BACHELOR

The cap-SET problem

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Abstract

In this report, the cap-SET problem is studied. This problem relates to the game SET and can be described through the vector space $\mathbb{F}_3^4$. The maximal cap-sizes for the first six dimensions are already known. In this report, we shall study the dimensions one to five thoroughly. We shall prove that these known maximal cap-sizes are optimal for the first five dimensions. For higher dimensions, we shall look at what lower bounds and upper bounds for the cap-size are known.

The report starts with an explanation of the game SET and how the game is translated to vector spaces. In the second chapter, a lot of counting will take place. Counting of points, lines, planes, solids and subspaces of all mentioned above in the vector spaces $\mathbb{F}_3^3, \mathbb{F}_q^3$ and also $\mathbb{F}_3^4$ eventually. Chapter 3 is about using Chapter 2 to prove the maximal cap-sizes of vector spaces in lower dimensions, to be exact, the dimension one to three. In Chapter 4 we take a look at the technique called 'Double Counting', which is then used to prove the maximal cap-size in the four-dimensional vector space. After that we take a look at Fourier transforms and obtain some upper bounds in Chapter 5. Here we also prove the maximal size of a cap in the five-dimensional vector space. This is followed by a Chapter 6, which looks into what is already know about lower and upper bounds. The last chapter, Chapter 7, is about the applications of this cap-SET problem.

Contents

1 Introduction

The purpose of this report is to study the maximal size of caps in the game SET and in several other dimensions. In order to do this, there will first be an introduction to the game SET along with some history about the game and an explanation of the game rules. After that, the main problem will be defined.

1.1 History

The game SET was invented by Marsha Jean Falco in 1974. Marsha Falco was a population geneticist. She developed the game while she was doing genetic research in Cambridge. Here she was researching if epilepsy with German Shepherds is hereditary. To study the genes and chromosomes in the cells, Marsha Falco made cards with information for every dog. As certain sections of information were identical on each card, she started drawing symbols that represented those sections with equal information instead of writing it out on every card. She used symbols with different properties, like different colours, different fillings and different quantities, to display the different combinations of genes. While explaining these combinations to other veterinarians she was working with, Marsha Falco realised the cards also served as a relaxing game and that is how SET was born. Marsha Falco played the game for years with friends and family before she founded the company Set Enterprises in 1990 in the US and made SET available for the general public. [8]

1.2 The Rules

The game SET is played with a special deck of cards. Each SET card displays a design with four attributes: number, shape, colour and filling. Each of these four attributes can take three possible values, listed in Figure 1 below:

Attribute	*Possible values*
Number	{ One, Two, Three}
Shape	{ Diamond, Oval, Squiggle}
Colour	{ Red, Green, Purple}
Filling	{ Open, Striped, Full}

Figure 1: Attributes of the SET cards

For each combination of attributes, there is exactly one card. The objective of the game is to quickly find a SET. A SET is collection of three cards whose attributes match. That means, for each of the four attributes, the three cards are either all equal or all different. See Figure 2 below for two examples of SETs:

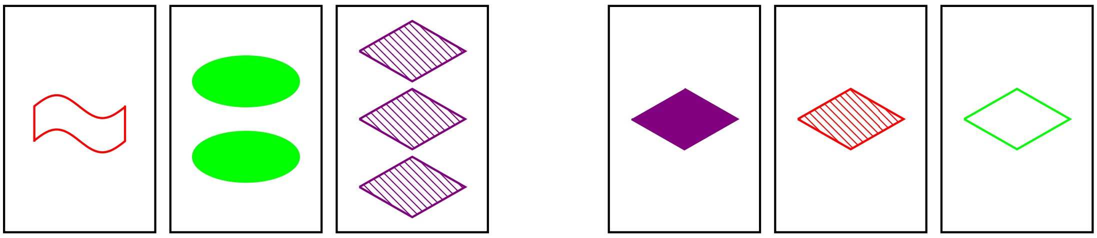

Figure 2: Two examples of SETs

The game begins by shuffling the deck and dealing twelve cards on the table, face-up. All players simultaneously search for SETs. The first player who spots a SET can take the corresponding three cards. The empty spots are then filled up with three new cards. It may happen that there is no SET among the cards that are on the table. In that case three additional cards are laid out. The game ends when all cards are used up and no SET can be made under the remaining cards on the table. The one who has collected the most SETs wins the game.

1.3 Problem Definition

When playing SET, it often happens that there is no SET among the twelve cards. Any such a collection of cards containing no SET is called a *cap*. Now we are interested in the maximum size that such a cap can take. Not just in the four-dimensional SET game, but also in higher and lower dimensions. All this will be explained in more detail later. For now, we define $a(n)$ as the maximum size of a cap in an n-dimensional SET game. The maximum cap-size $a(n)$ is known for dimension 1 up to dimension 6, with the know values listed in Table 1 below:

Dimension	1	2	3	4	5	6	7
Maximal cap-size $a(n)$	2	4	9	20	45	112	?

Table 1: Known maximal cap-sizes per dimension

The goal of this report is to understand how one came to these maximal cap-sizes. For the lower dimensions, we will look into these maximal cap-sizes and prove that they are indeed of maximum size. For the higher dimensions with unknown cap-sizes, we will look at what lower bounds and upper bounds exist for the cap-sizes.

2 Theory

In order for us to look into maximal cap-sizes in several dimensions, there is a lot of theory that needs to be understood first, and that is exactly what this chapter will cover. We will start with translating the game SET to vector spaces. After that, we will explain how we will distinguish between modulo 3 and modulo q, or better, the field $\mathbb{F}_q$ with q a prime number. This is followed by a lot of counting of points and lines in both $\mathbb{F}_3^3$ and $\mathbb{F}_q^3$. After that, we will look into planes and then step up by looking at the general vector space $\mathbb{F}_q^n$ and do a lot of counting of points, lines, planes, solids and subspaces in this vector space.

2.1 From SET to vector spaces

Let $\mathbb{F}_3$= {0,1,2} be the field with three elements. So addition, multiplication and all other operations are computed modulo 3. Note that although the field is defined as $\mathbb{F}_3 = \{0, 1, 2\}$, one could also define $\mathbb{F}_3 = \{-1, 0, 1\}$, which represents the same field. Throughout this report both fields will be alternated, depending on which one is practical or more logical for that specific situation.

Consider the vector space $\mathbb{F}_3^4$. Then a point of $\mathbb{F}_3^4$ is a 4-tuple of the form (x_1, x_2, x_3, x_4), where each coordinate assumes one of the three possible values in $\{0, 1, 2\}$. This way the SET cards correspond to the $3^4 = 81$ vectors in $\mathbb{F}_3^4$ and vice-versa. As example, it is clear that the three cards $(0, 1, 1, 2), (0, 2, 2, 2)$ and $(0, 0, 0, 2)$ form a SET. The first and last coordinate are after all the same within each vector. The second and third coordinates are however different within each vector. Together, these two observations confirm that the above three vectors form a SET. One could also state that, that in order for three cards to form a SET, the sum of the coordinates of each of the three cards must be zero, while calculating modulo 3. In our example, one then gets:

$$(0, 1, 1, 2) + (0, 2, 2, 2) + (0, 0, 0, 2) \stackrel{3}{=} (0, 3, 3, 6) \stackrel{3}{=} (0, 0, 0, 0) \tag{1}$$

Under this observation, one can state the following:

Statement 1 *Three cards form a SET if and only if the three cards are collinear.*

Proof
Take three arbitrary points $a, b, c \in \mathbb{F}_3^4$. These a, b and c form a SET if the coordinates of the vectors are all the same or if each of them is exactly one of $\{0, 1, 2\}$. In both cases, the sum of the coordinates is equal to zero for each coordinate, counting modulo 3. To prove that a, b and c are collinear, we need to prove that these three points lie on one line. We do that by showing that the direction vectors $a - b$ and $b - c$ are equal:

$$\begin{aligned} & a + b + c = 0 \\ \Leftrightarrow\ & a + b = -c \\ \Leftrightarrow\ & a + 2b = b - c \\ \Leftrightarrow\ & a - b = b - c \end{aligned} \qquad \square$$

A cap on the other hand is defined as a subset of $\mathbb{F}_3^4$ which does not contain any of these lines. So now SET's and cap's can be expressed in vector spaces, there is a lot of theory about vector spaces that we can dig into to understand more about the maximal cap-size problem.

2.2 Modulo 3 and modulo q

For each of the following topics, there are constantly two situations that will be discussed. The first situation will be about the vector space $\mathbb{F}_3^3$. After the theory is clear for the vector space $\mathbb{F}_3^3$, it will then be translated to the more general vector space $\mathbb{F}_q^3$. This is because the vector space $\mathbb{F}_3^3$ is easier to understand and picture than the vector space $\mathbb{F}_q^3$. However, eventually we should be able to express everything for any desired modulus.

Looking at the vector space $\mathbb{F}_3^3$, a point in $\mathbb{F}_3^3$ is of the form (x_1, x_2, x_3). Furthermore, one can state that the vector space $\mathbb{F}_3^3$ corresponds with the affine space $AG(3,3)$. In other words, $\mathbb{F}_3^3 \cong AG(3,3)$ as in an affine space, there is no distinguished point that serves as origin. However, the SET problem does not require an origin. It does not matter if the three points of a SET lie on one line that goes through the origin or somewhere else. If the three points would lie on one line somewhere far away from the origin, one could easily shift them to the origin with a transformation if preferred. Fact is that the three points still lie on one line and therefore they form a SET, whether they lie around the origin or not. So that is why one can state that $\mathbb{F}_3^3 \cong AG(3,3)$.

2.3 Counting points and lines in planes

So we start by looking at planes. The vector space $\mathbb{F}_3^2$ can be expressed as a plane modulo three. There are $3^2 = 9$ points in $\mathbb{F}_3^2$ and in general q^2 points in $\mathbb{F}_q^2$. Furthermore, the concept of three points lying on one line will be clarified here. Taking the vector space $\mathbb{F}_3^2$, one can represent the nine points in this vector space through a two-by-two grid. Now if we talk about three points lying on one line, the first lines that one will think of are probably the following blue, red, green and yellow lines in Figure 3:

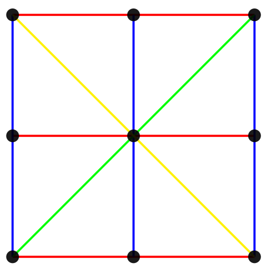

Figure 3: Obvious lines through three points

However, there are more less obvious lines that go through three points which will also count as lines in this case. These lines go through two points on one side of the grid and then cross all the way around to the opposite corner for the third point. These lines can be presented through curves and are shown through the following blue, red, green and yellow lines in Figure 4 below:

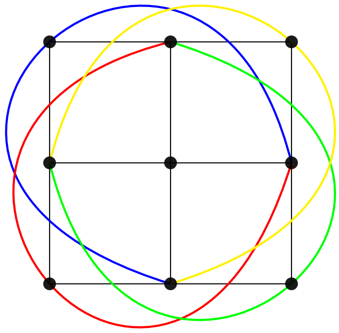

Figure 4: The remaining four lines

Looking at these two figures together, one can now see that the middle point is no more special than any of the other points. After all, there are exactly four lines that go through each point. In total we can thus conclude, by looking at both Figure 3 and Figure 4, that there are $9 + 3 = 12$ different lines in the vector space $\mathbb{F}_3^2$. Another way to see this, is by realising that there are four lines that go through each point, while there are $3^2 = 9$ points in total. However, each line goes through three points so in total you get: $(3^2 \cdot 4)/3 = 12$ lines in the vector space $\mathbb{F}_3^2$. In general one thus gets $(q^2 \cdot (q + 1))/q$ lines in the vector space $\mathbb{F}_q^2$.

2.4 Counting points and lines in solids

Method 1

Let us start with the vector space $\mathbb{F}_3^3$. Every line in $\mathbb{F}_3^3$ consists of three points. It is also known that through every two points, there goes exactly one line. This fact is used to count the number of ordered triples (p_1, p_2, l). Here p_1 stands for a point in $\mathbb{F}_3^3$, as does p_2. Furthermore, l stands for the line that goes through p_1 and p_2. There are two different ways to count the number of ordered triples (p_1, p_2, l). One can look at the fact that there are exactly 27 points in $\mathbb{F}_3^3$, so there are 27 points to choose from for p_1. That means that, minus the point chosen for p_1, there are 26 options left to choose from for p_2. Once these two points are chosen, there is exactly one line l that goes through them, so $l = 1$.

On the other hand, it is known that on every line there are a certain number of pairs of two points possible. So one could look at all the possible lines l and call this $\#l$. Now let us take an arbitrary line l in $\mathbb{F}_3^3$. Then this line l consists of three points, let us say x_1, x_2, x_3 as the vector space is three-dimensional. For the first point, x_1, one can choose from three different numbers, namely $x_1 \in \{0, 1, 2\}$, as this is how the vector space was defined for modulus three. For the second point, x_2, one can then only chose two numbers, as while looking at the second point x_2, one direction is already determined. With these two chosen, there is nothing left to chose for x_3 as x_3 is determined by x_1 and x_2.

Now combining these two ways of counting (p_1, p_2, l), the following equation can be made:

$$(p_1, p_2, l) = 27 \cdot 26 \cdot 1 = 3 \cdot 2 \cdot \#l \tag{2}$$

which gives us:

$$\#l = 117 \tag{3}$$

The same can be done for the number of lines in the vector space $\mathbb{F}_q^3$:

$$(p_1, p_2, l) = q^3 \cdot (q^3 - 1) \cdot 1 = q \cdot (q - 1) \cdot \#l \tag{4}$$

which gives us:

$$\#l = \frac{q^3 \cdot (q^3 - 1)}{q \cdot (q - 1)} \tag{5}$$

Method 2

Another way of looking at the number of lines, is by seeing every line as a vector in a vector space. So every line l can be described through a vector $\underline{x} = \underline{b} + \lambda \underline{d}$ in which $\underline{b}$ is the basis vector and $\underline{d}$ the direction vector. Now, the same can be done as before. On the left-hand side lies the line $\underline{x}$. For each $\underline{x}$, there are 27 points to choose from for the first point. That means there are 26 points left to choose form for the second point and with that the line is determined. On the right hand side, for the basis vector $\underline{b}$, one can choose three points as the dimension of the vector space is three. For the second vector, the direction vector, there are only two points left to choose from, as when looking at the second vector, you have already chosen one direction. This together determines the number of lines. So again the following equation can be made:

$$(p_1, p_2, l) = 27 \cdot 26 \cdot 1 = 3 \cdot 2 \cdot \#l \tag{6}$$

which again gives us:

$$\#l = 117 \tag{7}$$

The same can be done for the number of lines in the vector space $\mathbb{F}_q^3$:

$$(p_1, p_2, l) = q^3 \cdot (q^3 - 1) \cdot 1 = q \cdot (q - 1) \cdot \#l \tag{8}$$

which again gives us:

$$\Rightarrow \#l = \frac{q^3 \cdot (q^3 - 1)}{q \cdot (q - 1)} \tag{9}$$

2.4.1 Lines through a fixed point

Suppose you take one fixed point as the starting point of a line. As there are $3^3 = 27$ points in $\mathbb{F}_3^3$, that means there are 26 points left to choose from as endpoint of that line and thus 26 lines that go through that fixed point. However, lines that have opposite endpoints determine the same direction, so that is why there are actually only $26/2 = 13$ distinguished lines in $\mathbb{F}_3^3$ that go through one fixed point. Figure 5 below illustrates this. Suppose you look at the vector space $\mathbb{F}_3^3$ with $\mathbb{F}_3$= {-1,0,1} this time. The starting point of any direction vector is then defined as the origin, so in the point (0,0,0). That means that there are 26 points left in the cube that could be the endpoint of the line. However, the endpoints (1,0,1) and (-1,0,-1) indicate the same direction. Thus, in the vector space $\mathbb{F}_3^3$, one has $26/2 = 13$ distinguished lines and in the vector space $\mathbb{F}_q^3$ one has $(q^3 - 1)/2$ lines that go through one fixed point.

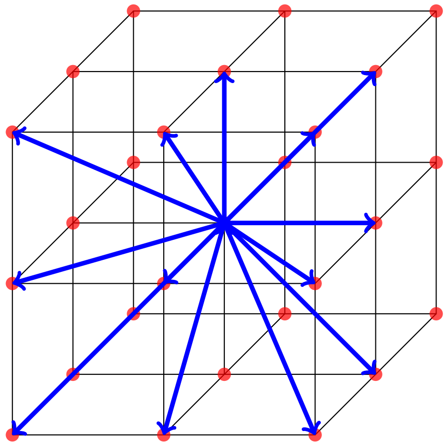

Figure 5: Unique lines through a fixed point

2.5 Overview of counting in $\mathbb{F}_3^3$ and $\mathbb{F}_q^3$

In the above sections, a lot of counting has taken place in both $\mathbb{F}_3^3$ and $\mathbb{F}_q^3$. Below in Table 2 is an overview of all the things that can be counted thus far for both vector spaces:

Counting	*The vector space* $\mathbb{F}_3^3$	*The vector space* $\mathbb{F}_q^3$
Points per line	3	q
Points per plane	$3^2 = 9$	q^2
Points per solid	$3^3 = 27$	q^3
Lines per plane	$\frac{9 \cdot 4}{3} = 12$	$\frac{q^2 \cdot (q+1)}{q}$
Lines per solid	$\frac{(3^3) \cdot (3^3 - 1)}{3 \cdot 2} = 117$	$\frac{q^3 \cdot (q^3 - 1)}{q \cdot (q-1)}$
Lines through a point	$\frac{26}{2} = 13$	$\frac{q^3 - 1}{q - 1}$

Table 2: Counting in $\mathbb{F}_3^3$ and $\mathbb{F}_q^3$

2.6 Planes

In the Euclidean space of any number of dimensions, a plane is uniquely determined by three non-collinear points. In other words, every set of three points that does not lie on one line, determines a plane. As seen before, the vector space $\mathbb{F}_3^3$ counts $3^2 = 9$ points and more general, the vector space $\mathbb{F}_q^3$ counts q^2 points. Now let us define a plane:

$$V : \underline{x} = \underline{s} + \lambda \cdot \underline{d}_1 + \mu \cdot \underline{d}_2 \tag{10}$$

In this plane, $\underline{s}$ is a starting vector, and $\underline{d}_1$ and $\underline{d}_2$ are direction vectors. Given this definition, the number of planes that goes through one point can be expressed in terms of q. As seen before, every two points determine exactly one line. In terms of planes, any two independent vectors determine exactly one plane. So given a vector $\underline{d}_1$ and a vector $\underline{d}_2$, a certain plane V is defined. Starting with the three-dimensional vector space $\mathbb{F}_3^3$, for the first vector $\underline{d}_1$ one can choose from the 27 points in $\mathbb{F}_3^3$. minus the possibility that $\underline{d}_1$ is the zero vector. This leaves 26 options for $\underline{d}_1$. For the second vector $\underline{d}_2$, one must take into account that $\underline{d}_2 \notin \langle \underline{d}_1 \rangle$ the span of $\underline{d}_1$. This leaves $27 - 3 = 24$ options for $\underline{d}_2$. With these two vectors, exactly one plane V is determined. On the other side, given all the planes that go through a certain starting vector $\underline{s}$, you can look at the possible vectors in that plane. As seen above, there are 9 points in a three-dimensional plane. So for the first vector $\underline{d}_1$ there are 8 points to choose from, as creating the zero vector is again not an option. For the second vector $\underline{d}_2$, there are $9 - 3 = 6$ options to choose from, as $\underline{d}_2 \notin \langle \underline{d}_1 \rangle$. Putting this together, one can count all possible $(\underline{d}_1, \underline{d}_2, V)$ and thus, one can make the following equation:

$$(\underline{d}_1, \underline{d}_2, V) = 26 \cdot (27 - 3) \cdot 1 = 8 \cdot (9 - 3) \cdot \#V \tag{11}$$

which gives us:

$$\#V = \frac{26 \cdot 24}{8 \cdot 6} = \frac{624}{48} = 13 \tag{12}$$

So in the vector space $\mathbb{F}_3^3$ there are 13 planes that go through one point, this one point defined as the starting vector. In $\mathbb{F}_q^3$ that means one gets the following equation:

$$(\underline{d}_1, \underline{d}_2, V) = (q^3 - 1) \cdot (q^3 - q) \cdot 1 = (q^2 - 1) \cdot (q^2 - q) \cdot \#V \tag{13}$$

which gives us:

$$\#V = \frac{(q^3 - 1) \cdot (q^3 - q)}{(q^2 - 1) \cdot (q^2 - q)} \tag{14}$$

2.7 Counting dimension n

In the previous section, everything was calculated for the three-dimensional vector space. However, one does not have to limit to the third dimension and can therefore take any dimension. So let us look at an n-dimensional vector space modulo q. In other words $V = \mathbb{F}_q^n$ and thus $|V| = q^n$. Now the number of k-dimensional subspaces will be defined as:

$$\begin{bmatrix} n \\ k \end{bmatrix} \text{ or } \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{15}$$

Suppose W is a k-dimensional subspace of V. Then W can be seen as the span of an independent system of k vectors and can be defined as follows:

$$W = \langle \underline{v}_1, \underline{v}_2, \ \dots \ , \underline{v}_k \rangle \tag{16}$$

The following step is to count all the possible tuples $(\underline{v}_1, \underline{v}_2, \ \dots \ , \underline{v}_k, W)$, just like before. As W is spanned by the vectors $\underline{v}_1$ through $\underline{v}_k$, these k vectors determine exactly one subspace W. For the first vector $\underline{v}_1$, one can chose any of the q^n points, minus zero, so $q^n - 1$. For the second vector $\underline{v}_2$, it holds that $\underline{v}_2 \notin \langle \underline{v}_1 \rangle$, so $\underline{v}_2 = q^n - q$. Similar for the third vector $\underline{v}_3$, it holds that $\underline{v}_3 \notin \langle \underline{v}_1, \underline{v}_2 \rangle$, so $\underline{v}_3 = q^n - q^2$ and etcetera. Therefore, one gets the following:

$$(\underline{v}_1, \underline{v}_2, \ \dots \ , \underline{v}_k, W) = (q^n - 1) \cdot (q^n - q) \cdot (q^n - q^2) \cdot (q^n - q^3) \cdot \ \cdots \ \cdot (q^n - q^{k-1}) \cdot 1 \tag{17}$$

On the other side, you can also count the number of k-dimensional subspaces and how many points there are to choose within each subspace, so for the same reasons, it also holds that:

$$(\underline{v}_1, \underline{v}_2, \ \dots \ , \underline{v}_k, W) = (q^k - 1) \cdot (q^k - q) \cdot (q^k - q^2) \cdot (q^n - q^3) \cdot \ \cdots \ \cdot (q^k - q^{k-1}) \ \cdot \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{18}$$

Combining these two together allows us to express the number of k-dimensional subspaces within V as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1) \cdot (q^n - q) \cdot (q^n - q^2) \cdot (q^n - q^3) \cdot \ \cdots \ \cdot (q^n - q^{k-1})}{(q^k - 1) \cdot (q^k - q) \cdot (q^k - q^2) \cdot (q^n - q^3) \cdot \ \cdots \ \cdot (q^k - q^{k-1})} \tag{19}$$

And this matches exactly with what was found previously. For example, let us determine the number of planes in the vector space $\mathbb{F}_q^3$. In section 2.6 we determined a formula for this number of planes called $\#V$. If we now compare that formula with (19) we see these two match:

$$\#V = \frac{(q^3 - 1) \cdot (q^3 - q)}{(q^2 - 1) \cdot (q^2 - q)} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = \frac{(q^3 - 1) \cdot (q^3 - q)}{(q^2 - 1) \cdot (q^2 - q)} \tag{20}$$

2.8 Looking at subspaces

So now we know how to express the number of k-dimensional subspaces within a vector space $V = \mathbb{F}_q^n$ in terms of q, k and n. However, there is more information that can be drawn from this. As W is a subspace of the vector space $V = \mathbb{F}_q^n$ one can state that $\{\underline{0}\} \subseteq W \subseteq V$ and thus $0 \subset k \subset n$.

Now if V is a n-dimensional space, W is a k-dimensional subspace of V and U is a m-dimensional subspace of W, then one can also state that: $U \subseteq W \subseteq V$ and thus $m \subset k \subset n$. Even more, one can state that, within an n-dimensional subspace, the number of k-dimensional subspaces that go through an m-dimensional subspace equals:

$$\begin{bmatrix} n-m \\ k-m \end{bmatrix} \text{ or } \begin{bmatrix} n-m \\ k-m \end{bmatrix}_q \tag{21}$$

Example

Suppose $V = \mathbb{F}_3^3 = AG(3,3)$. To determine how many planes in $AG(3,3)$ go through a line, we choose one fixed point on that line as origin. Now the number of two-dimensional subspaces in $\mathbb{F}_3^3$ that go through one point can be determined with:

$$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 3-1 \\ 2-1 \end{bmatrix}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 = \frac{q^2-1}{q-1} = \frac{8}{2} = 4 \tag{22}$$

which matches with what was found before.

2.9 Counting in $\mathbb{F}_3^4$

So now the formula for counting the number of k-dimensional subspaces W within an n-dimensional space V with $U \subseteq W \subseteq V$ for some m-dimensional space U is determined, there are a lot of things that can be calculated. For example, one can count how many planes go through one point, or how many solids go through a line. Below, in Table 3, is an overview of everything that can be counted using the formula (22) and (19) from the previous sections while looking at the vector space $\mathbb{F}_3^4$, the vector space that corresponds with the game SET:

Counting	*Dimensions*	*Formula*
lines through a point	$0 \subseteq 1 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_3 = \frac{(3^4-1)}{(3^1-1)} = \frac{80}{2} = 40$
planes through a point	$0 \subseteq 2 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_3 = \frac{(3^4-1)\cdot(3^4-3)}{(3^2-1)\cdot(3^2-3)} = \frac{80\cdot 78}{8\cdot 6} = 130$
solids through a point	$0 \subseteq 3 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_3 = \frac{(3^4-1)\cdot(3^4-3)\cdot(3^4-3^2)}{(3^3-1)\cdot(3^3-3)\cdot(3^3-3^2)} = \frac{80\cdot 78\cdot 72}{26\cdot 24\cdot 18} = 40$
planes through a line	$1 \subseteq 2 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_3 = \frac{(3^3-1)}{(3^1-1)} = \frac{26}{2} = 13$
solids through a line	$1 \subseteq 3 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_3 = \frac{(3^3-1)\cdot(3^3-3)}{(3^2-1)\cdot(3^2-3)} = \frac{26\cdot 24}{8\cdot 6} = 13$
solids through a plane	$2 \subseteq 3 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 = \frac{(3^2-1)}{(3^1-1)} = \frac{8}{2} = 4$

Table 3: Counting subspaces in $\mathbb{F}_3^4$

3 Maximal cap-sizes in lower dimensions

To recall, a cap is a collection of cards that does not contain a SET. So a cap is defined as a subset of the vector space $\mathbb{F}_3^n$ with no three collinear points. Now the big question is, what is the maximum possible size of a cap in $\mathbb{F}_3^n$ for different dimensions n? This question unfortunately cannot be answered for every dimension as the problem is unsolved yet for dimensions higher than six. However, the first couple of dimensions are interesting and we will look into those in the following sections. For the vector spaces $\mathbb{F}_3^1, \mathbb{F}_3^2$ and $\mathbb{F}_3^3$, the proofs are not too complicated and long yet. For the vector spaces higher than dimension three however, things get a lot more complicated. That is why these will be discussed in the next chapters.

3.1 The vector space $\mathbb{F}_3^1$

Dimension one is quite simple as dimension one only consists of one simple line. Let us define this line as x. As the vector space is one-dimensional and defined as $\mathbb{F}_3^1 = \{0, 1, 2\}$ this means that $x = \{0, 1, 2\}$. We see that this means that the maximum cap-size is two. After all, no matter which two values in $\mathbb{F}_3^1 = \{0, 1, 2\}$ you pick for the first two lines, you can always pick a third value that forms a SET with the first lines. After all, whether the first two lines were equal or different, you can always pick a third line that is equal with the first two lines, or different from the first two lines. In both cases you can thus create a set. Therefore the maximal cap-size of the vector space $\mathbb{F}_3^1$ is 2. Below, in Figure 6 is an representation of a possible one-dimensional version of the game SET using only the three different fillings for the three green ovals, followed by an illustration of a cap in this one-dimensional vector space, see Figure 7.

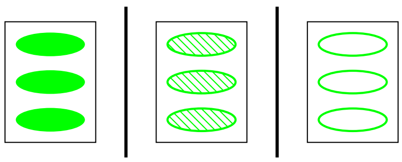

Figure 6: The game SET in dimension one

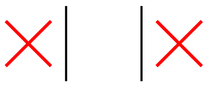

Figure 7: 1-cap

3.2 The vector space $\mathbb{F}_3^2$

In the two-dimensional vector space $\mathbb{F}_3^2$, every point is described by two coordinates. Therefore we get a plane. As seen before, any two independent vectors define a plane. A two-dimensional version of the game SET can be realized by playing with only the green ovals for example. This way, the vector space $\mathbb{F}_3^2$ can be graphically represented through Figure 8 below. In this figure, every three cards that lie on one line form a SET. Recall that the three points that form a SET do not have to lie on a obvious straight line, but that they could also meet at one edge of the grid and then loop around to the opposite edge, as explained before and shown in Figure 4.

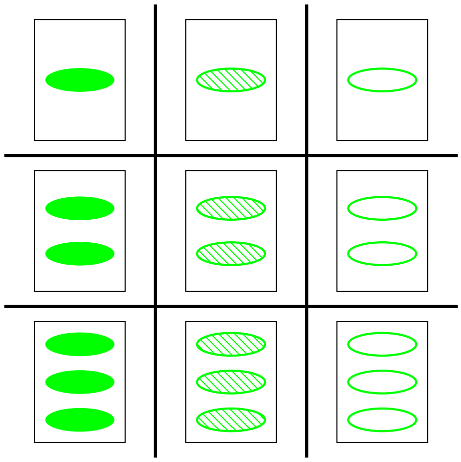

Figure 8: The game SET in dimension two

After taking a good look at the vector space $\mathbb{F}_3^2$ and especially at Figure 8, one might guess that the maximum cap-size is most possibly four. By taking the four corner cards for example, one cannot pick a fifth card without creating a SET. However, to be sure the maximum cap-size is indeed four, a proof will be given below.

Statement 2 *The maximum size of a cap in the vector space* $\mathbb{F}_3^2$ *is four.*

Proof
Suppose the maximum cap-size of the vector space $\mathbb{F}_3^2$ would be five. Then one has five points. Taking one of these five points, one can see that there must be four lines connected with this one point in order to create the set of five points. This means however, that every line has exactly two points or no points at all. It follows that the size of the cap is even and therefore has at most four points. Figure 9 below illustrates this:

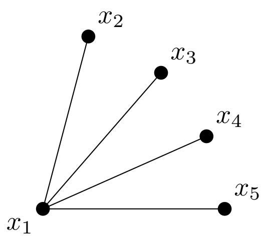

Figure 9: Points per line in a cap with five points

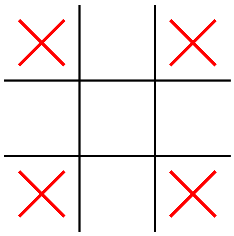

Figure 10: 2-cap

3.3 The vector space $\mathbb{F}_3^3$

In the three-dimensional vector space $\mathbb{F}_3^3$, every point is described through three coordinates. A three-dimensional version of the game SET can be realized by playing with the green, red and purple ovals, that appear in three different numbers and with three different fillings, see Figure 11 below.

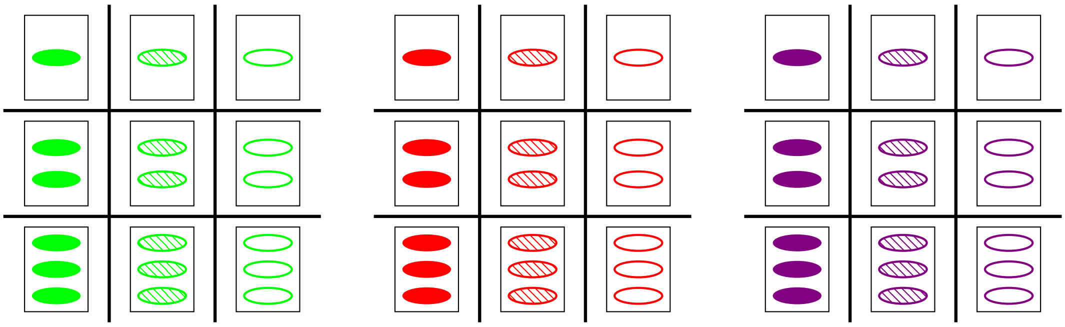

Figure 11: The game SET in dimension three

In the three-dimensional vector space the maximum cap-size is known to be nine. The question is, is a cap containing ten points really not possible? To show that this is not possible, a proof, that again works through contradiction, will be given here below.

Statement 3 *The maximum size of a cap in the vector space $\mathbb{F}_3^3$ is nine.*

Proof
First of all, suppose that there exists a cap of size ten. Now, let us look at the vector space $\mathbb{F}_3^3$ in a different way. As seen before, the vector space $\mathbb{F}_3^3$ can also be seen as the affine geometry space $AG(3,3)$, which can be represented through a cube shown in Figure 12 below:

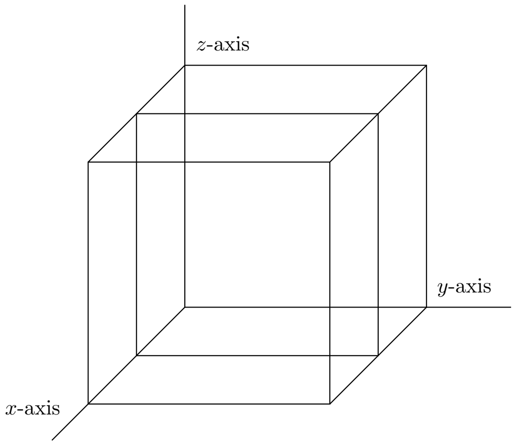

Figure 12: $AG(3,3)$

So the space consists of three parallel planes, where for each coordinate x, y and z one can choose one of the three values in $\{0,1,2\}$. Using what was proven for the vector space $\mathbb{F}_3^2$, one can conclude that every plane contains at most four points, as every plane on itself is a two-dimensional vector space. So if there is a cap-size of ten, then each plane in the cube above contains at most 4 points and at least 2 points, as $2+4+4=10$. Now we take two points from this cap, let us say a and

b, and define a line l that goes through a and b. Then there are four planes that intersect in the line l. An illustration of this is shown in Figure 13 below:

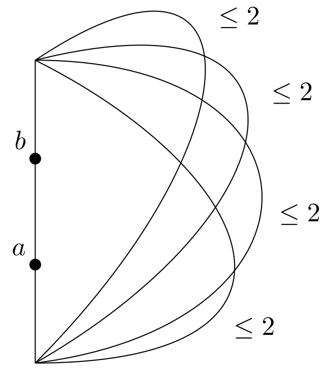

Figure 13: The four planes through line l

Now again, the known maximum size of a cap in a two-dimensional vector space is used, namely, that each plane can contain a maximum of four points. So as each of the four planes already contains two points, that is the two points a and b of line l, this means that each plane must contain two more points on top of these two. Otherwise, one can not achieve a cap of size ten. This however means that each plane with at least two points must directly contain four points in total. But, this does not work for our cube of three parallel planes in Figure 13. As it happens, to reach the cap-size of ten, the first parallel plane must contain at least two points, which results in four points according to what was stated before. The second plane must also contain more than two points, which also equals in containing four points. This brings us on a total of eight points so far, leaving exactly two points left for the last plane. However, every plane that contains two or more points, must necessarily contain four points. But this brings us on a total of twelve points which is definitely wrong. Hence, a cap of size ten does not exist and thus the maximum cap-size stays nine for the vector space $\mathbb{F}_3^3$. □

An illustration of a 3-cap of size nine is given in Figure 14:

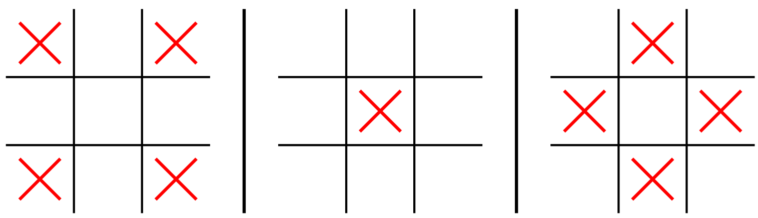

Figure 14: 3-cap

4 Double Counting

Unfortunately, the method above is not strong enough to prove that the maximum size of a cap in the vector space $\mathbb{F}_3^4$ is 20. In order to prove this, another combinatorial technique is used, namely double counting, also called, counting the same thing in two different ways. This technique shows that two expressions are equal by demonstrating that they are two ways of counting the size of one set. In this Chapter, as we are interested in maximal cap-sizes, we describe a finite set, X for example, from two perspectives. This leads to two distinct expressions for the size of this set X. Since both expressions equal the size of the same set, they equal each other. To introduce this method, another proof will be given to show that the maximum cap-size of the vector space $\mathbb{F}_3^3$ is nine, but now using the technique double counting. After that, double counting will be used to show that the maximum cap-size of the vector space $\mathbb{F}_3^4$ is 20.

4.1 Double counting in the vector space $\mathbb{F}_3^3$

So in section 3.3 it was shown that the maximum size of a cap in the vector space $\mathbb{F}_3^3$ is nine. Here, the same will be shown, but now proven through double counting.

Statement 3 *The maximum size of a cap in the vector space $\mathbb{F}_3^3$ is nine.*

Proof
The proof for this will again work through contradiction. So suppose that there exists a cap in $\mathbb{F}_3^3$, let us say C, that contains ten points. Now we know that the vector space $\mathbb{F}_3^3$ can decomposed as the union of three parallel planes, H_1, H_2 and H_3 in many different ways. This decomposition gives a triple of numbers:

$$\{|C \cap H_1|, |C \cap H_2|, |C \cap H_3|\} \tag{23}$$

Here, $\{C \cap H_i\}$ are called hyperplanes, that are subspaces of one dimension less than its ambient space. Furthermore, $|C \cap H_i|$ is the size of $C \cap H_i$, in other words, the number of points in $|C \cap H_i|$. Using what was proven for two-dimensional vector spaces, each of these planes separately contains at most four points. Therefore, the only two possible values for a hyperplane triple, in order to create a cap of size ten, are $(2, 4, 4)$ and $(3, 3, 4)$. Now suppose that the hyperplane triple $(2, 4, 4)$ occurs a times when decomposing $\mathbb{F}_3^3$ into parallel planes and that hyperplane triple $(3, 3, 4)$ occurs b times when decomposing $\mathbb{F}_3^3$. Then the numbers a and b can be calculated from a system of equations.

The first equation
First, there are two ways to calculate the many different ways to decompose $\mathbb{F}_3^3$ as the union of three planes. On the one hand, there are $a + b$ ways. On the other hand, there are thirteen ways to decompose the vector space $\mathbb{F}_3^3$ into three parallel planes. This can be explained as followed. If of the three parallel planes is fixed, then all three planes are fixed. Therefore, one only has to look at one plane, for example the plane through the origin. It was shown before that in the vector space $\mathbb{F}_3^3$, there are thirteen unique lines that go through the origin, see Figure 5 and section 2.4.1. Therefore there are also thirteen unique planes that go through the origin and thus also thirteen decompositions of three parallel planes. From this, the first equation can be derived, which is:

$$a + b = 13 \tag{24}$$

The second equation
The second equation is obtained by counting a certain set in two different ways. This set is the number planes that go through two points in C, our cap of size ten. Every element in that set can be defined as $(H, \{x, y\})$, where x and y are elements from C and H a plane that contains x and y.

In the first counting method, all planes that go through different pairs of points will be counted. In total there are $\binom{10}{2} = 45$ pairs of points. Furthermore, it was shown earlier, in Figure 12, that each pair of points intersects with four planes. Therefore, one counts $45 \times 4 = 180$ elements.

The second way of counting works exactly in the opposite way by first counting all the planes and then counting the pairs of points that lie within those planes. With the hyperplane triples of the form $(2, 4, 4)$, there are exactly $\binom{2}{2} + \binom{4}{2} + \binom{4}{2} = 13$ ways to chose a pair of points that lie within the same plane. With the hyperplane triples of the form $(3, 3, 4)$, there are however $\binom{3}{2} + \binom{3}{2} + \binom{4}{2} = 12$ ways to chose a pair of points that lie within the same plane. In total, one thus gets $13 \cdot a + 12 \cdot b$ elements in this set. So combining this with what was found first, the second equation becomes:

$$13 \cdot a + 12 \cdot b = 180 \tag{25}$$

Combining (24) and (25), one finds that $a = 24$ and $b = -11$. However, this is impossible, as a and b are natural numbers and can thus only take non-negative values. That means that our basic assumption was wrong and thus the maximum size of a cap in the vector space $\mathbb{F}_3^3$ cannot be ten and thus stays nine. □

4.2 Hyperplanes

In the prove above we needed to count the number of hyperplanes containing a fixed pair of points, or in other words, containing a fixed line. Now to apply this method to maximal 4-caps, we will need to solve a generalization of this problem which is what we will do in this section. First, let us define a k-flat to be a k-dimensional affine subspace of a vector space. Now, with using what we concluded before in (22) and (19), we can state the following:

Statement 4 *The number of hyperplanes containing a fixed k-flat in $\mathbb{F}_3^d$ is given by*

$$\begin{bmatrix} d-k \\ d-1-k \end{bmatrix}_3 = \begin{bmatrix} d-k \\ 1 \end{bmatrix}_3 = \frac{3^{d-k}-1}{3-1} = \frac{3^{d-k}-1}{2} \tag{26}$$

Proof
By affine transformation, we can move K to the origin. This way, K is a k-flat containing the origin. Then the natural map:

$$\mathbb{F}_3^{d-k} \to \mathbb{F}_3^d / K \cong \mathbb{F}_3^{d-k} \tag{27}$$

gives a bijection between hyperplanes of $\mathbb{F}_3^d$ containing K and hyperplanes of $\mathbb{F}_3^{d-k}$ containing the origin. This is because K is sent to 0. Each hyperplanes containing the origin is determined by a non-zero normal vector, and there are exactly two non-zero normal vectors determining each hyperplane. Thus, there are half as many hyperplanes as there are non-zero vectors. Since there are $3^{d-k} - 1$ non-zero vectors, there must be $(3^{d-k} - 1)/2$ hyperplanes containing the origin. □

4.3 Double counting in the vector space $\mathbb{F}_3^4$

In the proof of section 4.1, the number of planes containing a fixed pair of points, or in other words, containing a fixed line, was counted. Now for the vector space $\mathbb{F}_3^4$, a similar thing will be done, except now there are seven unknown values instead of two.

Statement 5 *The maximum size of a cap in the vector space* $\mathbb{F}_3^4$ *is* 20.

Proof
The proof for this again works through contradiction. Suppose that there exists a cap in $\mathbb{F}_3^4$, let us say C, that contains 21 points. Now the vector space $\mathbb{F}_3^4$ can again be decomposed as the union of three parallel solids, S_1, S_2 and S_3 in many different ways. This decomposition gives us again a triple of numbers:

$$\{|C \cap S_1|, |C \cap S_2|, |C \cap S_3|\} \tag{28}$$

Now, as the vector space is $\mathbb{F}_3^4$, the hyperplanes $\{C \cap S_1\}, \{C \cap S_2\}$ and $\{C \cap S_3\}$ are three-dimensional. That means that each hyperplane contains at most nine points. Define x_{ijk} as the number of $\{i, j, k\}$ hyperplane triples of C. Then there are seven possible hyperplane triples:

$$\{i, j, k\} = \{9, 9, 3\}, \{9, 8, 4\}, \{9, 7, 5\}, \{9, 6, 6\}, \{8, 8, 5\}, \{8, 7, 6\}, \{7, 7, 7\} \tag{29}$$

Here, x_{993} is the number of $\{9, 9, 3\}$ hyperplane triples, x_{984} is the number of $\{9, 8, 4\}$ hyperplane triples and etcetera. This gives us seven unknown values and the goal is to determine these seven values trough a system of equations.

The first equation
The vector space $\mathbb{F}_3^4$ is decomposed as the union of three parallel solids. We notice that if one of these solids is fixed, then all three the solids are fixed. Therefore, we only have to look at one solid, for example the solid that contains the origin. In $\mathbb{F}_3^4$ there are $3^4 - 1$ non zero points. Since each line contains two non-zero points, there are exactly $\dfrac{3^4 - 1}{2} = 40$ unique lines through the origin. Thus the first equation becomes:

$$x_{993} + x_{984} + x_{975} + x_{966} + x_{885} + x_{876} + x_{777} = 40 \tag{30}$$

The second equation
To obtain another equation in x_{ijk}, we count 2-marked hyperplanes. These are pairs of the form $(S, \{x, y\})$, where x and y are elements from C and S is a hyperplane that contains the points x and y. This set contains planes with pairs of points from C. In total, there are $\binom{21}{2} = 210$ pairs of points. Using Statement 4, we find that the number of hyperplanes containing a distinct pair of points is thirteen. Thus together, there are $210 \times 13 = 2730$ 2-marked hyperplanes. If we first decompose the vector space $\mathbb{F}_3^4$ into parallel solids and then count the total pairs of points within one solid, then we find:

$$\left[\binom{9}{2}+\binom{9}{2}+\binom{3}{2}\right]\cdot x_{993}+\left[\binom{9}{2}+\binom{8}{2}+\binom{4}{2}\right]\cdot x_{984}+\left[\binom{9}{2}+\binom{7}{2}+\binom{5}{2}\right]\cdot x_{975}+$$
$$\left[\binom{9}{2}+\binom{6}{2}+\binom{6}{2}\right]\cdot x_{966}+\left[\binom{8}{2}+\binom{8}{2}+\binom{5}{2}\right]\cdot x_{885}+\left[\binom{8}{2}+\binom{7}{2}+\binom{6}{2}\right]\cdot x_{876}+ \quad (31)$$
$$\left[\binom{7}{2}+\binom{7}{2}+\binom{7}{2}\right]\cdot x_{777}=2730$$

2-marked hyperplanes. Explicitly computing each coefficient above gives the following formula:

$$75\cdot x_{993}+70\cdot x_{984}+67\cdot x_{975}+66\cdot x_{966}+66\cdot x_{885}+64\cdot x_{876}+63\cdot x_{777}=2730 \quad (32)$$

The third equation
Analogue, we can obtain yet a third equation by counting all 3-marked hyperplanes. These are pairs of the form $(S,\{x,y,z\})$, where S is a hyperplane. Notice that, since $\{x,y,z\}\subset C$, the points x,y and z cannot be collinear. First, we count all triplets within the cap C, which are $\binom{21}{3}=1330$ triplets. Using Statement 4 again, we calculate that there are four hyperplanes containing three distinct non-collinear points. In total there thus are $1330\times 4=5320$ 3-marked hyperplanes. If we imitate our count of 2-marked hyperplanes and again first decompose the vector space $\mathbb{F}_3^4$ into three parallel solids and then count all triplets, we get:

$$\left[\binom{9}{3}+\binom{9}{3}+\binom{3}{3}\right]\cdot x_{993}+\left[\binom{9}{3}+\binom{8}{3}+\binom{4}{3}\right]\cdot x_{984}+\left[\binom{9}{3}+\binom{7}{3}+\binom{5}{3}\right]\cdot x_{975}+$$
$$\left[\binom{9}{3}+\binom{6}{3}+\binom{6}{3}\right]\cdot x_{966}+\left[\binom{8}{3}+\binom{8}{3}+\binom{5}{3}\right]\cdot x_{885}+\left[\binom{8}{3}+\binom{7}{3}+\binom{6}{3}\right]\cdot x_{876}+ \quad (33)$$
$$\left[\binom{7}{3}+\binom{7}{3}+\binom{7}{3}\right]\cdot x_{777}=5320$$

3-marked hyperplanes. Explicitly computing each coefficient above gives the following formula:

$$169\cdot x_{993}+144\cdot x_{984}+129\cdot x_{975}+124\cdot x_{966}+122\cdot x_{885}+111\cdot x_{876}+105\cdot x_{777}=5320 \quad (34)$$

We now have three equations, equation (30), (32) and (34), in seven variables, and so in principle, there could be infinitely many solutions. Fortunately, we are only interested in the non-negative integer solutions which reduces the list of possible solutions. Furthermore, we are lucky that the right hand sides of the equations (30), (32) and (34) are linearly dependent, as are the coefficients from x_{993} and the coefficients from x_{777}. Therefore, adding 693 times equation (30) to three times equation (34), and then subtracting off six times equation (32), gives the following equation:

$$5\cdot x_{984}+8\cdot x_{975}+9\cdot x_{966}+3\cdot x_{885}+2\cdot x_{876}=0 \quad (35)$$

the only non-negative solution to this equation is:

$$x_{984}=x_{975}=x_{966}=x_{885}=x_{876}=0 \quad (36)$$

However, taking equation (32) and subtracting 63 times equation (30) also gives the following equation:

$$12 \cdot x_{993} + 7 \cdot x_{984} + 4 \cdot x_{975} + 3 \cdot x_{966} + 3 \cdot x_{885} + x_{876} = 210 \tag{37}$$

Using what was found in equation (36) to solve (37), we get that $12 \cdot x_{993} = 210$. However, this gives $x_{993} = 17.5$ and thus contradicts x_{993} being a integer. That means that the basic assumption must be wrong and there cannot exist a cap of 21 points. The maximal size of a cap in the vector space $\mathbb{F}_3^4$ therefore stays 20. □

Below, in figure 15 is an illustration of a 4-cap.

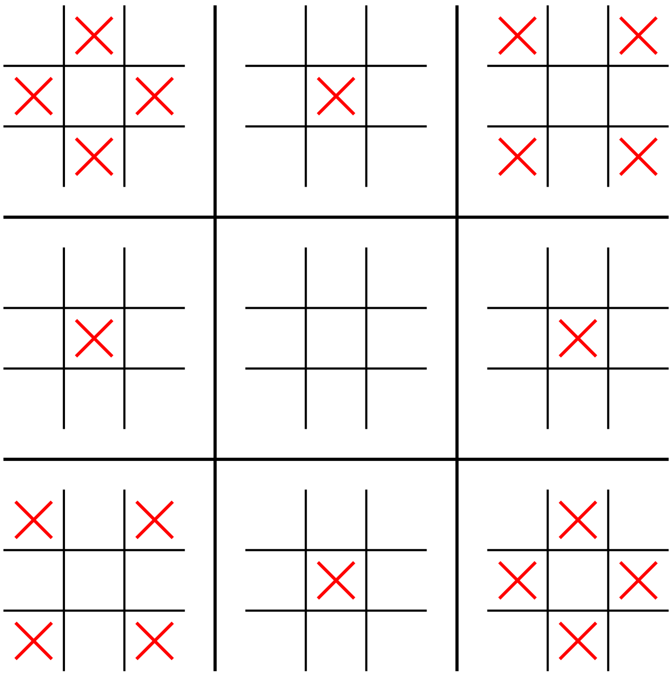

Figure 15: 4-cap

4.4 Permutations of Uniqueness of caps

In the above sections, the maximum sizes of caps in vector spaces up to dimension four have been calculated. These caps have been presented through Figures 7, 10, 14 and 15. However, these are not the only caps that one could make for each dimension, as one could simply create a new cap by permutating the points of a previous cap. This way, one could make many new caps from these original caps. These permutations are called affine transformations. An affine transformation is a function between affine spaces which preserves points, straight lines and planes. Now here, affine transformations are permutations of the vector space $\mathbb{F}_3^d$ of the form:

$$f(x) = Ax + b \tag{38}$$

where A is an invertible $d \times d$-matrix with entries in $\mathbb{F}_3^d$ and b is a vector in $\mathbb{F}_3^d$. Now if there exists an affine transformation from one cap to another, one says that these two caps are of the same type, or isomorphic. For example, consider the affine transformation $f(x, y) = (x + y,\ x - y - 1)$ taking a vector $(x, y) \in \mathbb{F}_3^2$ to another vector in $\mathbb{F}_3^2$. If this affine transformation is applied to a cap in the two-dimensional vector space, one gets another cap of the same type. In the Figure

16 below, this affine transformation is shown for the original 2-cap we saw in Figure 10. For this affine transformation, the centre of the grid is defined as the origin of the vector space $\mathbb{F}_3^2$ and the vector space $\mathbb{F}_3$ is defined as $\mathbb{F}_3 = \{-1, 0, 1\}$.

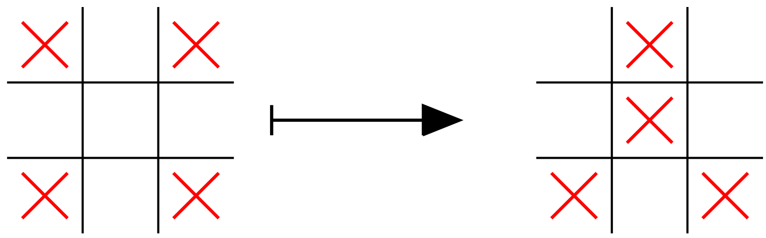

Figure 16: Affine transformation of a 2-cap

5 The Fourier Transform

In the sections 4.1 and 4.3 we made use of the combinatorial technique called 'Double Counting' in order to prove that the maximum size of a cap in the vector space $\mathbb{F}_3^4$ is 20. This method of counting marked hyperplanes via hyperplane triples is the shortest known way to prove this without using an exhaustive computer search. Unfortunately, a straightforward application of this method fails to show that the maximum size of a cap in the vector space $\mathbb{F}_3^5$ is 45. Part of the problem is that the new equations counting 4-marked hyperplanes require an additional variable to distinguish between the cases where four points are affinely dependent or independent. In the next section, we look into the Fourier Transform and into bounds for a_n. The theory explained can be used as a building stone to prove that the maximum size of a cap in the vector space $\mathbb{F}_3^5$ is 45. However, a part of the proof for this in the article by Maclagan and Davis [10] uses yet another article to complete the proof. We shall therefore look into the theory here, but then just follow Maclagan and Davis and their proof for a 5-cap with referring to the last needed article, but without also digging into that extra article. With that said, lets look at the first statement, one being about the bounds of cap-sizes.

5.1 Bounds

There is an interesting formula that gives an upper bound for the maximal size of a cap in a vector space $\mathbb{F}_3^n$. This formula goes as follows:

Statement 6 Let $C \subset \mathbb{F}_3^n$ be an n-cap such that any hyperplane intersects C in at most h points. Then it holds that:

$$p \leq \frac{1 + 3 \cdot h}{1 + \dfrac{h}{3^{n-1}}} \tag{39}$$

where p is the size of C.

Now a hyperplane is a subspace of exactly one dimension lower than its ambient space. Therefore, any hyperplane intersects an n-cap in an $(n-1)$-cap. As this cap is exactly one dimension lower, it contains at most that many points as the maximal size of a cap in that lower dimension. If we start with the fact that the maximal size of a cap in dimension one is two, we can inductively apply Statement 6 to obtain the maximal sizes of a cap for dimension two, by using this information. This process can be repeated per higher dimension. Let us define this maximal cap-size as a_n for dimension n. We can then compute the following:

$$\begin{aligned}
a_2 &\leq \frac{1 + 3 \cdot 2}{1 + \dfrac{2}{3^{2-1}}} = 7 \cdot \frac{3}{5} = 4.2 \\
a_3 &\leq \frac{1 + 3 \cdot 4}{1 + \dfrac{4}{3^{3-1}}} = 13 \cdot \frac{9}{13} = 9 \\
a_4 &\leq \frac{1 + 3 \cdot 9}{1 + \dfrac{9}{3^{4-1}}} = 28 \cdot \frac{27}{36} = 21 \\
a_5 &\leq \frac{1 + 3 \cdot 20}{1 + \dfrac{20}{3^{5-1}}} = 61 \cdot \frac{81}{101} \approx 48.9
\end{aligned} \tag{40}$$

Now we can see that this statement does not calculate the exact maximal size of a cap, but it does give an upper bound that is pretty close if not equal to the maximal size of the cap for the first couple of dimensions. It is also this statement with which we can calculate the upper bound of a_6. Applying Statement 6 while using $h = a_5 = 45$ and $d = 6$ gives an upper bound of $a_6 \leq 114$. Thus, for low-dimensional caps, Statement 6 gives nearly sharp bounds. In contrast to other methods, Statement 6 does not become more difficult to apply as the dimension grows larger which is the advantage of this upper bound formula.

5.2 Fourier Transform

Let us first give the general definition of the Fourier transform.

The Fourier Transform
Given a function $f : \mathbb{F}_3^n \to \mathbb{C}$, then the Fourier transform is defined as a new function $\hat{f} : \mathbb{F}_3^n \to \mathbb{C}$ defined by the formula:

$$\hat{f}(z) = \hat{f}(z_1, \ \ldots \ , z_n) = \sum_{x \in \mathbb{F}_3^d} f(x) \cdot \xi^{z \cdot x} \tag{41}$$

where ξ is defined as $\xi = e^{2\pi i/3}$. Note that this ξ is well defined because $\xi^3 = 1$.

Given a set $S \subset \mathbb{F}_3^n$, the characteristic function χ of S, with $\chi : \mathbb{F}_3^n \to \mathbb{C}$, is defined by the formula:

$$\chi(x) = \begin{cases} 1 & if \quad x \in S \\ 0 & if \quad x \notin S \end{cases} \tag{42}$$

So knowing the characteristic function is the same as knowing the set S. After all, the characteristic function is zero for all x outside the set S. This characteristic function of S gives us the following Fourier transform:

$$\tau(z) = \hat{\chi}(z) = \sum_{x \in \mathbb{F}_3^n} \chi(x) \cdot \xi^{z \cdot x} = \sum_{c \in S} \xi^{z \cdot c} \tag{43}$$

We notice that $\tau(0) = \hat{\chi}(0) = \sum_{c \in S} 1 = |S|$ and thus simply is the size of the set. Now let us assume that z is a non-zero vector, so $z \in \mathbb{F}_3^n \setminus \{0\}$, and consider the three parallel hyperplanes H_0, H_1, H_2 normal to z. We define these hyperplanes as follows:

$$H_j = \{x \in \mathbb{F}_3^n \mid z \cdot x = j, j \in \mathbb{F}_3\} \tag{44}$$

To each non-zero vector z we associate a hyperplane triple. We notice that $\mathbb{F}_3^n$ can be decomposed in three parallel hyperplanes, which gives us the following triple of numbers:

$$(h_0, h_1, h_2) = (S \cap H_0, S \cap H_1, S \cap H_2) \tag{45}$$

Statement 7 *The complex number $\tau(z)$ encodes the same data as the ordered hyperplane triple (h_0, h_1, h_2) associated to z. In particular:*

$$\tau(z) = h_0 \cdot \xi^0 + h_1 \cdot \xi^1 + h_2 \cdot \xi^2 = h_0 + h_1 \cdot \xi + h_2 \cdot \xi^2 \tag{46}$$

and

$$
\begin{aligned}
h_0 &= \frac{2}{3} \cdot u + \frac{1}{3} \cdot p \\
h_1 &= \frac{1}{3} \cdot (p-u) + \frac{1}{\sqrt{3}} \cdot v \\
h_2 &= \frac{2}{3} \cdot (p-u) - \frac{1}{\sqrt{3}} \cdot v
\end{aligned} \tag{47}
$$

where $\tau(z) \in \mathbb{C}$ *so we define* $\tau(z) = u + i \cdot v$ *and* $p = \tau(0) = |S| = h_0 + h_1 + h_2$ *and thus is the size of* S*.*

Here, τ is called the (ordered) hyperplane triple function of S. We also notice that we do not have to specifically look into hyperplanes, as hyperplanes now arise naturally via the Fourier transform.

Proof
Let us first rewrite $\tau(z)$. We know that $\xi = e^{2\pi i/3}$ and thus realise the following:

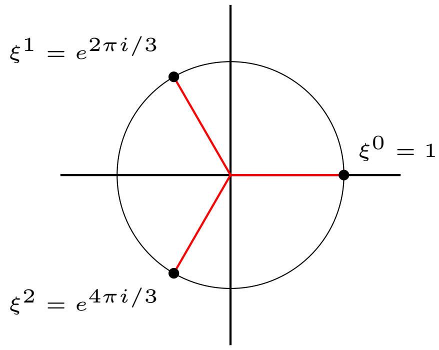

Figure 17: ξ in the complex plane

We use the above to rewrite ξ in the formula for $\tau(z)$ and get the following:

$$
\begin{aligned}
\tau(z) &= h_0 \cdot \xi^0 + h_1 \cdot \xi^1 + h_2 \cdot \xi^2 \\
&= h_0 \cdot 1 + h_1 \cdot (-\frac{1}{2} + \frac{1}{2} i\sqrt{3}) + h_2 \cdot (-\frac{1}{2} - \frac{1}{2} i\sqrt{3}) \\
&= h_0 - \frac{1}{2} \cdot (h_1 + h_2) + \frac{1}{2} i\sqrt{3} \cdot (h_1 - h_2)
\end{aligned} \tag{48}
$$

Here we see that with $\tau(z) = u + i \cdot v$ and thus $u = h_0 - \frac{1}{2} \cdot (h_1 + h_2)$. Furthermore, we use the fact that $p = h_0 + h_1 + h_2$ and so determine h_0 in terms of u and p:

$$
\begin{aligned}
& u = h_0 - \frac{1}{2} \cdot (h_1 + h_2) \\
\Rightarrow\ & u = h_0 - \frac{1}{2} \cdot (p - h_0) \\
\Rightarrow\ & u = \frac{3}{2} \cdot h_0 - \frac{1}{2} \cdot p \\
\Rightarrow\ & h_0 = \frac{2}{3} \cdot u + \frac{1}{3} \cdot p
\end{aligned} \tag{49}
$$

Now to determine h_1 and h_2 in terms of u, v and p, we use the fact that (48) shows us that $v = \frac{1}{2}\sqrt{3} \cdot (h_1 - h_2)$ notice that we can express $h_1 + h_2$ and $h_1 - h_2$ in the following ways:

$$
\begin{aligned}
h_1 + h_2 &= p - h_0 = \frac{2}{3} \cdot p - \frac{2}{3} \cdot u \\
h_1 - h_2 &= \frac{2}{3} \cdot v \cdot \sqrt{3} = \frac{2}{\sqrt{3}} \cdot v
\end{aligned} \tag{50}
$$

which gives us:

$$
2 \cdot h_1 = \frac{2}{3} \cdot p - \frac{2}{3} \cdot u + \frac{2}{\sqrt{3}} \cdot v \Rightarrow h_1 = \frac{1}{3} \cdot (p-u) + \frac{1}{\sqrt{3}} \cdot v \tag{51}
$$

and

$$
2 \cdot h_2 = \frac{2}{3} \cdot p - \frac{2}{3} \cdot u - \frac{2}{\sqrt{3}} \cdot v \Rightarrow h_2 = \frac{1}{3} \cdot (p-u) - \frac{1}{\sqrt{3}} \cdot v \tag{52}
$$

And with that, all of Statement 7 is proven. □

The following statement is about an interesting formula to count the number of lines in a set S. The formula goes as follows:

Statement 8 Let S be a subset of $\mathbb{F}_3^n$ that contains p points and l lines. Then:

$$
p + 6l = \frac{1}{3^d} \sum_{z \in \mathbb{F}_3^n} \tau^3(z) \tag{53}
$$

where τ is the hyperplane triple function of S as defined in (43) and (46).

Proof
Let us first look into the hyperplane triple function $\tau^3(z)$. We know that:

$$
\sum_{z \in \mathbb{F}_3^n} \tau^3(z) = \sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in \mathbb{F}_3^n} \chi(u)\xi^{z \cdot u} \cdot \chi(v)\xi^{z \cdot v} \cdot \chi(w)\xi^{z \cdot w} = \sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in S} \xi^{z \cdot (u+v+w)} \tag{54}
$$

Now there are two options, either $u + v + w = 0$ or $u + v + w \neq 0$. In the latter case, we get that $u + v + w$ is a non-zero vector and thus $z \cdot (u + v + w) = j$ with $j \in \mathbb{F}_3$. Note that each $j \in \mathbb{F}_3$ occurs an equal number of times, namely 3^{n-1} times. This gives us the following:

$$
\sum_{z \in \mathbb{F}_3^n} \xi^{z \cdot (u+v+w)} = (\xi^0 + \xi^1 + \xi^2) \cdot 3^{n-1} = 0 \tag{55}
$$

So the case $u+v+w \neq 0$ does not give a contribution. That leaves us with the case $u+v+w = 0$. We can again distinguish between two situations for for this case. Either, $u + v + w = 0$ because $u = v = w$ or $u + v + w = 0$ because u, v and w are not equal. In the first case we get:

$$
\sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in S} \xi^{z \cdot (u+v+w)} = \sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in S} \xi^0 = \sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in S} 1 = |\mathbb{F}_3^n||S| = 3^n p \tag{56}
$$

The second case works similar. However, now u, v and w lie on one line and for every line there are $3 \cdot 2 \cdot 1$ points to chose within that line. Therefore this gives us:

$$
\sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in S} \xi^{z \cdot (u+v+w)} = \sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in S} \xi^0 = \sum_{z \in \mathbb{F}_3^n} \sum_{u,v,w \in S} 1 = |\mathbb{F}_3^n| \cdot 6l = 3^n \cdot 6l \tag{57}
$$

So together this gives exactly what we were looking for:

$$\sum_{z\in\mathbb{F}_3^n} \tau^3(z) = 3^n p + 3^n 6l \Rightarrow \frac{1}{3^n}\sum_{z\in\mathbb{F}_3^n} \tau^3(z) = p + 6l \tag{58}$$

□

5.3 Proof of Statement 6

In the above sections, three different statements were introduced. Statement 6, 7 and 8, along with the proofs of Statement 7 and Statement 8. However, we did not yet give a proof of Statement 6. According to the paper of Davis and Maclagan, Statement 6 is proved by using Statement 8 and some clever estimates of $|\tau^3(z)|$. In the sections below, we shall give a proof for Statement 6 based on the paper by Bierbrauer and Edel. In that paper, the proof is quite long as it is given by proving several Lemma's first, which then build up to the proving Statement 6 in the end. Here we will go through these Lemma's and their proofs and so understand why Statement 6 is true.

5.3.1 Definitions

Let us first go through some definitions. In the paper by Bierbrauer and Edel, Statement 6 is proven not just for the space $\mathbb{F}_3^n$ but in the most general case, the vector space $\mathbb{F}_q^n$. We are however actually only interested in the first situation where $q = 3$ and that is why we shall go through the papers proof using $q = 3$. We therefore define a cap $A \subset \mathbb{F}_3^n$. We also define C_n as the maximum cardinality of a cap in $\mathbb{F}_3^n$. The density of the set A is then defined as c_n and goes as follows:

$$c_n = \frac{C_n}{|\mathbb{F}_3^n|} = \frac{C_n}{3^n} \tag{59}$$

With this defined, Bierbrauer and Edel now state a slightly different looking version of Statement 6, namely the following:

Statement 9 Let $q = 3$, if $n \geq 3$, then

$$c_n \leq \frac{3^{-n} + c_{n-1}}{1 + c_{n-1}} \tag{60}$$

equivalently

$$(1 - c_n) \cdot (c_{n-1} - c_n) \geq c_n^2 - 3^{-n} \tag{61}$$

And even though (60) looks slightly different than our Statement 6, using the definition of $c_n(q) = C_n(q)/q^n$ and rewriting, we get exactly the same as stated in equation (81):

$$\begin{aligned} &\Rightarrow c_n \leq \frac{3^{-n} + c_{n-1}}{1 + c_{n-1}} \\ &\Rightarrow \frac{C_n}{3^n} \leq \frac{3^{-n} + C_{n-1} \cdot 3^{-n+1}}{1 + C_n} \\ &\Rightarrow \ C_n \leq \frac{1 + 3 \cdot C_{n-1}}{1 + \frac{C_{n-1}}{3^{n-1}}} \end{aligned} \tag{62}$$

for $C_{n-1} = h$ and $C_n = p$. So therefore we will go along with the paper and try to prove Statement 9 from now on.

5.3.2 Lemmas

So we defined $q = 3$. Let $x \cdot y$ be the ordinary dot product defined on $V = \mathbb{F}_3^n = AG(n, 3)$, with values in $\mathbb{F}_3$ and $Q = |V| = |\mathbb{F}_3^n| = 3^n$. Finally, ξ is defined as $\xi = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, just like before. Now consider the complex number:

$$S = \sum_{y \in V \setminus \{0\}} \sum_{a_1, a_2, a_3 \in A} \xi^{\sum_i a_i \cdot y} \tag{63}$$

Lemma 1 $S = |A|(Q - |A|^2)$

Proof
We first notice the following:

$$S = \sum_{y \in V} \sum_{a_1, a_2, a_3 \in A} \xi^{\sum_i a_i \cdot y} - |A|^3 \tag{64}$$

After all, the case where $y = 0$ gives us the following:

$$\sum_{0 \in V} \sum_{a_1, a_2, a_3 \in A} \xi^{\sum_i a_i \cdot 0} = \sum_{a_1, a_2, a_3 \in A} 1 = \sum_{a_1 \in A} 1 \cdot \sum_{a_2 \in A} 1 \cdot \sum_{a_3 \in A} 1 \cdot = |A| \cdot |A| \cdot |A| \cdot = |A|^3 \tag{65}$$

Now whenever $\sum_{i=1}^{3} a_i \neq 0$, the corresponding $y \in V$ vanishes, as we saw in (55). As A is a cap, we know that this will always be the case, unless $a_1 = a_2 = a_3$. This means that:

$$\sum_{y \in V} \sum_{a_1, a_2, a_3 \in A} \xi^{\sum_i a_i \cdot y} = \sum_{y \in V} \sum_{a_1, a_2, a_3 \in A} 1 = |V||A| \tag{66}$$

Therefore we indeed find that $S = |V||A| - |A|^3 = |A|(Q - |A|^2)$. □

Definition 1 Let $0 \neq y \in V$. Consider the complex number $U_y = \sum_{a \in A} \xi^{a \cdot y}$. Let $u_y = |U_y|$. We define a real vector u of length $Q - 1 = 3^n - 1$ whose coordinates are parametrized by the $0 \neq y \in V$, the corresponding entry being u_y.

Lemma 2 Let $0 \neq y \in V$. Then:

$$u_y \leq 3 \cdot C_{n-1} - |A| = c_{n-1}Q - |A| \tag{67}$$

Proof
Denote by ν_c, the number of elements $a \in A$ such that $a \cdot y = c$, so $\nu_c = |\{a \in A | a \cdot y = c\}|$. Now as y is a non-zero vector, we can consider the following hyperplane: $H_c = \{v \in \mathbb{F}_3^n | v \cdot y = c\}$, with $c \in \mathbb{F}_3$. Therefore, $\nu_c \leq C_{n-1}$. Now using all these definitions it follows that:

$$u_y = |U_y| = |\sum_{a \in A} \xi^{a \cdot y}| = |\sum_{c \in \mathbb{F}_3} \nu_c \xi^c| \tag{68}$$

Now again, as stated in (55), we know that $\sum_{c \in \mathbb{F}_3} \xi^c = 0$ and thus $\sum_{c \in \mathbb{F}_3} \nu_c \xi^c = 0$ and also $\sum_{c \in \mathbb{F}_3} C_{n-1} \xi^c = 0$ so it follows that:

$$|\sum_{c\in\mathbb{F}_3}\nu_c\xi^c| = |\sum_{c\in\mathbb{F}_3}-\nu_c\xi^c| = |\sum_{c\in\mathbb{F}_3}(C_{n-1}-\nu_c)\xi^c| \tag{69}$$

Using the triangle inequality, the fact that $\xi = e^{2\pi i/3}$ and thus $|\xi| = 1$ and the fact that $\nu_c \leq C_{n-1}$, we obtain:

$$\begin{aligned}|\sum_{c\in\mathbb{F}_3}(C_{n-1}-\nu_c)\xi^c| &\leq \sum_{c\in\mathbb{F}_3}|C_{n-1}-\nu_c||\xi^c| \\ &= \sum_{c\in\mathbb{F}_3}|C_{n-1}-\nu_c| \\ &= \sum_{c\in\mathbb{F}_3}C_{n-1}-\nu_c \\ &= C_{n-1}\sum_{c\in\mathbb{F}_3}1-\sum_{c\in\mathbb{F}_3}\nu_c \\ &= 3\cdot C_{n-1}-|A| \\ &= 3^n\cdot c_{n-1}-|A| \\ &= Q\cdot c_{n-1}-|A|\end{aligned} \tag{70}$$

□

Lemma 3 $||u||^2 = |A|(Q-|A|)$

Proof By using and rewriting the definition of the norm for complex vectors we get the following:

$$||u||^2 = \sum_{y\in V\setminus\{0\}}U_y\bar{U}_y = \sum_{y\in V}U_y\bar{U}_y - U_0\bar{U}_0 \tag{71}$$

Using the definition of U_y for $y = 0$ gives $U_0 = \sum_{a\in A}\xi^{a\cdot 0} = \sum_{a\in A}1 = |A|$ and thus $U_0\bar{U}_0 = |A|^2$. The same way we also obtain:

$$\sum_{y\in V}U_y\bar{U}_y = \sum_{y\in V}\sum_{a\in A}\xi^{a\cdot y}\overline{\sum_{a\in A}\xi^{a\cdot y}} = \sum_{y\in V}\sum_{a\in A}\xi^{a\cdot y}\sum_{a\in A}\xi^{-a\cdot y} \tag{72}$$

Here, we can distinguish between two cases. The value for a in both the sums is either equal or not equal. In the case they are not equal, we define $a \neq b$, combine the sums to one sum and then get the following: $\sum_{y\in V}\sum_{a,b\in A}\xi^{(a-b)\cdot y} = 0$ as we have seen before. Now in the case that the a's in both sums are equal, we take $\xi^{a\cdot y}$ and $\xi^{-a\cdot y}$ together and that gives exactly what we were looking for:

$$\sum_{y\in V}\sum_{a\in A}\xi^{a\cdot y}\sum_{a\in A}\xi^{-a\cdot y} = \sum_{y\in V}\sum_{a\in A}\xi^{(a-a)\cdot y} = \sum_{y\in V}\sum_{a\in A}1 = |V||A| = Q|A| \tag{73}$$

So continuing with equation (71) and using what was found in (72) and (73) we find the desired result, namely that:

$$||u||^2 = \sum_{y\in V}U_y\bar{U}_y - U_0\bar{U}_0 = Q|A| - |A|^2 = |A|(Q-|A|) \tag{74}$$

□

Comparison of Lemmas 2 and 3 yields a first lower bound on $c_{n-1} - c_n$ as follows. Choose $|A| = C_n = c_n \cdot 3^n$. Then from Lemma 2 it follows that:

$$u_y \leq c_{n-1}Q - |A| = c_{n-1}3^n - c_n 3^n = 3^n(c_{n-1} - c_n) \tag{75}$$

so the entries of u are positive bounded numbers as $c_{n-1} \leq c_n$. By definition and with using (75) it follows that:

$$||u||^2 = \sum_{i=1}^{Q-1} |u_{y_i}|^2 \leq (Q-1) \cdot \max|u_{y_i}|^2 = (3^n - 1) \cdot \max\, u_{y_i}^2 \leq (3^n - 1) \cdot (3^n \cdot (c_{n-1} - c_n))^2 \tag{76}$$

Now combining what was found in (76) with Lemma 3 we get the following:

$$\begin{aligned} &(3^n - 1) \cdot (3^n(c_{n-1} - c_n))^2 \geq |A|(Q - |A|) \\ &\Rightarrow (3^n - 1) \cdot (3^n(c_{n-1} - c_n))^2 \geq 3^n c_n(3^n - 3^n c_n) \\ &\Rightarrow (3^n - 1) \cdot 3^{2n}(c_{n-1} - c_n)^2 \geq 3^{2n} c_n(1 - c_n) \\ &\Rightarrow (c_{n-1} - c_n)^2 \geq \frac{c_n(1 - c_n)}{3^n - 1} \end{aligned} \tag{77}$$

This result is called **Theorem 3:** $(c_{n-1} - c_n)^2 \geq c_n(1 - c_n)/(3^n - 1)$.

The following lemma is an obvious consequence of definitions.

Lemma 4 We have $S = \sum_{y \neq 0} U_y^3$, in particular $|S| \leq \sum_{y \neq 0} u_y^3$

With this the proof for Statement 9 can be completed. First, we use Lemma 2 to obtain an upper bound on u_y. Then Lemma 3 is used on $\sum_{y \in \mathbb{F}_3^n \setminus 0} u_y^2 = ||u||^2$ which gives the following:

$$\begin{aligned} |S| &\leq \sum_{y \neq 0} u_y^3 \\ &\leq (c_{n-1}Q - |A|) \cdot \sum_{y \in \mathbb{F}_3^n \setminus 0} u_y^2 \\ &= (c_{n-1}Q - |A|) \cdot ||u||^2 \\ &= (c_{n-1}Q - |A|) \cdot |A|(Q - |A|) \\ &= (c_{n-1}3^n - |A|) \cdot |A|(3^n - |A|) \end{aligned} \tag{78}$$

Like before, we take $|A| = C_n = c_n \cdot 3^n$. Furthermore, Lemma 1 is used, in which it was stated that $S = |A|(Q - |A|^2) = c_n \cdot 3^n \cdot (3^n - c_n^2 \cdot 3^{2n})$. This implies that S is an negative integer. Together with (78), this gives the following:

$$\begin{aligned} &|S| \leq (c_{n-1}3^n - |A|) \cdot |A|(3^n - |A|) \\ &\Rightarrow -c_n \cdot 3^n \cdot (3^n - c_n^2 \cdot 3^{2n}) \leq (c_{n-1} \cdot 3^n - c_n \cdot 3^n) \cdot c_n \cdot 3^n \cdot (3^n - c_n \cdot 3^n) \\ &\Rightarrow c_n^2 \cdot 3^n - 1 \leq (c_{n-1} - c_n) \cdot 3^n \cdot (1 - c_n) \\ &\Rightarrow c_n^2 - 3^{-n} \leq (c_{n-1} - c_n) \cdot (1 - c_n) \end{aligned} \tag{79}$$

□

And with this we see that Statement 9 is proven and that means that also the original Statement 6 is proven.

5.4 The vector space $\mathbb{F}_3^5$

So as stated at the beginning of this chapter, we shall here follow the article of Davis and Maclagan for the proof of the maximal size of a cap in the vector space $\mathbb{F}_3^5$, which also does get to the bottom as it states that exhaustive computer search is needed in order to complete it.

Statement 10 *The maximum size of a cap in the vector space $\mathbb{F}_3^5$ is 45.*

Proof
The proof for this again works through contradiction. Suppose that there exists a cap in $\mathbb{F}_3^5$, let us say C, that contains 46 points. By the Fourier analysis bound of Statement 6 we can conclude that if every hyperplane intersects C in at most 18 points, then C can have at most 45 points. That means that in our case, where we assume that there is a cap C containing 46 points, that there must be a hyperplane H intersecting C in 19 or 20 points. If we know delete a point of C that does not lie in H, we produce a 5-cap with 45 points such that H is a hyperplane intersecting in 19 or 20 points. However, in [14] it is shown that every 5-cap with 45 points has no hyperplanes intersecting in 19 or 20 points and we thus encounter a contradiction in our cap with 46 points. Therefore the maximum size of a cap in the vector space $\mathbb{F}_3^5$ stays 45. □

The proof in this article exploits an ingenious identity in the equations for counting marked hyperplanes, together with an exhaustive computer search, which we shall not further look into here. Below in Figure 18 is an illustration of a 5-cap.

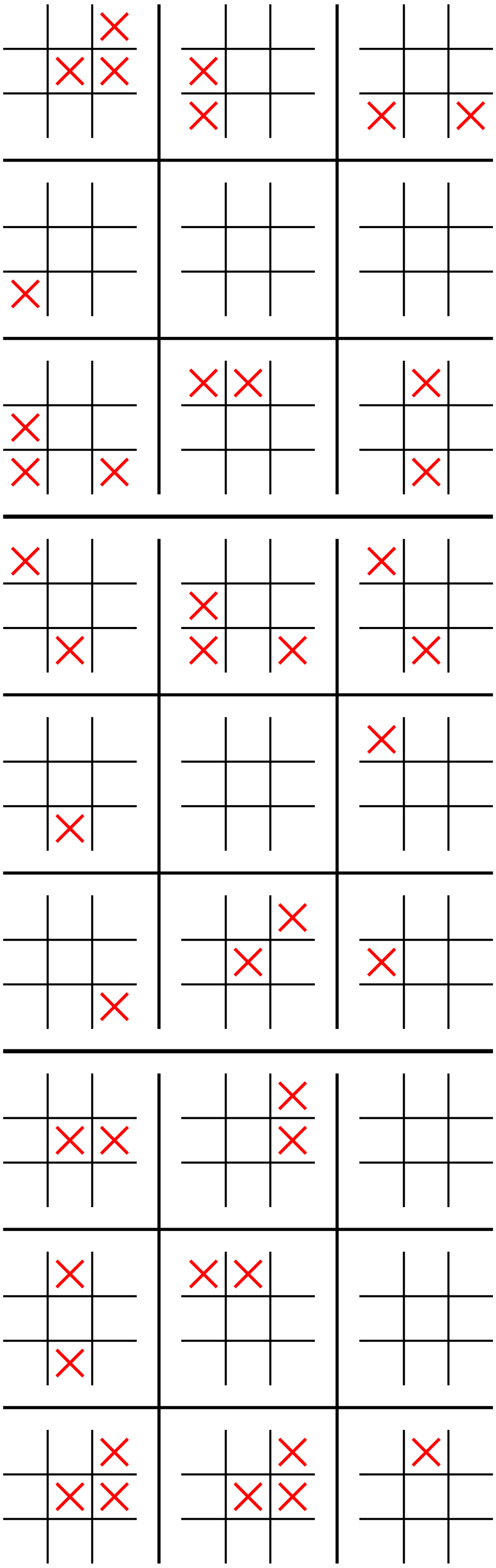

Figure 18: 5-cap

6 Asymptotic behaviour

In the above sections, the maximal cap-sizes in lower dimensions have been studied. Let a_n be the maximal number of points in a cap in $AG(3, n)$. Then we saw in the previous sections that the only values that are known are $a_1 = 2, a_2 = 4, a_3 = 9, a_4 = 20$ and $x_5 = 45$. Even though it was not proved in this report, it is also known that $a_6 = 112$. However, apart from these exact values, we are also interested in the asymptotic behaviour of the sequence (a_n). The following conclusions, about the asymptotic behaviour of a_n, come from an article about the cap-Set problem by Aart Blokhuis and Dion Gijswijt [3]. In this article, Blokhuis and Gijswijt discuss the recently new discovered upper bound, found by Jordan Ellenberg and Dion Gijswijt independently. The article includes an overview of which bounds existed prior to this new upper bound for $a(n)$ and below we have summarized this part of the article.

Let A_1 be a cap in $AG(n, 3)$ and A_2 be a cap in $AG(3, m)$, then the product $A_1 \times A_2 = \{(a_1, a_2)|a_1 \in A_1, a_2 \in A_2\}$ is a cap in $AG(n + m, 3)$. From this observation and via Fekete's Lemma it follows that the following limit exists:

$$a = \lim_{n\to\infty} \sqrt[n]{a_n} \tag{80}$$

Furthermore, $a \geq \sqrt[n]{a_n}$ for every n. With this we can determine lower bounds for a. If we for example take take the maximal cap-size of the vector space $\mathbb{F}_3^4$, so $a_4 = 20$, then we get the following lower bound: $a \geq \sqrt[4]{20} \geq 2,11$. The best lower bound that is known is comes from a construction of a cap in $AG(480, 3)$ by Edel [5] and gives $a \geq 2,21739$.

There is also an obvious upper bound, namely $a \leq 3$ as $AG(3, n)$ contains 3^n points. In 1982, Brown and Buhler [4] gave the first non-trivial upper bound $a_n = o(3^n)$. By making use of ideas from the Fourier Analysis, Meshulam improved this upper bound in 1995 to $a_n = O(3^n/n)$ [11]. The best upper bound until last year was a result of Bateman and Katz [1]. The proof for this result is again based on Fourier Analysis and gives $a_n = O(3^n/(n^{1+\epsilon}))$ for a certain positive but very small ϵ. For the constant a however, these upper bounds do not give an improvement. Therefore, the following problem, formulated by Brown and Buhler, still remains:

Problem 1 *Does it hold that $a < 3$?*

A breakthrough came in may 2016 when Ernie Croot, Seva Lev and Péter Pach [13] solved a very much related problem. They proved that every subset of $(\mathbb{Z}/4\mathbb{Z})^n$, that does not contain any arithmetic sequences of length 3, cannot contain more than γ^n elements for some constant $\gamma \approx 3,60$ so smaller than 4. Within a couple of weeks, Jordan Ellenberg and Dion Gijswijt independently, both succeeded into applying the technique of Croot, Lev and Pach in such a way that it worked for vector spaces $\mathbb{F}_q^n$ with q an odd prime-power. The case $q = 3$ corresponds exactly with the cap-SET problem. A nice exposition of this result can be found in [12]. Using this value for q, they found the following upper bound for a.

Let $A \subseteq \mathbb{F}_3^4$ be a cap. Then it holds that $|A| \leq 3 \cdot 2,756^n$ and thus $a \leq 2,756$

This result, along with its proof, can be found in an article that Dion Gijswijt and Jordan Ellenberg published together this year [7].

7 Applications

One might wonder what the use has been of all this calculating of maximal cap-sizes and if there are any applications of using these maximal cap-sizes. The most important application of maximal cap-sizes is for linear codes in Coding Theory. Coding theory is the study of the properties of codes and their respective fitness for specific applications. Codes are used for cryptography, error-correction and networking for example. We are mainly interested in error-detecting and error-correcting codes. During data transmission, errors may be introduced during the transmission from the source to a receiver. Error detection techniques allow detecting such errors, while error correction enables reconstruction of the original data in many cases. The general idea for achieving error detection and correction is to add some redundancy to a message, that is some extra data, which receivers can use to check consistency of the delivered message, and to hopefully recover data that has been corrupted.

7.1 Error detecting

Let us first look at binary codes, so codes that only consist of 0's and 1's. A binary string of eight bits looks like 00111001 for example. However, this is just an example. In total, there are 2^8 possible binary strings of eight bits long and in general thus 2^k k-bit binary strings. A simple way to help the receiver detect whether there is an error in the received binary message is for the sender to add one extra bit to the binary string that makes the number of 1's even. However, this only lets the receiver know that there is an error somewhere in the message. Furthermore, it can only detect one error, as soon as the there are more errors, this method becomes useless. Also, in order for the receiver to be able to fix the error, the sender must add more bits. This process is explained in the next section.

7.2 Error Correcting

First, let us define a matrix A. This matrix A is defined as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Now, the data that you want to send, must be perpendicular (mod 2) to each of the rows of the matrix above. Suppose we want to send be a 4-bit code word. Then there are $2^4 = 16$ different words we could make. Let us choose 1101 for example. Then, in order for 1101 be perpendicular to each row, the fourth row becomes 0011101. One then gets the following code, which we shall call matrix B:

$$\mathrm{B} = \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & [1 & 1 & 0 & 1] \end{array}\right]$$

Where we define $\mathbf{w}^T = (0, 0, 1, 1, 1, 0, 1)$. So we have that $\mathbf{w}^T$ is a code word if $A\mathbf{w} = 0$ holds. Another way of expressing this is $A\mathbf{w} = \sum_i w_i \cdot \mathbf{a}_i = 0$. Here w_i is the bit at position i in the vector $\mathbf{w}^T$ and $\mathbf{a}_i$ is the i^{th} column vector of the matrix A.

The important thing here, is that two different code words differ in at least 3 positions. After all, suppose that a code word differs in only one bit. As the matrix A does not contain a column with only zeros, there is no vector a_i one can add or subtract from $\sum_i w_i \cdot \mathbf{a}_i$ such that this sum still equals zero. Same holds for a code word that differs from the original code word in only two positions. No matter which two column vectors you chose, you cannot chose two column vectors a_i and a_j with $i \neq j$ such that $a_i \pm a_j = 0$. However, once a code word differs in three positions, $A\mathbf{w} = 0$ does still hold.

7.3 Translation to SET

To make the transition to the game SET and the caps we looked at before, we first change the binary codes to ternary codes, so codes with the numbers 0, 1 and 2. A ternary string of eight trits now looks like 01121102 for example. There are 3^k possible ternary strings of k-trits long. We now define our matrix C. The columns of this matrix are vectors from a CAP, so vectors in the vector space $\mathbb{F}_3^4$. So matrix C looks something like:

$$\text{C} = \begin{bmatrix} 0 & 0 & \dots & & 1 \\ 1 & 2 & \dots & & 0 \\ 1 & 2 & \dots & & 1 \\ 2 & 0 & \dots & & 0 \end{bmatrix}$$

Now the total code is defined as the matrix C we saw above, but with one extra row added to the bottom of matrix C. That is a row with only 1's in it. This extra row of 1's is necessary in order for the matrix to contain the same properties as the matrix in the previous section. We define this matrix C plus the extra row of 1's at the bottom as $\hat{C}$. Just like before, the data you want to send is added in the last row. Again this data must be perpendicular to each of the rows above of the matrix. Together this gives a matrix that looks like this:

$$\left[\begin{array}{cccccc} 0 & 0 & 1 & \dots & & 1 \\ 1 & 2 & 0 & \dots & & 0 \\ 1 & 2 & 2 & \dots & & 1 \\ 2 & 0 & 1 & \dots & & 0 \\ 1 & 1 & 1 & \dots & & 1 \\ \hline 0 & 2 & 1 & \dots & & 0 \end{array}\right]$$

So matrix $\hat{C}$ actually contains vectors from the vector space $\mathbb{F}_3^5$, so $(x_1, x_2, x_3, x_4, x_5)$ with $x_5 \neq 0$. So vectors that look like the following, if we divide all vectors by x_5 for example, $(y_1, y_2, y_3, y_4, 1)$ So (y_1, y_2, y_3, y_4) respond with vectors in a cap.

Thanks to the row of ones, there is now no column with only zeros in $\hat{C}$. Secondly, there are no two columns in $\hat{C}$ that are equal or where one column is two times the other column. Lastly, there are also no three columns that are linear dependent from one and another. So here, two different code words differ in at least four positions so the minimum distance, the hamming distance, is 4. The code can correct exactly one error and detect up to two errors. After all, in the case of three errors, one does not know for sure which code was the original code as there are more codes that could be created while trying to correct three errors.

To conclude why maximal caps are so important, the bigger the cap is, the bigger the message is that one can send. And people want to send as much information as possible within the smallest number of bits.

Conclusion

The cap-SET problem is the problem of finding the size of the largest possible cap-SET as a function of n. A cap-SET is defined as a subset of $\mathbb{F}_3^n$ with no three elements in a line. More generally, a cap is a subset of a finite affine space with no three in a line. Even though the maximal cap-sizes are known for the first six dimensions, the goal of this report was mainly to understand how one came to these maximal cap-sizes and to be able to prove them ourselves. In this report, we thoroughly looked at the first five dimensions and gave the proofs for their maximal cap-sizes. The results of this are shown in Table 4.

Dimension	1	2	3	4	5	6
Maximal cap-size $a(n)$	2	4	9	20	45	112

Table 4: Maximal cap-sizes per dimension

Before we came to these proofs and conclusions, we did a lot of theory research. We translated the SET problem to vector spaces and then first looked into all the subspaces we could count in $\mathbb{F}_3^3$ and $\mathbb{F}_q^3$. So how many lines per solid there were for example. The results for these two vector spaces are shown in the Table 5.

Counting	*The vector space* $\mathbb{F}_3^3$	*The vector space* $\mathbb{F}_q^3$
Points per line	3	q
Points per plane	$3^2 = 9$	q^2
Points per solid	$3^3 = 27$	q^3
Lines per plane	$\frac{9 \cdot 4}{3} = 12$	$\frac{q^2 \cdot (q+1)}{q}$
Lines per solid	$\frac{(3^3) \cdot (3^3 - 1)}{3 \cdot 2} = 117$	$\frac{q^3 \cdot (q^3 - 1)}{q \cdot (q-1)}$
Lines through a point	$\frac{26}{2} = 13$	$\frac{q^3 - 1}{q - 1}$

Table 5: Counting in $\mathbb{F}_3^3$ and $\mathbb{F}_q^3$

After that, we looked into the general vector space $\mathbb{F}_q^n$ and also started counting subspaces. For the vector space that corresponds with the game set, $\mathbb{F}_3^4$, we made an overview of all the k-dimensional subspaces within a n-dimensional space in $\mathbb{F}_3^4$. So the number of planes through a point in $\mathbb{F}_3^4$ or the number of solids through a line in $\mathbb{F}_3^4$ for example. An overview of this counting is shown in Table 6 below.

Counting	*Dimensions*	*Formula*
lines through a point	$0 \subseteq 1 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_3 = \frac{(3^4-1)}{(3^1-1)} = \frac{80}{2} = 40$
planes through a point	$0 \subseteq 2 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_3 = \frac{(3^4-1)\cdot(3^4-3)}{(3^2-1)\cdot(3^2-3)} = \frac{80\cdot 78}{8\cdot 6} = 130$
solids through a point	$0 \subseteq 3 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 4 \\ 3 \end{bmatrix}_3 = \frac{(3^4-1)\cdot(3^4-3)\cdot(3^4-3^2)}{(3^3-1)\cdot(3^3-3)\cdot(3^3-3^2)} = \frac{80\cdot 78\cdot 72}{26\cdot 24\cdot 18} = 40$
planes through a line	$1 \subseteq 2 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_3 = \frac{(3^3-1)}{(3^1-1)} = \frac{26}{2} = 13$
solids through a line	$1 \subseteq 3 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_3 = \frac{(3^3-1)\cdot(3^3-3)}{(3^2-1)\cdot(3^2-3)} = \frac{26\cdot 24}{8\cdot 6} = 13$
solids through a plane	$2 \subseteq 3 \subseteq 4$	$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_3 = \frac{(3^2-1)}{(3^1-1)} = \frac{8}{2} = 4$

Table 6: Counting subspaces in $\mathbb{F}_3^4$

After this, we were able to prove the maximal sizes of a cap in the first three dimensions, modulo three. Using the technique of 'Double Counting', we also managed to prove the size of a cap in a four-dimensional vector space, $\mathbb{F}_3^4$. After that we looked into Fourier Transforms to prove the maximal size of a cap in the five-dimensional vector space $\mathbb{F}_3^5$. Here we also looked at a formula that provides upper bounds. We saw for example that for an n-cap $C \subset \mathbb{F}_3^n$ which any hyperplane intersects in at most h points, that the following formula gives nearly sharp upper bounds for p, with p being the size of C:

$$p \leq \frac{1 + 3 \cdot h}{1 + \frac{h}{3^{n-1}}} \tag{81}$$

For higher dimensions, we shortly looked at the upper bounds that are known for the cap-sizes. In particular, the new discovered upper bound $|C| \leq 3 \cdot 2,756^n$ is mentioned, recently discovered by Jordan Ellenberg and Dion Gijswijt.

8 Discussion

As the maximum size of a cap is already known for the dimensions one up to six, there is little to discuss about the accuracy of our research, as our end results match these known values. However, it was going through these proofs for the lower dimensions, and seeing how every proof became more and more complicated per higher dimension, that made it clear why the maximum cap-sizes are only known for lower dimensions. It also clarified why people are still searching for better upper bounds in higher dimensions. This brings us to the next topic, upper bounds.

There are some things that we would looked deeper into if we would have had more time and the most important one of them is upper bounds for caps in higher dimensions. In the end, we only looked into one formula that determined upper bounds for caps using hyperplanes and we looked at the what upper bounds for caps already exist. If we would have taken more time, we could have looked into other methods or formulas for determining upper bounds. Then we could have compared compared these methods or formulas with each other to see which one gave sharper upper bounds. Now, we only had that one formula, and for lower dimensions, that one gave pretty sharp if not accurate upper bounds. If we had looked into more of these kind of upper bound formula's, we could have compared them for higher dimensions, where there is more uncertainty. Also, if we had more time, we could have paid more attention to the known upper bounds. We could have looked into how one determined the upper bounds that already exist. In particularly, the new upper bound recently discovered by both Jordan Ellenberg and Dion Gijswijt, would have been interesting to look into and maybe even to try to understand.

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