap A'$ may not be compact. Hence we need that $C$ is closed in the topology on $\mathcal{X}$.

Also note that this is the same reason that the localized sublevel sets of $J_\beta$, $\{x \,|\, x \in A', J_\beta(x) \leq a\}$, are pre-compact — since they are shown to be contained in $K_M$ — but not necessarily compact.

**Remark 3.7.** As mentioned in the beginning of the proof of Theorem 3.17, the step of pointwise estimates to localized and unnormalized LDPs are unnormalized versions of Theorems 4.1.11 and 4.1.18, and Lemma 1.2.18 of [DZ10], in the sense that

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta N \mathcal{E}(z^N)} 1'_A \right], \\ &\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta N \mathcal{E}(z^N)} 1'_A \right], \end{aligned} \tag{3.4.74}$$

both exist, but may not be equal.

Moreover, even when $A' = \mathcal{X}$ it is not a priori guaranteed that they are the same and hence a genuine limit exists, and only follows *after* we have established a unnormalized LDP.

In particular, from the pointwise estimates for the unnormalized exponential integrals on $\mathcal{X}$ one can *not* always conclude that $Q_\beta^N$ also satisfies certain pointwise estimates itself and then just use Theorems 4.1.11 and 4.1.18, and Lemma 1.2.18 of [DZ10], since

$$\begin{aligned} \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N \big( 1_{B_\epsilon(x)} \big) &= -J_\beta + \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta N \mathcal{E}(z^N)} \right], \\ \lim_{\epsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log Q_\beta^N \big( 1_{B_\epsilon(x)} \big) &= -J_\beta + \liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta N \mathcal{E}(z^N)} \right], \end{aligned} \tag{3.4.75}$$

but as said, it is not a priori known that the two terms are equal.

Interestingly enough, in [BG99], the main inspiration for our framework of quasi-continuity, they did not notice this distinction. In going from pointwise estimates to a LDP they made the error of engaging in a circular argument, i.e. simply stating that the large-deviation principles hold by unknowingly assuming that the limit of $\lim_{N\to\infty} \frac{1}{N} \log Z_\beta^N$ exist and applying Theorems 4.1.11 and 4.1.18 of [DZ10], and afterwards claiming that hence the limit always exists and is equal to $J_\beta(\mathcal{X})$.

In addition, they omitted the consideration of $x \notin D(I)$ from their pointwise estimates, which are strictly necessary to make the proof work. As shown in the proof of Theorem 3.17, we prove this by using a certain exponential moment condition, which also implied exponential tightness — which was not necessary for the case studied of [BG99] since the underlying space was compact.

Concluding, without any prior knowledge about the limit of $\lim_{N\to\infty} \frac{1}{N} \log Z_\beta^N$, one simply has to go the route of unnormalized LDPs first, and we hope that our Theorem 3.17 settles this question.

## 3.5 Convergence of quasi-continuous functions

In the remaining sections, to show that a function $\mathcal{E}$ is quasi-continuous, we will actually not explicitly verify Definition 3.7, i.e. that for every $x \in D(I)$ and every $\beta \in \mathbb{R}$ it holds that $|\mathcal{E}(x)| < \infty$ and

$$\lim_{\epsilon\to 0} [\![\beta(\mathcal{E} - \mathcal{E}(x))]\!]_{B_\epsilon(x)} = 0. \tag{3.5.1}$$

Instead, we will approximate $\mathcal{E}$ by a sequence of continuous functions $\mathcal{E}_\lambda$ in an appropriate way. Moreover, we will show that the linear space of quasi-continuous functions is even *closed* with respect to this convergence, and show how the latter is actually a quasi-counterpart of uniform convergence of continuous functions. In particular, our results will directly imply Theorems 3.2 and 3.3.

For additive energies, as shown in Section 4, and which encompasses the case for invariant measures in Section 5 and empirical processes in Section 6, convergence can be established easily for quite a general class of functions, and is also well-suited for discretization arguments for systems with logarithmic potentials, as shown in Section 5.

Let us first define this quasi-analogue for uniform convergence, with a similar distinction in locality as with quasi-boundedness of Section 3.3.

**Definition 3.18.** *Let $\mathcal{E}$ be such that for every $x \in D(I)$ it holds that $\mathcal{E}(x) < \infty$.*

*Then, the sequence of functions $\mathcal{E}_\lambda$ converges* locally quasi-uniformly *(l.q.u) to $\mathcal{E}$ if*

1. *$\mathcal{E}_\lambda$ converges to $\mathcal{E}$ pointwise at all $x \in D(I)$.*
2. *For every $x \in D(I)$ and all $\beta \in \mathbb{R}$,*

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0} [\![\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{B_\epsilon(x)} = 0. \tag{3.5.2}$$

*Moreover, it converges* quasi-uniformly *(q.u) to $\mathcal{E}$ if*

1. *$\mathcal{E}_\lambda$ converges to $\mathcal{E}$ pointwise at all $x \in D(I)$.*
2. *For every $\beta > 0$ it holds that*

$$\lim_{\lambda\to 0} [\![\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{\mathcal{X}} = 0. \tag{3.5.3}$$

As with quasi-boundedness, note the difference between the two definitions, and note the similarities to the definition of (local) uniform convergence for bounded functions, where $[\![\cdot]\!]_A$ is again simply replaced by the uniform norm $\|\cdot\|_{A,\infty}$.

**Lemma 3.19.** *Quasi-uniform convergence implies local quasi-uniform convergence.*

*Proof.* Similar as for local quasi-boundedness, recall Lemma 3.10, (3.3.17), which stated that for any $\beta \geq 1$ and any $O \subset A$ with $I(O^o) < \infty$, that

$$[\![\mathcal{E}]\!]_O \leq \frac{1}{\beta}\left([\![\beta\mathcal{E}]\!]_A + \limsup_{N\to\infty}\frac{1}{N}\log\frac{P^N(A)}{P^N(O)}\right). \tag{3.5.4}$$

Now, suppose $\mathcal{E}_\lambda$ converge quasi-uniformly to a function $\mathcal{E}$. Then $\mathcal{E}_\lambda$ converge pointwise to $\mathcal{E}$ on $D(I)$. Moreover, for any $x \in D(I)$, any $\epsilon > 0$ and any $\beta \geq 1$ it follows by (3.5.4) that

$$[\![\mathcal{E} - \mathcal{E}_\lambda]\!]_{B_\epsilon(x)} \leq \frac{1}{\beta}\left([\![\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{\mathcal{X}} - \liminf_{N\to\infty}\frac{1}{N}\log P^N\big(B_\epsilon(x)\big)\right). \tag{3.5.5}$$

Hence, by quasi-uniform convergence and the large-deviation principle for $P^N$,

$$\begin{aligned} \lim_{\lambda\to 0} [\![\mathcal{E} - \mathcal{E}_\lambda]\!]_{B_\epsilon(x)} &\le \frac{1}{\beta} I(B_\epsilon(x)) \\ &\le \frac{I(x)}{\beta}. \end{aligned} \tag{3.5.6}$$

Since the latter is true for any $\beta \ge 1$, it follows that

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0} [\![\mathcal{E} - \mathcal{E}_\lambda]\!]_{B_\epsilon(x)} = 0. \tag{3.5.7}$$

Similarly, for any $\beta' \in \mathbb{R}$, replacing $\mathcal{E}$ and $\mathcal{E}_\lambda$ by $\beta'\mathcal{E}$ and $\beta'\mathcal{E}_\lambda$ in the derivation above, local quasi-uniform convergence follows. □

Now, as mentioned, we have the following quasi-analogue of the closure of bounded and continuous functions under uniform convergence.

**Theorem 3.20.**

*Let $\mathcal{E}$ and $\mathcal{E}_\lambda$ for all $\lambda$ be finite on $D(I)$.*

*Suppose $\mathcal{E}_\lambda$ converges locally quasi-uniformly to $\mathcal{E}$. Then,*

1. *If all $\mathcal{E}_\lambda$ are locally quasi-bounded, then so is $\mathcal{E}$.*
2. *If all $\mathcal{E}_\lambda$ are quasi-continuous, then so is $\mathcal{E}$.*

*Moreover, suppose $\mathcal{E}_\lambda$ converges quasi-uniformly to $\mathcal{E}$. Then,*

1. *If all $\mathcal{E}_\lambda$ are quasi-bounded, then so is $\mathcal{E}$.*
2. *If all $\mathcal{E}_\lambda$ are quasi-bounded and quasi-continuous, then so is $\mathcal{E}$.*

Using the $[\![\cdot]\!]_A$-notation and the related calculus developed in Section 3.3, the proof is straightforward.

*Proof.* We will first prove the part concerning quasi-boundedness. Suppose the quasi-bounded functions $\mathcal{E}_\lambda$ converge quasi-uniformly to $\mathcal{E}$. Then, by convexity of $[\![\cdot]\!]_A$, see Lemma 3.9, it follows that for every $\beta \in \mathbb{R}$ and every $\lambda$,

$$\begin{aligned} [\![\beta\mathcal{E}]\!]_{\mathcal{X}} &= [\![\beta(\mathcal{E} - \mathcal{E}_\lambda) + \beta\mathcal{E}_\lambda]\!]_{\mathcal{X}} \\ &\le \frac{1}{2} [\![2\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{\mathcal{X}} + \frac{1}{2} [\![2\beta\mathcal{E}_\lambda]\!]_{\mathcal{X}}. \end{aligned} \tag{3.5.8}$$

By quasi-uniform convergence the first term converges to zero as $\lambda \to 0$, and hence there is a small enough $\lambda$ such that $[\![2\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{\mathcal{X}} < \infty$, and hence

$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} < \infty. \tag{3.5.9}$$

Similarly, suppose the locally quasi-bounded functions $\mathcal{E}_\lambda$ converge locally quasi-uniformly to $\mathcal{E}$. Then for every $x \in D(I)$ and every $\beta \in \mathbb{R}$, there exists an $\epsilon > 0$ and a $\lambda > 0$ such that both $[\![2\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{B_\epsilon(x)} < \infty$ and $[\![2\beta\mathcal{E}_\lambda]\!]_{B_\epsilon(x)} < \infty$, and hence

$$\begin{aligned} [\![\beta\mathcal{E}]\!]_{B_\epsilon(x)} &= [\![\beta(\mathcal{E} - \mathcal{E}_\lambda) + \beta\mathcal{E}_\lambda]\!]_{B_\epsilon(x)} \\ &\le \frac{1}{2} [\![2\beta(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{B_\epsilon(x)} + \frac{1}{2} [\![2\beta\mathcal{E}_\lambda]\!]_{B_\epsilon(x)} \\ &< \infty. \end{aligned} \tag{3.5.10}$$

Finally, suppose $\mathcal{E}_\lambda$ are quasi-continuous and converge locally quasi-uniformly to $\mathcal{E}$. Then note that for any set $A$, any $\lambda$, and $x \in D(I)$,

$$\begin{aligned}
[\![\mathcal{E} - \mathcal{E}(x)]\!]_A &= [\![(\mathcal{E} - \mathcal{E}_\lambda) + (\mathcal{E}_\lambda - \mathcal{E}_\lambda(x)) + (\mathcal{E}_\lambda(x) - \mathcal{E}(x))]\!]_A \\
&\le \frac{1}{3}[\![3(\mathcal{E} - \mathcal{E}_\lambda)]\!]_A + \frac{1}{3}[\![3(\mathcal{E}_\lambda - \mathcal{E}_\lambda(x))]\!]_A + |\mathcal{E}_\lambda(x) - \mathcal{E}(x)|.
\end{aligned} \tag{3.5.11}$$

Note that the left-hand side is independent of $\lambda$, and hence for any fixed $\lambda'$,

$$\begin{aligned}
[\![\mathcal{E} - \mathcal{E}(x)]\!]_A &\le \inf_\lambda \left\{ \frac{1}{3}[\![3(\mathcal{E} - \mathcal{E}_\lambda)]\!]_A + \frac{1}{3}[\![3(\mathcal{E}_\lambda - \mathcal{E}_\lambda(x))]\!]_A + |\mathcal{E}_\lambda(x) - \mathcal{E}(x)| \right\} \\
&\le \inf_{\lambda_1} \left\{ \frac{1}{3}[\![3(\mathcal{E} - \mathcal{E}_{\lambda_1})]\!]_A \right\} + \inf_{\lambda_2} \left\{ \frac{1}{3}[\![3(\mathcal{E}_{\lambda_2} - \mathcal{E}_{\lambda_2}(x))]\!]_A \right\} + \inf_{\lambda_3} \left\{ |\mathcal{E}_{\lambda_3}(x) - \mathcal{E}(x)| \right\} \\
&\le \lim_{\lambda \to 0} \frac{1}{3}[\![3(\mathcal{E} - \mathcal{E}_\lambda)]\!]_A + \frac{1}{3}[\![3(\mathcal{E}_{\lambda'} - \mathcal{E}_{\lambda'}(x))]\!]_A + \lim_{\lambda \to 0} |\mathcal{E}_\lambda(x) - \mathcal{E}(x)|.
\end{aligned} \tag{3.5.12}$$

Thus, for all $\lambda'$,

$$\begin{aligned}
\lim_{\epsilon \to 0} [\![\mathcal{E} - \mathcal{E}(x)]\!]_{B_\epsilon(x)} &\le \lim_{\epsilon \to 0} \lim_{\lambda \to 0} \frac{1}{3}[\![3(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{B_\epsilon(x)} + \lim_{\epsilon \to 0} \frac{1}{3}[\![3(\mathcal{E}_{\lambda'} - \mathcal{E}_{\lambda'}(x))]\!]_{B_\epsilon(x)} + \lim_{\lambda \to 0} |\mathcal{E}_\lambda(x) - \mathcal{E}(x)| \\
&= 0,
\end{aligned} \tag{3.5.13}$$

where the latter follows local quasi-uniform convergence of $\mathcal{E}_\lambda$ to $\mathcal{E}$, i.e. (3.5.2) and pointwise convergence on $D(I)$, and from quasi-continuity of $\mathcal{E}_{\lambda'}$. □

Finally, we have the following equivalent formulation of local quasi-uniform convergence.

**Lemma 3.21.** *Suppose the sequence of functions $\mathcal{E}_\lambda$ converges pointwise to a function $\mathcal{E}$, with $\mathcal{E}$ finite on $D(I)$.*

*Moreover, suppose that there exists a constant $K$ such that for every $x' \in D(I)$ and every $\beta' \in \mathbb{R}$ there exists a open neighbourhood $A'$ of $x'$ such that*

$$\limsup_{\lambda \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{\beta' N |\mathcal{E} - \mathcal{E}_\lambda|(z^N)} 1_{A'} \right] \le K. \tag{3.5.14}$$

*Then $\mathcal{E}_\lambda$ converges locally quasi-uniformly to $\mathcal{E}$.*

*Proof.* Fix any $x' \in D(I)$ and any $\beta' \in \mathbb{R}$. Then by assumption we can find a open neighbourhood $A'$ of $x'$ such that (3.5.14) holds. Now, take any $\epsilon > 0$ such that $B_\epsilon(x') \subset A'$. Then, using the LDP for $P^N$,

$$\begin{aligned}
[\![\beta'(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{B_\epsilon(x')} &= \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{\beta' N |\mathcal{E} - \mathcal{E}_\lambda|(z^N)} \middle| 1_{B_\epsilon(x')} \right] \\
&\le \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{\beta' N |\mathcal{E} - \mathcal{E}_\lambda|(z^N)} 1_{B_\epsilon(x')} \right] - \liminf_{N \to \infty} \frac{1}{N} \log P^N\big( B_\epsilon(x') \big) \\
&\le \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{\beta' N |\mathcal{E} - \mathcal{E}_\lambda|(z^N)} 1_{A'} \right] + I(B_\epsilon(x')).
\end{aligned} \tag{3.5.15}$$

Since $I(B_\epsilon(x')) \le I(x')$ it follows that

$$\limsup_{\epsilon \to 0} \limsup_{\lambda \to 0} [\![\beta'(\mathcal{E} - \mathcal{E}_\lambda)]\!]_{B_\epsilon(x')} \le K + I(x') \tag{3.5.16}$$

Note that $K$ and $x'$ are independent of $\beta'$, and since $I(x') < \infty$ by assumption, $K + I(x')$ is finite.

Now, by Lemma 3.10, (3.3.17), we have that for any $\beta \ge 1$ and any set $A$,

$$[\![\beta'(\mathcal{E} - \mathcal{E}_\lambda)]\!]_A \le \frac{1}{\beta}[\![\beta\beta'(\mathcal{E} - \mathcal{E}_\lambda)]\!]_A \tag{3.5.17}$$

Hence,

$$\limsup_{\epsilon\to 0}\limsup_{\lambda\to 0} [\![\beta'(\mathcal{E}-\mathcal{E}_\lambda)]\!]_{B_\epsilon(x')} \le \frac{K+I(x')}{\beta} \tag{3.5.18}$$

Letting $\beta\to\infty$, it follows that for any fixed $x'\in D(I)$ and $\beta'\in\mathbb{R}$ that

$$\lim_{\epsilon\to 0}\lim_{\lambda\to 0} [\![\beta'(\mathcal{E}-\mathcal{E}_\lambda)]\!]_{B_\epsilon(x')} = 0, \tag{3.5.19}$$

and thus we have established local quasi-uniform convergence. □

#### Comments

**Remark 3.8.** In the case for a $\mathcal{E}$ that is strictly locally quasi-bounded, and continuous $\mathcal{E}_\lambda$, it is clear that quasi-convergence can *only* be locally, since for some sets $A$ it can hold that $[\![\beta(\mathcal{E}-\mathcal{E}_\lambda)]\!]_A = \infty$ for high enough $\beta$, as mentioned in Remark 3.4.

This is the reason that for such a case we have to rely on a different approach. In Section 5.4 for example, we will use a discretization technique and apply Lemma 3.21 — which only requires us to find for every $x\in D(I)$ and $\beta\in\mathbb{R}$ an appropriate set $A$ such that a certain uniform bound holds. These sets $A$ will actually be constructed by placing restrictions on how many particles can be close together, which will act as a kind of approximate absolute continuity of $z^N$ with respect to the Lebesgue measure.

**Remark 3.9.** Note that for quasi-uniform convergence we require pointwise convergence of $\mathcal{E}_\lambda$ on $D(I)$, which might seem counter-intuitive However, as noted in Remark 3.1, recall that if quasi-continuity is not yet established, the behaviour of $\mathcal{E}(z^N)$ and $\mathcal{E}(x)$ with $x\in D(I)$ can be completely unrelated and we need a condition for convergence of $\mathcal{E}_\lambda$ to $\mathcal{E}$ on both $D(I)$ and the supports of $P^N$, as is also clear from the proof of Theorem 3.20. However, in our applications pointwise convergence on $D(I)$ is usually easily verified, or derived from the same exponential estimates — see Section 4.

**Remark 3.10.** Just as for quasi-continuity and quasi-boundedness, there is also a stochastic interpretation of uniform quasi-convergence. The idea is that the probability of $\mathcal{E}(z^N)$ being much different from $\mathcal{E}_\lambda(z^N)$ becomes super-exponentially small as $\lambda\to 0$.

Namely, one can show that quasi-uniform convergence implies that for any $\delta>0$,

$$\lim_{\lambda\to 0}\limsup_{N\to\infty}\frac{1}{N}\log P^N\Big(\,\big|\mathcal{E}(z^N)-\mathcal{E}_\lambda(z^N)\big|>\delta\,\Big) = -\infty. \tag{3.5.20}$$

This looks similar to a notation used in [DZ10], where the random variables $z_\lambda^N$ with law $P_\lambda^N$ are *exponential approximations* of the r.v. $z^N$ with law $P^N$ if there exists a joint law $\tilde{P}_\lambda^N$ such that

$$\lim_{\lambda\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\tilde{P}_\lambda^N\big(\big|z^N-z_\lambda^N\big|>\delta\big) = -\infty. \tag{3.5.21}$$

In particular, note that the variables $\mathcal{E}_\lambda(z^N)$ are exponential approximations of $\mathcal{E}(z^N)$. However, quasi-uniform convergence does not seem to imply that the induced measures $Q_{\beta,\lambda}^N$ are exponential approximations of $Q_\beta^N$ — which is rather restrictive. Moreover, if $Q_{\beta,\lambda}^N$ are exponential approximations of $Q_\beta^N$ for only one *single* fixed $\beta$, it does not follow that $\mathcal{E}_\lambda$ converges quasi-uniformly.

Hence, the key difference between the two concepts is that for quasi-uniform convergence we require conditions on $Q_{\beta,\lambda}^N$ for *all* $\beta\in\mathbb{R}$ simultaneously.

### 3.6 Properties of the unnormalized rate functions $J_\beta$

While it is clear that quasi-continuity is a quite restrictive condition, it is also interesting to ask what this entails for the unnormalized rate functions $J_\beta$. Again, we will see that $\mathcal{E}$ is as 'continuous' with respect to $I$ as we need for our purposes, and that it satisfies quite a strong growth condition.

First, recall the definition of $J_\beta$, namely

$$J_\beta(x) := \begin{cases} \beta\mathcal{E}(x) + I(x) & x \in D(I), \\ +\infty & x \notin D(I). \end{cases} \tag{3.6.1}$$

Note that if $|\mathcal{E}(x)|$ is finite on $D(I)$ then we can simply state $J_\beta := \beta\mathcal{E} + I$.

**Lemma 3.22.** *Suppose $\mathcal{E}$ is quasi-continuous. Then $\mathcal{E}$ is finite on $D(I)$, and*

1. *$J_\beta$ is lower semi-continous at all $x \in D(I)$ for all $\beta \in \mathbb{R}$.*
2. *$\mathcal{E}$ is continuous on the compact sets $\{x \,|\, I(x) \le M\}$.*
3. *For any set $A$, $x \in D(I) \cap A$ and $\beta > 0$ it holds that*
$$|\mathcal{E}(x)| \le \frac{1}{\beta}\Big(I(x) + [\![\beta\mathcal{E}]\!]_A\Big). \tag{3.6.2}$$
*In particular, $[\![\beta\mathcal{E}]\!]_A < \infty$ implies that for all sequences $x_n \in D(I) \cap A$ with $\mathcal{E}(x_n) \to \infty$ that*
$$\limsup_{n\to\infty} \frac{|\mathcal{E}(x_n)|}{I(x_n)} \le \beta^{-1}. \tag{3.6.3}$$
4. *In addition, $[\![\gamma\beta\mathcal{E}]\!]_A < \infty$ for some $\gamma > 1$ implies that $J_\beta$ is lower semi-continuous on $A$ with pre-compact sublevel sets $\{x \,|\, x \in A, J_\beta(x) \le M\}$ .*
5. *In particular, when $\mathcal{E}$ is quasi-continuous and quasi-bounded, $J_\beta$ is lower semi-continuous on $\mathcal{X}$ with compact sublevel sets for all $\beta \in \mathbb{R}$.*

Moreover, we will also briefly investigate tools to establish the properties $\mathcal{E}$ of above in a purely *deterministic* way, by establishing uniform convergence of continuous functions $\mathcal{E}_\lambda$ on the sublevel sets of $I$ instead of using limits of exponential integrals, and see how they are deterministic analogs of quasi-uniform and local quasi-uniform convergence.

Since the properties of Lemma 3.22 are *necessary* conditions for quasi-continuity, these tools provide us with possible candidates for quasi-continuity, as shown in Section 4.5. Moreover, uniform convergence of $\mathcal{E}_\lambda$ to $\mathcal{E}$ on the sublevel sets of $I$ shed valuable insight on minimizers of $J_\beta$.

*Proof.* 1. Fix any $x \in D(I)$. By Theorem 3.13 it follows that for all $\beta \in \mathbb{R}$ there exists an open set $A$ such that $J_\beta$ is lower semi-continuous on $A$, and in particular at $x$. Hence, $J_\beta$ is l.s.c. at $x$ for *all* $\beta \in \mathbb{R}$.

2. Fix $M$ and define $C_M := \{y \,|\, I(y) \le M\}$, which by the properties of the rate function $I$ are closed sets. Now, for any sequence $x_n \in C_M$ that converges to a $x \in C_M$, we have by the l.s.c. of $J_\beta$ for all $\beta \in \mathbb{R}$ that
$$\liminf_{n\to\infty} \beta\mathcal{E}(x_n) + I(x_n) \ge \beta\mathcal{E}(x) + I(x). \tag{3.6.4}$$
Dividing by $\beta$, we derive
$$\begin{aligned} \liminf_{n\to\infty} \mathcal{E}(x_n) + \frac{I(x_n)}{\beta} &\ge \mathcal{E}(x) + I(x), && \beta > 0, \\ \limsup_{n\to\infty} \mathcal{E}(x_n) + \frac{I(x_n)}{\beta} &\le \mathcal{E}(x) + \frac{I(x)}{\beta}, && \beta < 0. \end{aligned} \tag{3.6.5}$$

But $I(x) \le M$ and $I(x_n) \le M$ for all $n$. Thus, letting $\beta \to \pm\infty$, it follows that

$$\begin{aligned} \liminf_{n\to\infty} \mathcal{E}(x_n) &\ge \mathcal{E}(x), \\ \limsup_{n\to\infty} \mathcal{E}(x_n) &\le \mathcal{E}(x), \end{aligned} \tag{3.6.6}$$

and therefore $\mathcal{E}(x_n) \to \mathcal{E}(x)$. Hence, $\mathcal{E}$ is continuous on the sublevel sets of $I$.

3. From Lemma 3.10, (3.3.17), we have for any $\beta \in \mathbb{R}$,

$$[\![\beta\mathcal{E}]\!]_O \le [\![\beta\mathcal{E}]\!]_A + \limsup_{N\to\infty} \frac{1}{N} \log \frac{P(A)}{P(O)} \qquad \text{for any } O \subset A, \tag{3.6.7}$$

Setting $O := B_\epsilon(x)$, exploiting quasi-continuity of $|\mathcal{E}|$ and noting $P(A) \le 1$, it follows that in the limit $\epsilon \to 0$

$$|\beta\mathcal{E}(x)| \le \Big( [\![\beta\mathcal{E}]\!]_A + I(x) \Big). \tag{3.6.8}$$

4. Note that this follows directly from Theorem 3.17.

5. Similarly, this follows directly from Theorem 3.4.

□

Now, we will give a deterministic analog of $[\![\cdot]\!]_{\mathcal{X}}$.

$$\langle\!\langle \mathcal{E} \rangle\!\rangle_{\mathcal{X}} := \sup_{x\in D(I)} |\mathcal{E}(x)| - I(x)\,. \tag{3.6.9}$$

It can be shown that $\langle\!\langle \mathcal{E} \rangle\!\rangle_{\mathcal{X}}$ has many of the same properties of $[\![\cdot]\!]_{\mathcal{X}}$, and could similarly be extended to for example $\langle\!\langle \mathcal{E} \rangle\!\rangle_A$ via a variational statement over any set $A$. However, we do not treat those here, and merely provide a few equivalence and convergence results.

**Lemma 3.23.** *Suppose $\mathcal{E}$ is quasi-bounded and quasi-continuous. Then for any $\beta \in \mathbb{R}$, it holds that*

$$[\![\beta\mathcal{E}]\!]_{\mathcal{X}} = \langle\!\langle \beta\mathcal{E} \rangle\!\rangle_{\mathcal{X}}\,. \tag{3.6.10}$$

*Proof.* Note that $f : \mathbb{R} \to \mathbb{R}$, with $f(a) := |a|$, is Lipschitz continuous. Thus by Lemma 3.11 it follows that $|\mathcal{E}|$ is quasi-continuous and quasi-bounded. Moreover, note that by the assumption of quasi-continuity $|\mathcal{E}|(x) < \infty$ for any $x \in D(I)$.

Hence for any fixed $\beta \in \mathbb{R}$ it follows by Theorem 3.17, that the Laplace principle holds for $\beta|\mathcal{E}|$, and by the definition of $[\![\cdot]\!]_{\mathcal{X}}$, the result follows, since

$$\begin{aligned} [\![\beta\mathcal{E}]\!]_{\mathcal{X}} &= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \Big[ \, e^{N|\beta\mathcal{E}(z^N)|} \, \Big] \\ &= \sup_{x\in\mathcal{X}} |\beta\mathcal{E}(x)| - I(x) \\ &= \langle\!\langle \beta\mathcal{E} \rangle\!\rangle_{\mathcal{X}}\,. \end{aligned} \tag{3.6.11}$$

□

**Theorem 3.24.**
*Let $\mathcal{E}_\lambda$ be a sequence of functions that are finite on $D(I)$, and let $\mathcal{E}$ be some function.*

1. *Suppose that for all $\beta \in \mathbb{R}$ it holds that*

$$\lim_{\lambda\to 0} \langle\!\langle \beta(\mathcal{E} - \mathcal{E}_\lambda) \rangle\!\rangle_{\mathcal{X}} = 0. \tag{3.6.12}$$

*Then $\mathcal{E}_\lambda$ converge uniformly to $\mathcal{E}$ on $\{x \,|\, I(x) \le M\}$.*

2. *Suppose there exists a constant $K$, such that for every $x' \in D(I)$ and every $\beta' \geq 0 \in \mathbb{R}$ there exists a open neighbourhood $A'$ of $x'$ such that,*

$$\limsup_{\lambda\to 0} \sup_{x\in A'} \left\{ \beta' |\mathcal{E}(x) - \mathcal{E}_\lambda(x)| - I(x) \right\} \leq K. \tag{3.6.13}$$

*Then $\mathcal{E}_\lambda$ converge uniformly to $\mathcal{E}$ on the sublevel sets of $I$.*

*In particular, $\mathcal{E}$ is finite on $D(I)$ and $\mathcal{E}_\lambda$ converges pointwise to $\mathcal{E}$ on $D(I)$.*

*Proof.*

For the first part, as before, fix $M$ and define $C_M := \{y \,|\, I(y) \leq M\}$, which by the properties of the rate function $I$ are closed sets.

For any $\beta > 0$, then,

$$\begin{aligned}
\sup_{x\in C_M} |\mathcal{E}(x) - \mathcal{E}_\lambda(x)| &= \beta^{-1} \sup_{x\in C_M} \beta|\mathcal{E}(x) - \mathcal{E}_\lambda(x)| \\
&\leq \beta^{-1} \left( \sup_{x\in D(I)} \left\{ \beta|\mathcal{E}(x) - \mathcal{E}_\lambda(x)| - I(x) \right\} + M \right) \\
&= \frac{\langle\!\langle \beta(\mathcal{E} - \mathcal{E}_\lambda) \rangle\!\rangle_{\mathcal{X}} + M}{\beta}
\end{aligned} \tag{3.6.14}$$

Hence, by assumption of (3.6.12),

$$\lim_{\lambda\to 0} \sup_{x\in C_M} |\mathcal{E}(x) - \mathcal{E}_\lambda(x)| \leq \frac{M}{\beta}, \tag{3.6.15}$$

and thus uniform convergence on $C_M$ follows when we let $\beta \to \infty$.

For the second part, fix any $x' \in D(I)$ and any $\beta' \in \mathbb{R}$.

Now, by assumption, we can find an open set $A'$ containing $x'$ such that (3.6.13) holds. Hence, for any small enough $\epsilon$ such that $B_\epsilon(x) \subset A'$,

$$\begin{aligned}
\sup_{x\in C_M\cap B_\epsilon(x')} |\mathcal{E}(x) - \mathcal{E}_\lambda(x)| &= \frac{1}{\beta'} \sup_{x\in C_M\cap B_\epsilon(x')} \beta'|\mathcal{E}(x) - \mathcal{E}_\lambda(x)| \\
&\leq \frac{1}{\beta'} \left( \sup_{x\in B_\epsilon(x')} \left\{ \beta'|\mathcal{E}(x) - \mathcal{E}_\lambda(x)| - I(x) \right\} + M \right) \\
&\leq \frac{1}{\beta'} \left( \sup_{x\in A'} \left\{ \beta'|\mathcal{E}(x) - \mathcal{E}_\lambda(x)| - I(x) \right\} + M \right),
\end{aligned} \tag{3.6.16}$$

and therefore

$$\limsup_{\epsilon\to 0} \limsup_{\lambda\to 0} \sup_{x\in C_M\cap B_\epsilon(x')} |\mathcal{E}(x) - \mathcal{E}_\lambda(x)| \leq \frac{K + M}{\beta'}. \tag{3.6.17}$$

Note that the $K + M$ is independent of $\beta'$, and thus, letting $\beta' \to \infty$, it follows that

$$\lim_{\epsilon\to 0} \lim_{\lambda\to 0} \sup_{x\in C_M\cap B_\epsilon(x')} |\mathcal{E}(x) - \mathcal{E}_\lambda(x)| = 0. \tag{3.6.18}$$

Note that this is simply local uniform convergence of $\mathcal{E}_\lambda$ to $\mathcal{E}$, and because the sublevel sets of $I$ are compact, this implies uniform convergence.

□

**Lemma 3.25.** *Let $\mathcal{E}_\lambda$ be a sequence of bounded and continuous functions.*

1. *Suppose there exists a function $\mathcal{E}$ such that for every $\beta \in \mathbb{R}$*

$$\lim_{\lambda\to 0} \langle\!\langle \beta(\mathcal{E} - \mathcal{E}_\lambda) \rangle\!\rangle_{\mathcal{X}} = 0. \tag{3.6.19}$$

*Then, $J_\beta$ is lower semi-continuous on $\mathcal{X}$ with compact sublevel sets, for all $\beta \in \mathbb{R}$.*

2. *Suppose there exists a constant $K$, such that for every $x' \in D(I)$ and every $\beta' \geq 0 \in \mathbb{R}$ there exists a open neighbourhood $A'$ of $x'$ such that,*

$$\limsup_{\lambda\to 0} \sup_{x\in A'} \beta' |\mathcal{E}(x) - \mathcal{E}_\lambda(x)| - I(x) \leq K. \tag{3.6.20}$$

*Then $J_\beta$ is lower semi-continuous at all $x \in D(I)$ and for all $\beta \in \mathbb{R}$.*

*Suppose that in addition $\langle\!\langle \beta\mathcal{E} \rangle\!\rangle_{\mathcal{X}} < \infty$ for all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$, then $J_\beta$ is lower semi-continuous with compact sublevel sets for all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$.*

*In both cases, $\mathcal{E}$ is continuous on the compact sublevel sets of $I$, and $\mathcal{E}_\lambda$ converge uniformly on these sets. In particular $\mathcal{E}_\lambda$ converges pointwise to finite $|\mathcal{E}|$ on $D(I)$.*

*Proof.* In both cases, note that continuity of $\mathcal{E}$ on the sublevel sets of $I$ directly follows from Theorem 3.24 since the bounded and continuous $\mathcal{E}_\lambda$ converge uniformly to $\mathcal{E}$ on these sets. In particular, $|\mathcal{E}(x)| < \infty$ on $D(I)$.

Now, for the first part, note that for any $\beta \in \mathbb{R}$ and any $\alpha > 0$.

$$\begin{aligned} J_\beta(x) &= \beta\mathcal{E}(x) + I(x) \\ &= \alpha\left(\frac{\beta\mathcal{E}(x)}{\alpha} + I(x)\right) + (1-\alpha)I(x) \\ &\geq \alpha\left(-\left|\frac{\beta\mathcal{E}(x)}{\alpha}\right| + I(x)\right) + (1-\alpha)I(x) \\ &\geq \alpha \inf_{x\in\mathcal{E}}\left\{\left|\frac{\beta\mathcal{E}(x)}{\alpha}\right| - I(x)\right\} + (1-\alpha)I(x) \\ &\geq \alpha \left\langle\!\!\left\langle \frac{\beta\mathcal{E}}{\alpha} \right\rangle\!\!\right\rangle_{\mathcal{X}} + (1-\alpha)I(x). \end{aligned} \tag{3.6.21}$$

Hence it follows that for every $M'$,

$$J_\beta(x) \leq M' \implies I(x) \leq \frac{1}{1-\alpha}\left(M' - \alpha\left\langle\!\!\left\langle \frac{\beta\mathcal{E}}{\alpha} \right\rangle\!\!\right\rangle_{\mathcal{X}}\right). \tag{3.6.22}$$

Since by assumption $\langle\!\langle \frac{\beta}{\alpha}\mathcal{E} \rangle\!\rangle_{\mathcal{X}}$ is finite for all $\beta \in \mathbb{R}$ and $0 < \alpha < 1$, it follows that the sublevel sets of $J_\beta$ are contained in compact the sublevel sets of $I$, hence pre-compact. However, since $\mathcal{E}$ is continuous on the sublevel sets of $I$, it follows that the sublevel sets of $J_\beta$ are closed and hence compact, and in particular $J_\beta$ is lower semi-continuous.

For the second part, the derivation is similar. Namely, note from (3.6.21) that $\langle\!\langle \beta\mathcal{E} \rangle\!\rangle_{\mathcal{X}} < \infty$ for all $\beta \in \mathbb{R}$ with $|\beta| < \beta^*$, by choosing $\alpha$ such that $|\frac{\beta}{\alpha}| < \beta^*$, implies that the sublevel sets of $J_\beta$ are contained in the sublevel sets of $I$, and the result follows. ∎

### Comments

**Remark 3.11.** In the problems that we consider in this thesis there always exist a $\beta^*$ such that $[\![\beta^*\mathcal{E}]\!]_{\mathcal{X}} < \infty$, meaning that the exponential rate of normalization constants $Z_\beta$ are bounded for at least some high enough temperature. Thus the latter implies that $\mathcal{E}(x_n)$ can never grow faster than $I(x_n)$ for all sequences $x_n \in D(I)$ (which also follows directly from the l.s.c of $J_{\beta^*}$). This gives you an immediate restriction on the growth of the singularities that $\mathcal{E}$ can have for $\mathcal{E}$ to be quasi-continuous.

**Remark 3.12.** For any $x' \in D(I)$ and quasi-continuous $\mathcal{E}$, note that $J_\beta$ is lower semi-continuous for all $\beta$ only *at* $x'$. Namely, for each $\beta'$ there does exists a set $A'$ containing $x'$, such that $J_{\beta'}$ is l.s.c. on $A'$, but for a strictly locally quasi-bounded $\mathcal{E}$, there does not always exist a *single* set $A$ such $J_\beta$ is l.s.c. for all $\beta$ simultaneously.

**Remark 3.13.** As mentioned, Theorem 3.24 provides us a way to verify directly that the unnormalized rate functions $J_\beta$ have the same properties as when $\mathcal{E}$ is quasi-continuous — and hence gives us possible candidates for $\mathcal{E}$ that induce large-deviation principles.

For example, in Section 4.5, we will see that the conditions of Theorem 3.24 can be established via an entropic inequality and use of certain finite-dimensional exponential estimates, which are the same ones that we use to derive quasi-continuity. However, the space of functions for which we can show the properties of Theorem 3.24 is strictly *larger* then the space for which we can show quasi-continuity, which begs the question if more advanced techniques could extend our results.

**Remark 3.14.** Finally, note that part (2.) of Theorem 3.24 is a deterministic analog to Lemma 3.21, which was introduced as an equivalent formulation of local quasi-uniform convergence that is well-suited for discretization arguments, see also Remark 3.8.

This similarity is of course intentional, and in Section 5 we will first introduce our discretization argument for logarithmic potentials in a simple deterministic way — by proving that (2.) of Theorem 3.24 applies, and afterwards showing the more involved stochastic argument to prove quasi-continuity.

## 3.7 Assymetric generalization

In [DZ10], Varadhan's Lemma is split into two, namely that lower semi-continuity of $\mathcal{E}$ implies a upper bound, and upper semi-continuity implies the lower bound. A similar approach works for quasi-continuity, however we do not show them here in full. However, we do employ the same rationale to show the following asymmetric generalization of Theorem 3.4, which is essential for proving LDPs for systems with purely repulsive potentials, see Section 4.

First recall,

$$\begin{aligned} \frac{dQ_\beta^N}{dP^N}(x) &:= \frac{e^{-\beta N\mathcal{E}(x)}}{Z_\beta^N} \\ ,\mathcal{F}_\beta(x) &:= J_\beta(x) - J_\beta(\mathcal{X}), \\ J_\beta(x) &:= \begin{cases} \beta\mathcal{E}(x) + I(x) & x \in D(I), \\ +\infty & x \notin D(I), \end{cases} \end{aligned} \tag{3.7.1}$$

**Theorem 3.26.**
*Suppose $\mathcal{E}_\lambda$ are bounded and continuous functions that converge pointwise on $D(I)$ to a $\mathcal{E}$, where $\mathcal{E}$ is possibly infinite but $\mathcal{E} \not\equiv +\infty$ on $D(I)$.*

*Moreover, suppose that for all $\beta \geq 0$ it holds that,*

$$\limsup_{\lambda\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\beta N(\mathcal{E}-\mathcal{E}_\lambda)(z^N)}\right] \leq 0, \tag{3.7.2a}$$

$$\limsup_{\lambda\to 0}\limsup_{N\to\infty}\mathbb{E}^N\Big[(\mathcal{E}-\mathcal{E}_\lambda)(z^N)\Big] \leq 0. \tag{3.7.2b}$$

*Then for all $\beta \geq 0$ it holds that $Q_\beta^N$ satisfy a LDP with rate function $\mathcal{F}_\beta$.*

*Proof.* Note from Theorem 3.17 that we can prove a unnormalized large-deviation principle on $\mathcal{X}$ for all $\beta \geq 0$, if for every $x \in D(I)$ and all $\beta \geq 0$ it holds that

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\beta N\mathcal{E}(z^N)}1_{B_\epsilon(x)}\right] \leq -J_\beta(x), \tag{3.7.3a}$$

$$\liminf_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-\beta N\mathcal{E}(z^N)}1_{B_\epsilon(x)}\right] \geq -J_\beta(x),, \tag{3.7.3b}$$

where

$$J_\beta(x) := \begin{cases} \beta\mathcal{E}(x) + I(x) & x \in D(I), \\ +\infty & x \notin D(I), \end{cases} \tag{3.7.4}$$

and additionally, if for all $\beta \geq 0$,

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-\beta N\mathcal{E}(z^N)}\Big] < \infty. \tag{3.7.5}$$

Now fix any $\beta \geq 0$. Note that for any $\lambda$, by Cauchy-Schwartz,

$$\begin{aligned} \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-\beta N\mathcal{E}(z^N)}\Big] &= \limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}_\lambda)+\beta N\mathcal{E}_\lambda(z^N)}\Big] \\ &\leq \frac{1}{2}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{-2\beta N(\mathcal{E}-\mathcal{E}_\lambda)(z^N)}\right] \\ &+ \frac{1}{2}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\left[e^{2\beta\mathcal{E}_\lambda(z^N)}\right] \end{aligned} \tag{3.7.6}$$

Since the first term is eventually bounded from above by (3.7.2a), and because $\mathcal{E}_\lambda$ is continuous and bounded for all $\lambda$, (3.7.5) immediately follows.

Additionally, fix any $x \in D(I)$. Then, similar to Theorem 3.20, (3.5.11), using Hölder's inequality it holds that for any $\lambda > 0$ and any $\epsilon > 0$ that

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] \\
&\qquad \le \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ \exp\Big\{ -\beta N\Big( (\mathcal{E}-\mathcal{E}_\lambda)(z^N) + (\mathcal{E}_\lambda(z^N) - \mathcal{E}_\lambda(x)) + \mathcal{E}_\lambda(x) \Big)\Big\}\, 1_{B_\epsilon(x)} \Big] \\
&\qquad \le \frac{1}{3} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-3\beta N(\mathcal{E}-\mathcal{E}_\lambda)(z^N)} \Big] + \frac{1}{3} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-3\beta N(\mathcal{E}_\lambda(z^N)-\mathcal{E}_\lambda(x))}\, 1_{B_\epsilon(x)} \Big] \\
&\qquad\quad - \beta \mathcal{E}_\lambda(x) + \frac{1}{3} \limsup_{N\to\infty} \frac{1}{N} \log P^N\big( B_\epsilon(x) \big).
\end{aligned} \tag{3.7.7}$$

Note that the left side is independent of $\lambda$. Hence, as in the proof of Theorem 3.20, (3.5.12), for any fixed $\lambda'$,

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] \\
&\qquad \le \frac{1}{3} \lim_{\lambda\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-3\beta N(\mathcal{E}-\mathcal{E}_\lambda)(z^N)} \Big] + \frac{1}{3} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-3\beta N(\mathcal{E}_{\lambda'}(z^N)-\mathcal{E}_{\lambda'}(x))}\, 1_{B_\epsilon(x)} \Big] \\
&\qquad\quad - \beta \lim_{\lambda\to 0} \mathcal{E}_\lambda(x) + \frac{1}{3} \limsup_{N\to\infty} \frac{1}{N} \log P^N\big( B_\epsilon(x) \big).
\end{aligned} \tag{3.7.8}$$

Thus, similar to the proof of Theorem, (3.5.13), the first term is zero due to (3.7.2a) and the third term is equal to $-\beta\mathcal{E}(x)$ (with $\mathcal{E}$ possibly $+\infty$) due to pointwise convergence of $\mathcal{E}_\lambda$ to $\mathcal{E}$ on $D(I)$. Hence, now taking $\epsilon \to 0$,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] \\
&\qquad \le \frac{1}{3} \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-3\beta N(\mathcal{E}_{\lambda'}(z^N)-\mathcal{E}_{\lambda'}(x))}\, 1_{B_\epsilon(x)} \Big] \\
&\qquad\quad - \beta \mathcal{E}(x) + \frac{1}{3} \lim_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log P^N\big( B_\epsilon(x) \big).
\end{aligned} \tag{3.7.9}$$

However, by continuity and boundedness of $\mathcal{E}_{\lambda'}$ the first term vanishes—since continuity implies a pointwise estimate such as (3.7.3) for $\mathcal{E}_{\lambda'}$ at all $x \in D(I)$, see Theorem 3.13 — and finally the pointwise estimate for $P^N$, see Lemma 2.3, it follows that that

$$\limsup_{\epsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)}\, 1_{B_\epsilon(x)} \Big] \le -\beta\mathcal{E}(x) + I(x) \tag{3.7.10}$$

and hence we have proven (3.7.3a), since the latter equality follows by definition of $J_\beta$.

For the lower bound,(3.7.3b), we use Jensen's inequality for the transformation $a \to e^{-\beta N a}$. Fix $x \in D(I)$. Then for any set $A$.

$$\mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} \,\Big|\, A \Big] \ge \exp\Big\{ -\beta N\, \mathbb{E}^N\Big[ \mathcal{E}(z^N) \,\Big|\, A \Big] \Big\} \tag{3.7.11}$$

Hence,

$$\frac{1}{N} \log \mathbb{E}^N\Big[ e^{-\beta N \mathcal{E}(z^N)} \,\Big|\, A \Big] \ge -\beta\, \mathbb{E}^N\Big[ \mathcal{E}(z^N) \,\Big|\, A \Big]. \tag{3.7.12}$$

However, for any fixed $N$, $\lambda$, and $\epsilon > 0$.

$$\mathbb{E}^N\Big[ \mathcal{E}(z^N) \,\Big|\, B_\epsilon(x) \Big] = \mathbb{E}^N\Big[ (\mathcal{E}-\mathcal{E}_\lambda)(z^N) \,\Big|\, B_\epsilon(x) \Big] + \mathbb{E}^N\Big[ \mathcal{E}_\lambda(z^N) \,\Big|\, B_\epsilon(x) \Big]. \tag{3.7.13}$$

Note that the left side is independent of $\lambda$. Hence, for any fixed $\lambda'$,

$$\mathbb{E}^N\Big[ \mathcal{E}(z^N) \,\Big|\, B_\epsilon(x) \Big] \le \limsup_{\lambda\to 0} \mathbb{E}^N\Big[ (\mathcal{E}-\mathcal{E}_\lambda)(z^N) \,\Big|\, B_\epsilon(x) \Big] + \mathbb{E}^N\Big[ \mathcal{E}_{\lambda'}(z^N) \,\Big|\, B_\epsilon(x) \Big] \tag{3.7.14}$$

Again, taking appropriate limits for the two terms, using (3.7.2b) and continuity of $\mathcal{E}_{\lambda'}$ we derive

$$\begin{aligned}\limsup_{\epsilon\to 0}\mathbb{E}^N\Big[\mathcal{E}(z^N)\,\Big|\,B_\epsilon(x)\Big] &\le \limsup_{\epsilon\to 0}\limsup_{\lambda\to 0}\mathbb{E}^N\Big[(\mathcal{E}-\mathcal{E}_\lambda)(z^N)\,\Big|\,B_\epsilon(x)\Big]+\limsup_{\lambda'\to 0}\limsup_{\epsilon\to 0}\mathbb{E}^N\Big[\mathcal{E}_{\lambda'}(z^N)\,\Big|\,B_\epsilon(x)\Big]\\ &\le 0+\limsup_{\lambda'\to 0}\mathcal{E}_{\lambda'}(x)\\ &= \mathcal{E}(x)\end{aligned} \tag{3.7.15}$$

Thus, we conclude that

$$\liminf_{\epsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[\,e^{-\beta N\mathcal{E}(z^N)}\,\Big|\,B_\epsilon(x)\Big]\ge -\beta\mathcal{E}(x), \tag{3.7.16}$$

which is equivalent to (3.7.3) when taking out the conditional expectation and using the pointwise estimate for $P^N$.

Finally, similar to Theorem 3.4 note that by (3.7.5) it follows that $J_\beta(\mathcal{X}) > -\infty$, and by assumption $\mathcal{E}\not\equiv +\infty$ on $D(I)$ and thus $J_\beta(\mathcal{X})<\infty$. Hence the normalizations constants are finite, and we can turn the unnormalized large-deviation principle into an actual large-deviation principle. See the proof of Theorem 3.4, page 35, for details. □

**Remark 3.15.** Note that in Theorem 3.26 we have actually proven that for such a $\mathcal{E}$ it holds that for every $x\in D(I)$ and $\beta\ge 0$,

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[\,e^{-\beta N(\mathcal{E}(z^N)-\mathcal{E}(x))}\,\Big|B_\epsilon(x)\Big]\le 0, \tag{3.7.17a}$$

$$\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}^N\Big[\mathcal{E}(z^N)-\mathcal{E}(x)\,\Big|B_\epsilon(x)\Big]\le 0. \tag{3.7.17b}$$

This can be interpreted as upper semi-continuity of $\mathcal{E}$ ‘in expectation’, and a quasi-analogue of lower semi-continuity.

## 3.8 Discussion

In this section we introduced the notion of a quasi-continuous function $\mathcal{E}$, for which the probability of a discontinuity is vanishingly small with respect to some sequence of reference measures $P^N$. This enabled us to establish a equivalence between quasi-continuous functions and functions that satisfy Varadhan's Lemma, i.e. induce large-deviation principles, as seen in Theorems 3.1, 3.13, and 3.4.

Moreover, we showed how they form a linear space and are closed under a type of convergence, dubbed (local) quasi-uniform convergence, see Theorem 3.20. In addition, quasi-continuity and convergence apparently imply strong restrictions on the unnormalized rate functions $J_\beta = \beta\mathcal{E} + I$, see Theorem 3.24. The latter provides us with a nice deterministic way to create a class of possible candidates of quasi-continuous functions.

These results will be applied in the remaining sections, and as seen in Section 4 — which encompasses the models for both invariant measures of Section 5 and those for empirical processes of Section 6 — can be established via an exponential estimate over only a *finite* number of particles.

Although in the case for systems induced by a logarithmic potential our method forces us to only consider local quasi-uniform convergence — since $\mathcal{E}$ can only be locally quasi-bounded — Lemma 3.21 provides a technique that is well suited for discretization arguments, as exploited in Section 5.4.

The power of this section is that going forward no additional lifting will be required — in the sense that these finite-dimensional estimates completely determine a large-deviation principle for the relevant empirical measures, and valuable information about the minimizers of their rate functions.

# 4 Gibbs measures of weakly interacting particles

## 4.1 Introduction

In the previous section we developed an expansive and rather abstract framework to derive large-deviation principles by approximating a function $\mathcal{E}$ in an appropriate way. In this section we will show that in the case of a certain class of Gibbs measures over $N$ independent particles, in which $P^N$ can be factorized and $\mathcal{E}$ is the sum over all $k$-particle interactions for some finite $k$ — this approximation can be easily established via a factorization argument.

The key principle is that we can establish quasi-uniform convergence — which consists of pointwise convergence of a sequence of functions $\mathcal{E}_\lambda$ to $\mathcal{E}$ on the domain of the rate function $I$ of the underlying reference measures $P^N$, and limits of exponential integrals involving $\mathcal{E} - \mathcal{E}_\lambda$, see Definition 3.18 — *purely* in terms of a finite $k$-particle estimate, as will be shown in Theorem 4.3.

**Outline** First, we will recall and expand on the relevant notation of Section 2, since we will consider in Section 4.2 an abstract and generalized version of the model mentioned previously, which encompasses both the case of invariant measures of Section 5 and those of empirical processes in Section 6, and the main results that we will use for both are stated in Section 4.3.

Next, in Section 4.4, we will outline a factorization argument that uniformly bounds the $N$-particle exponential integrals induced by a $k$-particle interaction potential $V$ by a similar estimate over only $k$ particles.

Finally, we will show in Section 4.5 how that same exponential estimate directly implies the necessary properties of the rate functions via an entropic inequality, and in particular pointwise convergence of $\mathcal{E}_\lambda$ to $\mathcal{E}$ on $D(I)$. Moreover, we will show how a slight generalization of this argument points to other possible candidates for quasi-continuous functions.

## 4.2 Notation and basic properties

Let $V$ be a — possibly asymmetric — $k$-particle interaction potential on a Polish space $S$, i.e. a measurable function $V : S \times S^{k-1} \to \mathbb{R}$, with $k$ some positive integer. Moreover, let $\mu_0 \in \mathcal{P}(S)$.

While $V$ is allowed to be singular, any unboundedness is assumed to be restricted mostly to the first argument, in the sense that we have the following asymmetric $k$-particle estimate,

$$\sup_{x_2,\dots,x_k \in S} \log \int_S e^{|V(x_1;x_2,\dots,x_k)|}\, d\mu_0(x_1) < \infty. \tag{4.2.1}$$

In particular, we allow $V$ to be such that self-interaction terms with respect to the first argument,

$$V(x;\dots,x,\dots), \tag{4.2.2}$$

are possibly infinite for any $x \in S$ and arbitrary values for the other arguments.

Now, for every $V$ and $N$, we define the energy $E_V^N : S^N \to \mathbb{R}$ of a $N$-particle vector $(x_1,\dots,x_N)$ by

$$E_V^N(x_1,\dots,x_N) \coloneqq \frac{1}{N^k} \sum_{i_2,\dots,i_k \neq i_1, i_1} V(x_{i_1};x_{i_2},\dots,x_{i_k}), \tag{4.2.3}$$

where the sum is shorthand for

$$\sum_{i_1=1,i_2=1,\dots,i_k=1\,|\,i_j \neq i_1 \forall j=2,\dots,k}^{N} V(x_{i_1};x_{i_2},\dots,x_{i_k}), \tag{4.2.4}$$

i.e. we do not include terms such as (4.2.2).

Note that in the case of $k = 2$, both the pair potentials $V(x,y)$ and $\frac{1}{2}V(x,y) + V(y,x)$ lead to the same symmetric energy $E_N(V)$ for any $N$. The reason for why we still require an asymmetric condition (4.2.1) is a technical one arising from Lemma 4.7, and can be relaxed to a symmetric condition, as shown in Remark 4.3.

Now, we consider for each $V$ and each $N$ two systems of particles, represented as $S$-valued variables $X_i^N$, $1 \le i \le N$, with either joint law $\tilde{P}^N \in \mathcal{P}(S^N)$, or a joint law $\tilde{Q}_V^N \in \mathcal{P}(S^N)$, where the latter will be defined in terms of $\tilde{P}^N$ and $V$, as seen below.

For the rest of this section, we assume that under the law $\tilde{P}^N$ the $X_i^N$ are identically distributed and independent, i.e. $\tilde{P}^N = (\mu_0)^{\otimes N}$, with $\mu_0 \in \mathcal{P}(S)$ their common law. We will refer to this as the non-interacting system.

Subsequently, for every $V$, we define $\tilde{Q}_V^N \in \mathcal{P}(S^N)$, with

$$\frac{d\tilde{Q}_V^N}{d\tilde{P}^N}(x_1,\dots,x_N) \coloneqq \frac{1}{Z_V^N} e^{-NE_V^N(x_1,\dots,x_N)}, \qquad x_i \in S,\ 1 \le i \le N, \tag{4.2.5}$$

with $Z_V^N$ normalization constants. We will refer to the variables $X_i^N$ under $Q_V^N$ as the interacting system.

We will study large deviations for the empirical measures, and hence recall the notation and framework of Section 2.1, page 10.

We equip $\mathcal{P}(S)$ with the weak topology, which turns $\mathcal{P}(S)$ into a Polish space. As before, let $\pi^N : S^N \to \mathcal{P}(S)$ be the continuous operator that projects the particles into their empirical measures, i.e. $\pi^N(x_1,\dots,x_N) \coloneqq z^N$, with

$$z^N = \sum_{i=1}^{N} \delta_{x_i}. \tag{4.2.6}$$

Denote by $P^N \in \mathcal{P}(\mathcal{P}(S))$ and $Q_V^N \in \mathcal{P}(\mathcal{P}(S))$ the laws of the empirical measures induced by $\tilde{P}^N$ and $\tilde{Q}_V^N$, i.e.

$$\begin{aligned} P^N &\coloneqq \tilde{P}^N \circ \pi^{-1}, \\ Q_V^N &\coloneqq \tilde{Q}^N \circ \pi^{-1}. \end{aligned} \tag{4.2.7}$$

Because of the continuity of $\pi^N$, absolutely continuity of $\tilde{Q}_V^N$ with respect to $\tilde{P}^N$ implies that $Q_V^N$ is absolutely continuous with respect to $P^N$, with

$$\frac{dQ_V^N}{dP^N}(\mu) := \frac{1}{Z_V^N} e^{-N\mathcal{E}_V(\mu)}, \tag{4.2.8}$$

where $\mu \in \mathcal{P}(S)$, and $\mathcal{E}_V : \mathcal{P}(S) \to \bar{\mathbb{R}}$ is the function defined as

$$\mathcal{E}_V(\mu) = \int_{(S^k)'} V(x_1; x_2, \dots, x_k) \prod_{i=1}^k d\mu(x_i). \tag{4.2.9}$$

Here $(S^k)'$ is $S^k$ but with the $x_1$-diagonals removed, i.e.

$$(S^k)' := \big\{(x_1, \dots, x_k) \in S^k \,|\, x_j \neq x_1, \ \forall j = 2, \dots, k\big\}. \tag{4.2.10}$$

Now, set $\mathcal{X} := \mathcal{P}(S)$. Note that $\mu_0, z^N \in \mathcal{X}$, $\mathcal{E}_V : \mathcal{X} \to \bar{\mathbb{R}}$, and $P^N, Q_V^N \in \mathcal{P}(\mathcal{X})$. Moreover, under the laws $P^N$ or $Q_V^N$, $z^N$ is a random element in $\mathcal{X}$.

Additionally, recall from Section 2.1 that $P^N$ satisfies a large-deviation principle with rate function $I : \mathcal{X} \to \bar{\mathbb{R}}$,

$$I(\mu) := R(\mu \| \mu_0), \tag{4.2.11}$$

i.e. the relative entropy of $\mu \in \mathcal{X}$ with respect to $\mu_0$, with

$$R(\nu \| \mu) := \begin{cases} \displaystyle\int_S \log \frac{d\nu}{d\mu}\, d\nu & \text{if } \nu \text{ absolutely continuous w.r.t. } \mu, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.2.12}$$

Note that for bounded and continuous $V$, $\mathcal{E}_V$ is bounded but may not be continuous. This is because of the missing $x_1$-diagonals in (4.2.9). However, similar as done in Section 2.1, under a certain condition they can be approximated uniformly by bounded and continuous functions, as will be shown below. Hence, we have the following result.

**Lemma 4.1.**

*Suppose $V : S \times S^{k-1} \to \mathbb{R}$ is continuous and bounded. Moreover, suppose that for any $\mu$ with $R(\mu \| \mu_0) < \infty$ we have*

$$\mathcal{E}_V(\mu) = \int_{S^k} V(x_1; x_2, \dots, x_k) \prod_{i=1}^k d\mu(x_i). \tag{4.2.13}$$

*Then the function $\mathcal{E}_V : \mathcal{X} \to \mathbb{R}$ is bounded and is quasi-continuous with respect to $P^N$.*

*In particular, $Q_{\beta V}^N$ satisfies a large deviation principle with rate function $\mathcal{F}_{\beta V}$ for any $\beta \in \mathbb{R}$, where*

$$\mathcal{F}_V(\mu) := J_V(\mu) - \inf_{\nu \in \mathcal{X}} J_V(\nu), \tag{4.2.14}$$

*with*

$$J_V(\mu) := \begin{cases} \mathcal{E}_V(\mu) + I(\mu), & \mu \in D(I), \\ +\infty & \textit{otherwise}, \end{cases} \tag{4.2.15}$$

*and $D(I) := \{\mu \,|\, R(\mu \| \mu_0) < \infty\}$.*

*Proof.* Define $\bar{\mathcal{E}} : \mathcal{X} \to \bar{\mathbb{R}}$ by

$$\bar{\mathcal{E}}_V(\mu) = \int_{S^k} V(x_1; x_2, \dots, x_k) \prod_{i=1}^k d\mu(x_i). \tag{4.2.16}$$

Note that for any continuous and bounded $V$ and any fixed empirical measure $z^N$, by counting the particles that are left in out in (4.2.9),

$$\begin{aligned}\left|\mathcal{E}_V(z^N) - \bar{\mathcal{E}}_V(z^N)\right| &= \frac{1}{N^k}\left|\sum_{i_2,\ldots,i_k \neq i_1, i_1}^{n} V(x_{i_1}; x_{i_2}, \ldots, x_{i_k}) - \sum_{i_1, i_2, \ldots, i_k}^{n} V(x_{i_1}; x_{i_2}, \ldots, x_{i_k})\right| \\ &\leq \frac{1}{N^k}\left(N^k - N(N-1)^{k-1}\right)\|V\|_\infty \\ &= \frac{1}{N^{k-1}}\left(N^{k-1} - (N-1)^{k-1}\right)\|V\|_\infty\end{aligned} \tag{4.2.17}$$

But note that for any $N$ and any $l \geq 0$ we have that

$$\begin{aligned}N^l - (N-1)^l &= N^l\left(1 - \left(1 - \frac{1}{N}\right)^l\right) \\ &\leq N^l\left(1 - \left(1 - \frac{l}{N}\right)\right) \\ &\leq lN^{l-1},\end{aligned} \tag{4.2.18}$$

and hence

$$\begin{aligned}\left|\mathcal{E}_\mathcal{V}(z^N) - \bar{\mathcal{E}}_V(z^N)\right| &\leq \frac{1}{N^{k-1}}(k-1)N^{k-2}\|V\|_\infty \\ &\leq \frac{k\|V\|_\infty}{N}.\end{aligned} \tag{4.2.19}$$

The latter goes to zero as $N \to \infty$. By Theorem 2.5 it follows that the measures induced by $\mathcal{E}_V$ satisfy a LDP with the same rate function as $\bar{\mathcal{E}}_V(z^N)$. Since by assumption $\mathcal{E}_V = \bar{\mathcal{E}}_V(z^N)$ on $D(I)$, Varadhan's Lemma applies for $\mathcal{E}_V$ and by Theorem 3.1 is therefore quasi-continuous. □

Thus, we have created a large class of quasi-continuous and bounded (hence quasi-bounded) functions $\mathcal{E}_V$, that all satisfy Varadhan's Lemma. Therefore, we are now in the position to apply the framework and main results of Section 3, and can investigate large deviations induced by arbitrary $V$ purely in terms of approximation of $\mathcal{E}$ by $\mathcal{E}_{V_\lambda}$, with $V_\lambda$ bounded and continuous. Hence, for the remainder of this section, we will only be concerned in verifying these estimates.

Finally, note the following statement, which follows directly from the definition of $\mathcal{E}_V$.

**Lemma 4.2.**

$$\mathcal{E}_{V_1} + \mathcal{E}_{V_2} = \mathcal{E}_{V_1+V_2}, \tag{4.2.20a}$$

$$|\mathcal{E}_V| \leq \mathcal{E}_{|V|}. \tag{4.2.20b}$$

**Remark 4.1.** Recall Remark 3.1, and note that one can loosen the restriction (4.2.13) by considering the couple $\{\mathcal{E}_V, \bar{\mathcal{E}}_V\}$.

## 4.3 Main results

Let $[V]_S$ be the following $k$-particle estimate induced by $V$,

$$[V]_S := \sup_{x_2,\dots,x_k \in S} \int_S e^{|V(x_1;x_2,\dots,x_k)|}\, d\mu_0(x_1). \tag{4.3.1}$$

We then have the following result.

**Theorem 4.3.** *Suppose there exist bounded continuous functions $V_\lambda : S \times S^{k-1} \to \mathbb{R}$ such that for all $\beta \in \mathbb{R}$,*

$$\lim_{\lambda \to 0} [\beta(V - V_\lambda)] = 0. \tag{4.3.2}$$

*Then $\mathcal{E}_V$ is quasi-bounded and quasi-continuous with respect to the sequence of measures $P^N$. In particular, for all $\beta \in \mathbb{R}$, the induced Gibbs measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

The main underlying technique is a comparison result, see below, which is directly implied by Lemmass 4.7 and 4.9 of Sections 4.4 and 4.5 respectively.

**Theorem 4.4.** *Let $V : S \times S^{k-1} \to \mathbb{R}$. Then,*

$$[\![\mathcal{E}_V]\!]_{\mathcal{X}} \le [V]_S, \tag{4.3.3}$$

*and*

$$\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}} \le [V]_S. \tag{4.3.4}$$

Note that while $[V]_S$ involves an estimate over only $k$-particles, $[\![\mathcal{E}_V]\!]_{\mathcal{X}}$ is defined as the limit of $N$-particle estimates, see Definition 3.6, with

$$[\![\mathcal{E}]\!]_A := \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{N|\mathcal{E}(z^N)|} \,\middle|\, A \right], \tag{4.3.5}$$

and $\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}}$, see (3.6.9), is defined by a variational statement over the domain of $I$,

$$\langle\!\langle \mathcal{E} \rangle\!\rangle_{\mathcal{X}} := \sup_{x \in D(I)} |\mathcal{E}(x)| - I(x)\,. \tag{4.3.6}$$

We will see below that Theorem 4.3 follows quite easily from Theorem 4.4 and the framework developed in Section 3.

*Proof of Theorem 4.3.*
Let $V, V_\lambda : S \times S^{k-1} \to \mathbb{R}$, with $V_\lambda$ bounded and continuous.

Recall from Lemma 4.1 that $\mathcal{E}_{V_\lambda}$ is bounded (hence quasi-bounded) and quasi-continuous. Now, because of linearity, see (4.2.20a), it follows by assumption of (4.3.2) and Theorem 4.4, (4.3.3), that for every $\beta \in \mathbb{R}$,

$$\begin{aligned} \lim_{\lambda\to 0} [\![\beta(\mathcal{E}_V - \mathcal{E}_{V_\lambda})]\!]_{\mathcal{X}} &= \lim_{\lambda\to 0} [\![\beta\mathcal{E}_{V-V_\lambda}]\!]_{\mathcal{X}} \\ &\le \lim_{\lambda\to 0} [\beta(V - V_\lambda)]_S \\ &= 0. \end{aligned} \tag{4.3.7}$$

Similarly, it follows by (4.3.4) that,

$$\begin{aligned} \lim_{\lambda\to 0} \langle\!\langle \beta(\mathcal{E}_V - \mathcal{E}_{V_\lambda}) \rangle\!\rangle_{\mathcal{X}} &\le \lim_{\lambda\to 0} [\beta(V - V_\lambda)]_S \\ &= 0. \end{aligned} \tag{4.3.8}$$

By Theorem 3.24, (4.3.8) implies pointwise convergence of $\mathcal{E}_{V_\lambda}$ on $D(I)$, and combined with (4.3.7) it follows from Theorem 3.2 that $\mathcal{E}_V$ is quasi-continuous and quasi-bounded.

In particular, by Theorem 3.1, $\mathcal{E}$ induces a large-deviation principle for every $\beta \in \mathbb{R}$ with rate function $\mathcal{F}_{\beta V}$, and we conclude the proof. □

Finally, using Theorem 3.26, we can generalize Theorem 4.3 to an asymmetric condition, which is of particular use for purely repulsive potentials.

**Theorem 4.5.** *Suppose $V : S \times S^{k-1} \to \mathbb{R}$ is such that there exist continuous and bounded functions $V_\lambda$ such that $\mathcal{E}_{V_\lambda}$ converges pointwise to $\mathcal{E}_V$ on $D(I)$, and such that for all $\beta \geq 0$*

$$\limsup_{\lambda\to 0} \sup_{x_2,\ldots,x_k \in S} \log \int_S e^{-\beta(V-V_\lambda)(x_1;x_2,\ldots,x_k)} d\mu_0(x_1) \leq 0, \tag{4.3.9}$$

*and additionally,*

$$\limsup_{\lambda\to 0} \int_{S^k} (V - V_\lambda)(x_1; x_2, \ldots, x_k) \prod_i^k d\mu_0(x_i) \leq 0. \tag{4.3.10}$$

*Then for all $\beta \geq 0$ the induced Gibbs measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

*Proof.* Using Lemma 4.8, it follows that for all $\beta \geq 0$

$$\begin{aligned}
\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta N(\mathcal{E}_V - \mathcal{E}_{V_\lambda})(z^N)} \right] &= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta N(\mathcal{E}_{V-V_\lambda})(z^N)} \right] \\
&\leq \log \sup_{x_2,\ldots x_k \in S} \int_S e^{-\beta(V-V_\lambda)(x_1;x_2,\ldots x_k)} \, d\mu_0(x_1) \\
&= \sup_{x_2,\ldots x_k \in S} \log \int_S e^{-\beta(V-V_\lambda)(x_1;x_2,\ldots x_k)} \, d\mu_0(x_1),
\end{aligned} \tag{4.3.11}$$

where again we used the monotonicity and continuity of the logarithm to switch the supremum.

Hence, by assumption of (4.3.9).

$$\limsup_{\lambda\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{-\beta N(\mathcal{E}_V - \mathcal{E}_{V_\lambda})(z^N)} \right] \leq 0, \tag{4.3.12}$$

Moreover, by assumption of (4.3.10),

$$\begin{aligned}
\limsup_{\lambda\to 0} \limsup_{N\to\infty} \mathbb{E}^N \Big[ (\mathcal{E} - \mathcal{E}_\lambda)(z^N) \Big] &= \limsup_{\lambda\to 0} \limsup_{N\to\infty} \tilde{\mathbb{E}}^N \Big[ \frac{1}{N^k} \sum_{i_2,\ldots,i_k \neq i_1, i_1}^{n} (V - V_\lambda)(X_{i_1}; X_{i_2}, \ldots, X_{i_k}) \Big] \\
&= \limsup_{\lambda\to 0} \int_{S^k} (V - V_\lambda)(x_1\,; x_2, \ldots, x_k) \prod_i^k d\mu_0(x_i). \qquad \leq 0.
\end{aligned} \tag{4.3.13}$$

Combined with pointwise convergence on $D(I)$, $\mathcal{E}_V$ and $\mathcal{E}_{V_\lambda}$ satisfy all the assumptions of Theorem 3.26, and it follows that $\mathcal{E}_V$ induces a large-deviation principle for all $\beta \geq 0$. □

**Comments**

**Remark 4.2.** Note that in Theorem 4.5 we have the seperate requirement that $\mathcal{E}_{V_\lambda}$ converges pointwise on $D(I)$, while in Theorem 4.3 this is *not* necessary — since it already is *implied* by Theorems 3.24 and Theorem 4.4.

Of course, in many of the cases of Section 5 and 6, we will establish $[V - V_\lambda]_L \to 0$ by making assumptions on $V_\lambda$ that already directly imply pointwise convergence of $\mathcal{E}_{V_\lambda}$ on $D(I)$ — such as pointwise convergence of $V$ on $S^k$ and removal of the supremum by using translation invariance.

Moreover, note that we only considered quasi-bounded $\mathcal{E}_V$, and hence Theorem 4.3 does not treat the case of strictly *local* quasi-uniform convergence. In section 5.4 we will give an explicit example of where we do not have quasi-boundedness, where a pairwise potential $V : \mathbb{R}^2$ has a logarithmic singularity. However, to make it work, we will require additional assumptions on $V$ and $\mu_0$, namely that the logarithmic singularity of $V$ is only at the diagonal and $\mu_0$ is absolutely continuous with respect to the Lebesgue measure.

**Remark 4.3.** Suppose $V(x,y) = W(x) + W(y)$, where $W : S \to \mathbb{R}$ is a unbounded function that satisfies the conditions of Theorem 4.3. Considering $W(x)$ and $W(y)$ as functions on $S^2$, and using the linearity of quasi-continuous functions, it follows that $\mathcal{E}_V$ is quasi-bounded and continuous.

However, it is easy to see that $[V]_S = \infty$, so Theorem 4.3 could not have been used directly. The problem is of course the supremum over $x_2, \dots x_k$ in the definition of $[V]_S$, and we circumvented it by splitting $V$ up into different functions.

By a similar argument and permutating the variables, it is easy to see the following result,

**Corollary 4.6.** *Consider $k$ functions $W^i : S^k \to \mathbb{R}$, $1 \le i \le k$, and set*

$$V := \sum_{i=1}^{k} W^i. \tag{4.3.14}$$

*Moreover, suppose there exists bounded continuous functions $W^i_\lambda$, $1 \le i \le k$, such that for every $i$ and every $\beta \in \mathbb{R}$,*

$$\lim_{\lambda \to 0} [\beta(W^i - W^i_\lambda)]^i_S = 0, \tag{4.3.15}$$

*where*

$$[W]^i_S := \sup_{x_j \in S \,|\, j \neq i} \int_S e^{|V(x_i\,;\,\dots x_j\,\dots)|}\, d\mu_0(x_i). \tag{4.3.16}$$

*Then $\mathcal{E}_V$ is quasi-bounded and quasi-continuous with respect to the sequence of measures $P^N$. In particular, the induced Gibbs measures satisfy a LDP with rate function $\mathcal{F}_V$.*

**Remark 4.4.** Finally, note that the space of functions $V : S^k \to \mathbb{R}$ such that there exists a sequence of bounded continuous functions $V_\lambda$ such that $V, V_\lambda$ satisfies the conditions of Theorem 4.3 is a *linear* space.

Moreover, because of (4.2.20a) and the fact that the space of quasi-continuous and quasi-bounded functions is linear itself, it follows that the space of $V$ such that $\mathcal{E}_V$ induce a large-deviation principle is also linear, and is contained within the one mentioned above.

The question is of course if the containment is strict, which is to be expected since the underlying reduction technique that establishes (4.3.3), see Lemma 4.8, is fairly simple and straightforward, and adapted from a technique that is decades old, see [CLMP92]. We will provide at least one possible class of candidates, see Remark 4.7, but it is possible that more advanced and modern technique could give more general results.

## 4.4 Bounds on $[\![\mathcal{E}_V]\!]_{\mathcal{X}}$

As mentioned, in this section we will prove the following inequality.

**Lemma 4.7.** *For any $V$ it holds that*

$$[\![\mathcal{E}_V]\!]_{\mathcal{X}} \le [V]_S \tag{4.4.1}$$

However, we will also prove a slight generalization, which we will use to prove Theorem 4.5.

**Lemma 4.8.**

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{-N\mathcal{E}_V(z^N)} \Big] \le \log \sup_{x_2,\dots x_k \in S} \int_S e^{-V(x_1;x_2,\dots x_k)}\, d\mu_0(x_1). \tag{4.4.2}$$

Note that Lemma 4.7 follows easily from Lemma 4.8.

*Proof of Lemma 4.7.* Recall the definition of $[\![\mathcal{E}]\!]_{\mathcal{X}}$ for a function $\mathcal{E}$, see (3.4.2) or below.

Now, by (4.2.20b) and Lemma 4.8,

$$\begin{aligned}
[\![\mathcal{E}_V]\!]_{\mathcal{X}} &= \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{N|\mathcal{E}_V|(z^N)} \Big] \\
&\le \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{N\mathcal{E}_{|V|}(z^N)} \Big] \\
&\le \sup_{x_2,\dots x_k \in \Lambda} \log \int_S e^{|V(x_1;x_2,\dots x_k)|}\, d\mu_0(x_1) \\
&\le [V]_S.
\end{aligned} \tag{4.4.3}$$

□

The proof of Lemma 4.8 is straightforward, and is adapted from a more simple version for a logarithmic pair-potential, see [CLMP92, p. 506]. It uses the generalized Hölder inequality to asymptotically bound the partition function of the interacting system by a system that only contains an external potential, where the latter can be factorized in the usual way because of the independence of the particles under $\tilde{P}^N$, see below.

First, for a $k$-particle potential $V$, note that

$$\mathbb{E}^N\Big[ e^{-N\mathcal{E}_V(z^N)} \Big] = \tilde{\mathbb{E}}^N\left[ \exp\left\{ \frac{1}{N^{k-1}} \sum_{i_2,\dots,i_k \neq i_1, i_1}^{N} V(x_{i_1}; x_{i_2}, \dots, x_{i_k}) \right\} \right]. \tag{4.4.4}$$

Here $\mathbb{E}^N$ is the expectation under the law $P^N$ of $z^N$, with $P^N \in \mathcal{P}(\mathcal{P}(S))$, and $\tilde{\mathbb{E}}^N$ is the expectation under the law $\tilde{P}^N$ of the particle vector $X^N = \{X_1^N, \dots, x_N^N\} \in S^N$, with $\tilde{P}^N \in \mathcal{P}(S^N)$. Moreover, recall that $\tilde{P}^N = (\mu_0)^{\otimes N}$, i.e. the particles are independent and identical under $\tilde{P}^N = (\mu_0)^{\otimes N}$, and in particular, they are exchangeable.

*Proof of Lemma 4.8.* We will first show that (4.4.2) holds for a pair potential $V : S^2 \to \mathbb{R}$, and then extend this to a $k$-particle potential.

Note that trivially,

$$\frac{1}{N} \sum_{i,j\neq i}^{N} V(x_i; x_j) = \frac{1}{N} \sum_{j}^{N} \left( \sum_{i \neq j} V(x_i; x_j) \right). \tag{4.4.5}$$

Hence, using the generalized Hölder's inequality with common exponent $N$, and the exchangeability named above,

$$\begin{aligned}\tilde{\mathbb{E}}^N\Big[e^{\frac{1}{N}\sum_{i,j\neq i}^N V(x_i;x_j)}\Big] &= \tilde{\mathbb{E}}^N\Big[e^{\frac{1}{N}\sum_j^N\left(\sum_{i\neq j}V(x_i;x_j)\right)}\Big]\\ &\leq \prod_j^N\Big(\tilde{\mathbb{E}}^N\Big[e^{\sum_{i\neq j}^N V(x_i;x_j)}\Big]\Big)^{1/N}\\ &= \tilde{\mathbb{E}}^N\Big[e^{\sum_{i\neq 1}^N V(x_i;x_1)}\Big].\end{aligned} \tag{4.4.6}$$

The latter looks almost like an external potential, were it not for $x_1$ — see also Remark 4.5. Thus, we can factorize as following,

$$\begin{aligned}\tilde{\mathbb{E}}^N\Big[e^{\sum_{i\neq 1}V(x_i;x_1)}\Big] &= \int_{S^N} e^{\sum_{i\neq 1}V(x_i;x_1)}\,d\mu_0(x_1)\prod_{i\neq 1}d\mu_0(x_i)\\ &= \int_{S^N}\Big(\prod_{i\neq 1}e^{V(x_i;x_1)}\Big)\,d\mu_0(x_1)\prod_{i\neq 1}d\mu_0(x_i)\\ &= \int_S\prod_{i\neq 1}\Big(\int_S e^{V(x_i;x_1)}\,d\mu_0(x_i)\Big)\,d\mu_0(x_1)\\ &= \int_S\Big(\int_S e^{V(x_2;x_1)}\,d\mu_0(x_2)\Big)^{N-1}\,d\mu_0(x_1)\\ &\leq \sup_{x_1\in S}\Big(\int_S e^{V(x_2;x_1)}\,d\mu_0(x_2)\Big)^{N-1}\\ &= \Big(\sup_{x_1\in S}\int_S e^{V(x_2;x_1)}\,d\mu_0(x_2)\Big)^{N-1}.\end{aligned} \tag{4.4.7}$$

Similarly, for a $k$-particle exponent, using the generalized Hölders inequality with common exponent $N^{k-1}$ and exchangeability,

$$\begin{aligned}\tilde{\mathbb{E}}^N\Big[\exp\Big\{\frac{1}{N^{k-1}}\sum_{i_2,\dots,i_k\neq i_1,i_1}^N V(x_{i_1};x_{i_2},\dots,x_{i_k})\Big\}\Big] &= \tilde{\mathbb{E}}^N\Big[\exp\Big\{\frac{1}{N^{k-1}}\sum_{i_2,\dots,i_k\neq i_1}^N\Big(\sum_{i_1}^N V(x_{i_1};x_{i_2},\dots,x_{i_k})\Big)\Big\}\Big]\\ &\leq \tilde{\mathbb{E}}^N\Big[\exp\Big\{\sum_{i\neq 2,\dots k}^N V(x_i;x_2,\dots x_k)\Big\}\Big]\\ &= \int_S\Big(\int_S e^{V(x_1;x_2,\dots x_k)}\,d\mu_0(x_1)\Big)^{N-k}\,d\mu_0(x_2)\cdots d\mu_0(x_k)\\ &\leq \Big(\sup_{x_2,\dots x_k\in S}\int_S e^{V(x_1;x_2,\dots x_k)}\,d\mu_0(x_1)\Big)^{N-k}.\end{aligned} \tag{4.4.8}$$

Hence, taking limits, we conclude

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{-N\mathcal{E}_V(z^N)}\Big] &= \limsup_{N\to\infty}\frac{1}{N}\log\tilde{\mathbb{E}}^N\Big[\exp\Big\{\frac{1}{N^{k-1}}\sum_{i_2,\dots,i_k\neq i_1,i_1}^N V(x_{i_1};x_{i_2},\dots,x_{i_k})\Big\}\Big]\\ &\leq \limsup_{N\to\infty}\frac{N-k}{N}\log\sup_{x_2,\dots x_k\in S}\int_S e^{V(x_1;x_2,\dots x_k)}\,d\mu_0(x_1)\\ &= \log\sup_{x_2,\dots x_k\in S}\int_S e^{V(x_1;x_2,\dots x_k)}\,d\mu_0(x_1).\end{aligned} \tag{4.4.9}$$

□

**Comments**

**Remark 4.5.** Note that in the proof of Lemma 4.8, in using the generalized Hölder inequality, we actually only use the following statement,

$$\mathbb{E}\Big[\prod_{i=1}^{N} Y_i\Big] \leq \mathbb{E}\big[Y^N\big], \tag{4.4.10}$$

where $Y_i$ are random variables with common marginal $Y$, but otherwise not specified. In the proof of 4.8, for a two-particle potential $V$, we have

$$Y_i = \frac{1}{N}\sum_{j\neq i}^{N} V(X_i, X_j) \tag{4.4.11}$$

Note that equality in (4.4.10) holds when $Y_i = Y$ for all $i$, i.e. if all variables are perfectly correlated. In other words, we approximate the exponential expectation of the interacting system in Lemma 4.8 by one where all the particles act under a fixed external potential $V^*(\cdot)$ with

$$V^*(\cdot) = \lim_{N\to\infty}\frac{1}{N}\sum_{j}^{N} V(\cdot\,, X_j), \tag{4.4.12}$$

and we subsequently factorize the system as we would do for usual external potentials.

This is the underlying idea of treating the system as a so-called *mean field* system. See also for example [Hol16]) and [WJ17], where the idea of mean-field limits are made precise, and propagation of chaos is defined in terms of factorizability of the laws of dynamic particle systems at different times.

**Remark 4.6.** Note that Lemma 4.8 and Lemma 4.7 imply the following estimate that is valid *for all* $N$,

$$\frac{1}{N}\log\mathbb{E}^N\Big[e^{N|\mathcal{E}_V|(z^N)}\Big] \leq [V]_S. \tag{4.4.13}$$

## 4.5 Bounds on $\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}}$ and the properties of the rate functions

Recall from Section 3.6 that quasi-continuity and quasi-uniform convergence of $\mathcal{E}$ implies rather strict conditions on the (convergence of) rate functions, and hence on $\mathcal{E}$ itself. In this section we will discuss the implications of this for $\mathcal{E}_V$.

First, we will establish an inequality between $\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}}$ and $[V]_S$ — without of course assuming quasi-continuity — which completes the proof of Theorem 4.3. It relies on representing finite-particle exponential integrals via a variational entropic inequality.

Moreover, as mentioned in Remark 3.13, one can use bounds on $\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}}$ to prove that the unnormalized rate functions satisfies the necessary conditions for quasi-continuity. We will briefly discuss a larger class of possible candidates for quasi-continuous $\mathcal{E}_V$ than that as defined by $[V]_S$ as used in Section 4.4, see Remark 4.7.

First, similar to $[V]_S$, let $\langle V \rangle_S$, for any $V : S^k \to \mathbb{R}$ be defined as follows:

$$\langle V \rangle_S := \int_{S^k} e^{|V(x_1;x_2,\dots,x_k)|} \prod_{i=1}^k d\mu_0(x_i). \tag{4.5.1}$$

Note that $\langle V \rangle_S \le [V]_S$. We now have the following inequality.

**Lemma 4.9.** *Let $V : S^k \to \mathbb{R}$ be a measurable k-particle potential. Then,*

$$\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}} \le [V]_S, \tag{4.5.2}$$

*and*

$$\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}} \le \frac{1}{k} \langle kV \rangle_S. \tag{4.5.3}$$

Note that from Lemma 4.9 and Theorem 3.24 we immediately have the following result.

**Theorem 4.10.** *Let $V : S^k \to \mathbb{R}$. Suppose there exists bounded continuous functions $V_\lambda$ such that for all $\beta \in \mathbb{R}$ it holds that either*

$$[\beta(V - V_\lambda)] \to 0, \tag{4.5.4}$$

*or,*

$$\langle \beta(V - V_\lambda) \rangle_S \to 0. \tag{4.5.5}$$

*Then $\mathcal{E}_V(\mu)$ is continuous on the sublevel sets $\{\mu \,|\, R(\mu\|\mu_0) < M\}$, and in particular for $|\mathcal{E}_V(\mu)| < \infty$ for all $\mu \in \mathcal{P}(S)$ with $R(\mu\|\mu_0) < \infty$.*

*Moreover, the functions $J_\beta := \beta\mathcal{E}_V(\mu) + R(\mu\|\mu_0)$ are lower semi-continuous on $\mathcal{P}(S)$ for all $\beta \in \mathbb{R}$, and have compact sublevel sets.*

*Finally, $\mathcal{E}_\lambda$ converges uniformly on the sets $\{\mu \,|\, R(\mu\|\mu_0) < M\}$, and in particular, pointwise for every $\mu$ with $R(\mu\|\mu_0) < \infty$.*

We will first state a simpler inequality than that of Lemma 4.9, namely for an external potential, which as mentioned follows from an entropic inequality.

**Lemma 4.11.** *Let $\mu, \mu_0 \in \mathcal{P}(\mathcal{S})$, with $\mathcal{S}$ a Polish space. Moreover, let $V : \mathcal{S} \to \mathbb{R}$ be a bounded and measurable function. Then,*

$$-\log \int_{\mathcal{S}} e^{-V} d\mu_0(x) = \inf_{\mu \in \mathcal{P}(\mathcal{S})} \Big\{ R(\mu\|\mu_0) + \int_{\mathcal{S}} V(x)\, d\mu(x) \Big\}. \tag{4.5.6}$$

*Proof.* See Lemma 6.2.13 of [DZ10] for all the technical details, where $R(\cdot\|\mu_0)$ is shown to be the Legendre transform of

$$\Lambda(V) := \log \int_{\mathcal{S}} e^{-V(x)} d\mu_0(x). \tag{4.5.7}$$

However, for clarity, we will also provide a direct proof. Namely, assume $\mu$ is absolutely continuous with respect to $\mu_0$ — recall that $R(\mu\|\mu_0) = +\infty$ otherwise by definition — and define with a little abuse of notation $e^{-V}\mu_0 \in \mathcal{P}(\mathcal{S})$ by

$$\frac{d\, e^{-V}\mu_0}{d\mu_0}(x) = \frac{1}{\int_{\mathcal{S}} e^{-V} d\mu_0(x)} e^{-V(x)}. \tag{4.5.8}$$

Since $V$ is bounded, we have $e^{-V}\mu_0 \sim \mu_0$, and thus $\mu \ll \mu_0 \ll e^{-V}\mu_0$ and hence

$$\frac{d\mu}{e^{-V}\mu_0} = \frac{d\mu}{d\mu_0}\frac{d\mu_0}{d\, e^{-V}\mu_0}. \tag{4.5.9}$$

Hence,

$$\begin{aligned} R(\mu\|\mu_0) + \int V(x)\, d\mu(x) &= \int_{\mathcal{S}} \log \frac{d\mu(x)}{d\mu_0(x)}\, d\mu(x) + \int_{\mathcal{S}} V(x)\, d\mu(x) \\ &= \int_{\mathcal{S}} \log\left(\frac{\int_{\mathcal{S}} e^{-V(x)} d\mu_0(x)}{e^{-V(x)}} \frac{d\mu(x)}{d\mu_0(x)}\right) d\mu(x) - \log \int e^{-V} d\mu_0(x) \\ &= R(\mu|e^{-V}\mu_0) - \log \int e^{-V} d\mu_0(x). \end{aligned} \tag{4.5.10}$$

Because of the positivity of the relative entropy it follows that $R(\mu|e^{-V}\mu_0) \geq 0$, and hence,

$$R(\mu\|\mu_0) + \int V(x)\, d\mu(x) \geq -\log \int e^{-V} d\mu_0(x). \tag{4.5.11}$$

Note that the inequality becomes an inequality for $\mu = e^{-V}\mu_0$. □

We are now able to extend this reasoning to the multi-particle potentials of Lemma 4.9.

*Proof of Lemma 4.9.*
Fix any $\mu \in \mathcal{P}(S)$ with $R(\mu\|\mu_0) < \infty$, and let $V$ be a pair potential. Then,

$$\begin{aligned} R(\mu\|\mu_0) - |\mathcal{E}_V(\mu)| &\geq R(\mu\|\mu_0) - |\mathcal{E}_{|V|}(\mu)| \\ &\geq R(\mu\|\mu_0) - \int_{S^2} |V(x,y)| d\mu(x) d\mu(y) \\ &\geq \inf_{y \in S} \Big\{ R(\mu\|\mu_0) - \int_S |V(x,y)| d\mu(x) \Big\}. \end{aligned} \tag{4.5.12}$$

Now fix any $y \in \mathcal{S}$. Note that $|V(\cdot,y)| \wedge n$ is a bounded measurable function on $S$ for all $n \geq 0$, and by Lemma 4.11, we have for any $n \geq 0$,

$$\begin{aligned} R(\mu\|\mu_0) - \int_S |V(x,y) \wedge n| d\mu(x) &\geq -\log \int_S e^{|V(x,y)| \wedge n} d\mu_0(x) \\ &\geq -\log \int_S e^{|V(x,y)|} d\mu_0(x), \end{aligned} \tag{4.5.13}$$

or, rewritten,

$$\int_S |V(x,y) \wedge n| d\mu(x) \leq \log \int_S e^{|V(x,y)|} d\mu_0(x) + R(\mu\|\mu_0). \tag{4.5.14}$$

Hence, letting $n \to \infty$ and using Fatou's Lemma,

$$\int_S |V(x,y)| d\mu(x) \leq \log \int_S e^{|V(x,y)|} d\mu_0(x) + R(\mu\|\mu_0). \tag{4.5.15}$$

Rearranging the terms back, it follows that,

$$R(\mu\|\mu_0) - \int_S |V(x,y)| d\mu(x) \geq -\log \int_S e^{|V(x,y)|} d\mu_0(x). \tag{4.5.16}$$

Thus, we can conclude,

$$\begin{aligned}\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}} &= -\inf_{\mu\in\mathcal{P}(S)} \Big\{ R(\mu\|\mu_0) - |\mathcal{E}_V| \Big\} \\ &\leq \sup_{y\in S} \log \int_S e^{|V(x,y)|} d\mu_0(x) \\ &= [V]_S.\end{aligned} \tag{4.5.17}$$

The proof for a $k$-particle potential $V$ is similar, in which case

$$\begin{aligned}\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}} &= -\inf_{\mu\in\mathcal{P}(S)} \Big\{ R(\mu\|\mu_0) - |\mathcal{E}_V| \Big\} \\ &\leq -\inf_{\mu\in\mathcal{P}(S)} \inf_{x_2,\ldots,x_k\in S} \Big\{ R(\mu\|\mu_0) - \int_S |V(x_1;x_2,\ldots,x_k)| d\mu(x_1) \Big\} \\ &\leq \sup_{x_2,\ldots x_k\in S} \log \int_S e^{|V(x_1;x_2,\ldots,x_k)|} d\mu_0(x_1) \\ &= [V]_S.\end{aligned} \tag{4.5.18}$$

Finally, to prove (4.5.3), first note that

$$R(\mu\otimes\mu\,\|\mu_0\otimes\mu_0) = 2R(\mu\|\mu_0) \tag{4.5.19}$$

Now, fix $\mu\in\mathcal{P}(S)$ with $R(\mu\|\mu_0)<\infty$ and a pair potential $V$. Then, similar as before, using $|V|\wedge n|$ and taking limits we can show that by using Lemma 4.11 with $\mathcal{S} := S^2$ that

$$\begin{aligned}\int_{S^2} |V(x,y)| d\mu(x)d\mu(y) &= \frac{1}{2}\int_{S^2} |2\,V(x,y)| d\mu(x)d\mu(y) \\ &\leq \frac{1}{2}\liminf_{n\to\infty} \int_{S^2} |2\,V(x,y)\wedge n| d\mu(x)d\mu(y) \\ &\leq \frac{1}{2}\Big( \log \int_{S^2} e^{|2\,V(x,y)|} d\mu_0(x)d\mu_0(y) - R(\mu\otimes\mu\|\mu_0\otimes\mu_0)\Big) \\ &= \frac{1}{2}\Big( \log \int_{S^2} e^{|2\,V(x,y)|} d\mu_0(x)d\mu_0(y) - 2R(\mu\|\mu_0)\Big).\end{aligned} \tag{4.5.20}$$

Hence, rearranging the terms,

$$R(\mu\|\mu_0) - \int_{S^2} |V(x,y)| d\mu(x)d\mu(y) \geq -\frac{1}{2}\log\int_{S^2} e^{|2\,V(x,y)|} d\mu_0(x)d\mu_0(y), \tag{4.5.21}$$

and it follows that

$$\begin{aligned}\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}} &= -\inf_{\mu\in\mathcal{P}(S)} \Big\{ R(\mu\|\mu_0) - |\mathcal{E}_V| \Big\} \\ &\leq \frac{1}{2}\log\int_{S^2} e^{|2V(x,y)|} d\mu_0(x) \\ &= \frac{1}{2}\langle 2V\rangle_S.\end{aligned} \tag{4.5.22}$$

Similarly, for a $k$-particle potential $V$,

$$\begin{aligned}\langle\!\langle \mathcal{E}_V \rangle\!\rangle_{\mathcal{X}} &= -\inf_{\mu\in\mathcal{P}(S)} \Big\{ R(\mu\|\mu_0) - |\mathcal{E}_V| \Big\} \\ &= \inf_{\mu\in\mathcal{P}(S)} \Big\{ \frac{1}{k} R(\mu^{\otimes k}\|\mu_0^{\otimes k}) - \mathcal{E}_{|V|} \Big\} \\ &\leq \frac{1}{k}\log\int_{S^k} e^{|kV(x_1;x_2,\ldots,x_k)|} d\mu_0(x_1)\, d\mu_0(x_2)\cdots d\mu_0(x_k) \\ &= \frac{1}{k}\langle kV\rangle_S.\end{aligned} \tag{4.5.23}$$

□

#### Comments

**Remark 4.7.** Recall, for the reduction argument of Section 4.4, Lemma 4.8, we used a relatively easy technique, namely only a generalized Hölder's inequality and simple factorization. It allowed us to establish quasi-continuity for $\mathcal{E}_V$ with $V$ approximated by continuous functions $V_\lambda$ in the sense that for all $\beta \in \mathbb{R}$,

$$[\beta(V - V_\lambda)]_S \to 0. \tag{4.5.24}$$

However, Theorem 4.3 would suggest that those $V$ that are defined similarly using $\langle V \rangle_S$ instead of $[V]_S$ might also induce quasi-continuous $\mathcal{E}_V$, since the necessary conditions on the unnormalized rate functions $J_\beta$ are satisfied. Interestingly enough, Lemma 4.9 implies the following clue.

**Corollary 4.12.** *Suppose $V : S^2 \to \mathbb{R}$ is such that $\mathcal{E}_V$ is quasi-continuous and quasi-bounded, and such that there exists bounded continuous functions $V_\lambda$ such that for every $\beta \in \mathbb{R}$,*

$$\langle \beta(V - V_\lambda) \rangle_S \to 0. \tag{4.5.25}$$

*Moreover, let $\{X_{i,j}\}_{i,j}$ be $N^2$ independent variables with common law $\gamma_0 = \mu_0 \otimes \mu_0$, and $\bar{\mathbb{E}}$ the expectation under $\gamma_0^{\otimes N^2}$. Then, for every $\beta \in \mathbb{R}$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \tilde{\mathbb{E}}^N \Big[ e^{\frac{-\beta}{N} \sum_{i,j\neq i}^N V(X_i, X_j)} \Big] \leq \limsup_{N\to\infty} \frac{1}{2N^2} \log \bar{\mathbb{E}}^N \Big[ e^{-\beta \sum_{i,j\neq i}^N 2V(X_{i,j})} \Big] \tag{4.5.26}$$

*Here $V(X_{i,j}) \coloneqq V((X_{i,j})_1, (X_{i,j})_2)$, with $(X_{i,j})_1$ the $S$-valued first coordinate of $X_{i,j}$.*

While there are of course numerous advanced and modern stochastic techniques that could be of use — like those as used in stochastic control and employed in [Fis14] to control certain estimates — it follows from Corollary 4.12 that possible coupling techniques might also be appropriate.

*Proof.* Note that by quasi-continuity and boundedness of $\mathcal{E}_V$ it follows from the induced Laplace principle that for every $\beta \in \mathbb{R}$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \tilde{\mathbb{E}}^N \Big[ e^{\frac{-\beta}{N} \sum_{i,j\neq i}^N V(X_i, X_j)} \Big] = - \inf_{\mu \in \mathcal{P}(S)} \Big\{ R(\mu \| \mu_0) + \beta \mathcal{E}_V(\mu) \Big\} \tag{4.5.27}$$

. However, because of (4.5.25), it follows that the right side also satisfies a Laplace principle, now on the space $\mathcal{P}(S^2)$ but with $V(\cdot)$ now acting as an 'external' potential, and hence, because of Lemma 4.11

$$\limsup_{N\to\infty} \frac{1}{2N^2} \log \bar{\mathbb{E}}^N \Big[ e^{-\beta \sum_{i,j\neq i}^N 2V(X_{i,j})} \Big] = -\frac{1}{2} \inf_{\gamma \in \mathcal{P}(S^2)} \Big\{ R(\gamma \| \mu_0 \otimes \mu_0) + 2\beta \mathcal{E}_V(\gamma) \Big\}. \tag{4.5.28}$$

Finally, the inequality follows similar as in the proof of (4.5.3) of Lemma 4.9,

$$\inf_{\mu \in \mathcal{P}(S)} \Big\{ R(\mu \| \mu_0) + \beta \mathcal{E}_V(\mu) \Big\} \geq \inf_{\gamma \in \mathcal{P}(S^2)} \Big\{ \frac{1}{2} R(\gamma \| \mu_0 \otimes \mu_0) + \beta \mathcal{E}_V(\gamma) \Big\} \quad . \tag{4.5.29}$$

□

## 4.6 Discussion

Using the framework of quasi-continuous functions as developed in Section 3, we showed how for the case of Gibbs measures for weakly interacting particles — where we tilted the distribution of the empirical measure via an energy functional defined by a $k$-particle potential on these particles — large-deviation principles could be established for quite a general class of potentials, as seen in Theorems 4.3 and 4.5.

The main technique was approximating the potentials $V$ by continuous and bounded potentials $V_\lambda$ and employing a reduction argument — as shown by Lemma 4.8 — where the $N$-particle exponential integrals required for establishing quasi-uniform convergence, as defined in Section 3, could be factorized and reduced to a purely $k$-particle exponential estimate.

Moreover, we showed in Section 4.5 what quasi-uniform convergence, and hence quasi-continuity, implied for the rate functions. Additionally, we provided a class of potentials $V$ such that the induced rate functions satisfy all these necessary conditions, and hence are likely candidates for extending our techniques for establishing large-deviation principles.

Both the invariant measures studied in Section 5, and the empirical processes in Section 6, are special cases of the results presented in this section. In particular, Theorems 4.3 and 4.5 directly imply Theorems 2.8 and 2.9.

However, in the case of local quasi-uniform convergence, which implies quasi-continuity but not quasi-boundedness, additional steps are needed to approximate the localized exponential integrals, techniques that depend strongly on $V$ and the underlying space $S$, and those are not treated here.

Yet, in the upcoming Section 5, we will show how on a compact subset of $\mathbb{R}^d$ a discretization argument can extend our results for a pair potential $V$ with a logarithmic singularity confined to the diagonal $x = y$, which will imply Theorem 2.7.

# 5 LDP for invariant measures

## 5.1 Introduction

One of the most important examples for Gibbs measures as described in the previous section is one where the underlying space is $\mathbb{R}^d$. It has a rich history in statistical physics for describing equilibrium phenomena, and questions about the latter actually played a instrumental role in the development of large deviation theory (see also [DZ10, p. 339]). In particular, these Gibbs measures arise as invariant measures — albeit sometimes only formally — of possibly reversible systems of stochastic differential equations such as (2.2.1), that act under a certain singular potential $V : \mathbb{R}^d \to \mathbb{R}$.

In proving large deviations for these invariant measures we will consider two important cases. Loosely speaking, we will denote a system as *sub-critical* if the induced Gibbs measures are well-defined for *all* temperatures, both positive and negative (where, recall, the latter merely means switching the sign of $V$). Similarly, we denote a system as *critical* if the Gibbs measures are only well-defined on a *finite range* of positive and negative temperatures.

This criticality plays an important role for systems involving attracting particles, where for example for a low enough temperature the attraction might be so strong that particles start to collide, or even that the system collapses to a single point.

For invariant measures, this phenomenon is dependent on the exponential integrability of the underlying potential, which will be made precise in Section 5.2. In particular, the potentials inducing critical systems contain some logarithmic singularities. In studying vortex systems, [BG99] framed the problem in this context, i.e. a system of dipoles interacting under a logarithmic potential, and it is their investigations that form the inspiration and starting point of this thesis.

Moreover, as mentioned in Remark 3.4 of Section 3, the phenomenon is related to the distinction between quasi-boundedness of the induced energy functional $\mathcal{E}_V$ and strictly *local* quasi-boundedness, where criticality is equivalent to the latter. Because of this we have to use different techniques to establish large-deviation principles for the sub-critical and critical case.

For the former, it might be possible to find continuous and potentials $V_\lambda$ such that $\mathcal{E}_{V_\lambda}$ converges *quasi-uniformly* to $\mathcal{E}_V$, see Section 3.5. In particular, all the relevant results of this section directly follow from the results in Section 4 for abstract Gibbs measures, see Theorem 4.3, where we showed that a simple finite-dimensional exponential estimate on $V - V_\lambda$ is enough.

However, for the critical case, one has to find another way to establish *local quasi-convergence*. Inspired by the technique used in [BG99] we will employ in this section a discretization approach, in which we will place a certain approximate absolute continuity restriction with respect to the Lebeque measure on the random empirical measure $z^N$.

Finally, we will discuss how this technique could be generalized even further, by applying it to the 'deterministic case' purely in terms of the possible rate functions, which provides a suitable class of candidates, similar as done in 4.5.

**Outline** First, we briefly recall the relevant notation in Section 5.2, and we list and discuss our main results in 5.3. Next, we will outline in Section 5.4 the discretization approach in establishing quasi-continuity for pair interactions, for which we will decomposition the space $\Lambda$ into a finite collection of cubes and reduce the problem to one of local estimates over these cubes.

Finally, in Section 4.5, we will give a multi-particle analogue of these results for rate functions via entropic inequalities.

## 5.2 Notation and basic properties

The case for invariant measures is a direct application of the framework of Section 4. Therefore, we refer for notation to Section 4.2. However, for convenience, we will briefly recall a few important concepts, and define criticality.

We set $S := \Lambda$, with $\Lambda \subset \mathbb{R}^d$, and consider for any $V : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$ the induced energies $\mathcal{E}_V : \mathcal{P}(\Lambda) \to \mathbb{R}$,

$$\mathcal{E}_V(\mu) := \int_{(\Lambda^k)'} V(x_1; x_2 \ldots x_k)\, d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{5.2.1}$$

Here $(\Lambda^k)'$ is $\Lambda^k$ but with the $x_1$-diagonals removed.

Moreover, we consider the random empirical measures $z^N \in \mathcal{P}(S)$ with laws $P^N \in \mathcal{P}(\mathcal{P}(\Lambda))$, which are derived from $N$ independent particles $X_i \in \Lambda$ with common law $\mu_0$. Recall that

$$\mathbb{E}^N\Big[e^{-N\mathcal{E}_V(z^N)}\Big] = \tilde{\mathbb{E}}^N\Big[e^{-N\sum_{i_2,\ldots,i_k \neq 1, i_1} V(X_1;X_2,\ldots,X_k)}\Big], \tag{5.2.2}$$

where $\mathbb{E}^N$ is the expectation over the law $P^N$, and $\tilde{\mathbb{E}}^N$ is the expectation over the particles $X_i$.

Also, recall the induced measures $Q_V^N$, and that $J_V(\mu) := \mathcal{E}_V(\mu) + R(\mu\|\mu_0)$ when $R(\mu\|\mu_0) < \infty$ and $+\infty$ otherwise, and $\mathcal{F}_V(\mu) = J_V(\mu) - \inf_{\nu\in\mathcal{P}(\Lambda)} J_V(\nu)$.

Finally, we will denote either $V$, $\mathcal{E}_V$ or the system in general as *critical* if there exists a $\beta_0 > 0$ such that

$$\sup_{x_2,\ldots,x_k\in\Lambda} \int_\Lambda e^{\beta_0|V|(x_1;x_2\ldots x_k)}\, d\mu_0(x_1) = \infty, \tag{5.2.3}$$

but such that for all $0 < \beta < \beta_0$,

$$\sup_{x_2,\ldots,x_k\in\Lambda} \int_\Lambda e^{\beta|V|(x_1;x_2\ldots x_k)}\, d\mu_0(x_1) < \infty. \tag{5.2.4}$$

Similarly, we will denote it as *sub-critical* if there is no such finite $\beta_0$, in which case 5.2.4 holds for all $\beta \in \mathbb{R}$.

## 5.3 Main results

We will now list our main results, which in particular will imply Theorems 2.7 and 2.8. We will separate them into the sub-critical, asymmetric and critical cases. Only the latter needs significantly different tools than developed in Section 4, and we will introduce them in Sections 5.4 and 5.5.

### 5.3.1 Sub-critical case

For sub-critical $V$, the sequence of induced invariant measures is well defined for *any* temperature, and thus we can directly apply Theorem 4.3 to characterize quasi-continuity of $\mathcal{E}_V$.

**Theorem 5.1.**
*Let $V_\lambda : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$ be continuous and bounded for all $\lambda$.*

*Suppose $V : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$ is such that for all $\beta \in \mathbb{R}$,*

$$\lim_{\lambda\to 0} \sup_{x_2,\dots,x_k\in\Lambda} \log \int_\Lambda e^{\beta|V-V_\lambda|(x_1;x_2,\dots,x_k)}\, d\mu_0(x_1) = 0. \tag{5.3.1}$$

*Then $\mathcal{E}_V$ is quasi-continuous and for all $\beta \in \mathbb{R}$ the induced measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

However, note that in (5.3.1) the convergence is required to be *uniform* over $x_2, \dots, x_k$, and hence pointwise convergence of $V_\lambda$ to $V$ is not enough. If $V$ or $\mu_0$ is highly discontinuous in all arguments, uniform convergence is not easily established — see for example Remark 5.1.

Yet, for relatively regular $V$, we can still establish the following.

**Corollary 5.2.** *Let $\Lambda$ be compact, and let $V_\lambda : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$ be continuous and bounded for all $\lambda$. Moreover, suppose that $V_\lambda(\,\cdot\,; x_2, \dots, x_k)$ converges for every $(x_2, \dots x_k) \in \Lambda^{k-1}$ pointwise and monotonously $\mu_0$-almost everywhere to $V : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$.*

*Additionally, suppose $V$ is continuous and bounded everywhere, except for the diagonals $x_1 = x_i$, with $2 \le i \le k$, and suppose that for all $\beta \in \mathbb{R}$,*

$$\sup_{x_2,\dots,x_k\in\Lambda} \int_\Lambda e^{\beta|V|(x_1;x_2,\dots,x_k)}\, d\mu_0(x_1) < \infty. \tag{5.3.2}$$

*Then $\mathcal{E}_V$ is quasi-continuous and for all $\beta \in \mathbb{R}$ the induced measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

Although the above conditions might certainly be replaced by requiring certain local differentiability or Lipschitz conditions on $V$ or $\mu_0$, note that Corollary 5.2 immediately establishes quasi-continuity for a wide range of potentials $V$.

*Proof.* Let for any $\beta \in \mathbb{R}$, $g_\lambda : \Lambda^{k-1} \to \mathbb{R}$ be defined as

$$g_\lambda(x_2, \dots, x_k) := \int_\Lambda e^{\beta|V-V_\lambda|(x_1\; x_2,\dots,x_k)}\, d\mu_0(x_1). \tag{5.3.3}$$

For every $\lambda$, it holds that $V - V_\lambda$ is continuous and bounded, except along the diagonals $x_1 = x_i$. Combined with the fact $g_\lambda < \infty$ by (5.3.2) and that $\mu_0$ is continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, it follows that $g_\lambda$ is continuous and bounded.

Now, note that for every $x_2, \dots, x_k$, by pointwise convergence $\mu_0$-almost everywhere of $V_\lambda$ to $V$ in $x_1$,

$$\lim_{\lambda\to 0} g_\lambda(x_2, \dots, x_k) = 0, \tag{5.3.4}$$

and, trivially, note that the zero function is continuous on $\Lambda$.

Hence, since $V$ also converges monotonously, it follows that $g_\lambda(x_2,\ldots,x_k)$ converges pointwise and monotonously to a continuous function on the compact space $\Lambda^{k-1}$. Thus, by Dini's theorem, the convergence will be uniformly on $\Lambda^{k-1}$. Thus, we have have therefore established (5.3.1), and quasi-continuity and the relevant large-deviation principles follow from Theorem 5.1. □

Moreover, in the case that pointwise convergence and boundedness can still be verified, but not the uniform convergence of (5.3.1), note that we can still verify certain properties of the possible rate functions by use of Theorem 4.3.

**Corollary 5.3.**
*Let $V_\lambda : \Lambda^k \to \mathbb{R}$ be continuous and bounded for all $\lambda$, and suppose that they converge pointwise $\mu_0^{\otimes k}$ almost everywhere to a $V : \Lambda^k \to \mathbb{R}$.*

*Moreover, suppose $V : \Lambda^k \to \mathbb{R}$ is such that for some fixed for all $\beta \in \mathbb{R}$,*

$$\int_{\Lambda^k} e^{\beta|V|(x_1,\ldots,x_k)|}\, d\mu_0(x_1)\cdots\mu_0(x_k) < \infty, \tag{5.3.5}$$

*and suppose that there exists positive constants $a_1$, $a_2$, such that*

$$|V - V_\lambda| \le \max\{a_1, a_2 V\}. \tag{5.3.6}$$

*Then $\mathcal{E}$ is continuous on the sets $\{\,\mu \,|\, R(\mu\|\mu_0) \le M\,\}$.*

*Moreover, for any $\beta \in \mathbb{R}$, the function $\mathcal{F}_\beta$ is a proper rate function, i.e. lower semi-continuous with compact sublevel sets.*

*Proof.* Recall from Theorem 4.3 that the result follows if for every $\beta \in \mathbb{R}$

$$\lim_{\lambda\to 0} \log \int_{\Lambda^k} e^{\beta|V-V_\lambda|(x_1,\ldots,x_k)|}\, d\mu_0(x_1)\cdots\mu_0(x_k) = 0. \tag{5.3.7}$$

However, note that by (5.3.6) and (5.3.5) it holds that there exists a function $V^* := \max\{a_1, a_2 V\}$. such that $|V - V_\lambda| < V$ on $\Lambda^k$ and

$$\int_{\Lambda^k} e^{\beta V^*(x_1,\ldots,x_k)|}\, d\mu_0(x_1)\cdots\mu_0(x_k) < \infty, \tag{5.3.8}$$

for all $\beta \in \mathbb{R}$. Hence, since $V_\lambda$ converge to $V$ pointwise almost everywhere, it follows by dominated convergence that

$$\lim_{\lambda\to 0} \int_{\Lambda^k} e^{\beta|V-V_\lambda|(x_1,\ldots,x_k)|}\, d\mu_0(x_1)\cdots\mu_0(x_k) = 1, \tag{5.3.9}$$

and we conclude the proof. □

**Remark 5.1.** We will give a simple example such that (5.3.1) is not easily verified.

Namely, let $\Lambda = [-2, 2]$, $\mu_0$ constant on $\Lambda$, and $V : \Lambda^2 \to \mathbb{R}$ to be 1 at the unit cube $[0,1]^2$ and zero otherwise. To be precise,

$$V(x,y) := 1_{0<x<1} 1_{0<y<1} \tag{5.3.10}$$

Now, let $V_\lambda$ be continuous and positive, and such that $V_\lambda = V$ on $(x,y) \in [0,1]^2$. Consider $g(y)$, with

$$g(y) := \int_{-2}^{2} e^{\beta|V-V_\lambda|(x)} dx. \tag{5.3.11}$$

Then it clear that $g(y) = 0$ for $0 < y < 1$, but because of the continuity of $V_\lambda$ and the fact that $V_\lambda = V$ on $[0,1]^2$ for *all* $\lambda$,

$$\begin{aligned} \lim_{y\uparrow 0} g(y) &= \lim_{y\uparrow 0} \int_{-2}^{2} e^{\beta|V-V_\lambda|(x;y)} dx \\ &\ge \lim_{y\uparrow 0} \int_{0}^{1} e^{\beta|V-V_\lambda|(x;y)} dx \\ &= \lim_{y\uparrow 0} \int_{0}^{1} e^{\beta|V_\lambda|(x;y)} dx \\ &= e^{\beta}. \end{aligned} \tag{5.3.12}$$

Hence, any such convergence can *never* be uniform in $\lambda$.

Note that by Corollary 5.3 we can establish the desired properties of the rate function, and therefore it is still probable that $V$ induces a quasi-continuous $\mathcal{E}_V$, and hence a LDP.

Finally, interestingly enough, note that $V := 1_{0<x<1} + 1_{0<y<1}$ does not satisfy (5.3.1) directly, but by linearity ís quasi-continuous, since $1_{0<x<1}$ and $1_{0<y<1}$ are. Whether a comparable and suitable decomposition exists for the previous $V$ is not known.

### 5.3.2 Assymetric case

Similarly as for the sub-critical case, we can directly apply the results from Section 4.3.

**Theorem 5.4.**
*Let $V_\lambda : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$ be continuous and bounded for all $\lambda$.*

*Suppose $V : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$ is such that for all $\beta \geq 0$,*

$$\limsup_{\lambda \to 0} \sup_{x_2,\dots,x_k \in \Lambda} \log \int_\Lambda e^{-\beta(V-V_\lambda)(x_1;x_2,\dots,x_k)}\, d\mu_0(x_1) \leq 0. \tag{5.3.13a}$$

$$\limsup_{\lambda \to 0} \int_{\Lambda^k} (V - V_\lambda)(x_1; x_2, \dots, x_k) \prod_i^k d\mu_0(x_i) \leq 0. \tag{5.3.13b}$$

*Then for all $\beta \geq 0$ the induced measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

This follows directly from Theorem 4.5.

However, because of the relaxed condition (5.3.13b) and the restriction to only positive $\beta$, it is far easier to establish an LDP than for the full range of real temperatures of the sub-critical case. Namely, we have the following analogue of Corollary 5.2.

**Corollary 5.5.** *Let $V_\lambda : \Lambda \times \Lambda^{k-1} \to \mathbb{R}$ be continuous and bounded for all $\lambda$.*

*Moreover, suppose that $V_\lambda \leq V$ on $\Lambda \times \Lambda^{k-1}$, and that $V_\lambda$ converges pointwise $\mu_0^{\otimes k}$-almost everywhere to $V$.*

*Additionally, suppose $V$ is such that*

$$\int_{\Lambda^k} V(x_1; x_2, \dots, x_k) \prod_i^k d\mu_0(x_i) \leq 0. \tag{5.3.14}$$

*Then for all $\beta \geq 0$ the induced measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

*Proof.* Note that (5.3.13a) is automatically satisfied since $V_\lambda \leq V$. Moreover, $V_\lambda$ and (5.3.14) imply that $V - V_\lambda$ is dominated by the integrable $V$, and hence by pointwise convergence of $V_\lambda$ to $V$ almost everywhere, (5.3.13b) follows. □

### 5.3.3 Critical case

As mentioned, the main inspiration for our extended framework of quasi-continuity was from [BG99], where they investigated large deviations for invariant measures induced by potentials with logarithmic singularities.

We will now give a generalization of their result for pair interactions, in which we require that the logarithmic singularity only plays a role when two particles *meet*, i.e. it is confined to the diagonal $x = y$. Outside this diagonal we simply require that the singularities are at worst sub-logarithmic interactions.

**Theorem 5.6.**
*Let $\Lambda$ be compact, let $V_\lambda : \Lambda\times\Lambda \to \mathbb{R}$ be continuous and bounded, and let $\mu_0$ be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

*Suppose $V : \Lambda \times \Lambda \to \mathbb{R}$ is such that for some fixed $\beta_0 > 0$,*

$$\lim_{\lambda\to 0} \sup_{y\in\Lambda} \log \int_\Lambda e^{\beta_0 |V - V_\lambda|(x;y)}\, d\mu_0(x) = 0, \tag{5.3.15}$$

*Additionally, suppose that for all $\beta \in \mathbb{R}$ and all $L > 0$,*

$$\limsup_{\lambda\to 0} \sup_{y\in\Lambda} \log \int_\Lambda e^{\beta|V - V_\lambda|(x;y)}\, 1_{|x-y|>L}\, d\mu_0(x) \le 0. \tag{5.3.16}$$

*Then $\mathcal{E}_V$ is quasi-continuous and for all $\beta \in \mathbb{R}$ with $|\beta| < \beta_0$, the induced measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

The proof follows by establishing strictly *local* quasi-uniform convergence (see Theorem 3.20). To do so we employ a discretization approach, inspired by [BG99], by dividing $\Lambda$ into cubes and separating the logarithmic singularity at the diagonal, see Section 5.4. It is is because of this procedure that we require compactness of $\Lambda$ and absolutely continuity of $\mu_0$ with respect to the Lebesgue measure.

Similar to the sub-critical case, the convergence of (5.3.15) and (5.3.15) is required to be uniform in $y \in \Lambda$. However, similar to Corollary 5.2, this is verified for simple cases of $V$, and we will omit the proof.

**Corollary 5.7.** *Let $\Lambda$ be compact, and let $V : \Lambda \times \Lambda \to \mathbb{R}$ be continuous and bounded everywhere, except at the diagonal $x = y$. Let $V_\lambda : \Lambda \times \Lambda \to \mathbb{R}$ be continuous and bounded, and suppose they converge pointwise and monotonously to $V$, except on the diagonal $x = y$.*

*Moreover, let $\mu_0$ be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, and suppose $V$ is such that for some fixed $\beta_0 > 0$,*

$$\sup_{y\in\Lambda} \int_\Lambda e^{\beta_0 |V|(x;y)}\, d\mu_0(x) < \infty \tag{5.3.17}$$

*Then $\mathcal{E}_V$ is quasi-continuous, and for all $\beta \in \mathbb{R}$ with $|\beta| < \beta_0$, the induced measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

Because of technical issues, Theorem 5.6 can not easily be extended to multi-particle interactions, see also Remark 5.6. However, out of interest, we include the following theorem, which characterizes the properties of the unnormalized rate functions $J_V$, and hence provides us with a nice class of candidates for quasi-continuity.

**Theorem 5.8.**
*Let $\Lambda$ be compact, $V_\lambda : \Lambda^k \to \mathbb{R}$ be continuous and bounded for all $\lambda$. $\mu_0^{\otimes k}$-almost everywhere. Moreover, let $\mu_0$ be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

*Suppose $V : \Lambda^k \to \mathbb{R}$ is such that for some fixed $\beta_0 > 0$,*

$$\lim_{\lambda\to 0} \int_{\Lambda^k} e^{\beta_0 |V - V_\lambda|(x_1;x_2,\dots,x_k)}\, d\mu_0(x_1)\, \dots\, d\mu_0(x_k) = 0. \tag{5.3.18}$$

*Additionally, suppose that for all $\beta \geq 0$ and all $L > 0$,*

$$\limsup_{\lambda\to 0} \int_{\Lambda^k/\Lambda^k_L} e^{\beta |V - V_\lambda|(x_1;x_2,\dots,x_k)}\, d\mu_0(x_1)\, \dots\, d\mu_0(x_k) \leq 0. \tag{5.3.19}$$

*Then $\mathcal{E}_V$ is continuous on the sets $\{\, \mu \,|\, R(\mu \| \mu_0) \leq M \,\}$, and hence $J_{\beta V}$ is lower semi-continuous at all $\mu$ with $R(\mu\|\mu_0) < \infty$.*

*Moreover, for any $\beta \in \mathbb{R}$ with $-\frac{\beta_0}{k} < \beta < \frac{\beta_0}{k}$, $J_{\beta V}$ is lower semi-continuous with compact sublevel sets and with $|J_{\beta V}(\mathcal{X})| < \infty$.*

Note again the separation between the logarithmic singularity at the diagonals and the rest of $\Lambda$, although this can be extended to more general singularities.

The proof is quite technical, see Section 5.5, but since there is no supremum over any $y \in \Lambda$ involved, the conditions of Theorem 5.8 itself are verified quite easily in the case of pointwise convergence, as for the sub-critical case. Namely, it is easy to see the following consequence.

**Corollary 5.9.** *Let $\Lambda$ be compact, $V_\lambda : \Lambda^k \to \mathbb{R}$ be continuous and bounded for all $\lambda$, and suppose that they converge pointwise to a $V : \Lambda^k \to \mathbb{R}$, $\mu_0^{\otimes k}$-almost everywhere. Moreover, let $\mu_0$ be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

*Suppose $V : \Lambda^k \to \mathbb{R}$ is such that for some fixed $\beta_0 > 0$,*

$$\int_{\Lambda^k} e^{\beta_0 |V|(x_1;x_2,\dots,x_k)}\, d\mu_0(x_1)\, \dots\, d\mu_0(x_k) < \infty. \tag{5.3.20}$$

*Additionally, suppose that for all $\beta \geq 0$ and all $L > 0$,*

$$\lim_{\lambda\to 0} \int_{\Lambda^k/\Lambda^k_L} e^{\beta |V|(x_1;x_2,\dots,x_k)}\, d\mu_0(x_1)\, \dots\, d\mu_0(x_k) < \infty, \tag{5.3.21}$$

*and, finally, that there exists positive constants $a_1$, $a_2$ such that*

$$|V - V_\lambda| \leq \max\{a_1, a_2 V\}. \tag{5.3.22}$$

*Then $\mathcal{E}$ is continuous on the sets $\{\, \mu \,|\, R(\mu\|\mu_0) \leq M \,\}$.*

*Moreover, for any $\beta \in \mathbb{R}$ with $-\frac{\beta_0}{k} < \beta < \frac{\beta_0}{k}$, the function $\mathcal{F}_\beta$ is a proper rate function, i.e. lower semi-continuous with compact sublevel sets.*

## 5.4 Critical singularities: quasi-continuity

In [BG99] they employ a discretization technique for their specific case of logarithmic interactions. We have significantly adapted and generalized this procedure, which we will outline in this section.

Recall the notion of strictly local quasi-uniform convergence of Theorem 3.5 and Lemma 3.21, where instead of $[\![\beta(\mathcal{E}-\mathcal{E}_\lambda)]\!]_{\mathcal{X}}$ for any $\beta \in \mathbb{R}$ we consider $[\![\beta(\mathcal{E}-\mathcal{E}_\lambda)]\!]_A$ for appropriate sets $A$, dependent on $\beta$. Recall, in the current framework for invariant measures,

$$[\![\mathcal{E}_V]\!]_A = \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[e^{N|\mathcal{E}_V|(z^N)}\, 1_A\Big], \tag{5.4.1}$$

and

$$N\mathcal{E}_V(z^N) = \frac{1}{N} \sum_{i,j\neq i} V(X_i;X_j). \tag{5.4.2}$$

The main idea behind the technique of finding appropriate $A$ for every $\beta \in \mathbb{R}$, is that we first isolate the logarithmic singularity at the diagonal $x_1 = x_2$, i.e. where two particles meet. By discretization and the restriction that too many particles can not be 'too close' to each other, we then reduce the question to sub-problems over small cubes in $\mathbb{R}^d$ — where for each one the effective $\beta$ is now low enough to employ conventional techniques such as the one for sub-critical $V$.

First, some preliminaries. Throughout the rest of this section we assume that $\mu_0 \in \mathcal{P}(\Lambda)$, with $\Lambda \subseteq \mathbb{R}^2$, is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.

Moreover, for each $L > 0$ we cover the space $\Lambda$ — not necessarily compact — by a countable set of disjoint and half-open $\mathbb{R}^d$-cubes $C_k$ with side-length $L$. Moreover, for any cube $C_k$, we denote by $\hat{C}_k$ the union of $C_k$ and all the $(3^d - 1)$ cubes that neighbour $C_k$.

We then define for each $L > 0$ the following local analogues of $[V]_\Lambda$ of Section 4.4,

$$[V]_L := \sup_k \sup_{y\in\hat{C}_k} \log \int_{C_k} e^{|V|(x;y)}\, d\tilde{\mu}_{0,k}(x), \tag{5.4.3a}$$

$$[V]'_L := \sup_k \sup_{y\in\Lambda/\hat{C}_k} \log \int_{C_k} e^{|V|(x;y)}\, d\tilde{\mu}_{0,k}(x), \tag{5.4.3b}$$

where for every $k$, $\tilde{\mu}_{0,k} \in \mathcal{P}(C_k)$ is a renormalized probability measure of $\mu_0$ on $C_k$, defined by

$$d\tilde{\mu}_{0,k}(x) = \frac{d\mu_0(x)}{\mu_0(C_k)}. \tag{5.4.4}$$

Note the similarities of $[V]_L$ and $[V]'_L$ to $[V]_\lambda$, such as non-negativity and convexity with respect to $V$. Moreover, note that in the case of $[V]_L$ the estimate is over neighbouring cubes — and hence only concerns with short-range interactions on the order of $L$, while $[V]'_L$ only concerns particles that are at least some multiple of $L$ apart.

Below we will show that by treating the two estimates separately, i.e. we allow the short-range interactions to be critical, but long-range interaction sub-critical, we can show a local analogue to the convergence result of Theorem 4.3, which established a sufficient condition for quasi-uniform convergence for energies induced by Gibbs measures in terms of $[V]_\lambda$.

**Theorem 5.10.** *Let $\Lambda$ be compact, $V_\lambda : \Lambda \times \Lambda \to \mathbb{R}$ be continuous and bounded for all $\lambda$, and consider $V : \Lambda \times \Lambda \to \mathbb{R}$.*

*Suppose that there exists a $\beta_0$, such that for all $L > 0$,*

$$\lim_{\lambda\to 0} [\beta_0(V - V_\lambda)]_L = 0, \tag{5.4.5}$$

*and additionally, that for each $\beta \in \mathbb{R}$ and $L > 0$,*

$$\lim_{\lambda\to 0} [\beta(V - V_\lambda)]'_L = 0. \tag{5.4.6}$$

*Then $\mathcal{E}_V$ is quasi-continuous, and $[\![\beta\mathcal{E}]\!]_{\mathcal{X}} < \infty$ for all $\beta \in \mathbb{R}$ with $|\beta| < \beta_0$.*

*In particular, for all $\beta \in \mathbb{R}$ with $|\beta| < \beta_0$, the induced measures $Q^N_{\beta V}$ satisfy a LDP with rate function $\mathcal{F}_{\beta V}$.*

First, we will show how Theorem 5.10 implies Theorem 5.6.

*Proof of Theorem 5.6.* Recall that by assumption of Theorem 5.6 that $\Lambda$ is compact, that there exists a $\beta_0 > 0$ such that

$$\lim_{\lambda\to 0} \sup_{y\in\Lambda} \log \int_\Lambda e^{\beta_0 |V - V_\lambda|(x,y)}\, d\mu_0(x) = 0, \tag{5.4.7}$$

and, additionally, it holds for all $\beta \in \mathbb{R}$ and all $L > 0$, that

$$\limsup_{\lambda\to 0} \sup_{y\in\Lambda} \log \int_\Lambda e^{\beta|V-V_\lambda|(x,y)}\, 1_{|x-y|>L}\, d\mu_0(x) \le 0. \tag{5.4.8}$$

First, note that for any $y \in \Lambda$, by Hölders inequality with any exponent $\alpha^{-1}$ with $0 < \alpha < 1$,

$$\begin{aligned}
&\log \int_\Lambda e^{\beta|V-V_\lambda|(x,y)}\, 1_{|x-y|>L}\, d\mu_0(x) \\
&\quad\le \alpha \log \int_\Lambda e^{\alpha^{-1}\beta|V-V_\lambda|(x,y)}\, 1_{|x-y|>L}\, d\mu_0(x) + (1-\alpha)\log\int_\Lambda 1_{|x-y|>L}\, d\mu_0(x)
\end{aligned} \tag{5.4.9}$$

Hence, by (5.4.8) it follows that for any $\beta \ge 0$ and $0 < \alpha < 1$,

$$\begin{aligned}
0 &\le \limsup_{\lambda\to 0} \sup_{y\in\Lambda} \log \frac{\int_\Lambda e^{\beta|V-V_\lambda|(x,y)}\, 1_{|x-y|>L}\, d\mu_0(x)}{\int_\Lambda 1_{|x-y|\ge L}\, d\mu_0(x)} \\
&\le \limsup_{\lambda\to 0} \sup_{y\in\Lambda} \Big( \alpha \log \int_\Lambda e^{\alpha^{-1}\beta|V-V_\lambda|(x,y)}\, 1_{|x-y|>L}\, d\mu_0(x) - \alpha \log \int_\Lambda 1_{|x-y|>L}\, d\mu_0(x) \Big) \\
&\le -\alpha \inf_{y\in\Lambda} \log \int_\Lambda 1_{|x-y|>L}\, d\mu_0(x).
\end{aligned} \tag{5.4.10}$$

Letting $\alpha \to 0$, we derive,

$$\limsup_{\lambda\to 0} \sup_{y\in\Lambda} \log \frac{\int_\Lambda e^{\beta|V-V_\lambda|(x,y)}\, 1_{|x-y|>L}\, d\mu_0(x)}{\int_\Lambda 1_{|x-y|\ge L}\, d\mu_0(x)} = 0 \tag{5.4.11}$$

We will show that (5.4.7) (5.4.11) imply that the assumptions of Theorem 5.10 hold, in which case quasi-continuity and relevant large-deviation principles follow.

Since $\Lambda$ is compact, we cover $\Lambda$ for any $L > 0$ by a finite number of half-open $\mathbb{R}^d$-cubes $C_k$, with $1 \le k \le K$ for some finite $K$. Now fix $L$, and note that, because of the continuity and monotonicity of the logarithm, (5.4.7) implies that

$$\lim_{\lambda\to 0} \sup_{y\in\Lambda} \int_\Lambda e^{\beta_0|V-V_\lambda|(x;y)}\, d\mu_0(x) = 1, \tag{5.4.12}$$

and hence,

$$\lim_{\lambda\to 0} \sup_{y\in\Lambda} \int_\Lambda \left(e^{\beta_0|V-V_\lambda|(x;y)} - 1\right) d\mu_0(x) = 0. \tag{5.4.13}$$

Note that $e^{\beta_0|V-V_\lambda|(x,y)} - 1 \ge 0$ for any $x, y \in \Lambda$, and thus for any cube $C_k$

$$\begin{aligned}
\sup_{y\in\hat{C}_k} \int_{C_k} \left(e^{\beta_0|V-V_\lambda|(x;y)} - 1\right) d\tilde{\mu}_{0,k}(x) &= \frac{1}{\mu_0(C_k)} \sup_{y\in\hat{C}_k} \int_{C_k} \left(e^{\beta_0|V-V_\lambda|(x;y)} - 1\right) d\mu_0(x) \\
&\le \frac{1}{\mu_0(C_k)} \sup_{y\in\Lambda} \int_\Lambda \left(e^{\beta_0|V-V_\lambda|(x;y)} - 1\right) d\mu_0(x),
\end{aligned} \tag{5.4.14}$$

and therefore

$$\lim_{\lambda\to 0}\sup_{y\in\hat{C}_k}\int_{C_k}\left(e^{\beta_0|V-V_\lambda|(x;y)}-1\right)d\tilde{\mu}_{0,k}(x)=0. \tag{5.4.15}$$

Rearranging the terms back, it follows that

$$\lim_{\lambda\to 0}\sup_{y\in\hat{C}_k}\log\int_{C_k}e^{\beta_0|V-V_\lambda|(x;y)}\,d\tilde{\mu}_{0,k}(x)=0, \tag{5.4.16}$$

and since there are only a *finite* number of cubes $C_k$, we conclude that

$$\begin{aligned}\lim_{\lambda\to 0}[\beta_0(V-V_\lambda)]_L&=\lim_{\lambda\to 0}\max_{1\le k\le K}\sup_{y\in\hat{C}_k}\log\int_{C_k}e^{\beta_0|V-V_\lambda|(x;y)}\,d\tilde{\mu}_{0,k}(x),\\&=\max_{1\le k\le K}\lim_{\lambda\to 0}\sup_{y\in\hat{C}_k}\log\int_{C_k}e^{\beta_0|V-V_\lambda|(x;y)}\,d\tilde{\mu}_{0,k}(x)\\&=0.\end{aligned} \tag{5.4.17}$$

Similarly, note that for any $\beta\ge 0$,

$$\begin{aligned}\sup_{y\in\Lambda/\hat{C}_k}\int_{C_k}\left(e^{\beta|V-V_\lambda|(x;y)}-1\right)d\tilde{\mu}_{0,k}(x)&=\frac{1}{\mu_0(C_k)}\sup_{y\in\Lambda/\hat{C}_k}\int_{C_k}\left(e^{\beta|V-V_\lambda|(x;y)}-1\right)d\mu_0(x)\\&\le\frac{1}{\mu_0(C_k)}\sup_{y\in\Lambda}\int_{\Lambda}\left(e^{\beta|V-V_\lambda|(x;y)}-1\right)1_{|x-y|\ge L}\,d\mu_0(x).\end{aligned} \tag{5.4.18}$$

And, hence, similar as for $[V]_L$, we see that (5.4.11) implies that

$$\lim_{\lambda\to 0}\sup_{y\in\Lambda/\hat{C}_k}\log\int_{C_k}e^{\beta|V-V_\lambda|(x;y)}\,d\tilde{\mu}_{0,k}(x)=0 \tag{5.4.19}$$

and thus,

$$\begin{aligned}\lim_{\lambda\to 0}[\beta(V-V_\lambda)]'_L&=\lim_{\lambda\to 0}\max_{1\le k\le K}\sup_{y\in\Lambda/\hat{C}_k}\log\int_{C_k}e^{\beta|V-V_\lambda|(x;y)}\,d\tilde{\mu}_{0,k}(x),\\&=0.\end{aligned} \tag{5.4.20}$$

□

Next, we list the following trivial but essential result, which we will use to factorize the exponential estimates.

**Lemma 5.11.** *Suppose $X$ and $Y$ are independent variables. Then for any two functions $f(\cdot,\cdot)$ and $g(\cdot)$ it holds that*

$$\mathbb{E}[f(X,Y)\cdot g(Y)]\le\sup_y\mathbb{E}[f(X,y)]\cdot\mathbb{E}[g(Y)]. \tag{5.4.21}$$

*Proof.* Let $X\in\mathcal{S}_1$, $Y\in\mathcal{S}_2$, for some spaces $S_1,S_2$, and let $\mu,\nu$ be the laws of $X$ and $Y$, with $\mu\in\mathcal{P}(\mathcal{S}_1)$ , $\nu\in\mathcal{P}(\mathcal{S}_2)$. Then,

$$\begin{aligned}\int_{\mathcal{S}_1\times\mathcal{S}_2}f(x,y)g(y)\,d\mu(x)\,d\mu(y)&=\int_{\mathcal{S}_2}\left(\int_{\mathcal{S}_1}f(x,y)\,d\mu(x)\right)g(y)\,d\nu(y)\\&\le\int_{\mathcal{S}_2}\left(\sup_{y\in\mathcal{S}_2}\int_{\mathcal{S}_1}f(x,y)\,d\mu(x)\right)g(y)\,d\nu(y)\\&\le\left(\sup_{y\in\mathcal{S}_2}\int_{\mathcal{S}_1}f(x,y)\,d\mu(x)\right)\int_{\mathcal{S}_2}g(y)\,d\nu(y).\end{aligned} \tag{5.4.22}$$

□

Now, for the actual discretization technique, let us define the sets $\bar{\Lambda}^2_L, \Lambda^2_L$ with $\Lambda^2_L \subset \bar{\Lambda}^2_L \subset \Lambda^2$, where

$$\Lambda^2_L := \Big\{ (x;y) \in \Lambda^2 \,\Big|\, \|x - y| \le L \Big\}, \tag{5.4.23}$$

and $\bar{\Lambda}^2_L$ is the union of all the ordered couples of cubes $(C_i, C_j)$ of cubes that either equal or neighbour each other.

Moreover, define $\alpha_L(\mu)$ for any $L > 0$ and $\mu \in \mathcal{P}(S)$ by

$$\alpha_L(\mu) := \sup_k \mu(C^L_k), \tag{5.4.24}$$

and the set $A_{L,\delta}$,

$$A_{L,\delta} := \{\mu \,|\, \alpha_L(\mu) < \delta\}. \tag{5.4.25}$$

Note that for any $z^N \in A_{L,\delta}$ there at most $\delta N$ particles in any cube $C_k$. Moreover, note that for any $\mu \ll \mu_0$ we have $\mu \ll \lambda$, with $\lambda$ the Lebesgue measure on $\mathbb{R}^d$, and hence for such a $\mu$ it holds that $\alpha_L(\mu) \to 0$ as $L \to 0$.

Finally recall that by Lemma 4.9

$$[\![\mathcal{E}_V]\!]_{\mathcal{X}} \le [V]. \tag{5.4.26}$$

We will now show a similar approximation result for $A_{L,\delta}$, where $[V]_\lambda$ will be replaced with $[V]_L$ and $[V]'_L$.

**Lemma 5.12.**
*For any $L, \delta > 0$,*

$$[\![\mathcal{E}_V]\!]_{A_{L,\delta}} \le \Big( \frac{1}{4}[4\delta V]_L + \frac{1}{4}[4\cdot 3^d \delta V]_L + \frac{1}{2}[2V]'_L \Big). \tag{5.4.27}$$

As mentioned, we will employ a discretization argument. Intuitively speaking, for every $z^N \in A_{L,\delta}$ there are at most $3^d\delta N$ particles in $\hat{C}_k$. But for every $\hat{C}_k$ the energy term $N\mathcal{E}_V$ in the exponential of (5.4.2) is still scaled by a factor $\frac{1}{N}$. Therefore $V$ will be effectively replaced by $3^d\delta N$ and hence we would expect a contribution of short-range interactions in the order of $[3^d\delta V]_L$. Moreover, for long range interactions there will be no such rescaling, and thus we expect a term in the order of $[V]'_L$.

We will now make this argument precise.

*Proof.* First, note that $|\mathcal{E}_V| \le \mathcal{E}_{|V|}$, and $0 \le [|V|]_L = [V]_L$, and similarly $0 \le [|V|]'_L = [V]'_L$. Thus, we can assume without loss of generality that $V$ is non-negative.

Now fix $L, \delta$, and $N$. Note that for every $z^N \in A_{L,\delta}$ it holds that $\alpha_L(z^N) < \delta$, and hence there exist a countable set of non-negative integers $n_k := Nz^N(C_k)$ for all $k$, with the restriction that the configuration $\{n_k\} \in \mathcal{N}$, with

$$\mathcal{N} := \Big\{ \{n_k\} \,\Big|\, \sup_k n_k < \delta N,\, \sum_k n_k = N \Big\}. \tag{5.4.28}$$

Moreover, note that for any $z^N$ the set of $n_k$ such that $n_k > 0$ is finite.

We can write the exponential estimate of (5.4.29) into a sum over all possible configurations $\{n_k\}$. Namely, recall that

$$[\![\mathcal{E}_V]\!]_{A_{L,\delta}} = \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N\Big[ e^{N|\mathcal{E}_V|(z^N)}\, 1_{A_{L,\delta}} \Big], \tag{5.4.29}$$

and

$$N\mathcal{E}_V(z^N) = \frac{1}{N} \sum_{i,j\neq i} V(X_i; X_j), \tag{5.4.30}$$

where the sum is over all $N$ i.i.d. variables $X_i$, which have common law $\mu_0$. Hence, because of this independence and exchangeability,

$$\begin{aligned}
&\mathbb{E}^N\Big[e^{N\mathcal{E}_V(z^N)}\,1_{A_{L,\delta}}\Big]\\
&=\sum_{\{n_k\}\in\mathcal{N}}\mathbb{E}^N\Big[e^{N\mathcal{E}_V(z^N)}\,1_{z^N(C_k)=n_k,\forall k}\Big]\\
&=\sum_{\{n_k\}\in\mathcal{N}}\frac{N!}{\prod_k n_k!}\prod_k\mu_0(C_k)^{n_k}\,\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{1}{N}\sum_{k,k'}\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i};X_{k',j})\Big)\Big].
\end{aligned}\tag{5.4.31}$$

Here the sum in the exponent is shorthand for

$$\sum_k\sum_{1\le i\le n_k,1\le j\le n_k|i\neq j}V(X_{k,i};X_{k,j})+\sum_{k,k'}\sum_{1\le i\le n_k,1\le j\le n_{k'}\,|\,k\neq k'}V(X_{k,i};X_{k',j}),\tag{5.4.32}$$

and the expectation $\tilde{\mathbb{E}}^{\{n_k\}}$ is over all the $\Lambda$-valued independent random variables $X_{k,i}$, where for every $k$ the $X_{k,i}$, with $1\le i\le n_k$, are i.i.d. variables conditioned to be in the set $C_k$, with the common renormalized law $\tilde{\mu}_{0,k}$ defined by

$$d\tilde{\mu}_{0,k}=\frac{d\mu_0}{\mu_0(C_k)}.\tag{5.4.33}$$

Now, we will prove that for any fixed configuration $\{n_k\}\in\mathcal{N}$ we have that

$$\frac{1}{N}\log\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{1}{N}\sum_{k,k'}\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i}\,;\,X_{k',j})\Big)\Big]\le\frac{1}{4}[4\delta V]_L+\frac{1}{4}[4\cdot 3^d\delta V]_L+\frac{1}{2}[2V]'_L.\tag{5.4.34}$$

Hence, by taking the supremum over all configurations $\{n_k\}\in\mathcal{N}$ in (5.4.31),

$$\begin{aligned}
&\mathbb{E}^N\Big[e^{N\mathcal{E}_V(z^N)}\,1_{A_{L,\delta}}\Big]\\
&\le\sum_{\{n_k\}\in\mathcal{N}}\frac{N!}{\prod_k n_k!}\prod_k\mu_0(C_k)^{n_k}\sup_{\{n_k\}\in\mathcal{N}}\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{1}{N}\sum_{k,k'}\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i}\,;\,X_{k',j})\Big)\Big]\\
&=P^N\big(z^N\in A_{L,\delta}\big)\sup_{\{n_k\}\in\mathcal{N}}\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{1}{N}\sum_{k,k'}\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i}\,;\,X_{k',j})\Big)\Big]\\
&\le\exp\Big\{N\Big(\frac{1}{4}[4\delta V]_L+\frac{1}{4}[4\cdot 3^d\delta V]_L+\frac{1}{2}[2V]'_L\Big)\Big\}.
\end{aligned}\tag{5.4.35}$$

From this Theorem 5.10 quickly follows, since by taking limits,

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[e^{N\mathcal{E}_V(z^N)}\,1_{A_{L,\delta}}\Big]\le\Big(\frac{1}{4}[4\delta V]_L+\frac{1}{4}[4\cdot 3^d\delta V]_L+\frac{1}{2}[2V]'_L\Big).\tag{5.4.36}$$

Henceforth, let the configuration $\{n_k\}$ be fixed. To show that (5.4.34) holds, we will separate out the part with short-range interactions. Namely, recall that $\bar{\Lambda}_L^2$ is the union of all ordered couples of cubes $(C_i,C_j)$ that are either equal or neighbour each other. Then,

$$\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i}\,;\,X_{k',j})=\sum_{(C_k,C_{k'})\subset\bar{\Lambda}_L^2}\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i}\,;\,X_{k',j})+\sum_{(C_k,C_{k'})\subset\Lambda^2/\bar{\Lambda}_L^2}\sum_{X_{k,i},X_{k',j}}V(X_{k,i};X_{k',j}).\tag{5.4.37}$$

Using Hölder's inequality,

$$\begin{aligned}
\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{1}{N}\sum_{(k,k')}\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i}\,;\,X_{k',j})\Big)\Big]&\le\left(\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{2}{N}\sum_{(C_k,C_{k'})\subset\bar{\Lambda}_L^2}\sum_{X_{k,i}\neq X_{k',j}}V(X_{k,i}\,;\,X_{k',j})\Big)\Big]\right)^{1/2}\\
&\cdot\left(\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{2}{N}\sum_{(C_k,C_{k'})\subset\Lambda^2/\bar{\Lambda}_L^2}\sum_{X_{k,i},X_{k',j}}V(X_{k,i}\,;\,X_{k',j})\Big)\Big]\right)^{1/2}.
\end{aligned}\tag{5.4.38}$$

We will treat both exponential estimates via another separation argument, where we factorize them into estimates over every cube $C_k$ separately

Namely, recall, $\hat{C}_k$ is the union of $C_k$ and all the cubes that neighbour $C_k$, and $\hat{n}_k := z^N(\hat{C}_k)$. We will show that for every cube $C_k$ we have that

$$\sup_{\{y_j \in \hat{C}_k / C_k\}} \tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\Big\{\frac{2}{N}\Big(\sum_{i,j} V(X_{k,i}\,;\,y_j) + \sum_{i,i'\neq i} V(X_{k,i}\,;\,X_{k,i'})\Big)\Big\}\right] \leq \exp\Big\{\frac{n_k}{2}\Big([4\cdot 3^d \delta V]_L + [4\delta V]_L\Big)\Big\}, \tag{5.4.39}$$

where the supremum is over all collections $\{y_j\}$ with $y_j \in \hat{C}_k/C_k$ and $1 \leq j \leq \hat{n}_k - n_k$, and

$$\sup_{\{y_j \in \Lambda/\hat{C}_k\}} \tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big\{\frac{2}{N}\sum_{i,j} V(X_{k,i}\,;\,y_j)\Big\}\Big] \leq \exp\Big\{n_k[2V]'_L\Big\}, \tag{5.4.40}$$

where the supremum is over all collections $\{y_j\}$ with $y_j \in \Lambda/\hat{C}_k$ and $1 \leq j \leq N - \hat{n}_k$.

Recall that $X_{k,i}$ and $X_{k',j}$ are all independent when $k \neq k'$ (or when $k = k'$ but $i \neq j$). Hence, applying Lemma 5.11, we can subsequently integrate and separate out all cubes $C_k$, and it follows that,

$$\begin{aligned}
&\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{2}{N}\sum_{(C_k,C_{k'})\subset\bar{\Lambda}_L^2}\ \sum_{X_{k,i}\neq X_{k',j}} V(X_{k,i};X_{k',j})\Big)\Big] \\
&\quad\leq \exp\Big\{\frac{n_1}{2}\Big([4\delta V]_L + [4\cdot 3^d\delta V]_L\Big)\Big\}\cdot \tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{2}{N}\sum_{k,k'>1,(C_k,C_{k'})\subset\bar{\Lambda}_L^2}\ \sum_{X_{k,i}\neq X_{k',j}} V(X_{k,i};X_{k',j})\Big)\Big] \\
&\quad\leq \prod_k \exp\Big\{\frac{n_k}{2}\Big([4\delta V]_L + [4\cdot 3^d\delta V]_L\Big)\Big\} \\
&\quad= \exp\Big\{\frac{N}{2}\Big([4\delta V]_L + [4\cdot 3^d\delta V]_L\Big)\Big\}.
\end{aligned} \tag{5.4.41}$$

Similarly,

$$\begin{aligned}
&\tilde{\mathbb{E}}^{\{n_k\}}\Big[\exp\Big(\frac{2}{N}\sum_{(C_k,C_{k'})\subset\Lambda^2/\bar{\Lambda}_L^2}\ \sum_{X_{k,i}\neq X_{k',j}} V(X_{k,i};X_{k',j})\Big)\Big] \\
&\quad\leq \prod_k \exp\Big\{n_k[2V]'_L\Big\} \\
&\quad= \exp\Big\{N[2V]'_L\Big\},
\end{aligned} \tag{5.4.42}$$

and hence (5.4.34) follows.

To establish (5.4.39) and (5.4.40) we employ a similar but localized version of the factorization argument as used in Section 4.4, where we showed that $[\![\mathcal{E}_V]\!]_{\mathcal{X}} \leq [V]$.

First, we split the expectation of (5.4.39) into two parts using Hölder's inequality, one involving only interactions with $\hat{C}_k/C_k$, and one with all the self-interactions of $C_k$. For the latter part, similar to the factorization argument named above, we use the generalized Hölder's inequality with exponent $(n_k - 1)$. Moreover, recall that the variables $X_{k,i}$, $q \leq i \leq n_k$, are independent with common law $\tilde{\mu}_{0,k}$. Hence,

$$\begin{aligned}
\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{1}{N}\sum_{i,j\neq i}V(X_{k,i}\,;\,X_{k,j})\right\}\right] &= \tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{1}{N}\sum_{j\neq i}\left(\sum_i V(X_{k,i}\,;\,X_{k,j})\right)\right\}\right] \\
&\leq \tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{n_k-1}{N}\sum_{i\neq 1}V(X_{k,i}\,;\,X_{k,1})\right\}\right] \\
&= \int_{C_k}\left(\int_{C_k} e^{\frac{n_k-1}{N}V(x_1;x_2)}\,d\tilde{\mu}_{0,k}(x_1)\right)^{n_k-1}\tilde{\mu}_{0,k}(x_2) \\
&\leq \left(\sup_{y\in C_k}\int_{C_k} e^{\delta V(x;y)}\,d\tilde{\mu}_{0,k}(x)\right)^{n_k} \\
&\leq \exp\left\{\,n_k[\delta V]_L\,\right\}.
\end{aligned} \tag{5.4.43}$$

Here we used the non-negativity of $V$ and the fact that $n_k \leq \delta N$ by assumption of $\{n_k\}\in\mathcal{N}$, which arose from $z^N\in A_{L,\delta}$.

Similarly, for the first part of (5.4.39),

$$\begin{aligned}
\sup_{\{y_j\in\hat{C}_k/C_k\}}\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{1}{N}\sum_{i,j}V(X_{k,i}\,;\,y_j)\right\}\right] &= \sup_{\{y_j\in\hat{C}_k/C_k\}}\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{1}{N}\sum_j\left(\sum_i V(X_{k,i}\,;y_j)\right)\right\}\right] \\
&\leq \sup_{y\in\tilde{C}_k}\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{\hat{n}_k}{N}\sum_{i\neq 1}V(X_{k,i}\,;y)\right\}\right] \\
&= \sup_{y\in\tilde{C}_k}\left(\int_{C_k} e^{\frac{\hat{n}_k}{N}V(x;y)}\,d\tilde{\mu}_{0,k}(x)\right)^{n_k} \\
&\leq \left(\sup_{y\in\tilde{C}_k}\int_{C_k} e^{3^d\delta V(x;y)}\,d\tilde{\mu}_{0,k}(x)\right)^{n_k} \\
&= \exp\left\{n_k[3^d\delta V]_L\right\}.
\end{aligned} \tag{5.4.44}$$

For the second inequality of above, recall that $\hat{n}_k = z^N(\hat{C}_k)$, which is the sum over all the particles of $C_k$ and the cubes $C_{k'}$ that neighbour $C_k$. Since for each cube $C_{k'}$ it holds by assumption that $n_{k'}\leq\delta N$, and since there are at most $3^d-1$ neighbouring cubes, it follows that $\hat{n}_k\leq 3^d\delta$.

Thus, combining the two estimate we derive (5.4.39), since by Hölder's inequality,

$$\begin{aligned}
&\sup_{\{y_j\}}\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{1}{N}\left(\sum_{i,j}V(X_{k,i}\,;\,y_j)+\sum_{i,j\neq i}V(X_{k,i}\,;\,X_{k,j})\right)\right\}\right] \\
&\quad= \left(\sup_{\{y_j\}}\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{2}{N}\sum_j\left(\sum_i V(X_{k,i}\,;y_j)\right)\right\}\right]\right)^{1/2}\left(\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{2}{N}\sum_{i,j\neq i}V(X_{k,i}\,;\,X_{k,j})\right\}\right]\right)^{1/2} \\
&\quad\leq \exp\left\{\,\frac{n_k}{2}\left([\delta V]_L+[2\cdot 3^d\delta V]_L\right)\right\}.
\end{aligned} \tag{5.4.45}$$

Finally, concluding the proof, we establish (5.4.40) in a similar way,

$$\begin{aligned}
\sup_{\{y_j\in\Lambda/\hat{C}_k\}}\tilde{\mathbb{E}}^{\{n_k\}}\left[\exp\left\{\frac{1}{N}\sum_{i,j}V(X_{k,i}\,;\,y_j)\right\}\right] &\leq \sup_{y\in\Lambda/\tilde{C}_k}\left(\int_{C_k} e^{\frac{N-\hat{n}_k}{N}V(x;y)}\,d\tilde{\mu}_{0,k}(x)\right)^{n_k} \\
&\leq \left(\sup_{y\in\Lambda/\tilde{C}_k}\int_{C_k} e^{V(x;y)}\,d\tilde{\mu}_{0,k}(x)\right)^{n_k} \\
&\leq \exp\left\{\,n_k[V]'_L\,\right\}.
\end{aligned} \tag{5.4.46}$$

□

Using Lemma 5.12, we can now find suitable open sets to apply Lemma 3.21, and finally prove Theorem 5.10.

*Proof of Theorem 5.10.* Recall the sufficient condition for quasi-convergence stated in Lemma 3.21, which, phrased in the framework of this section, states that quasi-continuity of $\mathcal{E}_V$ follows if there exists a constant $K$ such that for every $\mu^* \in \mathcal{P}(\Lambda)$ with $R(\mu^*\|\mu_0) < \infty$ and every $\beta^* \in \mathbb{R}$ there exists a open neighbourhood $A^* \subset \mathcal{P}(\Lambda)$ of $\mu^*$ such that

$$\limsup_{\lambda\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}^N\Big[\ e^{\beta^* N|(\mathcal{E}_V-\mathcal{E}_{V_\lambda})|(z^N)}\,1_{A^*}\ \Big] \le K. \tag{5.4.47}$$

Hence, if we can find such a set $A^*$ for every $\mu^*$ and every $\beta^*$ we have proven Theorem 5.10, since the statement that $[\![\beta\mathcal{E}_V]\!]_{\mathcal{X}} < \infty$ for all $\beta$ with $|\beta| < \beta_0$ follows from the assumption of (5.4.5) and convexity of $[V]_L$.

We will first show that the convergence of (5.4.47) holds with $K = 0$ on the set $A_L^* := A_{L,(1+\epsilon)\alpha_L(\mu^*)}$, for some $\epsilon > 0$ and some small enough $L$. Finally, while the set $A_L^*$ is not open, we will show that for every $L > 0$ it contains an open neighbourhood of $\mu^*$, concluding the proof.

Now, first, because of the compactness of $\Lambda$, with $\Lambda \subset \mathbb{R}^d$, the number of half-open $\mathbb{R}^d$-cubes $C_k$ covering $\Lambda$ is finite for every $L$, say $K$, and hence

$$\alpha_L(\mu) = \max_{1\le k\le K}\mu(C_i). \tag{5.4.48}$$

Now, fix any $\mu^* \in \mathcal{P}(\Lambda)$ with $R(\mu^*\|\mu_0) < \infty$. In particular $\mu^* \ll \mu_0$, and is therefore absolutely continuous with respect to the Lebesgue measure since $\mu_0$ is assumed to be, and hence, as mentioned previously, $\alpha_L(\mu^*) \to L$ as $L \to 0$.

Moreover, fix some $\epsilon > 0$, and define for every $L$ the set $A_L^* \subset \mathcal{P}(\Lambda)$ by

$$A_L^* := A_{L,(1+\epsilon)\alpha_L(\mu^*)}, \tag{5.4.49}$$

or,

$$A_L^* = \Big\{\mu \in \mathcal{P}(\Lambda)\ \Big|\ \sup_{1\le k\le K}\mu(C_k) < (1+\epsilon)\sup_{1\le k\le K}\mu^*(C_k)\Big\}. \tag{5.4.50}$$

Moreover, fix any $\beta^* \in \mathbb{R}$. Then, by Lemma 5.12 and linearity of $\mathcal{E}_V$ in $V$, for any $L > 0$,

$$\begin{aligned}
[\![\beta^*(\mathcal{E}_V - \mathcal{E}_{V_\lambda})]\!]_{A_L^*} &= [\![\mathcal{E}_{\beta^*(V-V_\lambda)}]\!]_{A_L^*}\\
&\le \frac{1}{2}\Big[4(1+\epsilon)\alpha_L(\mu^*)\beta^*(V-V_\lambda)\Big]_L + \frac{1}{2}\Big[4\cdot 3^d(1+\epsilon)\alpha_L(\mu^*)\beta^*(V-V_\lambda)\Big]_L\\
&\quad + \frac{1}{2}\Big[2\beta^*(V-V_\lambda)\Big]_L'.
\end{aligned} \tag{5.4.51}$$

Since $\alpha_L(\mu^*) \to L$ as $L \to 0$, it follows that there exists some small enough $L^* > 0$ such that $4\cdot 3^d(1+\epsilon)\alpha_{L^*}(\mu^*) < \beta_0$, in which case by assumption of (5.4.5) and (5.4.6)

$$\lim_{\lambda\to 0}[\![\beta^*(\mathcal{E}_V - \mathcal{E}_{V_\lambda})]\!]_{A_L^*} = 0. \tag{5.4.52}$$

Hence, the only remaining step is to show that for every $L$, $A_L^*$ contains a open set containing $\mu_0^*$.

Now, note that for any $A_{L,\delta}$,

$$\begin{aligned}
A_{L,\delta} &:= \Big\{\mu \in \mathcal{P}(\Lambda)\ \Big|\ \alpha_L(\mu) < \delta\Big\}\\
&= \Big\{\mu \in \mathcal{P}(\Lambda)\ \Big|\ \max_{1\le k\le K}\mu(C_k) < \delta\Big\}\\
&= \Big\{\mu \in \mathcal{P}(\Lambda)\ \Big|\ \int_\Lambda 1_{C_k}\,d\mu < \delta,\ \text{for all } 1\le k\le K\Big\}\\
&\supset \Big\{\mu \in \mathcal{P}(\Lambda)\ \Big|\ \int_\Lambda f_k(x)\,d\mu(x) < \delta,\ \text{for all } 1\le k\le K\Big\},\\
&= \bigcap_{1\le k\le K}\Big\{\mu \in \mathcal{P}(\Lambda)\ \Big|\ \int_\Lambda f_k(x)\,d\mu(x) < \delta\Big\}.
\end{aligned} \tag{5.4.53}$$

for all collections of $K$ continuous, bounded and non-negative functions $f_k : \Lambda \to R$ such that $f_k = 1$ on $C_k$.

The sets determined by $f_i$ are open, since they are the intersection over a finite number of open sets, even though $A_{L,\delta}$ is not open itself.

Now, again consider $A_L^* = A_{L,(1+\epsilon)\alpha_L(\mu^*)}$, and note that for any $L$ and any $k$ that

$$\int_\Lambda 1_{C_k}(x)\, d\mu^*(x) < (1+\epsilon)\alpha_L(\mu^*). \tag{5.4.54}$$

Therefore, because of the absolute continuity of $\mu^*$ with respect to the Lebesgue measure, we can find some non-negative, continuous and bounded function $f_k$ such that

$$\int_\Lambda f_k(x)\, d\mu^*(x) < (1+\epsilon)\alpha_L(\mu^*), \tag{5.4.55}$$

which implies that there exists a set of $K$ non-negative, continuous and bounded functions $f_k$ such that

$$\mu^* \in \bigcap_{1\le k\le K} \Big\{ \mu \in \mathcal{P}(\Lambda) \,\Big|\, \int_\Lambda f_k(x)\, d\mu(x) < \alpha_L(\mu^*) \Big\}, \tag{5.4.56}$$

by which it follows that $A_L^*$ contains a open neighbourhood of $\mu^*$, and Theorem 5.10 follows. □

**Comments**

**Remark 5.2.** Recall that Lemma 5.12 applies to countable collections of cubes $C_k$, and hence also non-compact $\Lambda_k$. Therefore, it is interesting to discuss how the requirement for compact $\Lambda$ in Theorem 5.10 and 5.6 could be lifted appropriately.

In the case for Theorem 5.10, the restriction to compact $\Lambda$ only arises since we do not know a priori if $A_{L,\delta}$ contains any appropriate open sets. Namely, the set defined by (5.4.53),

$$\bigcap_k \Big\{ \mu \in \mathcal{P}(\Lambda) \,\Big|\, \int_\Lambda f_k(x)\, d\mu(x) < \delta \Big\}, \tag{5.4.57}$$

is now an intersection over a countable family of open sets, and hence maybe not open itself. However, if it possible to show that these sets are actually open, Theorem 5.10 does apply.

Finally, the restriction to compact $\Lambda$ in Theorem 5.6 is because then $[V - V_\lambda]_L$ can be written as a maximum over a *finite number* of exponential estimates that all converge to zero — instead of the supremum over a countable collection of estimates — where the convergence for every $k$ follows from being bounded for every by a function composed of $\mu(C_k)$ and $[V - V_\lambda]$. Recall,

$$[V - V_\lambda]_L = \sup_k \sup_{y\in\hat{C}_k} \log \int_{C_k} e^{|V-V_\lambda|(x;y)}\, d\tilde{\mu}_{0,k}(x), \tag{5.4.58}$$

with $\tilde{\mu}_{0,k}$ the renormalized measure of $\mu_0$ on $C_k$, and where for every $k$ the exponential estimates go to zero as $\lambda \to 0$.

Hence, one can imagine that in the case for non-compact $\Lambda$, for certain $V, V_\lambda$ and with certain regularity restrictions on $\mu_0$ — i.e. such that $V$ and $\mu_0$ can be bounded appropriately on $C_k$ and uniformly in $k$ — Theorem 5.6 still holds if 5.10 holds for general $\Lambda$.

**Remark 5.3.** While in this section we only discussed $V$ that is critical near the $x_1$-diagonal, it is clear that the proof of Lemma 5.12 and Theorems 5.10 can be extended for any critical singularity that can be *localized*.

Namely, let $M_L$ for any $L > 0$ be the number such that for every $k$ there are at most $M_L$ cubes $C_{k'}$ such that the interaction between $C_k$ and $C_{k'}$ is critical, i.e. that

$$\sup_{y\in C_{k'}} \log \int_{C_k} e^{\beta|V|(x;y)}\, d\tilde{\mu}_{0,k}(x) = \infty, \tag{5.4.59}$$

for some $\beta \in \mathbb{R}$.

Then a clear inspection of the proof of Theorems 5.10 reveals that if $M_L$ is uniformly bounded in $L$, i.e. $M_L < M^*$ for some $M^*$, one can still prove quasi-continuity under certain restrictions. Note that in the case of Theorem 5.10 we have $M^* \leq 3^d$.

**Remark 5.4.** In the discretization technique used by [BG99], which was the inspiration for the results of this section, they proved (weak) quasi-continuity of $V$ for compact $\Lambda \subset \mathbb{R}^2$, with $\mu_0$ constant on $\Lambda$, $V := \log|x-y|$, and $V_L = V$ for $|x-y| \geq L$ and constant otherwise. In particular, $[V - V_L]'_L = 0$ in their case.

However, in proving a localized form of convergence, the function $V_L$ and corresponding set $A_L$ were set to the same parameter $L$, instead of separating it into $\lambda$ and $L$. Therefore, and also due to the asymptotic nature of their estimates and dependence on $|\Lambda|$, they had go extra lengths in showing that the end result actually converged to zero. However, as shown in Theorem 5.6, only pointwise convergence of and exponential integrability is necessary, which in this particular case is trivial.

**Remark 5.5.** Recall from Remark 3.4 about local quasi-boundedness that sometimes there exists a critical inverse temperature $\beta^*$ for every set $A$, i.e. a $\beta^* > 0$ such that $[\![\beta\mathcal{E}]\!]_A < \infty$ for all $0 < \beta < \beta^*$, but with $[\![\beta\mathcal{E}]\!]_A = \infty$ for $\beta \geq \beta^*$. We will now give a implicit characterization of such sets and critical temperatures.

In particular, consider the case of $\Lambda \subset \mathbb{R}^2$ compact, $V = \log|x-y|$, $\mu_0$ is constant, and the critical inverse temperature $\beta_0 = 2$. Then we have the following upper bound for the critical inverse temperature $\beta^*$ of any open set $A$,

**Lemma 5.13.**

$$\frac{\beta^*}{\beta_0} \leq \inf\left\{ n \,\middle|\, \exists\, x_1, \dots x_n : \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i} \in A \right\}. \tag{5.4.60}$$

Note that in the case where $A = \mathcal{P}(\Lambda)$ we recover the trivial fact that for $\mathcal{P}(\Lambda)$ it holds that $\beta^* \leq \beta_0$.

*Line of proof.* We will only sketch the arguments.

First, recall that for any $\mu \in \mathcal{P}(\Lambda)$ with $R(\mu\|\mu_0) < \infty$, it holds that $\mu$ is absolutely continuous with respect to the Lebesgue measure, and hence $\alpha_L(\mu) \to 0$ as $L \to 0$, which is key in the proof of Theorem 5.10.

Then by Lemma 5.12, for any $\mu$ with $R(\mu\|\mu_0) < \infty$ and $\beta \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} [\![\beta\mathcal{E}_V]\!]_{B_\epsilon(\mu)} < \infty, \tag{5.4.61}$$

i.e., in the framework of Section 3, $\mathcal{E}$ is locally quasi-bounded at $\mu$.

However, Lemma 5.12 and the proof of Theorem 5.10 imply that for *any* $\mu$ — even one that is not absolutely continuous with respect to the Lebesgue measure — and any $\beta$ with

$$|\beta| \leq \frac{\beta_0}{4 \cdot 3^d \alpha_{0^+}(\mu)}, \tag{5.4.62}$$

where $\alpha_{0^+}(\mu) := \lim_{L\to 0} \alpha_L(\mu)$, it holds that

$$\lim_{\epsilon \to 0} \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}^N \left[ e^{\beta|\mathcal{E}_V|(z^N)|} \,\middle|\, B_\epsilon(\mu) \right] < \infty. \tag{5.4.63}$$

Now suppose $\mu$ is the sum over some finite number, say $n$, of Dirac measures, i.e. $\mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ where all $x_i$ are different. It can then even be shown that the factor $4 \cdot 3^d$ can be dropped, that $\alpha_{0^+} = \frac{1}{n}$, and that the term blows up for $\beta \geq n\beta_0$.

Heuristically, as $\epsilon \to 0$, any empirical measure $z^N \in B_\epsilon(\mu)$ contains $n$ sub-systems, clustered around the different $x_i$, and hence the for every sub-system the scaling is effectively $\frac{n}{N}$ instead of $\frac{1}{N}$, making the critical inverse temperature on $B_\epsilon(\mu)$ with $\epsilon$ small enough scale as $n\beta_0$ instead of $\beta_0$.

Therefore, since if the limit of (5.4.63) is infinite as $\epsilon \to 0$ implies that it is infinite for every $A$ such that $\mu \in A$, and (5.4.60) follows. □

**Remark 5.6.** Finally, recall that in Lemma 5.12 that the bound contains both $\frac{1}{2}[4\delta V]_L$ — which arises from the self-interaction of cubes $C_k$, see (5.4.43) — and $\frac{1}{2}[4 \cdot 3^d \delta V]_L$, which arises from the interaction between cubes $C_k$ and neighbouring cubes $C_{k'}$, see (5.4.44).

As is clear from the proof, this distinction is necessary, and one can not just simply bound

$$\begin{aligned}
&\sup_{\{y_j \in \hat{C}_k / C_k\}} \tilde{\mathbb{E}}^{\{n_k\}}\Bigg[\exp\Bigg\{\frac{1}{2N}\Bigg(\sum_{i,j} V(X_{k,i}\,;\,y_j) + \sum_{i,i'\neq i} V(X_{k,i}\,;\,X_{k,i'})\Bigg)\Bigg\} bigg]\\
&\qquad \overset{?}{\leq} \sup_{\{y_j \in \hat{C}_k\}} \tilde{\mathbb{E}}^{\{n_k\}}\Bigg[\exp\Bigg\{\frac{1}{2N}\sum_{i,j} V(X_{k,i}\,;\,y_j)\Bigg\}\Bigg] \\
&\qquad \leq \exp\Big\{\frac{n_k}{2}\Big([2 \cdot 3^d \delta\Big)\Big\}.
\end{aligned} \tag{5.4.64}$$

This, combined with the fact that the estimate requires a supremum, is one of the reasons why the separation argument of (5.4.41) becomes quite involved for interactions other than pair-interactions. Although we did manage to derive an estimate similar to Lemma 5.12 for 3-particle interactions with certain restrictions on $V$, the proof involved a large number of ad-hoc steps and was far from insightful, and we will not repeat it here.

However, recall the context of Theorem 4.10, which only concerned questions about the rate functions $\mathcal{E}_V + I$ on $D(I)$ and therefore sidesteps many of the technical issues involved with $[\![\cdot]\!]_{\mathcal{X}}$ and $[V]_\lambda$, and only needed exponential integrability of $V$ in all its arguments. In the next section we will do something similar, and show how local convergence purely in terms of rate functions can be established for a wide variety of multi-particle potentials.

## 5.5 Critical singularities: properties of rate functions

Recall Theorem 4.10 of Section 4.5, in which we stated sufficient conditions on $V$ for $\mathcal{E}_V$ to be continuous on the sublevel sets of $I$ and hence $J_{\beta V} = \beta\mathcal{E}_V + I$ to locally be proper rate functions up to a constant — i.e. lower semi-continuous with compact sublevel sets. The latter is itself a necessary condition for $\mathcal{E}_V$ to be quasi-continuous.

Now, in this section we will give a localized version of Theorem 4.10 when $V$ is critical — i.e. it has a logarithmic singularity, drawing on the same discretization procedure as Section 5.4. Similar to Theorem 4.10, for the treatment of rate functions we do not run into the technical problems of the stochastic estimates of Section 5.4, and hence we can consider multi-particle systems instead of merely one with pair interaction.

Recall $\Lambda \subseteq \mathbb{R}^d$, and let $\Lambda^k_L \subset \Lambda^k$ for any $L > 0$ be the set

$$\Lambda^k_L := \Big\{ (x_1, \dots, x_k) \in \Lambda^k \;\Big|\; \exists i, j : |x_i - x_j| \leq L \Big\}. \tag{5.5.1}$$

Moreover, for any $L > 0$ we decomposition $\Lambda$ into a countable collection of $\mathbb{R}^d$-cubes $\{C_i\}$ with side-length $L$. Let $\bar{\Lambda}^k_L$ be the union of all the ordered sets of cubes $C_1 \times \dots \times C_k$ such that $C_i = C_j$ or $C_i$ neighbours $C_j$ for some $1 \leq i, j \leq k$. Note that by simple geometry it follows that $\Lambda^k_L \subset \bar{\Lambda}^k_L \subset \Lambda^k_{3\sqrt{d}L}$.

We then define following analogues of $[V]_L$ and $[V]'_L$,

$$\langle V \rangle_L := \sup_{(C_1, \dots, C_k) \subset \bar{\Lambda}^k_L} \log \int_{C_1 \times \dots \times C_k} e^{|V|(x_1, \dots, x_k)}\, d\tilde{\mu}_{0,1}(x_1) \cdots d\tilde{\mu}_{0,k}(x_k), \tag{5.5.2a}$$

$$\langle V \rangle'_L := \log \int_{\Lambda^k/\Lambda^k_L} e^{|V|(x_1, \dots, x_k)}\, \frac{d\mu_0(x_1) \cdots \mu_0(x_k)}{(\mu_0^{\otimes k})(\Lambda^k/\Lambda^k_L)}, \tag{5.5.2b}$$

where, as before, $\mu_{0,i}$ are the renormalized measures on $C_i$ defined by

$$d\tilde{\mu}_{0,k} = \frac{d\mu_0}{\mu_0(C_i)}. \tag{5.5.3}$$

Note that $\langle V \rangle_L$ and $\langle V \rangle'_L$ are non-negative, and satisfy some of the same properties as $[V]_L$ and $[V]'_L$, and of $\langle V \rangle_\Lambda$ of Section 4.5, although we will not repeat those here.

We now have the following result.

**Theorem 5.14.**
*Let $\Lambda$ be compact, $V_\lambda : \Lambda^k \to \mathbb{R}$ be continuous and bounded for all $\lambda$, and consider $V : \Lambda^k \to \mathbb{R}$.*

*Suppose that there exists a $\beta^* > 0$, such that for all $L > 0$,*

$$\lim_{\lambda \to 0} \langle \beta^*(V - V_\lambda) \rangle_L = 0, \tag{5.5.4}$$

*and additionally, that for each $\beta \in \mathbb{R}$ and $L > 0$,*

$$\lim_{\lambda \to 0} \langle \beta(V - V_\lambda) \rangle'_L = 0. \tag{5.5.5}$$

*Then $\mathcal{E}$ is continuous on the sets $\{\, \mu \,|\, R(\mu \| \mu_0) \leq M \,\}$, and hence $J_{\beta V}$ is lower semi-continuous at all $\mu$ with $R(\mu\|\mu_0) < \infty$.*

*Moreover, for any $\beta \in \mathbb{R}$ with $-\frac{\beta^*}{k} < \beta < \frac{\beta^*}{k}$, $J_{\beta V}$ is lower semi-continuous with compact sublevel sets and with $|J_{\beta V}(\mathcal{X})| < \infty$.*

The proof requires some highly technical and — frankly — quite convoluted estimates, but recall from Theorem 5.8 and especially Corollary 5.9 that the verification of the above conditions itself can be quite easy — stated merely in terms of pointwise convergence of $V_\lambda \to V$ almost everywhere, and exponential bounds.

This is in contrast to the problem of establishing quasi-continuity as done in Section 5.4, which arises from the fact that $[V]_L$ requires a supremum over some points $y \in \Lambda$, while $\langle V \rangle_L$ does not.

Now, we will first show how Theorem 5.14 implies Theorem 5.8.

*Proof of Theorem 5.8.* The proof is highly similar to the proof of Theorem 5.6 following from Theorem 5.10, see Section 5.4, and therefore we will only briefly repeat the relevant steps.

Recall, by assumption there exists a $\beta^*$ such that,

$$\lim_{\lambda \to 0} \log \int_{\Lambda^k} e^{\beta^* |V - V_\lambda|(x_1; x_2, \dots, x_k)} \, d\mu_0(x_1) \, \dots \, d\mu_0(x_k) < \infty \tag{5.5.6}$$

and that for all $\beta \in \mathbb{R}$ and all $L > 0$,

$$\limsup_{\lambda \to 0} \log \int_{\Lambda^k / \Lambda_L^k} e^{|V - V_\lambda|(x_1, \dots, x_k)} \, d\mu_0(x_1) \cdots \mu_0(x_k) \le 0, \tag{5.5.7}$$

where the latter implies (see the proof of Theorem 5.6),

$$\lim_{\lambda \to 0} \log \int_{\Lambda^k / \Lambda_L^k} e^{|V - V_\lambda|(x_1, \dots, x_k)} \, \frac{d\mu_0(x_1) \cdots \mu_0(x_k)}{(\mu_0^{\otimes k})(\Lambda^k / \Lambda_L^k)} = 0, \tag{5.5.8}$$

and (5.5.5) follows.

Now, note that for any collection of $k$ cubes $(C_1, \dots, C_k)$ that

$$\begin{aligned} &\lim_{\lambda \to 0} \int_{C_1 \times \cdots \times C_k} \left( e^{\beta |V - V_\lambda|(x_1, \dots, x_k)} - 1 \right) d\tilde{\mu}_{0,1}(x_1) \cdots d\tilde{\mu}_{0,k}(x_k) \\ &\quad \le \lim_{\lambda \to 0} \frac{1}{\mu_0^{\otimes k}(C_1 \times \cdots \times C_k)} \int_{\Lambda^k} \left( e^{\beta |V - V_\lambda|(x_1, \dots, x_k)} - 1 \right) d\mu_0(x_1) \cdots d\mu_0(x_k) \\ &\quad = 0, \end{aligned} \tag{5.5.9}$$

where the last step can be seen to follow from (5.5.6). Hence, since because of the compactness of $\Lambda$ there are only a *finite* number of cubes $C_i$,

$$\begin{aligned} \lim_{\lambda \to 0} \langle \beta (V - V_\lambda) \rangle_L &:= \lim_{\lambda \to 0} \max_{(C_1, \dots, C_k) \subset \bar{\Lambda}_L^k} \log \int_{C_1 \times \cdots \times C_k} e^{|V|(x_1, \dots, x_k)} \, d\tilde{\mu}_{0,1}(x_1) \cdots d\tilde{\mu}_{0,k}(x_k) \\ &= \max_{(C_1, \dots, C_k) \lim_{\lambda \to 0} \subset \bar{\Lambda}_L^k} \log \int_{C_1 \times \cdots \times C_k} e^{|V|(x_1, \dots, x_k)} \, d\tilde{\mu}_{0,1}(x_1) \cdots d\tilde{\mu}_{0,k}(x_k) \\ &= 0. \end{aligned} \tag{5.5.10}$$

□

Next, we will give a localized version of the entropic inequality stated in Lemma 4.11.

**Lemma 5.15.** *Consider $V : \mathcal{S}^k \to \mathbb{R}$, and $\mu, \mu_0 \in \mathcal{P}(S)$, with $\mu \ll \mu_0$.*

*Moreover, let $\{\mathcal{S}_i\}_i$ be a countable collection of measurable subsets of $\mathcal{S}$.*

*Then for any subset $\mathcal{S}_1$ with $\mu_0(\mathcal{S}_1) > 0$,*

$$0 \le \int_{\mathcal{S}_1} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_1) \log \frac{\mu(\mathcal{S}_1)}{\mu_0(\mathcal{S}_1)} \le R(\mu \| \mu_0), \tag{5.5.11}$$

*and moreover, if all $S_i$ are disjoint,*

$$0 \le \sum_i \left( \int_{\mathcal{S}_i} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_i) \log \frac{\mu(\mathcal{S}_i)}{\mu_0(\mathcal{S}_i)} \right) \le R(\mu \| \mu_0), \tag{5.5.12}$$

*where the sum $\sum_i$ is over all subsets $\mathcal{S}_i$ such that $\mu_0(\mathcal{S}_i) > 0$.*

*Finally, for any $k$ subsets $S_1, \ldots, S_k$, and any $\beta > 0$,*

$$\int_{\mathcal{S}_1 \times \cdots \times \mathcal{S}_k} |V(x_1, \ldots, x_k)| \, d\mu(x_1) \cdots \mu(x_k) \leq \frac{\mu(\mathcal{S}_1) \cdots \mu(\mathcal{S}_k)}{\beta} \sum_{j=1}^{k} \mu(\mathcal{S}_j) \left( \int_{\mathcal{S}_j} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_j) \log \frac{\mu(\mathcal{S}_j)}{\mu_0(\mathcal{S}_j)} \right) + \frac{\mu(\mathcal{S}_1) \cdots \mu(\mathcal{S}_k)}{\beta} \log \int_{\mathcal{S}_1 \times \cdots \times \mathcal{S}_k} e^{\beta |V|(x_1, \ldots, x_k)} \, d\tilde{\mu}_{0,1}(x_1) \cdots d\tilde{\mu}_{0,k}(x_k). \tag{5.5.13}$$

The desired consequence is of course that we will use this to bound $|\mathcal{E}_V|$ by decomposition. Namely, note that the two terms are positive, and hence we can easy see the following result.

**Corollary 5.16.** *Suppose $\mathcal{S} = \cup_i \mathcal{S}_i$, where the latter is the union over countable collection of subsets $\{\mathcal{S}_i \subseteq \mathcal{S}\}_i$, where the $\mathcal{S}_i$ are not necessarily disjoint.*

*Then for any collection of positive numbers $\{\beta_{i_1, \ldots, i_k}\}$*

$$\int_{\mathcal{S}^k} |V(x_1, \ldots, x_k)| \, d\mu(x_1) \cdots \mu(x_k) \leq \sum_{i_1, \ldots, i_k} \frac{\mu(\mathcal{S}_{i_1}) \cdots \mu(\mathcal{S}_{i_k})}{\beta_{i_1, \ldots, i_k}} \Big[ \sum_{j=1}^{k} \mu(\mathcal{S}_{i_j}) \left( \int_{\mathcal{S}_{i_j}} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_{i_j}) \log \frac{\mu(\mathcal{S}_{i_j})}{\mu_0(\mathcal{S}_{i_j})} \right) + \log \int_{\mathcal{S}_{i_1} \times \cdots \times \mathcal{S}_{i_k}} e^{\beta_{i_1, \ldots, i_k} |V|(x_1, \ldots, x_k)} \, d\tilde{\mu}_{0,i_1}(x_1) \cdots d\tilde{\mu}_{0,i_k}(x_k) \Big] \tag{5.5.14}$$

*Proof of Lemma 5.15.*
Recall from Lemma 4.11 and similar to (4.5.15) that for any $V : \mathcal{S} \to \mathbb{R}$, and any $\mu, \mu_0 \in \mathcal{P}(\mathcal{S})$.

$$\int_{\mathcal{S}} |V(x)| \, d\mu(x) \leq \Big( R(\mu \| \mu_0) + \log \int_{\mathcal{S}} e^{|V|(x)} \, d\mu_0(x) \Big). \tag{5.5.15}$$

We will repeatedly apply (5.5.15) to renormalized measures over any subset $\mathcal{S}_i$.

Namely, fix a subset $\mathcal{S}_1$ such that $\mu_0(\mathcal{S}_1) > 0$, and let $\tilde{\mu}, \tilde{\mu}_0 \in \mathcal{P}(\mathcal{S}_1)$ be the normalized measures with

$$\begin{aligned} d\tilde{\mu} &= \frac{d\mu}{\mu(\mathcal{S})}, \\ d\tilde{\mu}_0 &= \frac{d\mu_0}{\mu_0(\mathcal{S})}. \end{aligned} \tag{5.5.16}$$

Note that by assumption $\mu \ll \mu_0$, hence $\mu(\mathcal{S}_1) > 0$ and $\tilde{\mu} \ll \tilde{\mu}_0$, from which follows that

$$\begin{aligned} R(\tilde{\mu} \| \tilde{\mu}_0) &= \int_{\mathcal{S}_1} \log \frac{d\tilde{\mu}}{d\tilde{\mu}_0} d\tilde{\mu} \\ &= \int_{\mathcal{S}_1} \log \frac{d\mu/\mu(\mathcal{S}_1)}{d\mu_0/\mu_0(\mathcal{S}_1)} \, d\mu/\mu(\mathcal{S}_1) \\ &= \frac{1}{\mu(\mathcal{S}_1)} \Big( \int_{\mathcal{S}_1} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_1) \log \frac{\mu(\mathcal{S}_1)}{\mu_0(\mathcal{S}_1)} \Big). \end{aligned} \tag{5.5.17}$$

Since $R(\tilde{\mu} \| \tilde{\mu}_0) \geq 0$ by the properties of relative entropy, the positivity stated (5.5.11) and (5.5.12) follows.

Now, let $\{\mathcal{S}_j\}_j$ be a collection of disjoint subsets such that $\mathcal{S} = \cup_j \mathcal{S}_j$, then note that

$$\begin{aligned} \sum_j \int_{\mathcal{S}_j} \log \frac{d\mu}{d\mu_0} d\mu &= \int_{\mathcal{S}} \log \frac{d\mu}{d\mu_0} d\mu \\ &= R(\mu \| \mu_0), \\ \sum_j \mu(\mathcal{S}_j) \log \frac{\mu(\mathcal{S}_j)}{\mu_0(\mathcal{S}_j)} &= R\big( (\mu)_\Delta \, \big\| \, (\mu_0)_\Delta \big). \end{aligned} \tag{5.5.18}$$

where $(\mu)_\Delta$ is a discretization of measure $\mu$ on the subsets $S_j$, i.e. a discrete measure such that $(\mu)_\Delta(\mathcal{S}_j) = \mu(\mathcal{S}_j)$ for all $j$. Note that from positivity above we see an instance of the contraction property of relative entropy, see Appendix A of [Fis14], i.e.

$$R\big( (\mu)_\Delta \,\big\|\, (\mu_0)_\Delta \big) \le R(\mu\|\mu_0). \tag{5.5.19}$$

In particular, the sums involved are well defined, and hence we can state that if $R(\mu\|\mu_0) < \infty$,

$$\begin{aligned} 0 &\le \sum_j \Big( \int_{\mathcal{S}_j} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_j) \log \frac{\mu(\mathcal{S}_j)}{\mu_0(\mathcal{S}_j)} \Big) \\ &\le R(\mu\|\mu_0) - R\big( (\mu)_\Delta \,\big\|\, (\mu_0)_\Delta \big) \\ &\le R(\mu\|\mu_0), \end{aligned} \tag{5.5.20}$$

and moreover, for any collection $\{\mathcal{S}_i\}_i$ of disjoint subsets such that $\{\mathcal{S}_i\}_i \subset \{\mathcal{S}_j\}_j$, it follows that

$$0 \le \sum_j \Big( \int_{\mathcal{S}_j} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_j) \log \frac{\mu(\mathcal{S}_j)}{\mu_0(\mathcal{S}_j)} \Big) \le R(\mu\|\mu_0), \tag{5.5.21}$$

and hence we have proven (5.5.11) and (5.5.12), since the inequality is trivially true when $R(\mu\|\mu_0) = \infty$.

Now, for any $V : \mathcal{S} \to \mathbb{R}$, we apply (5.5.15) to the normalized measures $\tilde{\mu}$ and $\tilde{\mu}_0$,

$$\begin{aligned} \frac{1}{\mu(\mathcal{S}_1)} \int_{\mathcal{S}_1} |V(x)|\, d\mu(x) &= \int_{\mathcal{S}_1} |V(x)|\, d\tilde{\mu}(x) \\ &\le R(\tilde{\mu}\|\tilde{\mu}_0) + \log \int_{\mathcal{S}_1} e^{|V|(x)}\, d\tilde{\mu}_0(x) \\ &= \frac{1}{\mu(\mathcal{S}_1)} \Big( \int_{\mathcal{S}_1} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_1) \log \frac{\mu(\mathcal{S}_1)}{\mu_0(\mathcal{S}_1)} \Big) + \log \int_{\mathcal{S}_1} e^{|V|(x)}\, d\mu_0(x)/\mu_0(\mathcal{S}_1), \end{aligned} \tag{5.5.22}$$

and therefore

$$\int_{\mathcal{S}_1} |V(x)|\, d\mu(x) \le \Big( \int_{\mathcal{S}_1} \log \frac{d\mu}{d\mu_0} d\mu - \mu(\mathcal{S}_1) \log \frac{\mu(\mathcal{S}_1)}{\mu_0(\mathcal{S}_1)} \Big) + \mu(\mathcal{S}_1) \log \int_{\mathcal{S}_1} e^{|V|(x)}\, d\mu_0(x)/\mu_0(\mathcal{S}_1). \tag{5.5.23}$$

Similarly, for any $V : \mathcal{S}^k \to \mathbb{R}$ and $k$ subsets $S_1, \dots, S_k$, let $\tilde{\gamma}, \tilde{\gamma}_0 \in \mathcal{P}(\mathcal{S}_1 \times \cdots \times \mathcal{S}_k)$ such that

$$\begin{aligned} d\tilde{\gamma} &= \frac{d\mu^{\otimes k}}{\mu(\Sigma_1) \dots \mu_0(\Sigma_k)}, \\ &= d\tilde{\mu}_1 \otimes \cdots \otimes d\tilde{\mu}_k \\ d\tilde{\gamma}_0 &= \frac{d\mu_0^{\otimes k}}{\mu(\Sigma_1) \dots \mu_0(\Sigma_k)} \\ &= d\tilde{\mu}_{0,1} \otimes \cdots \otimes d\tilde{\mu}_{0,k} \end{aligned} \tag{5.5.24}$$

with $\tilde{\mu}_i$ and $\tilde{\mu}_{0,i}$ the normalized measures on $S_i$.

Then, applying (5.5.15) on the product space $\mathcal{S}_1 \times \cdots \times \mathcal{S}_k$,

$$\begin{aligned}
\frac{1}{\mu(\mathcal{S}_1)\cdots\mu(\mathcal{S}_k)} &\int_{\mathcal{S}_1\times\cdots\times\mathcal{S}_k} |V(x_1,\ldots,x_k)|\, d\mu(x_1)\cdots\mu(x_k) \\
&\int_{\mathcal{S}_1\times\cdots\times\mathcal{S}_k} |V(x_1,\ldots,x_k)|\, d\tilde{\gamma}(x_1,\ldots,x_k) \\
&\leq R(\tilde{\gamma}\|\tilde{\gamma}_0) + \log \int_{\mathcal{S}_1\times\cdots\times\mathcal{S}_k} e^{|V(x_1,\ldots,x_k)|}\, d\tilde{\gamma}_0(x_1,\ldots,x_k) \\
&= \sum_{j=1}^k R(\tilde{\mu}_j\|\tilde{\mu}_{0,j}) + \log \int_{\mathcal{S}_1\times\cdots\times\mathcal{S}_k} e^{|V(x_1,\ldots,x_k)|}\, d\tilde{\gamma}_0(x_1,\ldots,x_k) \\
&= \sum_{j=1}^k \mu(\mathcal{S}_j)\left(\int_{\mathcal{S}_j} \log\frac{d\mu}{d\mu_0}d\mu - \mu(\mathcal{S}_j)\log\frac{\mu(\mathcal{S}_j)}{\mu_0(\mathcal{S}_j)}\right) \\
&+ \log \int_{\mathcal{S}_1\times\cdots\times\mathcal{S}_k} e^{|V|(x_1,\ldots,x_k)}\, \frac{d\mu_0(x_1)}{\mu_0(\mathcal{S}_1)}\cdots\frac{d\mu_0(x_k)}{\mu_0(\mathcal{S}_k)}.
\end{aligned} \tag{5.5.25}$$

Finally, substituting $V := \beta V$ for some $\beta > 0$, rearranging the normalization constant and $\beta$ to the right, the result follows. □

By applying Lemma 5.15 to the collection of $\mathbb{R}^d$ cubes $C_i$ with side-length $L$ that cover $\Lambda$, we will derive two Lemma's that together form an analogue of Lemma 5.12. Namely, they will bound $\mathcal{E}_V$ on the set $A_{L,\delta}$, where, recall,

$$A_{L,\delta} := \left\{\mu \in \mathcal{P}(\Lambda) \;\middle|\; \alpha_L(\mu) < \delta\right\}. \tag{5.5.26}$$

and

$$\alpha_L(\mu) := \sup_i \mu(C_i), \tag{5.5.27}$$

Moreover, recall that if $\mu$ is absolutely continuous with respect to the Lebesgue measure, $\alpha_L(\mu) \to 0$ as $L \to 0$.

We then have the following estimate of $|\mathcal{E}_V|$ purely in terms of the relative entropy, $\alpha_L(\mu)$, and localized versions of the exponential integral $\langle V\rangle_S$ of Section 4.5.

**Lemma 5.17.**
*Let $\mu, \mu_0 \in \mathcal{P}(\Lambda)$, $L > 0$, and $V : \Lambda^k \to \mathbb{R}$ with $k \geq 2$. Then for every $\beta_1, \beta_2 > 0$,*

$$\int_{\Lambda^k} |V|(x_1,\ldots,x_k)\, d\mu(x_1)\cdots\mu(x_k) \leq \frac{\alpha_L(\mu)}{\beta_1}\Big(c_1\, R(\mu\|\mu_0) + c_2\,\langle\beta_1 V\rangle_L\Big) + \frac{1}{\beta_2}\Big(c_3\, R(\mu\|\mu_0) + \langle\beta_2 V\rangle'_L\Big), \tag{5.5.28}$$

*where*

$$\begin{aligned}
c_1 &:= \frac{3^d\, k^2(k-1)}{2}, \\
c_2 &:= \frac{3^d\, k(k-1)}{2}, \\
c_3 &:= k,
\end{aligned} \tag{5.5.29}$$

*Proof of Lemma 5.17.* Because of the non-negativity of $\langle V\rangle_L$ and $\langle V\rangle'_L$, we can assume without loss of generality that $R(\mu\|\mu_0) < \infty$, since the inequality is trivially satisfied otherwise. In particular, $\mu \ll \mu_0 \ll \lambda$, with $\lambda$ the Lebesgue measure $\mathbb{R}^d$. Moreover, we can also assume without loss of generality that $V$ is non-negative, and drop the absolute signs $|V|$.

We first consider the case of a pair potential $V : \Lambda^2 \to \mathbb{R}$. Recall, $\bar{\Lambda}^2_L$ is the union of all the ordered couples of cubes $C_i \times C_j$ such that $C_i = C_j$ or $C_i$ neighbours $C_j$ (and hence vice versa).

Then, since $\mu \ll \lambda$, we can decomposition the integral of $V$ over $\Lambda^2$ as follows,

$$\begin{aligned}
\int_{\Lambda^2} V(x,y)\,d\mu(x)\,d\mu(y) &= \sum_{(C_i,C_j)} \int_{C_i\times C_j} V(x,y)\,d\mu(x)\,d\mu(y) \\
&= \sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2} \int_{C_i\times C_j} V(x,y)\,d\mu(x)\,d\mu(y) + \sum_{(C_i,C_j)\subset\Lambda^2/\bar{\Lambda}_L^2} \int_{C_i\times C_j} V(x,y)\,d\mu(x)\,d\mu(y) \\
&\le \sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2} \int_{C_i\times C_j} V(x,y)\,d\mu(x)\,d\mu(y) + \int_{\Lambda^2/\Lambda_L^2} V(x,y)\,d\mu(x)\,d\mu(y),
\end{aligned} \tag{5.5.30}$$

where the last inequality follows from the fact that $\Lambda_L^2 \subset \bar{\Lambda}_L^2$ with, recall, $\Lambda_L^2 := \{(x,y)\in\Lambda^2, ||x-y| > L\}$, and hence $\Lambda^2/\bar{\Lambda}_L^2 \subset \Lambda^2/\Lambda_L^2$ - i.e. the the fact that $x\in C_i, y\in C_j$ for two non-neighbouring cubes $C_i, C_j$ implies that $|x-y|>L$.

For the second term of (5.5.30), let $\gamma,\gamma_0\in\mathcal{P}(S^2)$ with $\gamma := \mu\otimes\mu, \gamma_0 := \mu_0\otimes\mu_0$. Suppose that $(\mu_0\otimes\mu_0)(\Lambda/\Lambda_L^2) > 0$, then applying (5.5.13) of Lemma 5.15 on $\Lambda^2/\Lambda_L^2$, we have that for any $\beta>0$,

$$\begin{aligned}
&\int_{\Lambda/\Lambda_L^2} V(x,y)\,d\mu(x)\,d\mu(y) \\
&\quad= \int_{\Lambda/\Lambda_L^2} V(x,y)\,d\gamma(x,y) \\
&\quad\le \frac{1}{\beta}\left(\int_{\Lambda/\Lambda_L^2} \log\frac{d\gamma}{d\gamma_0}d\gamma - \gamma(\Lambda/\Lambda_L^2)\log\frac{\gamma(\Lambda/\Lambda_L^2)}{\gamma_0(\Lambda/\Lambda_L^2)}\right) + \frac{\gamma(\Lambda/\Lambda_L^2)}{\beta}\log\int_{\Lambda/\Lambda_L^2} e^{\beta V(x,y)}\,\frac{d\gamma_0(x,y)}{\gamma_0(\Lambda/\Lambda_L^2)} \\
&\quad\le \frac{1}{\beta}R(\gamma\|\gamma_0) + \frac{1}{\beta}\log\int_{\Lambda/\Lambda_L^2} e^{\beta V(x,y)}\,\frac{d\gamma_0(x,y)}{\gamma_0(\Lambda/\Lambda_L^2)} \\
&\quad= \frac{2}{\beta}R(\mu\|\mu_0) + \frac{1}{\beta}\log\int_{\Lambda/\Lambda_L^2} e^{\beta V(x,y)}\,\frac{d\mu_0(x)\mu_0(y)}{(\mu_0\otimes\mu_0)(\Lambda/\Lambda_L^2)} \\
&\quad= \frac{1}{\beta}\Big(c_3 R(\mu\|\mu_0) + \langle\beta V\rangle_L'\Big),
\end{aligned} \tag{5.5.31}$$

and note that the inequality is trivially satisfied when $(\mu_0\otimes\mu_0)(\Lambda/\Lambda_L^2) = 0$.

For the second term of (5.5.30), we have by (5.5.13) of Lemma 5.15 for any $i,j$,

$$\begin{aligned}
\int_{C_i\times C_j} |V(x,y)|\,d\mu(x)\mu(y) \le \frac{\mu(C_i)\mu(C_j)}{\beta}\Big[&\mu(C_i)^{-1}\left(\int_{C_i}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_i)\log\frac{\mu(C_i)}{\mu_0(C_i)}\right) \\
&+ \mu(C_j)^{-1}\left(\int_{C_j}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_j)\log\frac{\mu(C_j)}{\mu_0(C_j)}\right)\Big] \\
&+ \frac{\mu(C_i)\mu(C_j)}{\beta}\log\int_{C_i\times C_j} e^{\beta|V|(x,y)}\,\frac{d\mu_0(x)}{\mu_0(C_i)}\frac{d\mu_0(y)}{\mu_0(C_j)}.
\end{aligned} \tag{5.5.32}$$

Now note

$$\begin{aligned}
\mu(C_j)\log\int_{C_i\times C_j} e^{\beta|V|(x,y)}\,\frac{d\mu_0(x)}{\mu_0(C_i)}\frac{d\mu_0(y)}{\mu_0(C_j)} &\le \alpha_L(\mu)\sup_{i,j}\log\int_{C_i\times C_j} e^{\beta|V|(x,y)}\,\frac{d\mu_0(x)}{\mu_0(C_i)}\frac{d\mu_0(y)}{\mu_0(C_j)}, \\
&= \alpha_L(\mu)\langle\beta V\rangle_L,
\end{aligned} \tag{5.5.33}$$

and hence

$$\begin{aligned}
\sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2} \frac{\mu(C_i)\mu(C_j)}{\beta}\log\int_{C_i\times C_j} e^{\beta|V|(x,y)}\,\frac{d\mu_0(x)}{\mu_0(C_i)}\frac{d\mu_0(y)}{\mu_0(C_j)} &\le \frac{3^d\,\alpha_L(\mu)\langle\beta V\rangle_L}{\beta} \\
&= \frac{c_2\,\alpha_L(\mu)\langle\beta V\rangle_L}{\beta},
\end{aligned} \tag{5.5.34}$$

where the factor $3^d$ stems from the fact that any cube has at most $3^d - 1$ neighbours and thus for any fixed $i$ there are at most $3^d$ cubes $C_j$ such that $(C_i, C_j) \subset \bar{\Lambda}_L^2$.

Moreover,

$$\begin{aligned}
&\sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2} \frac{\mu(C_i)\mu(C_j)}{\beta}\Big[\mu(C_i)^{-1}\left(\int_{C_i}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_i)\log\frac{\mu(C_i)}{\mu_0(C_i)}\right) + \mu(C_j)^{-1}\left(\int_{C_j}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_j)\log\frac{\mu(C_j)}{\mu_0(C_j)}\right)\Big] \\
&= \sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2} \frac{2\mu(C_i)\mu(C_j)}{\beta}\Big[\mu(C_i)^{-1}\left(\int_{C_i}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_i)\log\frac{\mu(C_i)}{\mu_0(C_i)}\right)\Big] \\
&\le \sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2} \frac{2\alpha_L(\mu)}{\beta}\left(\int_{C_i}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_i)\log\frac{\mu(C_i)}{\mu_0(C_i)}\right) \\
&\le \frac{2\,3^d\,\alpha_L(\mu)}{\beta}\sum_i\left(\int_{C_i}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_i)\log\frac{\mu(C_i)}{\mu_0(C_i)}\right) \\
&\le \frac{c_1\,\alpha_L(\mu)}{\beta}R(\mu\|\mu_0),
\end{aligned} \tag{5.5.35}$$

where the first equality follows from symmetry and the last inequality from (5.5.11). The term $3^d$ again arises from the fact for any fixed $i$ there are at most $3^d$ cubes $C_j$ such that $(C_i, C_j) \subset \bar{\Lambda}_L^2$.

Combining (5.5.34) and (5.5.35),

$$\sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2}\int_{C_i\times C_j} V(x,y)\,d\mu(x)\,d\mu(y) \le \frac{\alpha_L(\mu)}{\beta}\Big(c_1 R(\mu\|\mu_0) + c_2\langle\beta V\rangle_L\Big) \tag{5.5.36}$$

Now, setting $\beta := \beta_2$ in (5.5.31), and $\beta := \beta_1$ in (5.5.36), we conclude for the positive pair potential $V$ that

$$\int_\Lambda V(x,y)\,d\mu(x)\,d\mu(y) \le \frac{\alpha_L(\mu)}{\beta_1}\Big(c_1\,R(\mu\|\mu_0) + c_2\,\langle\beta_1 V\rangle_L\Big) + \frac{1}{\beta_2}\Big(c_3\,R(\mu\|\mu_0) + \langle\beta_2 V\rangle_L'\Big). \tag{5.5.37}$$

Finally, the case for a multi-particle potential $V : \Lambda^k \to \mathbb{R}$ is similar.

First, denote $\gamma, \gamma_0 \in \mathcal{P}(\Lambda^k/\Lambda_L^k)$, with $\gamma := \mu^{\otimes k}, \gamma_0 := \mu_0^{\otimes k}$, then similar to (5.5.31),

$$\begin{aligned}
&\int_{\Lambda^k/\Lambda_L^k} V(x_1,\dots,x_k)\,d\mu(x_1)\cdots d\mu(x_k) \\
&= \int_{\Lambda^k/\Lambda_L^k} V(x_1,\dots,x_k)\,d\gamma(x_1,\dots,x_k) \\
&\le \frac{1}{\beta_2}R(\gamma\|\gamma_0) + \frac{1}{\beta_2}\log\int_{\Lambda^k/\Lambda_L^k} e^{\beta_2 V(x_1,\dots,x_k)}\,\frac{d\gamma_0(x,y)}{\gamma_0(\Lambda/\Lambda_L^2)} \\
&= \frac{c_3}{\beta_2}R(\mu\|\mu_0) + \frac{1}{\beta_2}\langle\beta_2 V\rangle_L'.
\end{aligned} \tag{5.5.38}$$

Extending (5.5.34),

$$\begin{aligned}
&\sum_{(C_{i_1},\dots,C_{i_k})\subset\bar{\Lambda}_L^k} \frac{\mu(C_{i_1})\cdots\mu(C_{i_k})}{\beta_1}\log\int_{C_{i_1}\cdots C_{i_k}} e^{\beta V(x_1,\dots,x_k)}\,\frac{d\mu_0(x_1)}{\mu_0(C_{i_1})}\cdots\frac{d\mu_0(x_k)}{\mu_0(C_{i_k})} \\
&\le \sum_{(C_{i_1},\dots,C_{i_k})\subset\bar{\Lambda}_L^k} \frac{\mu(C_{i_1})\cdots\mu(C_{i_k})}{\beta_1}\,\langle\beta_1 V\rangle_L \\
&= \frac{k(k-1)}{2}\sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2}\frac{\mu(C_i)\mu(C_j)}{\beta_1}\,\langle\beta_1 V\rangle_L \\
&= \frac{c_2\,\alpha_L(\mu)\langle\beta_1 V\rangle_L}{\beta_1}.
\end{aligned} \tag{5.5.39}$$

Here the factor $k^2$ is because that for any $(C_{i_1},\dots,C_{i_k}) \subset \bar{\Lambda}_L^k$ there are at least two $C_{i_a}, C_{i_b}$ such that $(C_{i_a}, C_{i_b}) \subset \bar{\Lambda}_L^2$, hence the sum can be decompositioned into $\frac{k(k-1)}{2}$ sums over pairs $(C_i, C_j) \subset \bar{\Lambda}_L^2$.

Similarly, extending (5.5.35),

$$\begin{aligned}
&\sum_{(C_{i_1},\dots,C_{i_k})\subset\bar{\Lambda}_L^k} \frac{C_{i_1})\cdots\mu(C_{i_k})}{\beta_1} \sum_{j=1}^{k} \mu(C_{i_j})^{-1}\left(\int_{C_{i_j}} \log\frac{d\mu}{d\mu_0}d\mu - \mu(C_{i_j})\log\frac{\mu(C_{i_j})}{\mu_0(C_{i_j})}\right)\\
&\quad = k \sum_{(C_{i_1},\dots,C_{i_k})\subset\bar{\Lambda}_L^k} \frac{\mu(C_{i_1})\cdots\mu(C_{i_k})}{\beta_1}\mu(C_{i_1})^{-1}\left(\int_{C_{i_1}} \log\frac{d\mu}{d\mu_0}d\mu - \mu(C_{i_1})\log\frac{\mu(C_{i_1})}{\mu_0(C_{i_1})}\right)\\
&\quad \le \frac{k^2(k-1)}{2} \sum_{(C_i,C_j)\subset\bar{\Lambda}_L^2} \frac{\mu(C_i)\mu(C_j)}{\beta_1}\Big[\mu(C_i)^{-1}\left(\int_{C_i}\log\frac{d\mu}{d\mu_0}d\mu - \mu(C_i)\log\frac{\mu(C_i)}{\mu_0(C_i)}\right)\Big]\\
&\quad \le \frac{c_1\alpha_L(\mu)}{\beta_1}R(\mu\|\mu_0),
\end{aligned} \tag{5.5.40}$$

Here the first equality follows from symmetry and the first inequality follows from the same argument as above, that for any $(C_{i_1},\dots,C_{i_k}) \subset \bar{\Lambda}_L^k$ there are at least two $C_{i_a}, C_{i_b}$ such that $(C_{i_a}, C_{i_b}) \subset \bar{\Lambda}_L^2$. Moreover, note that this rough bound by overcounting is only allowed because all the terms involved are *positive*.

Joining (5.5.38), (5.5.39) and (5.5.40), we conclude that

$$\begin{aligned}
\int_{\Lambda^k} V(x_1,\dots,x_k)\,d\mu(x_1)\cdots\mu(x_k) &\le \frac{\alpha_L(\mu)}{\beta_1}\Big(\frac{3^d\,k^2(k-1)}{2}\,R(\mu\|\mu_0) + \frac{3^d\,k(k-1)}{2}\,\langle\beta_1 V\rangle_L\Big)\\
&\quad + \frac{1}{\beta_2}\Big(k\,R(\mu\|\mu_0) + \langle\beta_2 V\rangle_L'\Big)\\
&= \frac{\alpha_L(\mu)}{\beta_1}\Big(c_1\,R(\mu\|\mu_0) + c_2\,\langle\beta_1 V\rangle_L\Big) + \frac{1}{\beta_2}\Big(c_3\,R(\mu\|\mu_0) + \langle\beta_2 V\rangle_L'\Big).
\end{aligned} \tag{5.5.41}$$

□

From Lemma 5.17 an estimate on $A_{L,\delta}$ quickly follows.

**Lemma 5.18.** *For any $\beta > 0, L > 0, \delta > 0$, and a $V : \Lambda^k \to \mathbb{R}$ with $k \ge 2$,*

$$\sup_{\mu\in A_{L,\delta}} \Big\{ \frac{\beta_\delta}{2k}|\mathcal{E}_V|(\mu) - R(\mu\|\mu_0) \Big\} \le \frac{1}{2k}\Big(\langle\beta V\rangle_L + \langle\beta_\delta V\rangle_L'\Big) \tag{5.5.42}$$

*where*

$$\beta_\delta := \delta^{-1}\left(\frac{2\beta}{3^d k(k-1)}\right). \tag{5.5.43}$$

*Proof.* Setting $\beta_1 := \beta$ and $\beta_2 := \beta_\delta$, then by Lemma (5.17), for any $\mu \in A_{L,\delta}$.

$$\begin{aligned}
\int_{\Lambda^k} |V|(x_1,\dots,x_k)\,d\mu(x_1)\cdots\mu(x_k) &\le \frac{\alpha_L(\mu)}{\beta_1}\Big(c_1\,R(\mu\|\mu_0) + c_2\,\langle\beta_1 V\rangle_L\Big) + \frac{1}{\beta_2}\Big(c_3\,R(\mu\|\mu_0) + \langle\beta_2 V\rangle_L'\Big)\\
&\le \frac{\delta}{\beta}\Big(c_1\,R(\mu\|\mu_0) + c_2\,\langle\beta V\rangle_L\Big) + \frac{1}{\beta_\delta}\Big(c_3\,R(\mu\|\mu_0) + \langle\beta_\delta V\rangle_L'\Big).
\end{aligned} \tag{5.5.44}$$

Now recall, that

$$
\begin{aligned}
c_1 &:= \frac{3^d\, k^2(k-1)}{2},\\
c_2 &:= \frac{3^d\, k(k-1)}{2},\\
c_3 &:= k,
\end{aligned}
\tag{5.5.45}
$$

Hence,

$$
\int_{\Lambda^k} |V|(x_1,\dots,x_k)\, d\mu(x_1)\cdots\mu(x_k) \le 2kR(\mu\|\mu_0) + \frac{1}{\beta_\delta}\langle\beta V\rangle_L + \frac{1}{\beta_\delta}\langle\beta_\delta V\rangle_L, \tag{5.5.46}
$$

and by dividing the result follows, while noting that $0 \le |\mathcal{E}_V| \le \mathcal{E}_{|V|}$. □

We are now finally able to prove Theorem 5.14.

*Proof of Theorem 5.14.* We will show that for every $\mu^* \in \mathcal{P}$ with $R(\mu\|\mu_0) < \infty$ and every $\beta > 0$ there exists a open set $A$ containing $\mu^*$ such that

$$
\lim_{\lambda\to 0}\sup_{\mu\in A}\Big\{\ \beta|\mathcal{E}_V - \mathcal{E}_{V_\lambda}|(\mu) - R(\mu\|\mu_0)\ \Big\} = 0. \tag{5.5.47}
$$

Theorem 5.14 then follows directly from Lemma 3.25, and the fact that $\langle\beta^* V\rangle_L < \infty$ and $\langle\beta^* V\rangle'_L < \infty$ imply that

$$
\int_{\Lambda^k} e^{\beta^*|V|(x_1,\dots,x_k)}\, d\mu(x_1)\cdots\mu(x_k) < \infty, \tag{5.5.48}
$$

and hence $\left\langle\!\!\left\langle \frac{\beta^*}{k}V\right\rangle\!\!\right\rangle_{\mathcal{X}} < \infty$, see Lemma 4.9. Now, fix any $\mu^* \in D(I)$, i.e. with $R(\mu^*\|\mu_0) < \infty$, and in particular, $\mu^*$ is absolutely continuous with respect to the Lebesgue measure. Hence, $\lim_{L\to 0}\alpha_L(\mu) \to 0$, and recall that because of the compactness of $\Lambda$,

$$
\alpha_L(\mu) := \max_{1\le i\le K}\mu(C_i), \tag{5.5.49}
$$

where the maximum is over $K$ disjoint $\mathbb{R}^d$ cubes with side-length $L$ that cover the compact $\Lambda$.

Hence, for any fixed $\epsilon > 0$, it is clear that for any $L > 0$ the set $A_L^* := A_{L,(1+\epsilon)\alpha_L(\mu^*)}$ contains $\mu^*$, with

$$
A_L^* = \Big\{\mu \in \mathcal{P}(\Lambda)\ \Big|\ \sup_{1\le i\le K}\mu(C_i) < (1+\epsilon)\sup_{1\le i\le K}\mu^*(C_i)\Big\}, \tag{5.5.50}
$$

and recall from the proof Theorem 5.14 that $A_L^*$ even contains for every $L$ an open neighbourhood of $\mu^*$. Now, for any $L > 0$, apply Lemma 5.18, where we set $\beta := \beta^*$, $\delta := (1+\epsilon)\alpha_L(\mu^*)$,

$$
\begin{aligned}
\sup_{\mu\in A_L^*}\Big\{\ \frac{\beta_\delta}{2k}|\mathcal{E}_V - \mathcal{E}_{V_\lambda}|(\mu) - R(\mu\|\mu_0)\ \Big\} &= \sup_{\mu\in A_L^*}\Big\{\ \frac{\beta_\delta}{2k}|\mathcal{E}_{V-V_\lambda}|(\mu) - R(\mu\|\mu_0)\ \Big\}\\
&= \sup_{\mu\in A_{L,\delta}}\Big\{\ \frac{\beta_\delta}{2k}|\mathcal{E}_{V-V_\lambda}|(\mu) - R(\mu\|\mu_0)\ \Big\}\\
&\le \frac{1}{2k}\Big(\ \langle\beta^* V\rangle_L + \langle\beta_\delta V\rangle'_L\ \Big),
\end{aligned}
\tag{5.5.51}
$$

and thus, by assumption,

$$
\lim_{\lambda\to 0}\sup_{\mu\in A_L^*}\Big\{\ \frac{\beta_\delta}{2k}|\mathcal{E}_V - \mathcal{E}_{V_\lambda}|(\mu) - R(\mu\|\mu_0)\ \Big\} = 0, \tag{5.5.52}
$$

where

$$\beta_\delta := \delta^{-1}\left(\frac{2\beta^*}{3^d k(k-1)}\right)$$
$$= (1+\epsilon)^{-1}\alpha_L(\mu^*)^{-1}\left(\frac{2\beta^*}{3^d k(k-1)}\right). \tag{5.5.53}$$

Recall that as $L \to 0$, $\alpha_L(\mu^*) \to 0$, which implies $\delta \to 0$ and thus $\beta_\delta \to \infty$. Hence, for every $\beta \geq 0$ we can find a $L$ small enough and a corresponding $A_L^*$ — which contains a open set containing $\mu^*$, such that

$$\lim_{\lambda\to 0} \sup_{\mu\in A_L^*} \left\{ \beta\,|\mathcal{E}_V - \mathcal{E}_{V_\lambda}|(\mu) - R(\mu\|\mu_0) \right\} = 0. \tag{5.5.54}$$

Therefore, we can finally apply Lemma 3.25, and the result follows. □

**Comments**

**Remark 5.7.** Similar as for the stochastic case of pair interactions in Section 5.4, the proof of Theorem 5.14 can be generalized to any *localized* logarithmic singularity, see also Remark 5.3. For example, suppose that there exists some number $M$ such that for any $L > 0$ and any collection $(i_1, \dots, i_k)$, there are at most $MN^{k-2}$ collections $(i_1, \dots, i_k)$ such that

$$\int_{C_1\times\cdots\times C_k} e^{\beta|V|(x_1,\dots,x_k)}\, d\tilde{\mu}_{0,1}(x_1)\cdots d\tilde{\mu}_{0,k}(x_k) = \infty \tag{5.5.55}$$

for some $\beta \geq \beta^*$. Then, with appropriate convergence conditions similar to those of Theorem 5.14 one can still establish the desired properties of $J_{\beta V}$.

**Remark 5.8.** Additionally, just as for the stochastic case of pair interactions in Section 5.4, note that Lemma's 5.17 and 5.18 are valid for countable collections of cubes $C_i$, in contrast to Theorems 5.14 and 5.8. It might possible that the restrictions for the latter can be lifted, see also Remark 5.2.

## 5.6 Discussion

In this section we investigated a special case of the framework for Gibbs measures as outlined in Section 4, with now the underlying space $\Lambda$ a subset of $\mathbb{R}^d$, and where the Gibbs measures formally arise as invariant measures of a system of a particles under a potential $V$ on $\Lambda$. We made the distinction between *critical*, and *sub-critical* systems, where the latter is defined for all temperatures, but the former is not.

In the case of sub-critical systems, most of the relevant results follow directly from Section 4, such as Theorem 5.1 which characterized large-deviation principles for the induced measures, but we also gave a sufficient condition where a relatively simple $V$ is approximated pointwise and monotonously by continuous and bounded functions $V_\lambda$.

For a critical system, the approach of Section 4 is not sufficient — related to the distinction between quasi-uniform convergence and strictly local quasi-uniform convergence of Section 3, and hence we developed and employed a generalized version of the discretization technique for pair potentials as introduced in [BG99]. A main consequence is Theorem 5.6, where local convergence — and therefore large-deviation principles — was established on the assumption that there is only a critical singularity when two particles are very close together.

As outlined in Section 5.4, notably Theorem 5.10, we show that the exponential estimate $[V]_\Lambda$ for sub-critical potentials, see Section 4, can be replaced by similar but localized and renormalized estimates $[V]_L$ defined over a set of cubes covering $\Lambda$ and with side-length $L$. We discuss how this approach could be generalized for non-compact $\Lambda$ or more general singularities.

Finally, we discussed the technical problems involved in extending this technique to multi-particle potentials, and showed that these does not apply when one is merely concerned with the rate functions, see Theorem 5.14, but recall that the desired properties are *necessary* conditions for quasi-continuity and hence any large-deviation principle to hold.

The underlying framework of discretization techniques — as outlined in Section 5.4 for the stochastic case and in Section 5.5 for the rate functions — was purposely stated in a rather general manner, in such a way that it involved very little assumptions on $\Lambda$ or the reference measure of the non-interacting system $\mu_0$. This is because it was initially developed with the intention of lifting the approach to *path spaces*. Unfortunately, this is to be left for future research, but in the next section we will sketch how our framework might be applied to path spaces, in which we will consider large deviations for the paths of Brownian motion acting under a singular but sub-logarithmic potential $V$.

# 6 LDP for empirical process

## 6.1 Introduction

In the previous section we discussed large deviations for Gibbs measures on $\mathbb{R}^d$ induced by a singular potential $V$, which formally arose as invariant measures for a system of particles evolving under certain stochastic dynamics, such as the following set of coupled stochastic differential equations,

$$dX_i^N(t) = -\frac{1}{N}\sum_{i,j\neq i} \nabla V(X_i^N - X_j^N)\, dt + dB_i^N(t), \qquad t \in [0,T]. \tag{6.1.1}$$

Now, in this section, we will briefly discuss large deviations for the stochastic dynamics themselves. Under certain assumptions on minimizers, large-deviation principles imply that in the many particle limit the system becomes *deterministic*, i.e. described by a partial differential equation. This deterministic macroscopic description of the stochastic particle-system has of course many applications, and it was actually this question that we first set out to answer.

However, we did not manage to finish that line of investigation, and therefore we only briefly note here how our approach might be extended to this case.

Namely, while in the previous section we discussed random empirical measures, we will now consider the *empirical process*, which is a random distribution over continuous paths. Under certain conditions on the potential $V$, the law of (6.1.1) might formally be written as a Gibbs measure over the laws of independent Brownian motions, which would allows us to use the framework of quasi-continuity of Section 3 and the convergence results for Gibbs measures outlined in Section 4.

We will therefore briefly list this representation, and show how the fruits of our current research might be applied.

## 6.2 Notation and basic properties

We consider the following system of SDEs.

$$
\begin{aligned}
dX_i^N(t) &= -\frac{1}{N}\sum_{i,j\neq i}\nabla V(X_i^N - X_j^N)\,dt + dB_i^N(t), \qquad t\in[0,T],\\
&X_i(0) \text{ satisfies law } (\rho_0)^N.
\end{aligned}
\tag{6.2.1}
$$

First, recall the notation and context for empirical processes as outlined in Section 2.2 (page 13), which we slightly modify here. Moreover, recall the notation for abstract empirical measures and Gibbs measures as stated in Section 4.2.

We set $S := C([0,T]\to\mathbb{R})$, the space of continuous paths in $\mathbb{R}^d$, and note that $X_i^N \in S$. Moreover, let $\tilde{Q}_V^N \in \mathcal{P}(S^N)$ be the law of the system $X^N$ defined by (6.2.1), and $\tilde{P}^N \in \mathcal{P}(S^N)$ the law of $N$ independent Brownian motions, i.e. $\tilde{P}^N := W^{\otimes N}$, where $W\in\mathcal{P}(S)$ is the Wiener measure.

We then consider the *empirical process* $z^N \in \mathcal{P}(S)$,

$$
z^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}, \tag{6.2.2}
$$

Moreover, let $\mathrm{Q}_V^N \in \mathcal{P}(\mathcal{P}(S))$ be the laws of the random variable $z^N$ induced by $\tilde{Q}_V^N$ , and similarly, $\mathrm{P}^N$ the law for the empirical process of the non-interacting system induced by $W^{\otimes N}$.

Now, we will show how this fall in the framework of abstract Gibbs measures of Section 4.

**Lemma 6.1.** *For any fixed $V$, set $\mathcal{V}_a : S\times S^2 \to \mathbb{R}$, $\mathcal{V}_b : S\times S\to\mathbb{R}$, $\mathcal{V}_c : S\times S\to\mathbb{R}$, with*

$$
\begin{aligned}
\mathcal{V}_a(x,y,z) &:= \int_0^T \nabla V(x(t)-y(t))\cdot\nabla V(x(t)-z(t))\,dt\\
\mathcal{V}_b(x,y) &:= -\int_0^T \Delta V(x(t)-y(t))\,dt,\\
\mathcal{V}_c(x,y) &:= (V(x(T)-y(T)) - V(x(0)-y(0)).
\end{aligned}
\tag{6.2.3}
$$

*Now suppose that $V$ is twice continuously differentiable. Then*

$$
\frac{d\tilde{Q}_V^N}{d\tilde{P}^N}(x_1,\dots,x_N) = e^{-NE_{\mathcal{V}}^N(x_1,\dots,x_N)}, \tag{6.2.4}
$$

*with*

$$
E_{\mathcal{V}}^N(x_1,\dots,x_N) := \sum_{j,k\neq i,i}\mathcal{V}_a(x_i;x_j,x_k) + \sum_{i,j\neq i}\mathcal{V}_b(x_i;x_j) + \sum_{i,j\neq i}\mathcal{V}_c(x_i;x_j). \tag{6.2.5}
$$

*Moreover,*

$$
\frac{d\mathrm{Q}_V^N}{d\mathrm{P}^N}(\mu) = e^{-N\mathcal{E}_{\mathcal{V}}(\mu)}, \tag{6.2.6}
$$

*with*

$$
\mathcal{E}_{\mathcal{V}}(\mu) := \int_{(S^3)'}\mathcal{V}_a(x,y,z)d\mu(x)\,d\mu(y)\,d\mu(z) + \int_{(S^2)'}(\mathcal{V}_a+\mathcal{V}_b)(x,y)\,d\mu(x)\,d\mu(y). \tag{6.2.7}
$$

*In particular, $\mathrm{Q}_V^N$ satisfies a LDP with rate function $\mathcal{F}$, where*

$$
\begin{aligned}
\mathcal{F}_{\mathcal{V}}(\mu) &:= J_V(\mu) - \inf_{\nu\in P(S)} J_V(\nu),\\
J_V(\mu) &:= \mathcal{E}_{\mathcal{V}}(\mu) + R(\mu\|W),
\end{aligned}
\tag{6.2.8}
$$

*Proof.* The proof follows straightforwardly from Section 2 (page 13).

Namely, recall, whenever $V$ is continuously differentiable (see [dH08, p. 112]), it is known that strong solutions to (6.2.1) exist, and that $\tilde{Q}^N$ is absolutely continuous with respect to $\tilde{W}^N$, with

$$\frac{d\tilde{Q}^N}{d\tilde{P}^N}(x_1,\dots,x_N) := e^{-NE_V^N(x_1,\dots,x_N)}, \tag{6.2.9}$$

with $E_V^N := E_{V,a}^N + E_{V,b}^N + E_{V,c}^N$, where

$$\begin{aligned} E_{V,a}^N(x_1,\dots,x_N) &:= -\int_0^T \sum_{i=1}^N \left| \sum_{j=1|j\neq i}^N \nabla V\left(x_i(t)-x_j(t)\right)\right|^2 dt, \\ &= \sum_{i=1,j=1,k=1|i\neq j,k}^N \int_0^T \nabla V\left(x_i(t)-x_j(t)\right)\cdot\nabla V\left(x_i(t)-x_k(t)\right)\,dt, \\ E_{V,b}^N(x_1,\dots,x_N) &:= \sum_{i=1,j=1|i\neq j}^N \int_0^T \Delta V(x_i(t)-x_j(t))\,dt, \\ E_{V,c}^N(x_1,\dots,x_N) &:= -\sum_{i=1,j=1|i\neq j}^N \Big(V\big(x_i(T)-x_j(T)\big) - V\big(x_i(0)-x_j(0)\big)\Big)\,dt \end{aligned} \tag{6.2.10}$$

Since $\tilde{Q}^N$ is absolutely continuous with respect to $\tilde{P}^N$, $\mathrm{Q}^N$ and $\mathrm{P}^N$ are absolutely continuous, and such that (see [dH08, p. 112]),

$$\frac{d\mathrm{Q}^N}{d\mathrm{P}^N}(\mu) = e^{NF_V(\mu)}, \qquad \mu\in\mathcal{P}(S), \tag{6.2.11}$$

where

$$\begin{aligned} F_{V,a}(\mu) &:= -\int_0^T \int_S \left|\int_S \nabla V(x(t)-y(t))\,d\mu(x)\right|^2 dt\,d\mu(y) \\ &= -\int_{S^3}\int_0^T \nabla V(x(t)-y(t))\cdot\nabla V(x(t)-z(t))\,dt\,d\mu(x)\,d\mu(y)\;d\mu(z), \\ F_{V,b}(\mu) &:= \int_{S^2}\int_0^T \Delta V(x(t)-y(t))\,dt\,d\mu(x)\,d\mu(y), \\ F_{V,c}(\mu) &:= -\int_{S^2}\Big(V(x(T)-y(T)) - V(x(0)-y(0))\Big)\,d\mu(x)\,d\mu(y). \end{aligned} \tag{6.2.12}$$

As mentioned in Section 2.1, it therefore follows that $\mathrm{Q}^N$ satisfies a large-deviation principle with rate function $\mathcal{F}_V$,

$$\begin{aligned} \mathcal{F}_V(\mu) &:= J_V(\mu) - \inf_{\nu\in\mathcal{X}} J_V(\nu), \\ J_V(\mu) &:= R(\mu\|W) - F_V(\mu). \end{aligned} \tag{6.2.13}$$

Recall, $R(\mu\|W)$ is the relative entropy of $\mu$ with respect to the law $W$ of a Brownian motion on $\Lambda$ for $t\in[0,T]$, and is the rate function of the sequence of measures $\mathrm{P}^N$.

Now, note that simply by the definition of $\mathcal{V}_a$, $\mathcal{V}_b$ and $\mathcal{V}_c$, we have that $\mathcal{E}_V^N = \mathcal{E}_{\mathcal{V}}^N$ and $\mathcal{F}_V = \mathcal{F}_{\mathcal{V}}$, and we conclude the proof.

□

## 6.3 Main result

From the representation in Lemma 6.1 and our current results, we can immediately state the following theorem. Note that we use a slightly different notation, namely $x_t$ instead of $x(t)$, because of brevity and clarity of the exponential integrals involved and the fact that we do not require indices.

**Theorem 6.2.** *Suppose that for every $N$ the system SDE's* (6.2.1)*, determined by the singular potential $V$, has a weak solution and that its law $\tilde{Q}_V^N \in \mathcal{P}(S^N)$ is absolutely continuous with respect to the law of $N$ independent Brownian motions, $W^{\otimes N}$, with*

$$\frac{d\tilde{Q}_V^N}{d\tilde{P}^N}(x_1, \dots, x_N) = e^{-NE_{\mathcal{V}}^N(x_1,\dots,x_N)}, \tag{6.3.1}$$

*where $E_{\mathcal{V}}^N$ is as defined in Lemma 6.1.*

*Moreover, suppose that there exists a sequence of twice continuously differentiable functions $V_\lambda$ such that for all $\beta \in \mathbb{R}$ we have*

$$\lim_{\lambda\to 0} \sup_{y,z\in S} \log \mathbb{E}\left[\exp\left\{\beta \int_0^T \Big(\nabla V(y_t - B_t)\cdot V(z_t - B_t) - \nabla V_\lambda(y_t - B_t)\cdot V_\lambda(z_t - B_t)\Big)\, dt\right\}\right] = 0. \tag{6.3.2a}$$

$$\lim_{\lambda\to 0} \sup_{y\in S} \log \mathbb{E}\left[\exp\left\{\beta \int_0^T (\Delta V - \Delta V_\lambda)(y_t - B_t)\, dt\right\}\right] = 0, \tag{6.3.2b}$$

$$\lim_{\lambda\to 0} \sup_{y\in S}, \log \mathbb{E}\left[\exp\left\{\beta(V - V_\lambda)(y_T - B_T) - (V - V_\lambda)(y_0 - B_0)\right\}\right] = 0. \tag{6.3.2c}$$

*Then the sequence of measures $\mathrm{Q}_V^N$ satisfies a large-deviation principle with rate function $\mathcal{F}_V$.*

*Proof.* Note that by Lemma 6.1 and linearity that for any bounded and twice continuously differentiable $V$, the induced $\mathcal{E}_{\mathcal{V}}(\mu)$ induces itself a LDP for all $\beta$, since $\beta V$ is also bounded and twice continuously differentiable. Hence, $\mathcal{E}_{\mathcal{V}}(\mu)$ is quasi-continuous and quasi-bounded with respect to $\mathrm{P}^N$ by Theorem 3.1, although it could also be established in a similar way as in Theorem 2.5 or Lemma 4.1.

Hence, by Theorem 3.20 and linearity, (3.5.11) implies that $\mathcal{E}_{\mathcal{V}_\lambda}$ converges quasi-uniformly to $\mathcal{E}_{\mathcal{V}}$, and therefore the latter is quasi-continuous and quasi-bounded. In particular, it induces a large-deviation principle with rate function $\mathcal{F}_{\mathcal{V}} = F_V$ (see Lemma 6.1). □

Now, to actually establish absolute continuity of $\tilde{Q}^N$ with respect to $\tilde{P}^N$, one would need to resort to other tools, such as Girsanov's Theorem, but we will not list those here, except for a brief glimpse of possibilities in Conjecture 2.9 of Section 2.3, which we hope to prove in further research.

## 6.4 Discussion

In this section we gave a brief overview of how our current framework of quasi-continuity might be applied to a system of SDEs, by representing the law of the system $\tilde{Q}_V^N$ as a Gibbs measure with respect to the law of independent Brownian motions $W^{\otimes N}$, see Lemma 6.1.

As shown in Theorem 6.2, the convergence results of Section 4 then imply that we can establish LDPs by approximation, reducing it to a finite particle exponential integral over Brownian motions, as seen in (3.5.11).

These integrals are no longer integrals over a simple space such as $\mathbb{R}^d$, but over *path space*, and one could for examples use techniques as listed in [HH14].

Moreover, note that the system of SDEs can be well defined in and of itself — *regardless* of whether it can be represented as a Gibbs measure with respect to a sequence of nice reference measures. In particular, if $\tilde{Q}^N$ would be singular with respect to the law of independent Brownian motions $W^{\otimes N}$, different techniques would be needed.

# 7 Conclusion

## 7.1 Summary of results

In Section 2 we first outlined the basic notions underlying large-deviation principles of interacting particle systems — such as for the paths or invariants measures of a set of coupled stochastic differential equations — when the interaction potential $V$ is assumed to be nicely bounded and continuous. The key is that an interacting system can be represented via a continuous change of measure by a non-interacting systems.

To lift this restriction of continuity, and hence study systems involving singular potentials, we used the notion of a *quasi*-continuous function. Here quasi-continuity is defined with respect to a sequence of reference measures, such that the probability of the discontinuity vanishes rapidly enough. In Section 3 we introduced this concept and presented a vast framework for studying and establishing quasi-continuity, in which we drew parallels to classic results for continuous functions, and revealed the intimate relationship between quasi-continuity and large deviations. Even more so: the class of functions for which Varadhan's Lemma applies for all real temperatures is shown to be *equivalent* to the class of quasi-bounded and quasi-continuous functions.

Moreover, we showed how quasi-continuity follows if the function can be approximated by (quasi-)continuous functions in an appropriate sense, namely *quasi-uniformly.* We applied this in Section 4, concerning rather abstract Gibbs measures in metric spaces, where we showed how this convergence can be established rather easily via an exponential estimate over a finite number of particles.

Next, in Section 5, we used these results to directly verify LDPs for invariant measures for a large class of potentials, either sub-logarithmic for attracting particles, or up to Riesz potentials for repelling particles. Moreover, we investigated the case for logarithmic singularities, and derived a large-deviation principle by an discretization argument and localized version of quasi-uniform convergence.

Finally, we discussed in Section 6 how our techniques could be applied to empirical processes, where we described large-deviation principles for the paths of the interacting particles. Unfortunately we did not succeed in completing this line of inquiry, but we briefly listed how if a certain representation theorem holds, the framework of quasi-continuity might also be of important use for dynamic interacting particle systems.

## 7.2 Future research

As mentioned, while in the end we did not manage to prove LDPs for interacting Brownian motions, we hope to resolve this question in further investigations. We are confident that our results can at least be extended in the so-called *sub-critical* case, in which the system is defined for any arbitrary temperature.

However, also of particular interest is the critical case, for when the potential has a logarithmic singularity, which has numerous applications throughout various applied fields. We already thoroughly studied the case of invariant measures with logarithmic potentials, and we purposely structured our proofs so that we might lift them to the dynamic case.

However, preliminary investigations suggest that this approach might work on a strictly smaller range of temperatures than for which the system tends to its hydrodynamic limit. The reason for this is because for the latter it is possible that the laws of the interacting system are no longer absolutely continuous with respect to the non-interacting systems, in which case our framework does not apply. Yet, it is still interesting to wonder how far our techniques can actually be pushed.

Finally, throughout this thesis we have framed various concepts in terms of linear spaces that are contained within each other, and an important question if these relations are *strict.* Namely, while one can create a quasi-continuous functions by approximation of continuous functions, it is not clear such an approximation exists for évery quasi-continuous functions. Similarly, we encountered two different exponential estimates in Section 4 — one which implies quasi-continuity, and one which implies that the rate functions share most of the properties of those of quasi-continuous functions, and we do know if these classes are equivalent or not.

While many of these questions might not be answered, we hope that this thesis at least provided a glimpse of the various possibilities, and the use of quasi-continuity in establishing large-deviation principles for interacting particles.

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