On a motivic interpretation of primitive, variable and fixed cohomology

ON A MOTIVIC INTERPRETATION OF PRIMITIVE, VARIABLE AND FIXED COHOMOLOGY

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1. Introduction

This note aims to address the motivic nature of some classical cohomological results of Lefschetz. The first is the Lefschetz decomposition of the cohomology of a smooth projective manifold. The second is a consequence of Lefschetz' hyperplane theorem, namely the splitting of the cohomology of a complete intersection into a summand which comes from the surrounding variety, the "fixed part", and a supplementary summand, the "variable" part. Explicitly, fix an $(d+r)$-dimensional projective manifold M and an ample line bundle $\mathscr{L}$ on M; let $X = H_1 \cap \cdots \cap H_r$ be a smooth complete intersection of r divisors $H_j \in |\mathscr{L}|$, $j = 1, \ldots, r$ and let $i : X \hookrightarrow M$ be the inclusion. With

$$\begin{aligned} H^d(X)_{\text{fix}} &:= \operatorname{Im}(i^* : H^d(M) \to H^d(X)) \\ H^d(X)_{\text{var}} &:= \operatorname{Ker}(i_* : H^d(X) \to H^{d+2r}(M)) \end{aligned} \tag{1}$$

there is an orthogonal direct sum decomposition

$$H^d(X) = H^d(X)_{\text{fix}} \oplus H^d(X)_{\text{var}}. \tag{2}$$

In general it seems hard to show the motivic nature of these results and some conditions will be needed, Clearly, a first ingredient one needs is the existence of a correspondence inducing the inverse of the Lefschetz operator on $H^*(M)$. This is Lefschetz' conjecture $B(M)$. The second comes from a concept introduced by Kimura [3] and O'Sullivan, the concept of finite-dimensionality for motives. They conjecture that all motives are finite-dimensional. The main result of this note is that the primitive decomposition for the cohomology of M as well as the splitting (2) is motivic is motivic provided these two conjectures hold for M. [1] In fact, only a consequence of finite dimensionality is used, namely a certain nilpotency result which is stated as (3).

It is known that both Kimura's conjecture and conjecture B are verified for example for M a projective space, or an abelian variety. For these examples the motive of M is well understood and the primitive decomposition is probably well known. See e.g. Diaz' explicit results [2] for abelian varieties. The motivic nature of the splitting (2) for complete intersections $X \subset M$ shows that the relevant motivic information is hidden in the variable motive.

This can be taken advantage of in situations where the motive of M is too large. Let me illustrate this with the Bloch conjecture [1] for surfaces. Recall that the latter states that if $p_g = 0$,

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[1] For the comfort of the reader some facts about Chow motives are placed together in Section 2.

for zero-cycles homological equivalence and rational equivalence coincide so that CH_0 is "small". In the present setting, assuming that one has a *complex* complete intersection surface $X \subset M$, such that $h^{2,0}(M) \neq 0$, then, by Lefschetz' theorem on hyperplane sections $h^{2,0}(X) \neq 0$, and then, by a result of Mumford [5], the Chow group of zero cycles on X is huge. However, it may happen that the variable submotive of X, or a smaller submotive thereof does satisfy the conditions for Bloch's conjecture. This observation can indeed be put to use as is shown in the examples of [4]; the present note sets up the proper theoretical framework.

Notation.
- H^* denotes Weil cohomology; CH_* denotes Chow groups with $\mathbf{Q}$-coeffcients.
- The degree d correspondences from X to Y are denoted $\mathrm{Corr}^d(X, Y)$.
- For a smooth projective manifold X, its Chow motive is denoted $\mathsf{h}(X)$.

2. MOTIVES

Recall that a *(Chow) correspondence of degree* k from a smooth projective variety X to a smooth projective variety Y is a cycle class $\mathsf{CH}^{\dim X+k}(X \times Y)$. It induces a morphism on Chow groups of the same degree and on cohomology groups (of double the degree).

Correspondences can be composed and these give the morphisms in the category of Chow motives. Let me elaborate briefly on this but refer to [6] for more details.

Precisely, *an effective Chow motive* consists of a pair (X, p) with X a smooth projective variety and p a degree zero correspondence which is a projector, i.e., $p^2 = p$. Morphism between motives are induced by degree zero correspondences compatible with projectors. Every smooth projective variety X defines a motive

$$\mathsf{h}(X) = (X, \Delta), \quad \Delta \in \mathsf{CH}^{\dim X}(X \times X) \text{ the class of the diagonal}$$

and a morphism $f : X \to Y$ between smooth projective varieties defines a morphism $h(Y) \to \mathsf{h}(X)$ given by the transpose of the graph of X. This procedure defines the category of effective Chow motives.

One can also use correspondences of arbitrary degrees provided one uses triples (X, p, k) where p is again a projector, but a morphism $f : (X, p, k) \to (Y, q, \ell)$ is a correspondence of degree $\ell - k$ compatible with projectors. Such triples define the *category of Chow motives*.

It should be recalled that motives, like varieties have their Chow groups and cohomology groups:

$$\mathsf{CH}^m(X, p, k) := \mathrm{Im}\left(\mathsf{CH}^{m+k}(X) \xrightarrow{p_*} \mathsf{CH}^{m+k}(X)\right),$$
$$H^m(X, p, k) := \mathrm{Im}\left(H^{m+2k}(X) \xrightarrow{p_*} H^{m+2k}(X)\right).$$

Kimura [3] has introduced the concept finite-dimensionality for motives and he has shown that it implies the following nilpotency result.

$$N \in \mathrm{Corr}^0(M, M) \text{ with trivial action on } H^*(M) \implies N \text{ is nilpotent.} \tag{3}$$

3. THE PRIMITIVE MOTIVE

3.1. **Primitive cohomology.** Let M be a smooth projective variety. First we recall some pertinent facts concerning the Lefschetz theorems. These are most easily described in terms of

$H_M = H^*(M)(\dim M)$, the rational cohomology of M centered in degree 0. The Lefschetz operator $L \in \mathrm{End}(H_M)$ is a degree 2 operator and Hard Lefschetz states that

$$L^j : H_M^{-j} \xrightarrow{\sim} H_M^j, \quad j = 0, \ldots, \dim M. \tag{4}$$

Using this, one can construct a linear map $\lambda : H_M \to H_M$ of degree -2 which is an inverse of L on the subspace $LH_M \subset H_M$:

$$\lambda \circ L = L \circ \lambda = 1 \text{ on the image of } L \implies L \circ \lambda \text{ is a cohomological projector.} \tag{5}$$

The Lefschetz decomposition gives the corresponding direct sum decomposition

$$H_M = H_M^{\mathrm{pr}} \oplus LH'_M, \quad H_M^{\mathrm{pr}} := \mathrm{Ker}(\lambda), \tag{6}$$

which is orthogonal with respect to the cup-product pairing. Indeed, we have $L \circ \lambda(u) = 0$ if u is primitive and if $u = Lu'$ we have $L \circ \lambda(u) = L \circ \lambda \circ L(u') = L(u') = u$. This shows that

$$\pi^{\mathrm{pr}} := \mathrm{id} - L \circ \lambda \tag{7}$$

gives a projector onto the primitive cohomology.

3.2. **Construction of the "primitive" Chow projector.** We next explain under what conditions these projectors can be lifted to correspondences. First note that $L \in \mathrm{Corr}^1(M, M)$.

Lefschetz' conjecture $B(M)$ states that there is a correspondence $\Lambda \in \mathrm{Corr}^{-1}(M, M)$ inducing λ. More will be needed, namely a lift of $L^r \circ \lambda^r$ to a (Chow) projector. Since Λ and L are not known to commute on the image of p, this motivates the following variant of the Lefschetz conjecture $B(M)$.

Conjecture 3.1. Property $B(M)^*$ holds if for all $r \geq 1$ there are correspondences Λ_r and $\widetilde{\Lambda}_r$ in $\mathrm{Corr}^{-r}(M, M)$ such that

- $L^r \circ \Lambda_r \in \mathrm{Corr}^0(M, M)$ is a projector inducing $L^r \circ \lambda^r$ in cohomology.
- $\widetilde{\Lambda_r} \circ L^r \in \mathrm{Corr}^0(M, M)$ is a projector inducing $\lambda^r \circ L^r$ in cohomology.

Lemma 3.2. *If* $\mathsf{h}(M)$ *is finite dimensional, then* $B(M)$ *implies* $B(M)^*$.

Proof: I shall follow the proof of [6, Lemma 5.6.10] in detail. First I shall construct Λ_r. Let $e = L^r \circ \Lambda^r \in \mathrm{Corr}^0(M, M)$. Since this is a cohomological projector, (3) implies that $e^2 - e$ is nilpotent, say $(e^2 - e)^N = 0$. Introduce

$$\begin{aligned} E := (1 - (1 - e)^N)^N &= (P(e) \cdot e)^N, \quad (P \text{ some polynomial}) \\ &= e^N \cdot P(e)^N \\ &= L^r \circ \Lambda^r \cdot e^{N-1} \cdot P(e)^N. \end{aligned}$$

In cohomology this induces the same operator as e. One has

$$E = (1 - (1 - e)^N)^N = 1 + \sum_{j=1}^{N} (-1)^j \binom{N}{j} (1 - e)^{jN}$$

and so, since $E = e^N \cdot P(e)^N = e^N \cdot P(e)^N$, for some polynomial Q one has

$$\begin{aligned} E\circ E &= E\circ(1 + \sum_{j=1}^{N}(-1)^j \binom{N}{j}(1-e)^{jN}) \\ &= E + P(e)^N\circ e^N\circ(1-e)^N\circ Q(e) \\ &= E \quad (\text{since } e^N\circ(1-e)^N = 0). \end{aligned}$$

This is thus a projector inducing the same operator as e in cohomology. Now set $\Lambda_r := \Lambda^r \cdot e^{N-1} \cdot P(e)^N$. By construction $E = L^r\circ\Lambda_r$ induces the same operator as e in cohomology, i.e. the operator $L^r\circ\lambda^r$.

To show the second claim, exchange the order of L^r and Λ^r. □

For $r = 1$, this yields:

Corollary 3.3. *Suppose that Lefschetz' conjecture $B(M)$ holds and that the Chow motive $\mathsf{h}(M)$ is Kimura finite dimensional. There is a correspondence $\Lambda_1 \in \mathrm{Corr}^{-1}(M, M)$ such that $\Pi^{\mathrm{pr}} := \Delta_M - L\circ\Lambda_1$ is a projector inducing the projector π^{pr} (see (7)) in cohomology.*

Corollary 3.4. *For any smooth projective variety M for which $B(M)$ holds and with $\mathsf{h}(M)$ finite dimensional, there is a motive (M, Π^{pr}) with $H^k(M, \Pi^{\mathrm{pr}}) = H^k_{\mathrm{pr}}(M)$, the primitive cohomology.*

4. The variable and fixed motive

4.1. **Construction of the projectors.** Let $i : X \hookrightarrow M$ be a d-dimensional smooth complete intersection of r hypersurfaces. Note that the graph $\Gamma_i \in X \times M$ of i induces the Lefschetz correspondence (we are ignoring multiplicative constants here)

$$L^r = i_*\circ i^* \in \mathrm{Corr}^r(M, M).$$

Set

$$p_r := L^r\circ\Lambda_r \in \mathrm{Corr}^0(M, M),$$

which by construction (cf. Lemma 3.2) is a projector. One has

Lemma 4.1. *Assume $B(M)$ and that $h(M)$ is finite dimensional. Then the correspondences*

$$\pi^{\mathrm{fix}} := i^*\circ\Lambda_r\circ p_r\circ i_* \in \mathrm{Corr}^0(X, X)$$

and

$$\pi^{\mathrm{var}} := \Delta_X - \pi^{\mathrm{fix}}$$

are commuting projectors.

Proof: It suffices to show that π^{fix} is a projector. Then

$$\begin{aligned} (\pi^{\mathrm{fix}})^2 &= i^*\circ\Lambda_r\circ p_r\circ L^r\circ\Lambda_r\circ p_r\circ i_* \\ &= i^*\circ\Lambda_r\circ p_r^3 i_* \\ &= i^*\circ\Lambda_r\circ p_r i_* \\ &= \pi^{\mathrm{fix}}. \quad \square \end{aligned}$$

4.2. **Cohomological action.** The inclusion $i : X \hookrightarrow M$ induces maps $i^* : H^*(M) \to H^*(X)$ of degree 0 and $i_* : H^*(X) \to H^*(M)$ of degree $2r$ with $i_* \circ i^* = L^r$ and $i^* \circ i_* = (L|X)^r$.

Lemma 4.2. *For the action on $H^d(X)$ one has $p_r \circ i_* = i_*$ and π^{fix} induces the projector $i^* \circ \lambda^r \circ i_*$.*

Proof: By definition of the fixed and variable cohomology (1), one has

$$i_* H^d(X) = i_* H^d_{\mathrm{fix}}(X) = i_* \circ i^* H^d(M) = L^r H^d(M).$$

Since by equality (5), L and λ are inverses on the image of L, in cohomology one has $p_r \circ i_* = L^r \circ \lambda^r \circ i_* = i_*$ and $\pi^{\mathrm{fix}} = i^* \circ \lambda^r \circ i_*$. □

Corollary 4.3. *The cohomological projectors π^{fix} and π^{var} induce projection onto the fixed and variable cohomology.*

Proof: Let $x \in H^d(X)$. Then $\pi^{\mathrm{fix}}(x) = i^*(\lambda^r \circ i_* x) \in H^d_{\mathrm{fix}}(X)$. Since $i_*(x - i^* \lambda^r i_* x) = i_* x - L^r \lambda^r i_* x = i_* x - i_* x = 0$, one has $x - \pi^{\mathrm{fix}} x \in H^d_{\mathrm{var}}(X)$. The result follows because of the direct sum decomposition (2). □

4.3. **The motives.** Now define the *fixed* and *variable submotive* of X by means of

$$\mathsf{h}(X)^{\mathrm{fix}} = (X, \pi^{\mathrm{fix}}), \quad \mathsf{h}(X)^{\mathrm{var}} = (X, \pi^{\mathrm{var}}).$$

Then, Lemma 4.1 and Corollary 4.3 can be summarized as follows.

Proposition 4.4. *Let M be a smooth projective manifold for which $B(M)$ holds and suppose that $\mathsf{h}(M)$ is finite dimensional. Let $X \subset M$ be a smooth d-dimensional complete intersection. Then π^{fix} is a projector inducing in cohomology projection onto the fixed part of the cohomology and π^{var} is a projector commuting with π^{fix} and inducing projection on the variable cohomology. There is a direct sum splitting of motives*

$$\mathsf{h}(X) = \mathsf{h}(X)^{\mathrm{fix}} \oplus \mathsf{h}(X)^{\mathrm{var}}.$$

Remark 4.5. Let X be a surface. Then [6, §6.3] there is a self dual Chow-Lefschetz decomposition of the diagonal

$$\Delta = \pi_0 + \pi_1 + \underbrace{\pi_2^{\mathrm{alg}} + \pi_2^{\mathrm{tr}}}_{\pi_2} + \pi_3 + \pi_4.$$

This decomposition is compatible with the splitting into variable and fixed motives. This is because one has a splitting

$$\pi_2^{\mathrm{alg}} = \pi_2^{\mathrm{alg,fix}} + \pi_2^{\mathrm{alg,var}}. \tag{8}$$

Indeed, the construction of the projector π_2^{alg} as given in loc. cit. proceeds by first taking an orthogonal basis for the algebraic classes of X, say $d_1, \ldots, d_\rho$ with $\pi_1(d_j) = 0$ for $j = 1, \ldots, \rho$, and then one sets

$$\pi_2^{\mathrm{alg}} = \sum_{i=1}^{\rho} \frac{1}{d_i^2} d_i \times d_i \in \mathrm{Corr}^0(X, X)$$

Since the splitting in variable and fixed parts is an orthogonal splitting, the splitting (8) follows. One then puts $\pi_2^{\mathrm{tr}} = \pi_2 - \pi_2^{\mathrm{alg}}$ and hence, defining $\pi_2^{\mathrm{tr,var}} := \pi^{\mathrm{var}} - \pi_2^{\mathrm{alg,var}}$ and $\pi_2^{\mathrm{tr,fix}} := \pi_2^{\mathrm{tr}} - \pi_2^{\mathrm{tr,var}}$, one gets a refinement of the above Chow-Lefschetz decomposition

$$\Delta = \pi_0 + \pi_1 + \underbrace{\pi_2^{\mathrm{alg,fix}} + \pi_2^{\mathrm{tr,fix}}}_{\pi_2^{\mathrm{fix}}} + \underbrace{\pi_2^{\mathrm{alg,var}} + \pi_2^{\mathrm{tr,var}}}_{\pi_2^{\mathrm{var}}} + \pi_3 + \pi_4.$$

Proposition 4.4 asserting the splitting into variable and fixed motives has the following consequence which states that the characterization for fixed and variable cohomology has a motivic analog:

Lemma 4.6. *Same assumptions as before.*
1. *For $k \leq d$ we have*

$$\mathsf{CH}_k(\mathsf{h}(X)^{\mathrm{var}}) = \mathrm{Ker}(p_r \circ i_* : \mathsf{CH}_k(X) \to \mathsf{CH}_k(M)).$$

2. *We have a surjective morphism*

$$i^* : \mathsf{CH}_{k+r}(M) \twoheadrightarrow \mathsf{CH}_k(\mathsf{h}(X)^{\mathrm{fix}}).$$

Proof: 1. By definition the left hand side consists of cycles of the form $y = z - i^*\Lambda_r p_r i_* z$ for some $z \in \mathsf{CH}_k(X)$. Clearly, if $p_r \circ i_* u = 0$, u is of this form and conversely, if y is of this form, we have $i_* y = i_* z - i_* i^* \Lambda_r p_r \circ i_* z = i_* z - L^r \circ \Lambda_r p_r \circ i_* z = i_* z - p_r \circ i_* z$ since p_r is a projector and applying p_r this vanishes.
2. This follows since the "fixed" cycles are all in the image of i^*. □

4.4. **Variants with group actions.** There are variants with group actions as follows. Suppose that a finite group G acts on M and that X is invariant under the action of G. In particular, g commutes with i and with L_X and L_M. Let Γ_g be the graph of the action of g on X. For $\chi = \sum_g \chi(g) \cdot g \in \mathbf{Q}[G]$ we set

$$\pi_\chi := \frac{1}{|G|} \sum \chi(g) \Gamma_g.$$

This is a projector and defines the motive (X, π_χ). Moreover, $\pi_\chi \circ \pi^{\mathrm{fix}} = \pi^{\mathrm{fix}} \circ \pi_\chi$ and hence, π^{fix} also commutes with π^{var} and both $\pi^{\mathrm{fix}} \circ \pi_\chi$ and $\pi^{\mathrm{fix}} \circ \pi_\chi$ are projectors. For any $\mathbf{Q}$-vector space on which G acts, setting

$$V^\chi := \{x \in V \mid g(x) = \chi(g)x \text{ for al } g \in G\},$$

one has

$$H^k(X, \pi_\chi) = H^k(X)^\chi.$$

Since $X \subset M$ is left invariant by the G-action, the variable and fixed motives are G-stable and one sets

$$\mathsf{h}(X, \pi_\chi)^{\mathrm{fix}} := (X, \pi^{\mathrm{fix}} \circ \pi_\chi), \quad \mathsf{h}(X, \pi_\chi)^{\mathrm{var}} := (X, \pi^{\mathrm{var}} \circ \pi_\chi).$$

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