On some subgroups of linear groups over $\mathbb{F}_2$ generated by elements of order 3

On some subgroups of linear groups over $\mathbb{F}_2$ generated by elements of order 3

Hans Cuypers
email: f.g.m.t.cuypers@tue.nl

July 10, 2017

Abstract

Let V be a vector space over the field $\mathbb{F}_2$. We investigate subgroups of the linear group $\mathrm{GL}(V)$ which are generated by a conjugacy class D of elements of order 3 such that all $d \in D$ have 2-dimensional commutator space $[V, d]$.

MSC 2010: 20D06, 20E45, 51A45, 51A50
Keywords: Linear groups, elements of order 3

1 Introduction

In his revision of Quadratic Pairs [2, 3], Chermak [2] classifies various subgroups of the symplectic groups $\mathrm{Sp}(2n, 2)$ generated by elements d of order 3 with $[V, d] = \{vd - v \mid v \in V\}$ being 2-dimensional, where V is the natural module of $\mathrm{Sp}(2n, 2)$. Besides the full symplectic group he encounters orthogonal and unitary groups over the field with 2 or 4 elements, respectively, as well as alternating groups. Chermak's proof of this classification is inductive and relies mainly on methods from geometric algebra.

By using discrete geometric methods we are able to classify subgroups of $\mathrm{GL}(V)$, where V is an $\mathbb{F}_2$-vector space of possibly infinite dimension, generated by elements d of order 3 with $[V, d]$ being 2-dimensional.

In particular, we prove the following. (For notation and definitions, the reader is referred to the next section.)

1.1 Theorem. *Let V be a vector space of dimension at least 3 over the field $\mathbb{F}_2$. Suppose $G \leq \mathrm{GL}(V)$ is a group generated by a conjugacy class D of elements of order 3 such that*

(a) *$[V, d]$ is 2-dimensional for all $d \in D$;*

(b) *$[V, G] = V$ and $C_V(G) = \{0\}$.*

Then, up to isomorphism, we have one of the following.

(a) *There exists a subspace Φ of V^* annihilating V such that $G = \mathrm{T}(V, \Phi)$; the class D is the unique class of elements of order 3 with 2-dimensional commutator on V.*

(b) *$\dim(V) = 3$ and $G = 7 : 3$; D is one of the two classes of elements of order 3 in G.*

(c) *$\dim(V) = 4$ and $G = \mathrm{Alt}_7$ (inside $\mathrm{Alt}_8 \simeq \mathrm{GL}(4, 2)$); the class D corresponds to the class of elements of order 3 which are products of two disjoint 3-cycles inside Alt_7.*

(d) *$\dim(V) \geq 6$, and $G = \mathrm{FSp}(V, f)$ with respect to some nondegenerate symplectic form f on V; the class D is the unique class of elements of order 3 with 2-dimensional commutator on V.*

(e) *$\dim(V) \geq 6$ and $G = \mathrm{F}\Omega(V, Q)$ for some nondegenerate quadratic form Q on V with trivial radical; the class D is the unique class of elements of order 3 with 2-dimensional commutator on V.*

(f) *$G = \mathrm{FAlt}(\Omega)$ for some set Ω of size at least 5; the class D corresponds to the class of 3-cycles, or in case $|\Omega| = 6$, the class of elements which are a products of two disjoint 3-cycles. The space V is the subspace of the space $\mathbb{F}_2\Omega$ generated by all vectors of even weight, or, in case $|\Omega|$ is even, the quotient of this subspace by the all one vector.*

(g) *V carries a G-invariant structure of an $\mathbb{F}_4$-space V_4 such that G equals $\mathrm{R}(V_4, \Phi)$, where Φ is a subspace of V_4^* annihilating V_4. The class D is the class of reflections in G.*

(h) *V carries a G-invariant structure (V_4, h) of an $\mathbb{F}_4$-space V_4 equipped with a nondegenerate Hermitian form h such that $G = \mathrm{FU}(V_4, h)$, the subgroup of $\mathrm{GU}(V_4, h)$ generated by all reflections. The class D is the class of reflections in G.*

As indicated above, our proof of this theorem is of geometric nature. By using methods similar to those developed in Cameron and Hall [1] and Cohen, Cuypers and Sterk [4] we are able to show that the subspaces $[V, d]$ with $d \in D$, are either all the lines of $\mathbb{P}(V)$ (leading to the cases (a)-(c) of the theorem) or of a cotriangular space embedded in $\mathbb{P}(V)$ (cases (d)-(f)) or these spaces are (part of) the one-dimensional subspaces of an $\mathbb{F}_4$-space induced on V (leading to the cases (g) and (h)). Although not entirely self contained (we rely on Jonathan Hall's classification of cotriangular spaces [5]), our proofs and methods are completely elementary.

In the following section we describe the examples occurring in the conclusion of Theorem 1.1 somewhat closer. The Sections 3 and 4 are devoted to the proof of Theorem 1.1. In particular, in Section 3 we consider the case where there are $d, e \in D$ with $[V, d] \cap [V, e]$ being one-dimensional, leading to the groups defined over $\mathbb{F}_2$, while the final section covers the remaining cases of groups defined over $\mathbb{F}_4$.

2 The examples and their geometries

Suppose V is a vector space over the field $\mathbb{F}_2$. If $\dim(V) < \infty$, then the generic example of a group G generated by a class D of elements d with $[V, d]$ of dimension 2 is the group $\mathrm{SL}(V)$. The elements in D correspond to conjugates of the element

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

In small dimensions we encounter two exceptional examples of groups satisfying the hypothesis of Theorem 1.1.

Suppose V has dimension 4. The group $\mathrm{SL}(V)$ is isomorphic to Alt_8. Under this isomorphism the elements in D correspond to those elements in Alt_8 that are products of two disjoint 3-cycles. The subgroup Alt_7 of Alt_8 also acts irreducibly on V and is of course generated by its elements of D.

Suppose $\dim(V) = 3$. Then the group $\mathrm{SL}(V)$ contains a subgroup $7:3$, a split extension of a group of order 7 by a group of order 3, which is irreducible on V. This subgroup is generated by its elements of order 3.

If V is infinite dimensional, then the set of all elements of $\mathrm{GL}(V)$ of order 3 with two dimensional commutator generates the subgroup $\mathrm{FSL}(V)$ of $\mathrm{GL}(V)$, consisting of all the finitary elements of determinant 1, see [1]. Inside this group we encounter more examples which we now describe. Let $0 \neq v \in V$ and $0 \neq \varphi \in V^*$ with $\varphi(v) = 0$. Then $t_{v,\varphi}$ denotes the *transvection*

$$t_{v,\varphi} : V \to V, w \in V \mapsto w + \varphi(w)v.$$

The group $G = \mathrm{T}(V, \Phi)$ where Φ is a subspace of V^* is defined to be the subgroup of $\mathrm{GL}(V)$ generated by all transvections $t_{v,\varphi}$ with $0 \neq v \in V$ and $0 \neq \varphi \in \Phi$ with $\varphi(v) = 0$. Suppose Φ annihilates V. Then one easily checks that $C_V(G) = \{0\}$. If $v, w \in V$ and $\varphi, \psi \in \Phi$ such that $\varphi(v) = \psi(w) = 0$ but $\psi(v) = \varphi(w) = 1$, then the product $d = t_{v,\varphi}t_{w,\psi}$ is an element of order 3 with commutator $[V, d] = \langle v, w\rangle$ of dimension 2. Let D denote the set of all such elements. It is straightforward to check that this set D is a conjugacy class of G generating G. Clearly, $[V, d]$ with d running through D is the set of all 2-spaces of V. So, $[V, G] = V$.

Next suppose that (V, f) is a symplectic space over $\mathbb{F}_2$. The *radical* $\mathrm{Rad}(f)$ is the subspace $\{v \in V | f(v, w) = 0 \text{ for all } w \in V\}$ of V. For any nonzero vector $v \in V \setminus \mathrm{Rad}(f)$ the transvection

$$t_v : V \to V, w \in V \mapsto w + f(v, w)v$$

is a nontrivial element of $\mathrm{Sp}(V, f)$. It is well known that the set of such transvections forms a conjugacy class of 3-transpositions in $\mathrm{Sp}(V, f)$. If we set D to be the set of all products $t_v t_w$ where $v, w \in V$ with $f(v, w) = 1$, then one readily checks that D is a conjugacy class of elements of order 3 in $G = \langle D \rangle$ with $[V, d]$ of dimension 2. The radical $\mathrm{Rad}(f)$ is contained in $C_V(G)$.

By $\mathcal{S}p(V, f)$ we denote the partial linear space (P, L) where P consists of all the nonzero vectors of V. A line in L is the set of three nonzero vectors in a 2-dimensional subspace W of V on which f does not vanish.

Notice that for each $d \in D$, the commutator $[V, d]$ determines a unique line of $\mathcal{S}p(V, f)$. The class D generates $\mathrm{FSp}(V, f)$ provided $\dim(V) \geq 6$. Here $\mathrm{FSp}(V, f)$ denotes the finitary symplectic group, i.e., the subgroup of finitary elements in the symplectic group $\mathrm{Sp}(V, f)$. If $\dim(V) = 4$ or 2, then D generates the derived groups $\mathrm{Alt}_6 \leq \mathrm{Sp}(4, 2)$ and $\mathrm{Alt}_3 \leq \mathrm{Sp}(2, 2)$, respectively.

Next consider a quadratic form Q on V whose associated bilinear form is the symplectic form f. The *radical* $\mathrm{Rad}(Q)$ of Q is defined to be the radical of f. The transvection t_v, where $v \in V \setminus \mathrm{Rad}(Q)$ with $Q(v) = 1$, is in the orthogonal group $\mathrm{O}(V, Q)$. Suppose $\dim(V) \geq 6$. Then the subset D_Q of D of all elements obtained as products $t_v t_w$ with $Q(v) = Q(w) = 1$ and $f(v, w) = 1$ is a conjugacy class of $\Omega(V, Q)$, the derived subgroup of $\mathrm{O}(V, Q)$. The set D_Q generates the finitary group $\mathrm{F}\Omega(V, Q)$. The radical of Q is centralized by $\mathrm{F}\Omega(V, Q)$.

The corresponding geometry $\mathcal{N}(V, Q)$ has as points the vectors $v \in V$ with $Q(v) = 1$. A typical line is the set of three nonzero vectors in an elliptic 2-space, i.e., a 2-space in which $Q(v) = 1$ for any nonzero vector v contained in it. Clearly, $\mathcal{N}(V, Q)$ is a subspace of $\mathcal{S}p(V, f)$.

There is yet another class of subgroups of $\mathrm{Sp}(V, f)$ generated by a subset of D. To describe this class we will start with a particular description of the symplectic space (V, f). Indeed, the symplectic space (V, f) might be obtained as follows. Suppose Ω is a (possibly infinite) set. Let $\mathbb{F}_2\Omega$ be the $\mathbb{F}_2$-vector space with basis Ω. By $E\mathbb{F}_2\Omega$ we denote the subspace of $\mathbb{F}_2\Omega$ generated by the vectors $\omega_1 + \omega_2$, where $\omega_1, \omega_2 \in \Omega$. Notice that the standard dot product on $\mathbb{F}_2\Omega$ induces a symplectic form on $E\mathbb{F}_2\Omega$. We can identify (V, f) with this symplectic space. The transpositions in $\mathrm{Sym}(\Omega)$ induce transvections on V. So, the 3-cycles in $\mathrm{Sym}(\Omega)$, (i.e., the products of two noncommuting transpositions) induce a subset D_Ω of D generating a subgroup of $\mathrm{GL}(V)$ isomorphic to the (finitary) alternating group $\mathrm{FAlt}(\Omega)$, the subgroup of $\mathrm{Sym}(\Omega)$ of all permutations with finite support that are even. The corresponding geometry $\mathcal{T}(\Omega)$ has as points the vectors $\omega_1 + \omega_2$, where $\omega_1 \neq \omega_2 \in \Omega$, a line being the triples of points of the form $\omega + \omega'$, where $\omega \neq \omega'$ are taken from some subset of size 3 of Ω.

The above geometries are all examples of *cotriangular spaces*. These are partial linear spaces with lines of size three and having the property that any point p not on a line l is collinear with no point or with exactly two points on l. A cotriangular space is called *irreducible* if it is connected and for any pair of points p, q we have that $p^\perp = q^\perp$ implies $p = q$. Here $p^\perp$ denotes the set consisting of p and all points *not* collinear with p. The spaces described above are characterized by the following result (rephrased to fit our purposes) of Jonathan Hall.

2.1 (J.I. Hall [5]) *Let V be a vector space over the field $\mathbb{F}_2$. Let $\Pi = (P, L)$ be an irreducible cotriangular space, where P is a subset of $V \setminus \{0\}$, and each line in L is a triple of points inside a 2-dimensional subspace of V. If P generates V and $\bigcap_{p\in P}\langle p^\perp\rangle = \{0\}$ we have one of the following:*

(a) *$\Pi = \mathcal{S}p(V, f)$ for some nondegenerate symplectic form f on V.*

(b) *$\Pi = \mathcal{N}(V, Q)$ for some nondegenerate quadratic form Q on V with trivial radical.*

(c) *There is a set Ω such that $\Pi = \mathcal{T}(\Omega)$ and $V = E\mathbb{F}_2(\Omega)$ or, in case $|\Omega|$ is even, $V = E\mathbb{F}_2(\Omega)/\langle\Sigma_{\omega\in\Omega}\,\omega\rangle$, the quotient of $E\mathbb{F}_2(\Omega)$ by the all one vector.*

Finally we shall discuss the examples coming from groups defined over $\mathbb{F}_4$. Let V_4 be a vector space over $\mathbb{F}_4$. For every $v \in V_4$ and $\varphi \in V_4^*$, with $\varphi(v) \neq 0, 1$ we define the map

$$r_{v,\varphi} : V \to V_4, w \in V_4 \mapsto w - \varphi(w)v.$$

The map $r_{v,\varphi}$ is a *reflection* with *center* $\langle v\rangle$ and *axis* $\ker\varphi$. A reflection has order 3. If Φ is a subspace of V_4^*, then denote by $\mathrm{R}(V_4, \Phi)$ the subgroup of $\mathrm{GL}(V_4)$ generated by all reflections $r_{v,\varphi}$ with $v \in V_4$, $\varphi \in \Phi$ and $\varphi(v) \neq 0, 1$. If $\dim(V_4)$ is finite-dimensional, then $\mathrm{R}(V_4, V_4^*) = \mathrm{GL}(V_4)$. Let V denote the space V_4 considered as an $\mathbb{F}_2$-space. The reflections provide examples of elements of order 3 having a 2-dimensional commutator on V.

If h is a nondegenerate Hermitian form on V_4, then for each vector $v \in V$ with $h(v, v) = 1$ and $\alpha \in \mathbb{F}_4$, $\alpha \neq 0, 1$, the map

$$r_v : w \in V_4 \mapsto w + \alpha h(w, v)v$$

is a reflection in the finitary unitary group

$$\mathrm{FU}(V_4, h) = \{g \in \mathrm{FGL}(V) \mid \forall_{x,y\in V} \quad h(xg, yg)) = h(x, y)\}.$$

In fact, all these reflections generate the finitary group $\mathrm{FU}(V_4, h)$.

Notice that in these examples over $\mathbb{F}_4$, the commutators $[V, r_1]$ and $[V, r_2]$, where r_1 and r_2 are reflections on V_4, either are equal or meet trivially.

3 Geometries with points and groups over $\mathbb{F}_2$

Let V be a vector space over $\mathbb{F}_2$, and suppose $G \leq \mathrm{GL}(V)$ is generated by a normal set D of elements $d \in G$ of order 3 such that $[V, d]$ is 2-dimensional. (Here normal means closed under conjugation.)

3.1 *Suppose $d \in D$. Then $V = [V, d] \oplus C_V(d)$.*

Proof. Suppose $v \in V$ with $[v, d] \in [V, d] \cap C_V(d)$. Then $0 = [vd + v, d] = vd^2 + vd + vd + v = vd^2 + v$. But then $[v, d] = (vd^2 + v)d = 0$. We have found that $[V, d] \cap C_V(d) = 0$. As each $v \in V$ equals $v = v + [vd, d] + [vd, d] = v + vd + vd^2 + [vd, d] \in C_V(d) + [V, d]$ we have proved that $V = [V, d] + C_V(d)$. $\square$

3.2 *A subspace W of V is invariant under $d \in D$ if and only if $W \leq C_V(d)$ or $[V, d] \leq W$.*

Proof. If W is centralized by d, it clearly is invariant. If $[V, d] \leq W$, then for $w \in W$ we have $wd = [w, d] + w \in W$ and we find W to invariant under d.

Now suppose $w \in W$ is invariant, but not centralized by d. Then $0 \neq [w, d] \in W \cap [V, d]$. But then $[V, d] = \langle [w, d], [w, d]d \rangle$ is contained in W. $\square$

A *D-line*, or just *line*, is a subspace of V of the form $[V, d]$ with $d \in D$. A *D-point*, or just *point*, is a 1-space of V which is the intersection of two distinct D-lines. Both points and lines are also considered to be points and lines of the projective space $\mathbb{P}(V)$.

Let $\mathcal{P}$ be the set of D-points and $\mathcal{L}$ the set of D-lines. The geometry $\Pi(D)$ is the pair $(\mathcal{P}, \mathcal{L})$, where incidence is symmetrized containment. A line is often identified with the set of points it contains.

If W is a subspace of V, then by $\Pi(D)_W$ we denote the pair $(\mathcal{P}_W, \mathcal{L}_W)$ where $\mathcal{L}_W$ is the set of D-lines contained in W, and $\mathcal{P}_W$ the set of intersection points of two distinct lines in $\mathcal{L}_W$ meeting nontrivially.

Let U be a subspace of V. Then by D_U we denote the set of all $d \in D$ with $[V, d] \leq U$. The subspace A_U of V is equal to $\bigcap_{d \in D_U} C_V(d)$.

3.3 *If l is a D-line, then it contains zero or three D-points.*

Proof. Since $[V, d] \cap C_V(d) = \{0\}$ by (3.1), we find that d is transitive on the three nonzero vectors in $[V, d]$. This clearly implies the result. □

3.4 *Suppose l and m are distinct D-lines intersecting at a point. Let W be the subspace $l + m$ of V. Then $\Pi(D)_W$ is either a dual affine plane or a projective plane in $\mathbb{P}(W)$.*

The group $\langle D_W \rangle$ is transitive on the lines in $\mathcal{L}_W$.

Proof. Let $d \in D$ with $[V, d] = l$. Then $\langle d \rangle$ fixes a unique point in $\mathbb{P}(W)$, call this point q, is transitive on the three points of l, and transitive on the three remaining points of $\mathbb{P}(W)$. The group $\langle d \rangle$ fixes l, is transitive on the three lines of $\mathbb{P}(W)$ on q and on the three remaining lines.

If q is a point on a D-line, then clearly $\langle D_W \rangle$ is transitive on the points and lines of $\mathbb{P}(W)$ and $\Pi(D)_W$ equals $\mathbb{P}(W)$.

If q not on any D-line, then $\Pi(D)_W$ equals the dual affine plane of all points of $\Pi(D)_W$ different from q and all lines not on q. Also in this case $\langle D_W \rangle$ is transitive on the points and lines of $\Pi(D)_W$. □

3.5 *If D is a conjugacy class in G, then G is transitive on $\mathcal{L}$ and $\mathcal{P}$.*

Proof. Transitivity of G on D implies transitivity on lines. As each $d \in D$ is transitive on the three 1-spaces of the line $[V, d]$, transitivity on points follows immediately. □

3.6 *If $l, m \in \mathcal{L}$ are in the same connected component Π_0 of $\Pi(D)$ and $l = l_0, \ldots, l_k = m$ is a path from l to m inside Π_0, with l_i and l_{i+1} intersecting at a point for $0 \leq i < k$, then there is a $g \in \langle D_{l_0}, \ldots, D_{l_k} \rangle$ with $lg = m$.*

Proof. By (3.4), there is for $i = 0, \ldots, k-1$ a $g_i \in \langle D_{l_0}, \ldots, D_{l_k} \rangle$ with $l_i g_i = l_{i+1}$. But then $g = g_0 \cdots g_{k-1}$ maps l to m. □

3.7 *Suppose $l \neq m \in \mathcal{L}$ are in the same G-orbit on $\mathcal{L}$. Then $l \subseteq A_m$ if and only if $m \subseteq A_l$.*

Proof. Suppose $l \subseteq A_m$, then l and therefore also A_l is invariant under each element $d \in D_m$. So, by (3.2) we either have $m \subseteq A_l$ or $A_l \subseteq A_m$. In the latter case the inclusion is proper since $l \in A_m$ but not in A_l.

Now suppose $m \not\subseteq A_l$. Then $A_l \subset A_m$. Let g be an element in G with $mg = l$, which is the product of a finite number of elements from D. Such an element exists by (3.6). Then $A_l = A_m g \subset A_m$. However, as $g \in G$ is the product of a finite number of elements from D, the subspace $C_V(g)$ has finite codimension in V. As g induces a bijective map $A_m/(A_m \cap C_V(g)) \to A_m g/(A_m g \cap C_V(g))$ between finite dimensional spaces, these spaces are equal. This implies $A_l = A_m g = A_m$ contradicting the above. $\square$

3.8 *Suppose $\mathcal{L}$ is a single G-orbit. Let $d, e \in D$. If $\dim(C_V(d) \cap [V,e]) = 1$, then $[V,d]$ and $[V,e]$ are in the same connected component of $\Pi(D)_{[V,d]+[V,e]}$.*

Proof. Let $d, e \in D$ with $\dim(C_V(d) \cap [V,e]) = 1$. Notice that $[V,e] \not\subseteq A_{[V,d]}$. So, by (3.7) we can assume, eventually after replacing e by some appropriate element in $D_{[V,e]}$, that $[V,d] \not\subseteq C_V(e)$.

The lines $[V,e]d$ and $[V,e]$ intersect in the point $C_V(d) \cap [V,e]$ and are contained in the 4-dimensional subspace $[V,d] + [V,e]$. The subspace $W = [V,e] + [V,e]d$ is 3-dimensional and $\Pi(D)_W$ is a dual affine or projective plane inside $\mathbb{P}(W)$.

The line $[V,d]$ intersects $\mathbb{P}(W)$ in a point p of $\mathbb{P}(W)$. If this point p is in $\mathcal{P}_W$ we are done. Thus assume that this point is not in $\mathcal{P}$. In this case $\Pi(D)_W$ is a dual affine plane and p is the unique 1-space of $\mathbb{P}(W)$ not in this dual affine plane. In particular, p equals $C_W(e)$.

By the same arguments we can assume that $[V,d]$ meets $W' = [V,e] + [V,e]d^2$ in the point $q = C_{W'}(e)$ which is the unique 1-space of $\mathbb{P}(W')$ not in $\Pi(D)_{W'}$. As $p \neq q$, we find that $[V,d] = \langle p, q\rangle \leq C_V(e)$, which contradicts our assumption. $\square$

3.9 *Suppose $C_V(G) = \{0\}$ and $\mathcal{P} \neq \emptyset$. Then Π is connected if and only if G is transitive on $\mathcal{L}$.*

Moreover, if Π is connected, then its diameter is at most 2.

Proof. First assume that Π is connected. Then by (3.6) G is transitive on $\mathcal{L}$.

Now suppose that Π is not connected, but G is transitive on $\mathcal{L}$. Since G is generated by D, there are $f, h \in D$ such that $[V,h]$ and $[V,h]f = [V,h^f]$ are in different components of Π. Then also $[V,h]f$ and $[V,h]f^2$ are in different components. Since h^f does not centralize both $[V,h]$ and $[V,h]f^2$, we can assume that there are noncommuting d and e in D with $[V,d]$ and $[V,e]$ in different connected components of Π. Fix such an element e. By (3.8) we can assume that $C_V(e) \cap [V,d] = \{0\}$.

Let $f \in D$ such that $[V,f]$ meets $[V,d]$ in a point and let W be the subspace $[V,d] + [V,f]$. If $\Pi(D)_W$ contains a line $[V,g]$ with $g \in D$ such that $C_V(e) \cap [V,g]$ is 1-dimensional, then (3.8) gives a contradiction with $[V,e]$ being in a connected component of $\Pi(D)$ different form the one containing

$[V,d]$. Hence $\Pi(D)_W$ is a dual affine plane, moreover, $C_V(e)$ meets W in the unique 1-space p not in that dual affine plane.

As $C_V(G)=\{0\}$, there is an element $h\in D$ not centralizing p. But then $C_V(h)$ meets at least one of the lines of the dual affine plane $\Pi(D)_W$ in a point. Without loss of generality we may assume this line to be $[V,d]$. Let $U=[V,d]+[V,h]$. If $[V,d]$ and $[V,h]$ meet nontrivially, then $\Pi(D)_U$ is a projective plane containing a line $l\in\mathcal{L}$ which meets $C_V(e)$ in a point. As above, this leads by (3.8) to a contradiction. Thus $\dim(U)=4$. But then $U_1=[V,d]+[V,d]h$ and $U_2=[V,d]+[V,d]h^2$ are two distinct 3-dimensional spaces on $[V,d]$. As above, for both $i=1$ or 2, we can assume that $C_V(e)$ meets U_i in a 1-space, which is the unique 1-space of U_i which is not in $\mathcal{P}_{U_i}$. But that implies that $C_U(e)=C_U(d)$.

By (3.6) the above reasoning also applies to $[V,h]$ and h, so that $C_U(e)=C_U(h)$. But $C_U(h)\neq C_U(d)$, which is a final contradiction. Hence we have shown that Π is connected.

It remains to prove that the diameter of Π is at most 2, provided Π is connected. So assume Π to be conected and let p,q,r,s be a path of length 3 in the collinearity graph of Π. Then both p and s have at least 2 neighbors on the line through q and r, see (3.4). But that implies that they have a common neighbor. So, indeed, the diameter of Π is at most 2. □

Let $p,q\in\mathcal{P}$ be points. We write $p\sim q$ if p and q are distinct collinear points of Π. By $p\perp q$ we mean that p and q are equal or noncollinear. By $p^\sim$ we denote the set of all points collinear to p (excluding p). The complement of $p^\sim$ in $\mathcal{P}$ is the set $p^\perp$.

If for p and q we have $p^\perp=q^\perp$, then we write $p\equiv q$. The relation $\equiv$ is obviously an equivalence relation.

3.10 *Suppose* Π *is connected. If* $p\neq q\in\mathcal{P}$ *with* $p\equiv q$*, then* $C_V(G)\cap p+q\neq\{0\}$*.*

Proof. Suppose $p\neq q\in\mathcal{P}$ with $p\equiv q$. Notice that $p\perp q$. Let r be the third point on the projective line through p and q. If $l\in\mathcal{L}$ is a line on p, then Π_W is a dual affine plane, where W is the subspace spanned by l and q. So, each $d\in D_l$ centralizes r.

If p_1 and q_1 are two noncollinear points in Π_W, then the projective line on p_1 and q_1 contains r. Moreover, if $s\in p_1^\perp$ but not in $q_1^\perp$, then either s is collinear to p but not to q, or vice verse. As this contradicts $p\equiv q$, we find that $p_1\equiv q_1$.

But that implies, by connectivity of Π, that r is in $C_V(d)$ for each $d\in D$. In particular, $C_V(G)\cap p+q\neq\{0\}$. □

3.11 *Suppose G is transitive on $\mathcal{L}$, $C_V(G) = \{0\}$, $[V,G] = V$ and $\mathcal{P} \neq \emptyset$.*

If $(\mathcal{P},\mathcal{L})$ contains no projective planes, then the subspace $\bigcap_{p\in\mathcal{P}} \langle p^\perp\rangle$ of V is equal to $\{0\}$.

Proof. Suppose $(\mathcal{P},\mathcal{L})$ does not contain projective planes. Let p be a point in $\mathcal{P}$ and l a line on p. Then there is an element $d \in D$ with $l = [V,d]$.

Let q be a point in $p^\perp$. If q is collinear with a point on l, then inside the 3-space $\langle q,l\rangle$ we find a unique 1-space r, which is not in the dual affine plane generated by q and l. This point r is centralized by d and on the line through p and q. But then $q \in \langle p,r\rangle \subseteq \langle p, C_V(d)\rangle$.

If q is not collinear to a point on l, then, by (3.9), we can find a point s collinear with both q and the point $p' = pd$ on l. We can assume that s is in $p^\perp$. (Indeed, if s is collinear to p, then we may replace it by the third point on the line through p' and s, which is not collinear with p.) Let t be the third point on the line through s and q. Notice that also t is in $p^\perp$ but collinear to p'. By the previous paragraph we find both $s,t \in \langle p, C_V(d)\rangle$ but then also $q \in \langle p, C_V(d)\rangle$. This shows that $\langle p^\perp\rangle$ is contained in $\langle p, C_V(d)\rangle$. Moreover, as $C_V(d)$ has codimension 2 in V, we find $\langle p^\perp\rangle$ to be a proper subspace of V not containing $l = [V,d]$.

This implies that the space $\bigcap_{p\in\mathcal{P}} \langle p^\perp\rangle$, which is invariant under each $e \in D$, does not contain any line from $\mathcal{L}$. But then (3.2) implies that $\bigcap_{p\in\mathcal{P}} \langle p^\perp\rangle$ is centralized by each $e \in D$ and hence is contained in $C_V(G) = \{0\}$, proving the result. □

3.12 Theorem. *Suppose G is transitive on $\mathcal{L}$, $C_V(G) = \{0\}$, $[V,G] = V$ and $\mathcal{P} \neq \emptyset$. Then, up to isomorphism, Π is one of the following spaces:*

(a) $\mathbb{P}(V)$.

(b) $\mathcal{S}p(V,f)$ *for some nondegenerate symplectic form f on V.*

(c) $\mathcal{N}(V,Q)$ *for some nondegenerate quadratic form Q on V with trivial radical.*

(d) *There is a set Ω such that $V = E\mathbb{F}_2(\Omega)$ or, in case $|\Omega|$ is even, $V = E\mathbb{F}_2(\Omega)/\langle\Sigma_{\omega\in\Omega}\,\omega\rangle$, the quotient by the all one vector, and $\Pi = \mathcal{T}(\Omega)$.*

Proof. By (3.4) and (3.9), the space Π is a connected partial linear space of order 3 in which any two intersecting lines generate a subspace isomorphic to a projective or a dual affine plane.

If all planes are projective, then clearly $\Pi = \mathbb{P}(V)$ and we are in case (a). If all planes are dual affine, then by (3.11) we can apply Theorem 2.1 and we are in one of the cases (b), (c) or (d).

Now to prove the theorem it suffices to show that Π cannot contain both a projective and dual affine plane. So, suppose it does. Let π be a projective plane and p a point outside π. We claim that p is collinear with all or all but one of the points of π. First assume that p is collinear with some point $q \in \pi$ and let r be the third point on the line through p and q. If $p^\perp \cap \pi$ is a line, then so is $r^\perp \cap \pi$. This would imply that there is a point in $\pi \cap p^\perp \cap r^\perp \subset q^\perp$, which is clearly impossible. So $p^\perp$ meets π in at most one point.

Now assume that $\pi \subseteq p^\perp$. As the diameter of Π is at most 2, there is a point q collinear to p and also to some point in π. Since by the above $q^\perp \cap \pi$ contains at most one point, there is a projective plane π' on q meeting π in a line. But then $p^\perp$ meets π' in a line, which contradicts the above, and we have proved our claim.

Now suppose p and q are noncollinear points. By (3.5) each line on p is in a projective plane and hence contains at most one point in $q^\perp$. So, all points collinear to p are also collinear to q. Similarly all points collinear to q are also collinear p. This implies that $p \equiv q$, and, by (3.10), contradicts that $C_V(G) = \{0\}$. □

3.13 Theorem. *Suppose* $\Pi = \mathbb{P}(V)$. *Then, up to isomorphism,* $G = \mathrm{T}(V, \Phi)$ *for some subspace* Φ *of* V^* *annihilating* V, *or* $\dim(V) = 4$ *and* $G = \mathrm{Alt}_7$, *or* $\dim(V) = 3$ *and* $G = 7:3$.

Proof. First assume that the group G contains a transvection τ. As G is transitive on the points in $\mathbb{P}(V)$, see (3.5), each point in $\mathbb{P}(V)$ serves as center of some transvection in G. Suppose H is a hyperplane of V serving as the axis of some transvection $\tau \in G$. Let p be the center of this transvection. If q is now a second point in H, then let e be an element of D with $p, q \subseteq [V, e]$. Then $q = pe$ or $q = pe^{-1}$ and the transvection with center q and axis H is a conjugate of τ in G.

Now let K be a second hyperplane of $\mathbb{P}(V)$ serving as transvection axis for some transvection in G. Then by the above we can find transvections τ and σ in G with the same center and with axis H and K respectively. But then $\sigma\tau$ is a transvection with axis the unique hyperplane L distinct from H and K containing $H \cap K$. So the elements of V^* serving as transvection axis for some transvection in G form the set of nonzero vectors of a subspace Φ of V^*. This implies that the transvections in G generate the subgroup $\mathrm{T}(V, \Phi)$ of G.

If $\dim(V) = 3$ or 4, and G contains a transvection, then by the above $G = \mathrm{SL}(V)$. So, assume that G does not contain any transvection.

If $\dim(V) = 3$, then any involution in $\mathrm{GL}(V)$ is a transvection. So, $|G|$ is odd and $|G| \mid 21$. On the other hand, G is transitive on the 7 lines, while an element $d \in D$ fixes a line. Hence G has order 21 and is isomorphic to $7 : 3$.

If $\dim(V) = 4$, then G has order divisible by $3 \cdot 15 \cdot 7$ as the stabilizer of a point-line flag has order at least 3. Indeed, an element $d \in D$, fixes a point-line-flag. An easy computation within $\mathrm{Alt}_8 \simeq \mathrm{GL}(4, 2)$ reveals that $G \simeq \mathrm{Alt}_7$.

Now assume that $\dim(V) \geq 5$. Fix an element $d \in D$ and the consider the line $[V, d]$. This line is contained in 5-dimensional subspace U of V. The intersection of U with $C_V(d)$ is 3-dimensional. Pick two lines l and m in $\mathcal{L}$ spanning $C_V(d) \cap \Delta$. The above shows that inside the subgroups generated by $D_{[V,d]+l}$ and $D_{[V,d]+m}$, respectively, we can find elements $e \in D_l$ and $f \in D_m$ not centralizing $[V, d]$. But then it is straightforward to check that among the conjugates of d under $\langle e, f\rangle$ we find two elements, d_1 and d_2 say, with $[V, d_1]$ and $[V, d_2]$ meeting at a point. Moreover, as both e and f leave $C_V(d)$ invariant, we have $C_V(d_1) = C_V(d_2) = C_V(d)$. But then either d_1d_2 or $d_1d_2^{-1}$ induces a transvection on $[V, d_1] + [V, d_2]$ with center $[V, d_1] \cap [V, d_2]$. But as $C_V(d)$ is centralized by d_1d_2 or $d_1d_2^{-1}$, we have found a transvection τ on V in G. Now notice that $d_1 \in \langle \tau, \tau^{d_1}\rangle$. So G is generated by its transvections. By the above we can conclude that G equals $\mathrm{T}(V, \Phi)$, where Φ is some subspace of V^*. Since $\bigcap_{\varphi\in\Phi} \ker\varphi$ is centralized by G, we can conclude that $\bigcap_{\varphi\in\Phi} \ker\varphi = \{0\}$ and Φ annihilates V. □

3.14 Theorem. *Suppose Π is a nondegenerate cotriangular space as in case (b), (c) or (d) of (3.12). Then, up to isomorphism, we have one of the following.*

(a) $G = \mathrm{FSp}(V, f)$ *for some nondegenerate symplectic form f.*

(b) $G = \mathrm{F\Omega}(V, Q)$ *for some nondegenerate quadratic form Q with trivial radical.*

(c) $G = \mathrm{FAlt}(\Omega)$ *for some set Ω, where $V = E\mathbb{F}_2\Omega$ or, in case $|\Omega|$ is even, $V = E\mathbb{F}_2(\Omega)/\langle\Sigma_{\omega\in\Omega}\ \omega\rangle$, the quotient by the all one vector.*

In all cases D is uniquely determined.

Proof. For each line l of Π, there is (up to taking inverses) at most one element $d \in \mathrm{GL}(V)$ with $[V, d] = l$ and centralizing the codimension 2 subspace $\bigcap_{p\in l}\langle p^\perp\rangle$ of V. So, this element is in D. But now it is straightforward to check that the theorem holds. □

The above results classify all the groups satisfying the hypothesis of Theorem 1.1 for which the set $\mathcal{P}$ is nonempty.

4 Pointless geometries and groups over $\mathbb{F}_4$

Suppose V is an $\mathbb{F}_2$-vector space and $G \leq \mathrm{GL}(V)$ a subgroup generated by a normal set D of elements $d \in G$ of order 3 with $[V,d]$ of dimension 2. We keep the notation of the previous section.

In this final section we consider the case where $\mathcal{P}$ is the empty set. Although the set $\mathcal{P}$ is empty, we will still be able to construct a useful geometry. However, now the elements of $\mathcal{L}$ will play the role of 'points' and certain 4-dimensional subspaces of V will play the role of 'lines'. We make this precise in the sequel of this section.

Assume throughout this section that the set $\mathcal{P}$ is empty.

4.1 *If $d, e \in D$, then $[V,d] \cap C_V(e) = \{0\}$ or $[V,d] \leq C_V(e)$.*

Proof. If $[V,d] \cap C_v(e) \neq \{0\}$, then $[V,d]e = [V,d^e]$ meets $[V,d]$ nontrivially. By the assumption that $\mathcal{P}$ is empty, we find $[V,d]e = [V,d]$. By (3.1) and (3.2) we find $[V,d] \leq C_V(e)$. □

A *spread* of a 4-dimensional subspace W of V is a set of 5 subspaces of W of dimension 2, pairwise intersecting in $\{0\}$.

4.2 *Suppose $d, e \in D$ with $[V,d] \neq [V,e]$. If $[V,e] \not\leq C_V(d)$, then $W := [V, \langle d, e\rangle]$ is a 4-dimensional subspace of V containing 4 or 5 lines from $\mathcal{L}$.*

If W contains 5 lines of $\mathcal{L}$, then these 5 lines form a spread in W.

If W contains exactly 4 lines from $\mathcal{L}$, then these 4 lines together with the 2-dimensional space $C_V(d) \cap W$ form a spread of W.

Proof. Let d, e be elements in D with $[V,d] \neq [V,e]$. Then, as by assumption $[V,d] \cap [V,e] = \{0\}$, the space $W := [V,d] + [V,e]$ is 4-dimensional.

If d does not centralize $[V,e]$, then W contains the four lines $[V,d]$, $[V,e]$, $[V,e]d$ and $[V,e]d^2$. So , it contains 4 or 5 lines from $\mathcal{L}$.

Clearly, if W contains 5 lines from $\mathcal{L}$, then these lines from a spread.

If W contains 4 lines from $\mathcal{L}$, then none of these 4 lines is in $C_V(d)$. So, by (4.1), these 4 lines together with $C_V(d)$ form a spread. □

A spread is called *full* if it contains 5 lines from $\mathcal{L}$. The set of all full spreads is denoted by $\mathcal{F}$. A subspace W as in (4.2) containing exactly 4 lines from $\mathcal{L}$ contains a unique spread containing these four lines. This spread is called a *tangent* spread. By $\mathcal{T}$ we denote the set of tangent spreads. The fifth 2-space in such a tangent spread is called the *singular* line of the spread. The set of all singular lines is denoted by $\mathcal{L}_S$.

If we identify each (full or tangent) spread with the set of lines from $\mathcal{L}$ contained in it, then $(\mathcal{L}, \mathcal{F} \cup \mathcal{T})$ is a partial linear space, which we denote by Δ.

4.3 *Let W be a 4-space containing a full or tangent spread S. If S is a tangent spread, then $\langle D_W \rangle$ induces Alt_4 on the lines of $\mathcal{L}$ in S. If S is full, then $\langle D_W \rangle$ induces Alt_5 on the lines of $\mathcal{L}$ in S.*

Proof. Let l be a line of S. An element $d \in D_l$ induces a 3-cycle on the lines in S and fixes l. So, if S is a tangent spread, then $\langle D_S \rangle$ induces the 2-transitive group Alt_4 on the 4 lines in S.

If S is full, then let m be the unique line of S different from l fixed by d. If there is an element $e \in D_m$ not fixing l, then $\langle d, e \rangle$ induces the 2-transitive group Alt_5 on S.

So, assume that $l \subseteq A_m$, then any line $k \neq l, m$ in S is not in A_k, see (3.7). Let k be such a line and $f \in D_k$ an element not fixing m. Then $\langle d, e, f \rangle$ induces the the 2-transitive group Alt_5 on S. □

4.4 *If S is a full or tangent spread and $d \in D$, then $C_V(d)$ either contains S or $C_V(d) \cap S$ is a line in $\mathcal{L} \cup \mathcal{L}_S$.*

Proof. This follows from (4.1). □

4.5 *If $l \in \mathcal{L}$ and $h \in \mathcal{L}_S$, then $h \cap l = \{0\}$.*

Proof. Let S be a tangent spread containing h and let W be the 4-space containing S. Fix an element $d \in D$ with $l = [V, d]$. Suppose l meets h nontrivially. Then $C_V(d) \cap W$ is a line $m \in \mathcal{L}$ of S. Now let $e \in D$ with $[V, e]$ a line of S distinct from m. Then $[V, e] + [V, d]$ contains a spread T and meets W in $[V, e] + (l \cap h)$. A point in $[V, e] + (l \cap h)$ not on $[V, e]$ or h is on a line of $\mathcal{L}$ inside S and on some line in $\mathcal{L} \cup \mathcal{L}_S$ of T. Since there is at most one line of $\mathcal{L}_S$ in T, there is a point in $[V, e] + (l \cap h)$ on two distinct lines of $\mathcal{L}$, which contradicts $\mathcal{P}$ being empty. □

4.6 *Suppose W is a 4-space containing a tangent spread with singular line h. Suppose $f \in D$ does not centralize h.*

Then there exists an involution $t \in \langle D_W \rangle$ with $h = [V, t]$ being the singular line of the spread, that does not centralize $[V, f]$.

Proof. Let S be the tangent spread in W and h its singular line. Let $f \in D$ be an element not centralizing h. Then, f centralizes a line, say $[V,d]$ of S, with $d \in D$. If d does not centralize $[V,f]$, then $[V,d]+[V,f]$ contains a full spread. So, after replacing d by a suitable element in $D_{[V,d]}$, we can assume that d centralizes $[V,f]$, see (3.7). Let e be an element in D with $[V,e]$ a line in S different from $[V,d]$. If e centralizes $[V,f]$, then $[V,f]+[V,e]$ is a full spread. So again, after replacing e by a suitable element in $D_{[V,e]}$, we can assume that e does not centralize $[V,f]$.

The group $\langle d,e\rangle$ induces Alt_4 on S. After replacing e by e^{-1}, if necessary, the element $g=de$ induces a product of two disjoint 2-cycles in Alt_4 on S. Then for $0 \neq v \in [V,f]$, we have $0 \neq [v,g]=[v,de]=[v,e] \in [V,e]$ and $[v,g^2]=[[v,g],g] \neq 0$ as $[V,e] \cap C_V(g)=\{0\}$.

Let t be the element $g^2 \neq 1$. Then t does not centralize $[V,f]$. The element t fixes all five lines of S. In particular, it fixes h pointwise. Any 3-space on a line $l \in S \setminus \{h\}$ meets all other lines form S, including h, in exactly one point. So, such 3-space is not only t-invariant, all its points not on l are fixed by t. By varying l and the 3-space, we find all points $\mathbb{W}$ to be fixed by t. So, $[V,t] \subseteq W \subseteq C_V(t)$, and we can deduce that t has order 2. Indeed, for all $v \in V$ we have $vt^2+vt=[v,t]t=[v,t]=vt+v$, and thus $vt^2=v$. Moreover, $C_V(t)$ contains $C_V(d) \cap C_V(e)$ and S and therefore is of codimension at most 2. Since t commutes with g, we find that $[V,t]$ is contained in h. Indeed, for all $v \in V$ we have $[[v,t],g]=(vt+v)g+(vt+v)=vgt+vg+vt+v=[vg+v,t]=[[v,g],t]=0$. So $[V,t] \subseteq C_V(g) \cap S=h$. Hence, either t is a transvection, or $[V,t]=h$. If t is a transvection, then there is a line k in $\mathcal{L}$ meeting the axis of t in a point. But then k and kt meet nontrivially, contradicting $\mathcal{P}$ to be empty. So t is indeed the element we are looking for. □

4.7 *Suppose $h \in \mathcal{L}_S$ and $d \in D$ not centralizing h. Then $[V,d]+h$ is a 4-dimensional space containing two lines from $\mathcal{L}$ and three from $\mathcal{L}_S$, pairwise nonintersecting.*

Proof. Fix a tangent spread S containing h and an element $d \in D$ not centralizing h. Let t be an involution as in (4.6). Then $[V,d]$ and $[V,d]t$ are two lines from $\mathcal{L}$ in $[V,d]+h$, and h, hd and hd^2 are three lines from $\mathcal{L}_S$ in S. By construction the three lines in $\mathcal{L}_S$ do not intersect. So, the result follows by (4.5). □

The five lines from $\mathcal{L} \cup \mathcal{L}_S$ in a subspace $[V,d]+h$, where $h \in \mathcal{L}_S$ and

$d \in D$ not centralizing h, form a spread of $[V, d] + h$ called a *hyperbolic* spread. The set of all hyperbolic spreads is denoted by $\mathcal{H}$.

4.8 *If $l, h \in \mathcal{L}_S$ are distinct, then $h \cap l = \{0\}$.*

Proof. Suppose h and l meet in a point p. Let S be a tangent spread containing l and suppose $f \in D$ is an element not centralizing l. By (4.6) there are involutions t_l and t_h in G with $[V, t_l] = l$ and $[V, t_h] = h$ not centralizing $[V, f]$. The space $[V, f] + h$ is a hyperbolic spread meeting $\mathcal{L}$ in the lines $[V, f]$ and $[V, f]t_h$, see (4.6), where $[V, f]t_h = C_V(f) \cap S$. Similarly $[V, f] + l$ is a hyperbolic spread T meeting $\mathcal{L}$ in the two lines $[V, f]$ and $[V, f]t_l$, where $[V, f]t_l = C_V(f) \cap T$. But since $([V, f] + l) \cap ([V, f] + h)$ is 3-dimensional, there is a point in $C_V(f) \cap S \cap T$, which has to be on two lines from $\mathcal{L}$. This contradicts our assumption that $\mathcal{P}$ is empty. $\square$

4.9 *If S and T are two spreads (full, tangent or hyperbolic), then $S \cap T$ is empty or a line in $\mathcal{L} \cup \mathcal{L}_S$.*

Proof. This follows immediately from the assumption that $\mathcal{P}$ is empty, (4.8) and (4.5). $\square$

4.10 Δ *is connected if and only if G is transitive on $\mathcal{L}$.*

If Δ is connected, then the diameter of its collinearity graph is at most 2.

Proof. Suppose d, e are elements from D with $[V, d]$ and $[V, e]$ lines not in a spread. Then d centralizes $[V, e]$ and e centralizes $[V, d]$. Since $G = \langle D \rangle$, transitivity on the set $\mathcal{L}$ implies the space Δ has to be connected.

Suppose Δ is connected. Then (4.3) implies G to be transitive on $\mathcal{L}$. Now suppose $[V, d], [V, f], [V, g], [V, e]$ is a path of length 3 in the collinearity graph of Δ. Let S be the spread in $[V, f] + [V, g]$. As both $[V, d]$ and $[V, e]$ are in a spread with at least 3 lines inside S, there is at least one line in S at distance 1 from both $[V, d]$ and $[V, e]$. So the distance between $[V, d]$ and $[V, e]$ is at most 2. This implies that the diameter of the collinearity graph of Δ is at most 2. $\square$

From now on assume that Δ is connected and hence also that G is tranisive on $\mathcal{L}$.

4.11 *If there exists a full spread, then there are no singular lines.*

Proof. Suppose S is a full spread and $d \in D$ with $[V, d] \in S$. If $\mathcal{L}_S$ is nonempty, then by transitivity of G on $\mathcal{L}$ there exists a hyperbolic spread T on $[V, d]$ containing a second line $[V, e]$ with $e \in D$ from $\mathcal{L}$ and three lines from $\mathcal{L}_S$. Let h be a singular line in T. Let t_h be an involution in G as in (4.6) with $[V, t_h] = h$. The involution t_h centralizes a line l in S distinct from $[V, d]$, but it maps $[V, d]$ to $[V, e]$. So, $l + [V, e]$ is also a full spread.

As the group $\langle D_{[V,e]}\rangle$ is transitive on the 4 lines in the full spread $l+[V, e]$, see (4.3), but fixes $[V, d]$, we find at least 4 full spreads on $[V, d]$ inside $S+T$. In particular, there are at least $2 + 16$ lines from $\mathcal{L}$ in $S + T$.

Now fix an element $f \in D_l$ not centralizing $[V, d]$, which exists by (4.3). Then f centralizes at most one of the three singular lines of T. So, we find at least $1 + 3 \cdot 2 = 7$ singular lines in $S + T$. As no two lines from $\mathcal{L} \cup \mathcal{H}$ intersect, there are at least $3\cdot(18+7) = 75$ projective points in $S+T$, which contradicts that $S + T$ has dimension 6, and thus only 63 points. Hence $\mathcal{L}_S$ is empty. □

4.12 Theorem. *Suppose Δ is connected and contains a full spread. Then Δ is isomorphic to a projective space of order* 4. *In particular, G preserves an $\mathbb{F}_4$-structure V_4 on V. Moreover, the group G is isomorphic to* $\mathrm{R}(V_4, \Phi)$ *for some subspace Φ of V_4^* annihilating V_4.*

Proof. Let $l, m \in \mathcal{L}$ be distinct lines not in a full spread. By (4.11), there is not tangent or hyperbolic spread on l and m and we have $l \subseteq A_m$ and $m \subseteq A_l$.

By (4.10) there are two full spreads S on l and T on m meeting at a line n. Let k be a line in S distinct from l and n. Then m and k are in a full spread R. Inside R we can find a line h which spans a full spread with n distinct from S and T. But then none of the lines in R is inside A_l and each of them spans a full spread together with l. So, l is on 5 full spreads each meeting T in a line. Hence, at least one of these spreads contains m. A contradiction. Thus any two lines from $\mathcal{L}$ are in a full spread.

Now $(\mathcal{L}, \mathcal{F})$ is a linear space of order 4. Moreover, it satisfies the Veblen and Young axiom. Indeed, suppose S_1, S_2 are two spreads on a line $l \in \mathcal{L}$, and T_1 and T_2 are two spreads meeting both S_1 and S_2 at lines distinct from l, then as subspaces of V, the intersection $T_1 \cap T_2$ is 2-dimensional and thus, by (4.9), a line of $\mathcal{L}$.

Thus V carries a G-invariant $\mathbb{F}_4$-structure V_4 and we can consider G to be a subgroup of $\mathrm{GL}(V_4)$. The elements in D induce reflections on V_4. Each 1-dimensional subspace of V_4 is in $\mathcal{L}$ and thus serves as center of a reflection. Let $l \in \mathcal{L}$. By (4.3), no element from $\mathcal{L}$ is in A_l. As in [4, 6.2], we can conclude that there is a subspace Φ of V_4^* annihilating V with $G \simeq \mathrm{R}(V, \Phi)$. □

From now one we can and do assume that $\mathcal{F}$ is empty.

4.13 *Suppose S is a tangent spread and h its singular line. If $d \in D$ does not centralize h, then the subspace W generated by S and $[V,d]$ contains 12 lines and 9 singular lines. These 21 lines and spreads in W form a projective plane.*

Proof. Let S be a tangent spread and h its singular line. Let $d \in D$ not centralizing h. Then there is a unique line m in S centralized by d. The space $[V,d]+m$ meets $\mathcal{L}$ in just $[V,d]$ and m, for otherwise it would be a full spread. So, on $[V,d]$ there are 3 tangent and one hyperbolic spread inside W. We now easily deduce that there are $1+9+2=12$ lines of $\mathcal{L}$ in W, each on three tangent spreads. Together they form a dual affine plane.

Inside W we find also 9 singular lines. As in the proof of the above theorem, we find that these lines in W together with the spreads form a projective plane. □

A set of five singular lines in a 2-dimensional subspace of V is called a *singular* spread. By $\mathcal{S}$ we denote the set of all singular spreads.

4.14 *Let S be a tangent spread with singular line h. If $d \in D$ with $[V,d]$ not in S and $C_V(f) \cap S$ equal to h, then $S+[V,f]$ contains 5 singular lines contained in a singular spread and 16 lines, together forming a projective plane of order 4.*

Proof. Every line l of S determines a unique tangent spread with $[V,d]$. So, $[V,d]$ is on at least 4 distinct spreads inside $W := S+[V,d]$. Thus there are at least 13 lines from $\mathcal{L}$ and 5 singular lines from $\mathcal{H}$, which are all in the 4-dimensional space $C_W(d)$. Thus the latter 5 lines form a singular spread T. By similar argument we find that all lines form $\mathcal{L}$ inside W are on 4 tangent spreads. But that implies that there are 16 lines from $\mathcal{L}$ in W forming an affine plane. The rest follows immediately. □

4.15 *If $d \in D$ centralizes $h \in \mathcal{L}_S$, then there is a tangent spread S containing $[V,d]$ and h.*

Proof. Let S be a tangent spread on h. If d does not centralize the spread (i.e. is collinear in Δ with some point of S), then we are done by (4.14). Since the graph Δ is connected, the result follows. □

4.16 *If $h, l \in \mathcal{L}_S$ are two singular lines, then there is a hyperbolic or singular spread containing h and l.*

Proof. Let S be a tangent spread on h and $d \in D$ with $[V, d] \in S$. Let t_h be an involution in G with $[V, t_h] = h$ as in (4.6). If d does not centralize l, then inside the subspace $[V, d]t_h + S$ we find that $h + l$ contains a hyperbolic spread, see (4.13).

If d centralizes h, then $[V, d] + h$ contains a tangent spread and we can apply (4.14) to find that $h + l$ contains a singular spread. □

4.17 *If W is a 6-space in V containing two spreads, then the lines from $\mathcal{L} \cup \mathcal{H}$ and spreads in W form a projective plane of order 4.*

Proof. Let S and T be two spreads in W intersecting at a line l. As each line $m \in S$ different from l forms a spread with each line of T, there are at least and hence exactly 21 lines in W. Clearly the spreads induce a projective plane of order 4 on these 21 lines. □

4.18 Theorem. *Suppose Δ is connected but does not contain full spreads. Then the geometry $(\mathcal{L} \cup \mathcal{L}_S, \mathcal{T} \cup \mathcal{H} \cup \mathcal{S})$ is a projective space of order 4. The set $\mathcal{L}_S$ is the set of absolute points of this projective space with respect to some Hermitian polarity.*

In particular, G preserves a nondegenerate Hermitian $\mathbb{F}_4$-structure (V_4, h). Moreover, G is isomorphic to $\mathrm{FU}(V_4, h)$, with D corresponding to the class of reflections in G.

Proof. By the above we find that the geometry $(\mathcal{L} \cup \mathcal{L}_S, \mathcal{T} \cup \mathcal{H} \cup \mathcal{S})$ is a linear space. As in the proof of (4.12) we can prove this space to be a projective space of order 4. Thus V carries the structure of a vector space V_4 over $\mathbb{F}_4$ invariant under G. In particular, we can consider G to be a subgroup of $\mathrm{GL}(V_4)$.

Let $d \in D$ and $l = [V, d]$, then $A_l = C_V(d)$ is a hyperplane of V_4.

Now consider a singular line h, and let t_h be an involution as in (4.6) with $h = [V, t_h]$. From (4.13) and (4.14) it is readily seen that $A_h := C_V(t_h)$ is a hyperplane of V_4 containing precisely those spreads on h that are tangent or singular. This shows that the map $l \in \mathcal{L} \cup \mathcal{L}_S \mapsto A_l$ is a nondegenerate Hermitian polarity on the projective space $\mathbb{P}(V_4)$. As each $d \in D$ induces a unitary reflection with center $[V, d]$ on V_4, the theorem readily follows. □

Now the Theorems 3.13, 3.14, 4.12 and 4.18 certainly imply Theorem 1.1. Actually, they provide a proof for a slightly more general result, in which the assumption on D being a conjugacy class can be replaced by $\{[V, d] \mid d \in D\}$ being a G-orbit.

References

[1] P.J. Cameron, J.I. Hall, Some groups generated by transvection subgroups, *J. Algebra* **140** (1991), 184–209.

[2] A. Chermak, Quadratic pairs without components. *J. Algebra* **258** (2002), no. 2, 442–476.

[3] A. Chermak, Quadratic Pairs, *J. Algebra* **277** (2004), no. 1, 36–72.

[4] A.M. Cohen, H. Cuypers, H. Sterk, Linear groups generated by reflection tori, *Canadian Journal of Mathematics* **56** (1999), 1149-1174.

[5] J.I. Hall, Graphs, geometry, 3-transpositions, and symplectic $\mathbb{F}_2$-transvection groups, *Proc. London Math. Soc.* (3) **58** (1989), 89-111.

Hans Cuypers
Department of Mathematics
Eindhoven University of Technology
P.O. BOX 513
5600 MB Eindhoven
The Netherlands
email: f.g.m.t.cuypers@tue.nl