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Mechanical performance of wall structures in 3D printing processes: Theory, design tools and experiments

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ABSTRACT

In the current contribution for the first time a mechanistic model is presented that can be used for analysing and optimising the mechanical performance of straight wall structures in 3D printing processes. The two failure mechanisms considered are elastic buckling and plastic collapse. The model incorporates the most relevant process parameters, which are the printing velocity, the curing characteristics of the printing material, the geometrical features of the printed object, the heterogeneous strength and stiffness properties, the presence of imperfections, and the non-uniform dead weight loading. The sensitivity to elastic buckling and plastic collapse is first explored for three basic configurations, namely (i) a free wall, (ii) a simply-supported wall and (iii) a fully-clamped wall, which are printed under linear or exponentially-decaying curing processes. As demonstrated for the specific case of a rectangular wall lay-out, the design graphs and failure mechanism maps constructed for these basic configurations provide a convenient practical tool for analysing arbitrary wall structures under a broad range of possible printing process parameters. Here, the simply-supported wall results in a lower bound for the wall buckling length, corresponding to global buckling of the complete wall structure, while the fully-clamped wall gives an upper bound, reflecting local buckling of an individual wall. The range of critical buckling lengths defined by these bounds may be further narrowed by the critical wall length for plastic collapse. For an arbitrary wall configuration the critical buckling length and corresponding buckling mode can be accurately predicted by deriving an expression for the non-uniform rotational stiffness provided by the support structure of a buckling wall. This has been elaborated for the specific case of a wall structure characterised by a rectangular lay-out. It is further shown that under the presence of imperfections the buckling response at growing deflection correctly asymptotes towards the bifurcation buckling length of an ideally straight wall. The buckling responses computed for a free wall and a wall structure with a rectangular lay-out turn out to be in good agreement with experimental results of 3D printed concrete wall structures. Hence, the model can be applied to systematically explore the influence of individual printing process parameters on the mechanical performance of particular wall structures, which should lead to clear directions for the optimisation on printing time and material usage. The model may be further utilised as a validation tool for finite element models of wall structures printed under specific process conditions.

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1. Introduction

Additive manufacturing, also known as 3D printing, is a revolutionary technology that over the past 10 years has found a wide range of applications in the automotive and aerospace industries (car, airplane and satellite components), biomedical engineering (dental implants, prosthetics, tissue scaffolding, bioprinting of organs), food industry (chocolate, pizza, meat), consumer goods industry (sporting goods, toys, electronics), arms industry (gun prototyping), architectural and civil engineering (structural elements, houses, bridges), pharmacokinetics (drug delivery devices), custom art and design (paintings, sculptures), among others, see [1–8] and references therein. The principle of 3D printing is to convert a digital design into a three-dimensional object by adding material in a layerwise fashion. For achieving this goal, a large number of additive manufacturing processes has been developed, which mainly differ in the printing materials applied, and in the way the layers are laid down on one another to create so-called wall structures. Some techniques liquefy or soften the printing material for constructing the layers (selective laser sintering, electron beam melting), whereas other techniques cure liquid materials using advanced technologies (stereolithography, inkjet printing, laminated object manufacturing, fused deposition modelling) [1,3,5–7,9]. The advantages of additive manufacturing over traditional manufacturing are that the product is easy to customise with an enormous flexibility in shape, quick prototyping is possible, product waste is reduced, manufacturing costs are low, and product storage costs are eliminated [1,4–6,9,10].
Despite the ongoing success of 3D printing technologies, little is known about the influence of the manufacturing parameters and conditions on the objects’ mechanical performance during the printing process. This for a large part is due to the complexity and diversity of the process parameters, such as the curing characteristics of the printing material, the geometrical features of the printed object, the heterogeneous strength and stiffness properties of the printing material, the non-uniform dead weight loading, the presence of imperfections, and the printing velocity. As a consequence, adequate process parameters commonly are determined by means of a trial-and-error procedure, whereby it remains unclear if the optimal parameter set eventually has been found under the conditions and requirements imposed. This makes product development by 3D printing more expensive and time-consuming than necessary, especially when the size of the printed object is relatively large, such as in architectural and civil engineering applications.

In order to improve on this aspect, accurate mechanistic models need to be developed, which not only predict the influence of individual printing process parameters on the mechanical performance of the object, but also reveal how the printing process can be optimised in terms of manufacturing time and amount of printing material used. Considering the initially low strength and stiffness values typical of a soft, viscous printing material, the failure resistance of the object during manufacturing may be more critical than during operation; consequently, minimising the amount of printing material required for maintaining the objects’ strength and stability in a printing process may substantially reduce the production costs.

In this contribution for the first time a mechanistic model is developed for the determination of the mechanical performance of straight wall structures in 3D printing processes. The modelling approach incorporates the two relevant failure mechanisms, which are i) elastic buckling, and ii) plastic collapse. The competition between these two failure mechanisms is analysed for three basic wall types, namely i) a free, unconstrained wall, ii) a simply-supported wall, and iii) a fully-clamped wall, with the adjective used on the wall type (in italics) describing the type of boundary condition applied in the horizontal direction of the wall. For a wall structure with a rectangular lay-out it is demonstrated that the simply-supported wall and fully-clamped wall are representative of global buckling of the whole structure and local buckling of an individual wall, respectively, thereby providing lower and upper bounds for the critical buckling length of the wall structure during 3D printing. The model for elastic wall buckling is derived from the equilibrium equation and boundary conditions for a rectangular heterogeneous plate subjected to non-uniform in-plane forces. The buckling model is reduced to an ordinary fourth-order differential equation, where the contribution of an arbitrary rotational stiffness furnished by the supporting wall structure is accounted for via a constraint factor and the number of half-waves defining the horizontal buckling mode of the wall. The printing velocity and curing characteristics of the printing material enter the model after transforming the mathematical formulation in vertical wall direction from Lagrangian to dimensionless Eulerian coordinates. The two basic curing processes considered are linear curing and exponentially-decaying curing, where the latter type is representative of accelerated curing obtained under the application of an external stimulus, e.g., UV light or heat [11,12], or through a modification of the chemical composition [13]. The effect of geometrical imperfections is added to the model formulation, and the combined analytical-numerical solution procedure is presented. Writing the model equations in dimensionless form enables to uniquely describe the failure behaviour of a wall by a minimum of 5 independent, dimensionless (time and length scale) parameters, with 3 parameters characterising the elastic buckling behaviour of a wall structure, and 2 parameters defining plastic collapse. When geometrical imperfections are accounted for, 2 additional (length scale) parameters need to be considered.

The resistance against elastic buckling and plastic collapse is first analysed for the three basic wall types by means of design graphs and failure mechanism maps. These graphical representations summarise the results of a large number of simulations performed for a wide range of process parameters, thereby providing a useful practical tool for the design and optimisation of 3D printing processes of straight wall structures. The experimental validation of the model is performed by considering two types of geometries constructed with 3D concrete printing, namely a free wall and a rectangular wall-lay-out. It is shown how the design graphs and formulas constructed for the three basic wall types can be used to provide a first useful estimate of the critical failure length of these geometries. The comparison is complemented with accurate model predictions of the buckling length and buckling mode, which turn out to be in good agreement with the experimental results.

The manuscript is organised as follows. In Section 2 the equilibrium equation and boundary conditions are derived for a rectangular heterogeneous plate subjected to non-uniform in-plane forces. These equations form the basis for the buckling model of a wall structure, which, together with the formulation for plastic collapse, is presented in Section 3. Section 4 provides numerical results for the three basic configurations, i.e., the free wall, the simply-supported wall and the fully-clamped wall. These results are used in Section 5 for a first comparison with the outcome of 3D concrete printing experiments on a free wall and a rectangular wall-lay-out. The comparison is subsequently extended with results of a refined analysis. Section 6 presents the main conclusions and some suggestions for future research.

2. Rectangular heterogeneous plate subjected to non-uniform in-plane forces

During 3D printing the mechanical properties of a wall structure are heterogeneous in space due to the curing behaviour of the printing material. In addition, the loading experienced by the wall structure is caused by its dead weight, which is non-uniform in the vertical direction of the wall (= the direction of gravitation). In correspondence with these aspects, the model for wall buckling should be based on the equilibrium equation and boundary conditions for a rectangular heterogeneous plate subjected to non-uniform in-plane forces. To the best of the author’s knowledge, these relations are not available in the literature, and therefore are derived in the current section from the formulation and minimization of the potential energy of the plate.

2.1. Potential energy

Consider a rectangular, heterogeneous plate of length $l$, width $b$ and thickness $h$ subjected to non-uniform in-plane forces (per unit length) $\eta = \eta(x, y)$ acting in the mid-plane of the plate, see Fig. 1. The heterogeneity of the plate is characterised by the dependence of the elastic properties on the in-plane coordinates $x$ and $y$. The origin of the in-plane coordinates is located at the lower right corner of the plate. The components of the in-plane forces are $\eta = \eta_0(x, y) = \{\eta_{xx}, \eta_{xy}, \eta_{yy}\}$, which should satisfy the localised equilibrium equations in $x$- and $y$-directions:

$$\begin{align*}
\eta_{xx} + \eta_{xy} + b_x &= 0 \quad \text{where} \quad b_x = -\rho g h, \\
\eta_{xy} + \eta_{yy} &= 0.
\end{align*}$$

(1)

Here, $b_x$ represents the body force per unit area, which depends on the volumetric mass density $\rho$, the gravitational acceleration $g = 9.81\,\text{m/s}^2$, and the plate thickness $h$. Further, $(\_)_x$ and $(\_)_y$ denote partial derivatives in $x$- and $y$-directions, respectively. The in-plane forces follow from the integration of the Cauchy stresses, $\sigma = \sigma(x, y, z)$, across the plate thickness $h$:

$$\eta = \eta(x, y) = \int_{z=-h/2}^{z=h/2} \sigma dz = h \sigma,$$

(2)

The final result in the right-hand side of Eq. (2) assumes that under in-plane loading conditions the Cauchy stresses are constant across the plate thickness. The in-plane forces are conservative and correspond to the initial, uncurved configuration. Since the in-plane shear stresses are symmetric, $\sigma_{xy} = \sigma_{yx}$, so are the in-plane forces, $\eta_{xy} = \eta_{yx}$. 

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The in-plane forces found after solving Eq. (1) for the specific loading conditions applied serve as input for a buckling analysis. For the description of buckling it is assumed that the plate bends into a curved configuration without any stretching in the mid-plane, i.e., membrane effects are ignored. Correspondingly, the potential energy \( V \) generated by the in-plane forces may be expressed as

\[
V = \frac{1}{2} \int \int \left[ \eta_{xx}(w_{xx})^2 + \eta_{yy}(w_{yy})^2 + 2\eta_{xy}w_{xx}w_{yy} \right] dydx,
\]

where \( w = \hat{w}(x,y) \) is the deflection in the out-of-plane direction \( z \). In addition, the elastic strain energy of a rectangular plate subjected to bending reads

\[
U = \frac{1}{2} \int \int \left[ D_s \left( (w_{xx} + w_{yy})^2 - 2(1 - \nu_s)(w_{xx}w_{yy} - w_{xy}^2) \right) \right] dydx,
\]

whereby the homogeneous bending stiffness is given by

\[
D_s = D_s(x,y) = \frac{E_s h^3}{12(1-\nu_s^2)} \quad \text{with} \quad E_s = \hat{E}_s(x,y), \quad \text{and} \quad \nu_s = \hat{\nu}_s(x,y),
\]

with \( E_s \) and \( \nu_s \) the spatially-variable stiffness modulus and Poisson’s ratio, respectively. Here, the asterisk subindex indicates that the material parameters are heterogeneous in space. Combining Eqs. (3) and (4), the total potential energy \( \Pi \) of the plate follows as

\[
\Pi = U + V = \frac{1}{2} \int \int \left[ D_s \left( (w_{xx} + w_{yy})^2 - 2(1 - \nu_s)(w_{xx}w_{yy} - w_{xy}^2) \right) \right] dydx
\]

\[
+ \frac{1}{2} \int \int \left[ \eta_{xx}(w_{xx})^2 + \eta_{yy}(w_{yy})^2 + 2\eta_{xy}w_{xx}w_{yy} \right] dydx.
\]

2.2. Equilibrium and boundary conditions

The equilibrium conditions of the heterogeneous plate can be derived by minimising the potential energy in Eq. (6). This is done by requiring the potential energy to be stationary:

\[
\delta \Pi = \int \int \left[ \int \int D_s \left( w_{xx}\delta w_{xx} + w_{yy}\delta w_{yy} + w_{xy}\delta w_{xy} + w_{yy}\delta w_{xy} \right)
\]

\[
- \int \int \left[ \eta_{xx}(w_{xx}^2\delta w_{xx} + w_{xx}w_{yy}\delta w_{xy} + w_{xy}^2\delta w_{xy}) \right] dydx
\]

\[
\int \int \left[ \eta_{xy}w_{xx}\delta w_{yy} + \eta_{yy}w_{yy}\delta w_{xy} - (1 - \nu_s)w_{xx}\delta w_{xy} + w_{xy}\delta w_{xy} - 2w_{yy}\delta w_{xy} \right] dydx = 0.
\]

Applying integration by parts, after an extensive mathematical procedure, the Euler-Lagrange equation describing the localised equilibrium under plate buckling becomes

\[
V^2(D_s V^2) - ((1 - \nu_s)D_s w_{xx})_{yy} - ((1 - \nu_s)D_s w_{yy})_{xx}
\]

\[
+ 2(1 - \nu_s)D_s w_{xy} - \eta_{xx}w_{xx} - \eta_{yy}w_{yy} - (\eta_{xy}w_{xx})_{y} - (\eta_{xy}w_{yy})_{y} = 0,
\]

with the Laplacian given by \( V^2V = (\zeta_{xx} + \zeta_{yy}) \). In addition, the homogeneous (natural and essential) boundary conditions along the plate edges \( x = 0 \) and \( x = l \) are

\[
D_s(w_{xx} + w_{yy}) = 0 \quad \text{or} \quad w_x = 0, \quad \text{and} \quad (D_s w_{xx})_{y} + (v_s D_s w_{xx})_{y} + 2(1 - \nu_s)D_s w_{xy} - \eta_{xx}w_{xx} - \eta_{yy}w_{yy} = 0
\]

\[
\text{or} \quad w_y = 0.
\]

and along the plate edges \( y = 0 \) and \( y = b \) are

\[
D_s(w_{yy} + w_{xx}) = 0 \quad \text{or} \quad w_y = 0, \quad \text{and} \quad (D_s w_{yy})_{x} + (v_s D_s w_{yy})_{x} + 2(1 - \nu_s)D_s w_{xy} - \eta_{xx}w_{xx} - \eta_{yy}w_{yy} = 0
\]

\[
\text{or} \quad w_x = 0.
\]

Further, the four plate corners, \( (x, y) \in \{(0, 0), (l, 0), (0, b), (l, b)\} \), should satisfy the condition

\[
2(1 - \nu_s)D_s w_{xy} = 0 \quad \text{or} \quad w = 0.
\]

Note that the natural boundary conditions \((9)_1 \) and \((10)_1 \) relate to the bending moment at the plate edge, while \((9)_2 \) and \((10)_2 \) prescribe the shear force.

For the specific case of a homogeneous rectangular plate with uniform elastic properties \( D_s = D = \) constant and in-plane forces \( \eta_{xx}, \eta_{yy}, \eta_{xy} \), the equilibrium equation, Eq. (8), reduces to

\[
D \left( w_{xx} + w_{yy} \right) = 0 \quad \text{or} \quad w_x = 0, \quad \text{and} \quad D_{w_{xxx}} + 2(1 - \nu_s)D_{w_{xyy}} - \eta_{xx}w_{xx} - \eta_{yy}w_{yy} = 0 \quad \text{or} \quad w = 0,
\]

at \( x = 0 \) and \( x = l \), and to

\[
D_{w_{yy}} + 2(1 - \nu_s)D_{w_{xxx}} - \eta_{xx}w_{xx} - \eta_{yy}w_{yy} = 0 \quad \text{or} \quad w = 0,
\]

at \( y = 0 \) and \( y = b \). Eqs. (12) to (14) indeed are in agreement with the common expressions, see e.g., [14]. It can be observed that for the homogeneous plate the structure of the natural boundary condition related to the shear force, Eqs. (13) and (14)\(_2 \), is different than for the heterogeneous plate, Eqs. (9)\(_3 \) and (10)\(_3 \). For the other three types of boundary conditions the expressions for the two cases are similar.

3. Wall failure during 3D printing: elastic buckling and plastic collapse

The two failure mechanisms that may occur during 3D printing are elastic buckling and plastic collapse. In this section the governing equations for these two mechanisms are established, whereby the formulation for elastic buckling is based on the equilibrium equations and boundary conditions derived for a heterogeneous plate subjected to non-uniform in-plane forces, see Section 2. First, three basic geometries are
considered, which are i) a free wall, ii) a simply-supported wall and iii) a fully-clamped wall. The equilibrium formulation for bifurcation buckling of these wall types is reduced to an ordinary fourth-order differential equation, which in turn is enriched to include the general case of a wall of which the rotation at the lateral supports is constrained by a rotational spring stiffness. The rotational spring stiffness may be non-uniform along the vertical direction, and is characterised by the properties of the supporting wall structure. This aspect is analysed in detail by deriving the rotational spring stiffness for a rectangular wall lay-out. The buckling model is completed by accounting for time effects related to the curing process of the printing material and the printing velocity. The influence of wall imperfections is added to the formulation, and the combined analytical-numerical solution procedure of the buckling model is treated. The section ends with the formulation of criteria for the competition between elastic buckling and plastic collapse.

3.1 Wall failure by elastic buckling

The equilibrium equation and boundary conditions given by Eqs. (8)–(10) refer to a plate with heterogeneous stiffness properties in both the x- and y-directions. However, it is realistic to assume that during the 3D printing of an individual layer the stiffness properties in that layer do not significantly change. In other words, the characteristic time associated to the curing process of the printing material is assumed to be larger than the period $T_l$ related to the printing of an individual layer. This condition is essential from a structural point of view, as it warrants a good bonding between the actual layer and the layer below, printed in the previous cycle. Hence, the stiffness variation in the wall prominently occurs along the direction of increasing wall length, i.e., the x-direction, which allows for writing the heterogeneous stiffness modulus as $E_x = E_0(x)$. Further, for reasons of simplicity the Poisson’s ratio is taken as constant, $\nu_x = \nu$. Under these conditions the heterogeneous bending stiffness, Eq. (5), turns into

$$D_x = \dot{D}_x(x) = \frac{E_0(x)h^3}{12(1 - \nu^2)}. \tag{15}$$

The in-plane forces generated by gravitational effects in x-direction are described by the following non-uniform biaxial loading condition

$$\eta_{xx} = \ddot{\eta}_{xx}(x) = pg h(x - l), \quad \eta_{yy} = \ddot{\eta}_{yy}(x) = K_{yx} \eta_{xx}(x) pg h(x - l), \quad \eta_{xy} = 0, \tag{16}$$

whereby $K_{yx}$ represents the coefficient of lateral stress, which, in principle, may depend on the x-coordinate, $K_{yx} = K_{yx}(x)$, as indicated by the asterisk subindex. It can be easily confirmed that the above expressions for the in-plane loading forces satisfy the equilibrium conditions given by Eq. (1). The coefficient of lateral stress may be estimated from the constitutive formulation for an elastic continuum, in correspondence with

$$\epsilon_{yy} = \frac{1}{E_x} (\sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz})). \tag{17}$$

Using that the out-of-plane normal stress in the wall is zero, $\sigma_{zz} = 0$, with Eq. (2) the above constitutive expression turns into

$$\sigma_{yy} = \nu \eta_{xx} + E_x \nu_{yy}. \tag{18}$$

Assuming the supports at $y = 0$ and $y = b$ as fully constrained in the lateral direction results in $\sigma_{yy} = 0$, which brings Eq. (18) to $\eta_{yy} = \nu \eta_{xx}$, so that, from Eq. (16), the coefficient of lateral stress becomes $K_{yx} = \nu$. On the contrary, when these supports are unconstrained in the lateral direction, the lateral stress $\eta_{yy} = 0$, by which Eq. (18) leads to $\epsilon_{yy} = -\nu \eta_{xx}/(E_x)h$, and Eq. (16) provides the coefficient of lateral stress as $K_{yx} = 0$. Hence, the range for the coefficient of lateral stress is

$$0 \leq K_{yx} \leq \nu \quad \text{for} \quad -\frac{\nu g h (x - l)}{E_x} \geq \epsilon_{yy} \geq 0, \tag{19}$$

which indicates that $K_{yx}$ increases from zero to its maximum value $\nu$ when the deformation $\epsilon_{yy}$ in the lateral direction decreases from its maximum value $-\nu g h (x - l)/E_x$ (which is non-negative since $0 \leq x \leq l$) to zero.

3.1.1 Wall buckling without time effects

In the y-direction, initially three types of two-sided boundary conditions are considered, in correspondence with i) a free, unconstrained wall, ii) a simply-supported wall, and iii) a fully-clamped wall, see Fig. 2. The in-plane force is constant along the y-direction of the wall, see Eq. (16), whereby the critical buckling mode under the in-plane force may be described by a function $f(y)$ that is symmetrical with respect to the centreline $y = b/2$, and is characterised by the boundary conditions in y-direction. Along this way, the partial differential equation, Eq. (8), can be solved by subjecting the displacement response to a separation of variables:

$$w = \tilde{w}(x, y) = \tilde{w}(x)f(y), \tag{20}$$

where $\tilde{w}(x)$ is the out-of-plane displacement in the x-direction, measured along the symmetry line $y = b/2$, and $f(y)$ denotes the normalised displacement in y-direction. For the three types of boundary conditions introduced above the normalised displacement can be expressed as

Free wall:

$$f(y) = 1, \tag{21}$$

Simply-supported wall:

$$f(y) = \sin(\pi y/b), \tag{21}$$

Fully-clamped wall:

$$f(y) = (1 - \cos(2\pi y/b))/2. \tag{21}$$

Inserting Eq. (21) into Eq. (20) shows that the deflection $w$ together with the in-plane loading conditions, Eq. (16), indeed satisfy the boundary conditions, Eq. (10), at $y = 0$ and $y = b$, which for the free wall specify into

$$w_y = 0 \quad \text{and} \quad \left(D_x w_{xy}\right)_y + \nu \left(D_x w_{xx}\right)_y + 2(1 - \nu) \left(D_x w_{xy}\right)_x = 0, \tag{22}$$

where $w_{xy}$ and $w_{xx}$ are the first and second partial derivatives of $w$ with respect to $x$ and $y$. The solutions $w(x, y)$ are obtained by inserting Eq. (21) into Eq. (20) and solving the resulting fourth-order differential equation for each boundary condition type.
for the simply-supported wall read
\[ w = 0 \quad \text{and} \quad D_s (w_{,yy} + v w_{,xx}) = 0, \] (23)
and for the fully-clamped wall are
\[ w = 0 \quad \text{and} \quad w_y = 0. \] (24)

Note that in Eqs. (22)–(24) the first boundary condition also ensures that Eq. (11) is satisfied at the four corners of the wall. Inserting the combination of Eqs. (20) and (21), together with the loading condition, Eq. (16), into the differential equation, Eq. (8), the equilibrium condition in terms of the deflection \( \bar{w}^e(x) \) measured along the wall centreline \( y = h/2 \) becomes
\[ \left( D_s w_{,xx}^e \right)_{,xx} - k_1 w_{,x}^e + k_2 w^e = 0, \] (25)
with the functions \( k_1 \) and \( k_2 \) as
\[ k_1 = \bar{k}_1(x) = \rho g h(x - l) + 2c_1 \left( \frac{n_p \pi}{b} \right)^2 D_s, \]
\[ k_2 = \bar{k}_2(x) = c_1 \left( \frac{n_p \pi}{b} \right)^2 (\rho g h(x - l) - \nu D_s)_{,xx}. \] (26)

where \( n_p \) represents the number of half-waves characterising the (critical) buckling mode in \( y \)-direction, and \( c_1 \) is a constraint factor that is determined by the boundary conditions applied in \( y \)-direction. For the three basic wall types the above procedure results in the following values for \( n_p \) and \( c_1 \):

Free wall: \( n_p = 0, \quad c_1 = 0 \).

Simply-supported wall: \( n_p = 1, \quad c_1 = 1 \).

Fully-clamped wall: \( n_p = 2, \quad c_1 = 0.5 \).

During the printing process the wall may be considered as fully clamped at the bottom \( x = 0 \) and unconstrained at the top \( x = l \). Accordingly, inserting Eqs. (20), (21) and the loading condition, Eq. (16), into the boundary conditions, Eq. (9), after evaluating the result at \( y = h/2 \), gives at \( x = 0 \)
\[ w^e = 0, \]
\[ w_y^e = 0, \] (28)
while at \( x = l \) it follows that
\[ D_s \left( w_{,xx}^e - c_1 \left( \frac{n_p \pi}{b} \right)^2 \frac{w^e}{b} \right) = 0, \] (29)
\[ \left( D_s w_{,xx}^e \right)_{,x} - c_1 \left( \frac{n_p \pi}{b} \right)^2 \left( v (D_s w^e)_{,x} + 2(1 - v)D_s w_{,xx}^e \right) = 0. \]

3.1.2. General rotational stiffness at boundary walls in \( y \)-direction

As discussed above, the influence by the boundary conditions in \( y \)-direction on the buckling behaviour is accounted for in Eqs. (25) and (26) by the number of half-waves \( n_p \) and the constraint factor \( c_1 \), which are specified in Eq. (27) for the three basic wall types. These specifications, however, can be generalised towards the case whereby the rotation at the supports \( y = 0 \) and \( y = h \) is constrained by a rotational stiffness \( \bar{k}_r \). Under a zero deflection at the supports, the boundary conditions at \( y = 0 \) then turn into
\[ w = 0 \quad \text{and} \quad D_s (w_{,yy} + v w_{,xx}) = 0, \] (30)
while at \( y = h \) they become
\[ w = 0 \quad \text{and} \quad D_s (w_{,yy} + v w_{,xx}) + k_r w_y = 0. \] (31)

Note that in the limit cases of \( k_r = 0 \) and \( k_r \rightarrow \infty \) the Robin-type boundary conditions, Eqs. (30)\(_2\) and (31)\(_2\), indeed reduce to the boundary conditions for the simply-supported wall and the fully-clamped wall, represented by Eqs. (23)\(_2\) and (24)\(_2\), respectively.

It is convenient to introduce a dimensionless form of the rotational stiffness \( k_r \) as
\[ \bar{k}_r = \frac{k_r b}{D_s}. \] (32)

Depending on the type of support structure present at \( y = 0 \) and \( y = h \), the rotational stiffness \( k_r \) may vary along the \( x \)-axis, \( k_r = \bar{k}_r(x) \). This is also the case for the heterogeneous bending stiffness, \( D_s = D_s(x) \), and, through Eq. (32), for the dimensionless rotational stiffness, \( \bar{k}_r = \bar{k}_r(x) \). Since the dimensionless rotational stiffness determines the values of the number of half-waves and the constraint factor, both these parameters in principle may also depend on the \( x \)-coordinate, i.e., \( n_p = \tilde{n}_p(\bar{k}_r(x)) \) and \( c_1 = c_1(\bar{k}_r(x)) \), and must thus be used as such in the equilibrium equation, Eq. (25), and the natural boundary conditions, Eq. (26). The derivation of these functions, however, is postponed until Section 3.1.5 when addressing the influence on the buckling of wall \( b \) by the supporting wall structure.

As an initial assumption, the dependency of the stiffness parameters on the \( x \)-coordinate is omitted by examining a wall segment of width \( b \) in the \( y \)-direction and unit length in the \( x \)-direction, i.e., an “elemental strip” with constant rotational stiffness \( k_r \) and bending stiffness \( D_s = D \) (and thus, via Eq. (32), a constant dimensionless rotational stiffness \( \bar{k}_r \)). Since there are no conditions imposed on the out-of-plane displacement \( u^e \), this parameter in principle may depend on the \( x \)-coordinate, i.e., \( u^e = u^e(x) \); however, it will be demonstrated below that this will not have an effect on the computational result. The general solution for the buckling mode in \( y \)-direction, caused by an in-plane force that is constant in \( y \)-direction, see Eq. (16)\(_2\), reads [14]
\[ f = \tilde{f}(y) = A_1 + A_2 y + A_3 \sin \left( \frac{n_p \pi y}{b} \right) + A_4 \sin \left( \frac{n_p \pi y}{b} \right). \] (33)

Here, the amplitudes \( A_1 \) to \( A_4 \) are proportionally scaled such that \( \tilde{f}(y) \) has a maximum value of 1 at \( y = h/2 \). Combining Eqs. (33) and (20) and substituting the result into the boundary conditions, Eqs. (30) and (31), leads to a system of four homogeneous, algebraic equations, which in matrix-vector form reads
\[ [\mathbf{D}] [\mathbf{A}] = [\mathbf{0}], \] (34)
with \([\mathbf{A}] = [A_1, A_2, A_3, A_4]^T\) and the components of the \( 4 \times 4 \) matrix \([\mathbf{D}]\) given by
\[
\begin{align*}
D(1,1) &= \nu w^e_{,x}, \\
D(1,2) &= -\frac{\bar{k}_r}{b^2} w^e, \\
D(1,3) &= -\frac{w^e n_p^2 \pi^2}{b^2} + v w_{,xx}^e, \\
D(1,4) &= -\frac{\bar{k}_r}{b^2} n_p \pi. \\
D(2,1) &= w^e, \\
D(2,2) &= 0, \\
D(2,3) &= w^e, \\
D(2,4) &= 0, \\
D(3,1) &= v w^e_{,x}, \\
D(3,2) &= v w^e_{,xx} + \frac{\bar{k}_r}{b^2} w^e, \\
D(3,3) &= -\frac{w^e n_p^2 \pi^2 \cos(n_p \pi)}{b^2} + v w^e_{,x} \cos(n_p \pi) - \frac{\bar{k}_r}{b^2} n_p \pi \sin(n_p \pi), \\
D(3,4) &= -\frac{w^e n_p^2 \pi^2 \sin(n_p \pi)}{b^2} + v w^e_{,xx} \sin(n_p \pi) + \frac{\bar{k}_r}{b^2} n_p \pi \cos(n_p \pi), \\
D(4,1) &= w^e, \\
D(4,2) &= w^e, \\
D(4,3) &= w^e \cos(n_p \pi), \\
D(4,4) &= w^e \sin(n_p \pi).
\end{align*}
\] (35)
The non-trivial solution of Eq. (34) is obtained by det[D] = 0, which results in the characteristic equation

\[
2\kappa_y^2 - 2\kappa_y^2 \cos(n_y \pi) + 2n_y \pi \kappa_y \sin(n_y \pi) + (n_y \pi)^2 \sin(n_y \pi) - 2(n_y \pi)^2 \kappa_y \cos(n_y \pi) - n_y \pi \kappa_y \sin(n_y \pi) = 0.
\]  
(36)

Note that the above equation only depends on the number of half-waves \( n_y \), and the dimensionless rotational stiffness \( \kappa_y \), and indeed is independent of the out-of-plane displacement \( u' = \ddot{u}'(x) \). For a given value of \( \kappa_y \), the minimal number of half-waves \( n_y \) characterising the critical buckling mode can be determined from Eq. (36) by using an iterative (Newton–Raphson) solution procedure. The result is plotted in Fig. 3 for a wide range of stiffness values \( 0.01 \leq \kappa_y \leq 1000 \). It can be observed that the number of half-waves defining the buckling shape in \( y \)-direction increases with increasing rotational stiffness, and that in the limits \( \kappa_y \rightarrow 0 \) and \( \kappa_y \rightarrow \infty \) the values \( n_y = 1 \) and \( n_y = \frac{1}{2} \) are retrieved, respectively, corresponding to a simply-supported wall and a fully-clamped wall, see Eq. (27). The exact solution depicted in Fig. 3 can be closely approximated by the expression

\[
n_y = \hat{n}_y(\kappa_y) = 1.9841 \left[ 1 - \exp\left(-0.360\kappa_y + 0.430\kappa_y^{0.452}\right) \right].
\]  
(37)

Eq. (37) has been calibrated from a least-squares procedure with \( R^2 = 0.9995 \), indicating a very high accuracy. With the solution \( n_y = \hat{n}_y(\kappa_y) \) at hand, the coefficients \( A_4 \) to \( A_4 \) defining the critical buckling mode in the \( y \)-direction, see Eq. (33), can be calculated from Eq. (34) for arbitrary values of the rotational stiffness \( \kappa_y \). Substituting the buckling mode into Eq. (20), followed by inserting the result, together with the biaxial loading condition, Eq. (16), into the differential equation, Eq. (9), allows to distill from the structure of the resulting expressions, Eqs. (25) and (26), the value of the constraint factor \( c_y \) as a function of the value of \( \kappa_y \). This relation is depicted in Fig. 4, showing that \( c_y \) decreases with increasing \( \kappa_y \). The limit values at \( \kappa_y \rightarrow 0 \) and \( \kappa_y \rightarrow \infty \) are \( c_y = 1 \) and \( c_y = 0.5 \), respectively, which correspond to the simply-supported and fully-clamped walls, see Eq. (27). The approximation plotted in Fig. 4 is given by the closed-form expression

\[
c_y = \xi_y(\kappa_y) = 0.5 + 0.309 \exp(-0.854\kappa_y) + 0.192 \exp(-0.183\kappa_y),
\]  
(38)

which has been found from a least-squares approach with \( R^2 = 0.9998 \). It is emphasised that Eqs. (37) and (38) for \( n_y \) and \( c_y \) are also applicable when the dimensionless rotational stiffness is a function of the \( x \)-coordinate, \( \kappa_x = \kappa_x(x) \). In Section 3.1.5 these relations will be applied as such for the specific case of a rectangular wall lay-out.

3.1.3. Time evolution of elastic stiffness during curing

Now that the equilibrium equation, Eq. (25), and the boundary conditions, Eqs. (28) and (29), are expressed in terms of the \( x \)-coordinate only, the next step is to account for the increase in wall length \( l \) during the printing process. The growth of a wall of length \( l \) during printing is simplified by modelling it as a continuous process that takes place at a constant wall growth velocity \( \dot{l} \) in the \( x \)-direction. The velocity \( \dot{l} \) is dependent of printing process parameters via

\[
\dot{l} = \frac{Q}{\nu_h \delta T_f},
\]  
(39)

with \( Q \) the material volume discharged from the printing nozzle per unit time, \( \nu_h \) the (horizontal) velocity of the printing nozzle, \( T_f \) the period required for printing an individual material layer, and \( h \) the thickness of the wall. The printing process can be conveniently formulated by adopting an Eulerian coordinate system that is attached to the end of the printing nozzle. In accordance with Fig. 5, the Eulerian coordinate \( X \) is
related to the Lagrangian coordinate $x$ as

$$X = \hat{X}(x, t) = x - l = x - lt,$$

which uses

$$l = \bar{l}(t) = \bar{h},$$

whereby $t$ represents time. Consider now a material point at $x = 0$, for which the time evolution of the stiffness modulus as a result of the curing process may be formally expressed as

$$\hat{E}_0(x, 0) = \hat{E}_0(0, 0).$$

(42)

with $\hat{g}_x = \hat{g}_x(t)$ the characteristic curing function and $E_0$ the initial stiffness of the printing material, measured at the moment the material is discharged from the printing nozzle. In the present communication two representative types of curing functions will be considered, namely a linear curing function

$$\hat{g}_x(t) = \hat{g}_x(0) + \hat{g}_x(t),$$

and an exponentially-decaying curing function

$$\hat{g}_x(t) = \hat{g}_x(0) + \hat{g}_x(t) + \exp(-\hat{g}_x(t)),$$

where

$$\hat{g}_x(t) = \hat{g}_x(0) + \hat{g}_x(t) + \exp(-\hat{g}_x(t))$$

(43)

and

$$\hat{g}_x(t) = \hat{g}_x(0) + \hat{g}_x(t) + \exp(-\hat{g}_x(t))$$

(44)

in which $\hat{g}_x(t)$ and $\hat{g}_x(t)$ are the curing rates for the elastic modulus (with dimension of time$^{-1}$) in the linear and exponential evolutions, respectively. Further, $\hat{g}_x(t)$ is the ratio between the final stiffness $E_0$, obtained when $t \to \infty$, and the initial stiffness $E_0$ in the exponential curing process. The characteristics of the linear and exponentially-decaying curing functions are illustrated in Fig. 6. Exponentially-decaying curing is representative of curing processes that are accelerated by the application of an external stimulus, such as UV light or heat [11,12], or through a modification of the chemical composition [13]. Combining Eq. (40) with Eqs. (42) to (44) allows to reformulate the stiffness change in terms of a dimensionless Eulerian coordinate $\tilde{X}$:

$$\hat{E}_0(\tilde{X}) = \hat{g}_x(\tilde{X}) E_0,$$

with the linear curing function

$$\hat{g}_x(\tilde{X}) = \hat{g}_x(\tilde{X}) = \frac{\hat{g}_x}{\hat{g}_x},$$

(45)

and the exponentially-decaying curing function

$$\hat{g}_x(\tilde{X}) = \hat{g}_x(\tilde{X}) = \frac{\hat{g}_x}{\hat{g}_x}$$

(46)

and

$$\hat{g}_x(\tilde{X}) = \hat{g}_x(\tilde{X}) = \frac{\hat{g}_x}{\hat{g}_x}$$

(47)

The superimposed bar used on parameters in the above equations emphasises that these are dimensionless.

3.1.4. Wall buckling including time effects

For incorporating the time effects related to the curing rate $\xi_c$ and printing velocity $v$ into the buckling equation, from Eqs. (40), (46) and (47) the dimensionless Eulerian coordinate $\tilde{X}$ is mapped to the Lagrangian coordinate $x$ as

$$\tilde{X} = \tilde{X}(x, t) = \frac{\xi_c}{\xi_c} X = \frac{\xi_c}{\xi_c} x - \xi_c t$$

with $\xi_c \in \{\xi_c, \xi_c\}$. (48)

Applying the above coordinate transformation to Eq. (25) and invoking Eq. (41) turns the equilibrium condition into a dimensionless, Eulerian form:

$$\left(\bar{g}_x \bar{w}_X \bar{X} \right) - \left(\bar{g}_x \bar{w}_X \bar{X} \right) + \bar{g}_x \bar{w}_X = 0,$$

(49)

with the dimensionless deflection specified as $\tilde{w} = \xi_c \tilde{w}_t / h$, and the dimensionless forms of the functions $\bar{k}_1$ and $\bar{k}_2$ in Eq. (26) given by

$$\bar{k}_1 = \bar{k}_1(\tilde{X}) = \tilde{X} + \gamma_1 \left(\frac{n_x}{v_c} \right)^2 \tilde{w}_x,$$

$$\bar{k}_2 = \bar{k}_2(\tilde{X}) = \gamma_2 \left(\frac{n_x}{v_c} \right)^2 \tilde{w}_x + \gamma_2 \left(\frac{n_x}{v_c} \right)^2 \left(K_{xy} \tilde{X} - \gamma_3 \tilde{w}_x \right),$$

(50)

whereby the parameters $\lambda$ and $\gamma$ read

$$\lambda = \frac{\bar{g}_x \bar{h}}{\bar{D}_0}, \quad \gamma = \frac{\xi_c \lambda h}{\bar{D}_0}.$$

(51)

The initial bending stiffness $D_0$ used in the above expression straightforwardly follows from combining Eqs. (42) and (15), i.e.,

$$D_0 = \frac{E_0 h^3}{12(1 - \nu^2)}.$$ (52)

In addition to the equilibrium equation, the boundary conditions, Eqs. (28) and (29), need to be formulated in terms of the dimensionless Eulerian coordinate $\tilde{X}$. The locations of the boundaries are $\tilde{X} = -\kappa$ and $\tilde{X} = 0$, with

$$\kappa = \frac{\xi_c}{\xi_c} l,$$

(53)

Combining Eqs. (28) and (29) with Eq. (48), and accounting for Eqs. (45) and (52), furnishes the boundary conditions at $\tilde{X} = -\kappa$ as

$$\bar{w}_X = 0,$$

(54)

and at $\tilde{X} = 0$ as

$$\left(\bar{g}_x \bar{w}_X \bar{X} \right) - \left(\bar{g}_x \bar{w}_X \bar{X} \right) = 0,$$

(55)

Notice that in Eqs. (50) and (55) the possible non-uniformity of $n_x$ and $c_x$ in the $\bar{X}$-direction has been indicated by adding an asterisk subindex to these parameters, i.e.,

$$n_x = \tilde{n}_x(\tilde{X}) = \tilde{n}_x(\tilde{X}),$$

(56)

and

$$c_x = \tilde{c}_x(\tilde{X}) = \tilde{c}_x(\tilde{X}).$$

(57)

In order to establish the functions $\tilde{n}_x(\tilde{X})$ and $\tilde{c}_x(\tilde{X})$, the rotational stiffness $\tilde{k}_x(\tilde{X})$ needs to be derived and subsequently substituted into Eqs. (37) and (38), respectively. In the next section it is demonstrated for a rectangular wall lay-out how to derive the function $\tilde{k}_x(\tilde{X})$.

3.1.5. Failure of a rectangular wall lay-out — local versus global buckling

Consider a rectangular wall lay-out composed of two primary walls of width $b$, thickness $h$ and bending stiffness $D_b$, and two supporting walls of width $d$, thickness $h'$ and bending stiffness $D_h$, with $b \geq d$, see Fig. 7. The superindex $s$ refers to “supporting”, but for notational convenience this superindex is omitted on the wall width $d$. The expression
for \( \hat{f}(X) \) depends on the mechanical behaviour of the supporting wall \( d \), whereby a distinction can be made between local buckling of wall \( b \) and global buckling of the whole structure. For the identification of these two buckling phenomena, it is instructive to first analyse the case whereby the walls have equal initial bending stiffness, \( D_s'/D_b = 1 \). The analysis starts by taking a rectangular wall segment of unit length in \( \overline{X} \)-direction, located at infinite vertical distance from the clamped support \( \overline{X} = -\kappa \), i.e., \( \kappa \rightarrow \infty \). The response of this wall segment thus characterises the limit to which the response of the rectangular wall lay-out asymptotes under increasing wall length \( L \). Under the assumption that the lateral in-plane forces in wall parts \( b \) and \( d \) are equal, i.e., \( \eta_{y} = \eta_{0} = K_{s}/K_{l} \), the critical buckling mode of the rectangular wall segment may be described by an harmonic function for which the number of half-waves along the distance \( b + d \) is equal to 2, in correspondence with the buckling response depicted in Fig. 7 (dashed line). Combining this proportionality with the number of half-waves along distance \( b \) being equal to \( n_{y, \infty} \), leads to the surprisingly simple expression

\[
n_{y, \infty} = \frac{2b/d}{1 + b/d} \quad \text{with} \quad b/d \geq 1,
\]

where \( n_{y, \infty} \) thus should be interpreted as the limit value of the number of half-waves at infinite vertical distance from the clamped support \( \overline{X} = -\kappa \), with \( \kappa \rightarrow \infty \). In the limit case of a square lay-out with \( b/d = 1 \), Eq. (57) results in \( n_{y, \infty} = 1 \), which is in agreement with an asymptotic rotational stiffness \( \overline{K}_{r, \infty} = 0 \), see Fig. 3, and thus with a wall \( b \) that is simply supported. This is a logical result, since for a square lay-out the wall segments \( b \) and \( d \) buckle simultaneously, so that the supporting wall part \( d \) is unable to provide rotational resistance to wall part \( b \). In other words, the rectangular wall segment fails by global buckling. Conversely, when \( b/d \rightarrow \infty \), Eq. (57) leads to \( n_{y, \infty} \rightarrow 2 \), which indeed corresponds to a fully-clamped wall with \( \overline{K}_{r, \infty} \rightarrow \infty \), see Fig. 3, characterising failure by local buckling. In summary, for aspect ratios \( 1 \leq b/d < \infty \) the asymptotic rotational stiffness corresponds to the range \( 0 \leq \overline{K}_{r, \infty} < \infty \), with the limit values \( \overline{K}_{r, \infty} = 0 \) and \( \overline{K}_{r, \infty} \rightarrow \infty \) being representative of global buckling and local buckling, respectively. With the number of half-waves \( n_{y, \infty} \) computed from Eq. (57), the asymptotic rotational stiffness can be obtained via Fig. 3, or via the inverse of Eq. (37), i.e.,

\[
\overline{K}_{r, \infty} = \overline{K}_{r, \infty}(n_{y, \infty}) = \ln\left(1 - \frac{n_{y, \infty}}{1.984}\right) \div 0.360 - 0.430.
\]

The above analysis can be extended for \( D_b'/D_b > 1 \), which is representative of a rectangular wall lay-out of which the thickness \( h' \) of the supporting wall \( d \) is larger than the thickness \( h \) of the primary wall \( b \). In contrast to the relatively simple case \( D_b'/D_b = 1 \), this analysis requires the formulation of the complete boundary value problem, since the expression for the critical buckling mode now is more complicated and also includes the non-harmonic terms present in Eq. (33). At the connection between the two walls, equilibrium requires that the shear force in the supporting wall \( d \) equals the normal force in wall \( b \), and vice versa. Moreover, since the normal forces in walls \( b \) and \( d \) are assumed to be equal, the shear forces at the wall connection should also be equal. Additionally, from continuity requirements the bending moments and rotations about the \( \overline{X} \)-axis of the wall connection must be equal. These conditions allow for a convenient modification of the rectangular geometry, by rotating wall part \( d \) at the connection with wall part \( b \) about the \( \overline{X} \)-axis over an angle of 90° in the counterclockwise direction, thereby simplifying the analysis to that of a straight wall segment of length \( b/2 + d/2 \), see Fig. 8. The boundary and continuity conditions of this wall segment are based on Eq. (10); here, the derivatives of the out-of-plane displacement in \( x \)-direction are omitted, since these do not influence the value of the dimensionless rotational stiffness \( \overline{K}_{r, \infty} \), see Eq. (36). Choosing a reference system whereby the origin of the \( y \)-coordinate lies at the centre \( y = b/2 \) of the deformed wall segment \( b \), see Fig. 8, the boundary conditions at \( \tilde{y} = 0 \) become

\[
f = 0 \quad \text{and} \quad f_{\tilde{y}} = 0,
\]

with the normalised out-of-plane displacement \( f = \tilde{f}(\tilde{y}) \) of wall part \( b/2 \) given by Eq. (33), in which the coordinate \( y \) is replaced by \( \tilde{y} \). The boundary conditions at \( \tilde{y} = b/2 + d/2 \) are

\[
f_{\tilde{y}} = 0 \quad \text{and} \quad D_0 f_{\tilde{y} \tilde{y}} = \eta_{y} f_{\tilde{y}} = 0.
\]
with $f$ the normalised out-of-plane displacement of wall part $d/2$. Additionally, the four continuity conditions at $y = b/2$ are

$$f = f^*, \quad f_y = f_y^*, \quad D_0 f_y = D_0 f_y^*, \quad D_0 f_{yy} = -\eta_{yy} f_y^* - \eta_y f^*_y.$$  
(61)

In accordance with the general expression for the buckling load in $y$-direction

$$\eta_{yy} = \eta_y = -\frac{\pi^2}{b^2} D_0,$$

the solution for the normalised out-of-plane displacement of wall part $d/2$ may be written as

$$f^* = f^f(y) = A_4 + A_6 \cos \left( \sqrt{\frac{D_0}{D_0 - b^2}} \eta_y \pi y \right) + A_8 \sin \left( \sqrt{\frac{D_0}{D_0 - b^2}} \eta_y \pi y \right).$$  
(63)

Inserting the solutions, Eqs. (33) (with $n_y = n_{yy}$) and (63), and the buckling load, Eq. (62), into Eqs. (59) to (61) leads to a system of eight equations of the form Eq. (34), for which the homogeneous solution follows from equating the determinant of the corresponding matrix [D] to zero:

$$\beta \cos \left( \frac{\eta_y \pi}{2} \right) \left[ \sin \left( \frac{\beta n_y \pi (d + b)}{2b} \right) \cos \left( \frac{\beta n_y \pi}{2} \right) \right] - \beta \cos \left( \frac{\eta_y \pi}{2} \right) \left[ \sin \left( \frac{\beta n_y \pi (d + b)}{2b} \right) \sin \left( \frac{\beta n_y \pi}{2} \right) \right] + \sin \left( \frac{\eta_y \pi}{2} \right) \left[ \sin \left( \frac{\beta n_y \pi (d + b)}{2b} \right) \sin \left( \frac{\beta n_y \pi}{2} \right) \right] + \cos \left( \frac{\eta_y \pi}{2} \right) \cos \left( \frac{\beta n_y \pi (d + b)}{2b} \right) = 0,$$

with $\beta = \sqrt{\frac{D_0}{D_0 - b^2}}$.  
(64)

This characteristic equation was solved numerically and the solution for the number of half-waves $n_y, \infty$ as a function of the aspect ratio $b/d$ is depicted in Fig. 9 for various stiffness mismatches $D_y^\infty / D_0$. Observe that the number of half-waves gradually grows towards the asymptotic value $n_y, \infty = 2$ (= fully-clamped support) with increasing stiffness mismatch $D_y^\infty / D_0$ and with increasing aspect ratio $b/d$. It can be further confirmed that the curve calculated for $D_y^\infty / D_0 = 1$ indeed corresponds to the closed-form expression, Eq. (57).

The dashed curves presented in Fig. 9 were calculated by assuming that the supporting wall part $d$ is insensitive to buckling, whereby it provides a rotational stiffness $k_y, \infty = 2 D_y^\infty / d$ at its connection to wall part $b$. In other words, wall part $b$ is supposed to fail by local buckling. In correspondence with the deformed configuration sketched in Fig. 8 by the dashed line, the above stiffness value has been calculated by modelling wall part $d$ as simply supported, and subjecting it to two equal, but oppositely directed (dimensionless) bending moments $M$ at the supports. The rotational stiffness follows from the ratio between the applied moment and the rotation at the supports. The dashed curves in Fig. 9 were constructed by substituting this rotational stiffness, together with $D_x = D_0$, into Eq. (32), giving

$$T_{r, \infty} = \frac{2 b D_y^\infty}{D_0},$$  
(65)

and inserting the result into the characteristic equation, Eq. (36), that subsequently was solved numerically. The comparison of the curves calculated with the local buckling assumption to those of the exact solution illustrates that the mechanism of local buckling only becomes operative at a sufficiently high value of $b/d$, which is larger at a smaller mismatch of the initial bending stiffnesses $D_y^\infty / D_0$. In specific, adopting a relative difference between the solid and dashed curves of less than 1% in order to consider these as equal, it may be concluded that local buckling occurs if $b/d$ is larger than about 7, 5, 2 and 1.5 for $D_y^\infty / D_0 = 1, 2, 5$ and 10, respectively. For lower values of $b/d$ global buckling effects of the rectangular wall segment start to play a role, which thus can be taken into account in the stability analysis of wall part $b$ via Eq. (64).

The asymptotic value of the rotational stiffness $T_{r, \infty}$ computed from the above analysis can be used for constructing the function $\tilde{k}_y(X)$ via the relation

$$\tilde{k}_y(X) = \frac{T_{r, \infty}}{F(X)} \quad \text{with} \quad -\kappa \leq X \leq 0,$$  
(66)

with $F(X)$ a function that has values between 0 at the clamped support $X = -\kappa$ and 1 at infinite vertical distance from the clamped support (corresponding to $\kappa \to \infty$). In order to find a manageable expression for this function, the out-of-plane response of the supporting wall $d$ is simplified by ignoring torsional effects and constructing the bending response from the coupled bending responses $\tilde{w}^{\infty}$ and $\tilde{w}_t^{\infty}$ of wall segments $s1$ and $s2$ with unit widths in the y- and X-directions, respectively, see Fig. 10. More specifically, the bending response $\tilde{w}_t^{\infty}(X, y)$ of a cantilever wall segment $s1$, generated under a (dimensionless) uniform line load $\tilde{q}$, at $X = X_s = \kappa$ with $-\kappa \leq X_s \leq 0$, is translated into a continuous transversal spring support $\tilde{k}_y(X_s) = \tilde{q} / \tilde{w}_t^{\infty}(X_s)$ for wall segment $s2$ with simple supports at $y = 0$ and $y = b$. Subsequently, the bending response $\tilde{w}_t^{\infty}(X_s, y)$ of this wall segment is calculated under two equal, but oppositely directed (dimensionless) moments $\tilde{M}$ applied at the supports, whereby the rotational stiffness $\tilde{k}_y(X_s)$ follows from the ratio between the applied moment and the rotation generated at these supports, which at $y = 0$ renders $\tilde{k}_y(X_s) = \tilde{M} / \tilde{w}_t^{\infty}(X_s, y)|_{y=0}$. Note that under this specific loading condition the asymptotic value of the rotational stiffness $\tilde{k}_y, \infty$ is equal to Eq. (65), since at infinite vertical distance from the wall bottom the transversal spring stiffness approaches to zero, $\tilde{k}_y \rightarrow 0$. Inserting the values for $\tilde{k}_y(X_s)$ and $\tilde{k}_y, \infty$ into Eq. (66) finally results into the function value $F(X_s)$. This procedure applies to all coordinates $X = X_s$ across the wall length $-\kappa \leq X_s \leq 0$, by which it furnishes the function $F(X)$.

The above procedure thus starts with the computation of the dimensionless transversal stiffnesses $\tilde{k}_y(X)$ representing the deflection resistance at $X = X_s$ of a cantilever wall segment $s1$ of unit width in y-direction under a (dimensionless) uniform line load $\tilde{q}$, imposed at $X = X_s$. The displacement response $\tilde{w}_t^{\infty}$ of this wall segment needs to
satisfy the boundary conditions Eqs. (54) and (55), whereby the contributions from the y-direction are left out of consideration by imposing \( c_y = 0 \). Due to the presence of the curing function \( \zeta \), in the equilibrium expression Eq. (49) and the natural boundary condition Eq. (55), the displacement response \( \tilde{w}^{s1} \) can only be computed in an approximate form. Adopting a deflection response of the polynomial form of the sixth degree, for a linear curing process, Eq. (46), the solution of the weak form of the equilibrium equation leads to the following expression for the dimensionless translational stiffness (see Appendix A for more details):

\[
\tilde{k}_i(\bar{X}) = \left[ 3\bar{X}^2 - 12\bar{X} + 5\pi^2 - 8\pi\bar{X} + 28 \right] (\bar{X} - 1) \\
\times \left[ -73\bar{X}^5 - 15\pi^2 \bar{X} + 390\bar{X}^2 - 1950\bar{X}^3 + 5300\bar{X}^4 - 3k^5 \right] \\
+ 350\bar{X}^6 + 840\bar{X}^5 - 420\bar{X}^4 - 280\pi\bar{X}^3 - 210\bar{X}^2 \\
- 280\bar{X}^7 - 840\bar{X}^6 - 840\bar{X}^5 - 840\bar{X}^4 - 840\bar{X}^3 - 280\bar{X}^2 \right]\end{equation}

For an exponential curing process, Eq. (47), a comparable, but more extensive expression can be derived, which is omitted here for reasons of brevity. The translational stiffness, Eq. (67), defines the continuous elastic support of a wall segment \( s_2 \) of unit width in \( \bar{X} \)-direction, which is simply supported at \( y = 0 \) and \( y = d \) and is subjected to two equal, but opposite (dimensionless) moments \( \bar{M} \) at the supports, see Fig. 10. The bending deflection \( \bar{w}^{s2} \) of this configuration can be computed in closed form [15], in accordance with

\[
\bar{w}^{s2}(\bar{X}, y) = \exp\left(\frac{\psi y}{b}\right) \left( A_1 \cos\left(\frac{\psi y}{b}\right) + A_2 \sin\left(\frac{\psi y}{b}\right) \right) \\
+ \exp\left(\frac{\psi y}{b}\right) \left( A_3 \cos\left(\frac{\psi y}{b}\right) + A_4 \sin\left(\frac{\psi y}{b}\right) \right),
\]

where \( \psi = \psi(\bar{X}) = \left( \frac{\tilde{k}_i(\bar{X})}{4} \right)^{1/4}, \quad \) (68)

where \( \tilde{k}_i(\bar{X}) \) is given by Eq. (67). Solving the coefficients \( A_i \) with \( i \in \{ 1, 2, 3, 4 \} \) from the four boundary conditions at \( y = 0 \) and \( y = d \) and computing the rotational stiffness via \( \tilde{k}_i(\bar{X}) = \bar{M}/\bar{w}^{s2}(\bar{X}, y) \) at \( y = 0 \), together with Eqs. (65) and (66) leads to the following function \( \tilde{F}(\bar{X}) \):

\[
\tilde{F}(\bar{X}) = \left[ 4\sin(\psi) \cos(\psi) \exp(\psi) + \exp(-\psi) \exp(\psi) \right] \\
+ 2\cos(\psi) \exp(2\psi) - \exp(-2\psi) \\
- 2\sin(\psi) \exp(2\psi) + \exp(-2\psi) - 4\sin(\psi) \exp(3\psi) - \exp(3\psi) \\
- \exp(\psi) \exp(-\psi) \left[ \psi \left( 4\cos^2(\psi) \exp(\psi) + \exp(-\psi) \right) - \exp(3\psi) - \exp(-3\psi) - 3 \left( \exp(\psi) + \exp(-\psi) \right) \right]^{-1},
\]

where \( \psi = \psi(\bar{X}) = \left( \frac{\tilde{k}_i(\bar{X})}{4} \right)^{1/4}, \quad \) (69)

The function \( \tilde{F}(\bar{X}) \) is illustrated in Fig. 11 for different values of \( \kappa \). It can be confirmed that for all values of \( \kappa \) the function \( \tilde{F}(\bar{X}) \) is zero at the
where \( \omega \) is the rate of \( n \)-damped vibrations, \( \alpha \) the damping coefficient, \( c \) and \( k \) the density and stiffness of the wall, respectively. Using the method of complex potentials, the governing equation of the problem can be written as:

\[
\frac{d^2 \phi}{dz^2} + k^2 \phi = 0 \]

where \( k = \sqrt{\frac{c}{\rho}} \). The solution to this equation is given by

\[
\phi(z) = A \cos(kz) + B \sin(kz)
\]

subject to the boundary conditions \( \phi(0) = \phi(L) = 0 \), where \( L \) is the length of the wall. The natural frequency \( f_n \) is given by

\[
f_n = \frac{1}{2L} \sqrt{\frac{c}{\rho}}
\]

The deflection \( w(z) \) of the wall is then found from the relation

\[
w(z) = -\frac{c}{k^2} \phi(z)
\]

The maximum deflection occurs at the midpoint of the wall, and the maximum stress is given by

\[
\sigma = \frac{E}{1-\nu^2} \left( \frac{d^2 w}{dz^2} \right)_{z=L/2}
\]

The von Mises yield criterion is used to determine the plastic deformation of the wall. The effective plastic strain \( \varepsilon_p \) is calculated as

\[
\varepsilon_p = \sqrt{\frac{2}{3} \left[ (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2 \right]}
\]

where \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are the principal strains. The plastic strain energy density \( U_p \) is

\[
U_p = \frac{1}{2} \int_{0}^{L} \sigma \varepsilon_p dz
\]

The fracture energy \( G \) is then determined from the integral of the plastic strain energy density.

\[
G = \frac{1}{2} \int_{0}^{L} \sigma \varepsilon_p dz
\]

The fracture toughness \( K_c \) is related to the fracture energy by

\[
K_c = \sqrt{\frac{E}{1-\nu^2} \rho G}
\]

For the wall, the plastic strain energy density \( U_p \) is given by

\[
U_p = \frac{1}{2} \int_{0}^{L} \sigma \varepsilon_p dz
\]

where \( \sigma \) is the stress and \( \varepsilon_p \) is the plastic strain. The plastic strain energy \( U_p \) is then determined from the integral of the plastic strain energy density.

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\]

The fracture energy \( G \) is then determined from the integral of the plastic strain energy density.

\[
G = \frac{1}{2} \int_{0}^{L} \sigma \varepsilon_p dz
\]
are maximal, see Eq. (16). The yield criterion for plastic collapse at the bottom of the wall can be generally formulated as

$$\rho gl = |\sigma_p|,$$  
(76)

with $|\cdot|$ denoting the absolute value of the yield strength $\sigma_p$ (to compensate for a negative value sometimes reported for the compressive strength), and the wall length $l$ given by Eq. (41). The value of the yield strength is dependent of the type of failure criterion adopted for the printing material. Two representative failure criteria are: compressive failure described by the maximal stress theory and pressure-dependent shear failure in accordance with the Mohr-Coulomb theory. In the first case plastic collapse is reached when the compressive yield strength is equal to the maximal principal stress under biaxial compression ($\approx \sigma_{xx}$, see Eq. (16)). Accordingly, the yield strength $\sigma_p$ in Eq. (76) is represented by the uniaxial compressive strength $\sigma_c$, i.e.,

$$\sigma_p = \sigma_c,$$  
(77)

In the second case plastic collapse under biaxial loading conditions may be formulated as [16]

$$\frac{1}{2}(\sigma_{xx} - \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} + \sigma_{yy})\sin(\phi) - \cos(\phi) = 0,$$  
(78)

where $\phi$ is the friction angle of the printing material and $c$ is the cohesion. Inserting Eqs. (2) and (16) into the Mohr-Coulomb criterion Eq. (78) and comparing the result to Eq. (76) gives for the yield strength

$$\sigma_p = \frac{2c}{1-K_y}(\cos(\phi))$$  
(79)

When assuming that at the bottom the wall is fully constrained in the lateral direction, $\gamma_{yy} = 0$, in Eq. (79) the coefficient of lateral stress may be set equal to the Poisson’s ratio, $K_y = v$, see also Eq. (19). For a printing material without frictional resistance ($\phi = 0$), Eq. (79) reduces to

$$\sigma_p = \frac{2c}{1-K_y},$$  
(80)

which essentially reflects Tresca’s yield criterion based on reaching the maximal shear stress at the wall bottom. Material tests on the specific printing material used should point out which expression for the yield strength $\sigma_p$ is most representative. Additional examples of yield criteria at material point level can be found in [17].

Similar to the elastic stiffness, as a result of the curing process the yield strength is heterogeneous in the direction of increasing wall length, i.e., the $x$-direction, which can be accounted for by replacing $\sigma_p$ in the above expressions by $\sigma_{p\cdot x}$. The time evolution of the yield strength in a material point at the wall bottom is given by

$$\delta_p(x = 0, t) = \hat{h}_p(t)|\sigma_{p,0}|,$$  
(81)

where $\sigma_{p,0}$ is the yield strength at the onset of the printing process $t = 0$, and $\hat{h}_p = \hat{h}_p(t)$ is the curing function characterizing the time evolution of the yield strength. The evolution of strength with deformation, known as strain hardening, may be left out of consideration here, as this behaviour generally is not very relevant for a soft, viscous printing material.

Analogous to the curing functions, Eqs. (43) and (44), selected for the elastic stiffness, the two types of curing functions considered for the yield strength are a linear curing function

$$\hat{h}_p(t) = 1 + \varepsilon_p^l t,$$  
(82)

and an exponentially-decaying curing function

$$\hat{h}_p(t) = \gamma_e + (1-\gamma_e)\exp(-\varepsilon_p^e t),$$  
with $\gamma_e = \frac{\sigma_{p,\infty}}{\sigma_{p,0}},$  
(83)

where $\varepsilon_p^l$ and $\varepsilon_p^e$ are the curing rates for the strength in the linear and exponential evolutions, respectively. The parameter $\sigma_{p,\infty}$ appearing in the strength ratio $\gamma_e$ in Eq. (83) represents the final, asymptotic strength obtained under exponential curing when $t \to \infty$. Eq. (81) can be expressed in terms of a dimensionless Eulerian coordinate $\bar{X}$ by invoking a coordinate transformation analogous to Eq. (48), with the curing rate $\varepsilon_p$ replaced by $\xi_e$. This leads to

$$\delta_p(\bar{X}) = \frac{\hat{h}_p(\bar{X})\sigma_{p,0}}{\bar{X} - \delta},$$  
(84)

whereby the linear curing function, Eq. (82), turns into

$$\hat{h}_p(\bar{X}) = 1 - \frac{\varepsilon_p^l}{\bar{X}},$$  
(85)

and the exponentially-decaying curing function, Eq. (83), becomes

$$\hat{h}_p(\bar{X}) = \gamma_e + (1-\gamma_e)\exp(\frac{\varepsilon_p^e \bar{X}}{\bar{X} - \delta}),$$  
(86)

With Eq. (84), the yield condition, Eq. (76), for failure at the bottom of the wall, $\bar{X} = \delta$, with $\delta$ given by

$$\delta = \frac{\varepsilon_p^l l}{\bar{X}},$$  
where $\varepsilon_p^l \in \{\varepsilon_p^e, \varepsilon_p^c\},$  
(87)

results in

$$\rho gl = \left(1 + \frac{\varepsilon_p^l l}{\bar{X}}\right)|\sigma_{p,0}| = 0.$$  
(88)

The length at which the wall fails under yielding can be simply derived from Eq. (89) as

$$l = \frac{|\sigma_{p,0}|}{\rho g - \varepsilon_p^l |\sigma_{p,0}|/l}.$$  
(90)

In order to further develop this expression, a dimensionless length $\bar{\iota}_p$ and curing rate $\bar{\xi}_e$ are introduced:

$$\bar{\iota}_p = \frac{\rho g l}{|\sigma_{p,0}|},$$  
(91)

$$\bar{\xi}_e = \frac{\rho g l}{|\sigma_{p,0}|}$$  
with $\varepsilon_p^l \in \{\varepsilon_p^e, \varepsilon_p^c\}.$

By taking the wall length equal to the wall length at plastic collapse, $l = \iota_p$, an explicit expression for the dimensionless critical length at yielding can be obtained after combining Eqs. (90) and (91), i.e.,

$$\bar{\iota}_p = \frac{1}{1 - \bar{\xi}_e}$$  
with $0 \leq \bar{\xi}_e < 1.$

As indicated by Eq. (91), the dimensionless curing rate $\bar{\xi}_e$ can be computed a priori from the material properties and printing process data. Note from Eq. (92) that in the case of $\bar{\xi}_e \rightarrow 1$ the dimensionless length $\bar{\iota}_p$ approaches infinity. Hence, for a linear curing process this value of the curing rate should be considered as the upper limit for plastic collapse. Essentially, for values $\bar{\xi}_e > 1$ the growth in yield strength caused by the curing rate $\bar{\xi}_e$ goes faster than the growth in stress governed by the printing velocity $l$, see Eq. (92), as a result of which the stress at the wall bottom is no longer able to reach the yield strength, and plastic collapse does not occur.

For the exponential curing process, Eq. (86), the yield condition, Eq. (88), evaluated at the wall bottom, $\bar{X} = \delta$, with $\delta$ presented by Eq. (87), results in

$$\rho gl = \left[\gamma_e + (1-\gamma_e)\exp\left(-\frac{\varepsilon_p^e \bar{X}}{\bar{X} - \delta}\right)\right]|\sigma_{p,0}| = 0.$$  
(93)

Eq. (93) can be made dimensionless by invoking the parameters presented in Eq. (91), which, with $l = \iota_p$, leads to

$$\bar{\iota}_p = \left[\gamma_e + (1-\gamma_e)\exp\left(-\frac{\varepsilon_p^e \bar{\iota}_p}{\bar{\iota}_p - \bar{\xi}_e}\right)\right] = 0.$$  
(94)
The above (transcendental) equation does not have a closed-form solution for \( \tilde{t}_p \), but can be solved by means of a numerical procedure. The applicability of the yield functions, Eqs. (92) and (94), in predicting the failure response of a printed wall is treated in Section 4.

3.2.1. Competition between elastic buckling and plastic collapse

The wall will fail by yielding if the wall length for plastic collapse is smaller than the critical buckling length, \( t_p < t_{cr} \). Conversely, when \( t_p > t_{cr} \) the wall will fail by elastic buckling. This criterion for the determination of the possible failure mechanism can be conveniently expressed in terms of geometrical, material and printing process data by invoking Eqs. (75) and (91), leading to

\[
\frac{\tilde{t}_w}{t_p} < \bar{X} : \text{elastic buckling,}
\]

\[
\frac{\tilde{t}_w}{t_p} > \bar{X} : \text{plastic collapse,}
\]

with \( \bar{X} = \left( \frac{b}{D_0} \right)^{\frac{3}{2}} \left( \frac{\sigma_{pl}}{\mu g} \right) \) and \( \tilde{t}_p = \tilde{t}_p(\bar{X}) \). \( \bar{X} \) is independent of the wall width \( b \) and the specific boundary conditions applied in the \( y \)-direction. Wall failure during 3D printing, as characterised by either elastic buckling or plastic collapse, thus essentially is determined by the 5 independent, dimensionless parameters presented in Eqs. (75) and (91). In Section 4 the functions \( \tilde{t}_p(\bar{X}) \) will be computed and visualised for a broad range of parameter values, thereby providing a useful tool for the practical design and optimisation of 3D printing processes.

4. Numerical results

In the analysis of the results, first the case of bifurcation buckling is considered by computing the critical buckling length \( \tilde{t}_p(\bar{X}, \tilde{t}_w, \tilde{t}_E) \) from the weak form of equilibrium, Eqs. (73) and (74), and the corresponding boundary conditions, Eqs. (54) and (55). This is done successively for the free wall, the simply-supported wall and the fully-clamped wall depicted in Fig. 2. For bifurcation buckling the imperfections vanish: \( \delta w^{(0)} = 0 \), \( w' = \tilde{w}' \), turning the weak form, Eq. (73), into an eigenvalue problem. The lowest eigenvalue represents the critical buckling length \( \tilde{t}_p \), which is calculated by assuming a displacement profile of the polynomial form

\[
\tilde{w}' = \sum_{n=1}^{N} C_n \bar{X}^{n-1}
\]

with \( C_n \) representing the unknown, generalised coordinates. Preliminary simulations have indicated that for the present problem polynomial basis functions are preferential above harmonic basis functions; although their level of accuracy is comparable for a similar number of terms \( N \), after carrying out the integration procedure, Eq. (73), the polynomial basis functions lead to shorter, more elegant algebraic equations, which are relatively easy to incorporate in the subsequent numerical solution procedure. The combination of Eqs. (20), (21) and (96), together with the coordinate transformation Eq. (48), illustrates that the overall process thus is formulated in an Eulerian-Lagrangian \( \tilde{X} - y \) coordinate system, using combined polynomial-trigonometric basis functions. Applying the criterion given by Eq. (95), the critical buckling length, \( \tilde{t}_p(\bar{X}, \tilde{t}_w, \tilde{t}_E) \), can be evaluated against the wall length \( \tilde{t}_w(\bar{X}) \) for plastic collapse. This is done for both the linear and exponential curing processes. Subsequently, the effect of geometrical imperfections on the buckling response of a free wall is analysed for different imperfection profiles, whereby the asymptotic buckling length reached under increasing wall deflection is compared against the bifurcation buckling length for an ideally straight wall.

As a working assumption, in the simulations the coefficient of lateral stress is taken as constant and set equal to the Poisson’s ratio, i.e., \( K_y = K_x = \nu = 0.3 \). Correspondingly, at \( y = 0 \) and \( y = b \) the wall is supposed to be fully constrained in the lateral direction, \( \psi_y = 0 \), see Eq. (19). Since this assumption maximises the lateral stress and thus minimises the critical buckling length, from a design point of view the computational results presented here are conservative and therefore on the safe side. At the end of this section it will be demonstrated for a specific configuration that the influence of \( K_y \) on the buckling response is relatively small for a broad range of curing rates \( \tilde{t}_E \).

4.1. Free wall printed under a linear curing process

In the case of a free wall the constraint factor reflecting the boundary conditions in \( y \)-direction equals \( c_y = 0 \), see Eq. (27), by which the residual, Eq. (74), after inserting Eqs. (50) and (51), reduces to

\[
\tilde{R} = \tilde{R}(\bar{X}) = \left( \tilde{g} \bar{w}'_{XX} \right)_{XX} - \frac{gh}{D_0} \left( \tilde{t}_E \right)^3 \left( \bar{X} \tilde{w}' \right)_{XX}\tilde{X}'
\]

and the natural boundary conditions at \( \bar{X} = 0 \), Eq. (55), simplify to

\[
\bar{w}'_{XX} = 0, \quad \left( \tilde{g} \bar{w}'_{XX} \right)_{XX} = 0.
\]

The essential boundary conditions at \( \bar{X} = -1 \) are given by Eq. (54).

The accuracy of the numerical solution for bifurcation buckling of a free wall printed under a linear curing process is evaluated by subsequently adopting polynomial basis functions of the fourth, fifth and sixth degree, as represented in Eq. (96) by \( N = 5, 6 \) and 7, respectively. Fig. 12 sketches the critical buckling length \( \tilde{t}_p \), as a function of the curing rate \( \tilde{t}_E \), with \( \tilde{t}_p(\bar{X},, \tilde{t}_w) \) given by Eq. (75),. Clearly, the value of \( \tilde{t}_p \) increases with increasing \( \tilde{t}_E \). This is, since a higher value of \( \tilde{t}_E \) relates to a higher curing rate \( \tilde{t}_p(\bar{X},, \tilde{t}_w) \) and/or a lower printing velocity \( l \), see Eq. (75), which enable the wall to better develop its bending stiffness.
during the printing process, and thereby its resistance against elastic buckling. It can be further observed that for a polygonal basis with \( N = 5 \) the critical buckling length is somewhat overestimated, but that for \( N = 6 \) and \( 7 \) the numerical solutions for the critical buckling length are virtually identical over the whole range of \( \xi_k \), and thus seem to have converged towards the exact solution.

In the rate-independent limit \( \xi_k \to 0 \), an analytical solution can be established for the critical buckling length \( \tilde{\xi} \), which may be used as an additional check on the accuracy of the approximations computed for \( N = 5, 6 \) and \( 7 \). The derivation of the analytical solution starts from the rate-independent differential equation, Eq. (25), which, by inserting \( c_r = 0 \) and the rate-independent bending stiffness \( D_c = D_0 \), turns into

\[
D_0 w_{\text{xxx}} - \left( \frac{pg\hbar(x-l)}{D_0} w_{xx} \right)_{,x} = 0.
\]

(99)

Dividing this equation by \( D_0 \), and integrating once renders

\[
w_{\text{xxx}} - \frac{pg\hbar}{D_0} (x - l) w_{xx} = A_1.
\]

(100)

From the boundary condition \( w_{\text{xxx}} = 0 \) at \( x = l \) it follows that the integration constant \( A_1 = 0 \). Adopting the coordinate transformation \( x = l - \xi \), Eq. (100) becomes

\[
w_{\text{xxx}} + \frac{pg\hbar}{D_0} \xi w_{xx} = 0.
\]

(101)

The differential equation given by Eq. (101) has a form similar to that describing bifurcation buckling of a prismatic column under its own weight. As pointed out in [18], this mathematical form can be elegantly developed by applying the substitutions

\[
\xi^2 \ddot{w}_{\xi\xi} + \ddot{\xi} w_{\xi\xi} + (r^2 \xi^2 - s^2) \ddot{w} = 0,
\]

(103)

with the coefficients \( r^2 \) and \( s^2 \) as

\[
r^2 = \frac{4pg\hbar}{D_0}, \quad s^2 = \frac{1}{9}.
\]

(104)

The solution of Eq. (103) is given by [19]

\[
\ddot{w}_s = \ddot{w}_s(\xi) + \ddot{w}_s(\xi) = J_s(\xi) + J_s(\xi),
\]

(105)

with \( J_s \) a Bessel function of the first kind of order \( s \),

\[
J_s = J_s(r, \xi) = \sum_{k=0}^{\infty} \frac{(-1)^s (r\xi)^{s+2k}}{2^{s+2k} k! (s+k+1)}
\]

(106)

where \( \Gamma \) represents the gamma function and \( k \) is a positive integer. Combining Eq. (105) with Eqs. (104) and (102) now leads to

\[
w_{\text{ss}} = \frac{1}{\xi^2} (A_s \dot{J}_s(r \xi^2) + A_s J_s(r \xi^2)).
\]

(107)

The boundary condition \( w_{\text{ss}} = 0 \) at \( \xi = 0 \) is only satisfied if \( A_1 = 0 \), so that Eq. (107) becomes

\[
w_{\text{ss}} = A_s \frac{1}{\xi^2} \dot{J}_s(r \xi^2).
\]

(108)

In addition, combining the boundary condition \( w_{\text{ss}} = 0 \) at \( \xi = l \) with Eq. (108) renders

\[
\dot{J}_s(l r^2) = 0.
\]

(109)

The smallest root of this equation can be found as

\[
rl = 1.866351.
\]

(110)

Inserting Eq. (104), into Eq. (110) then gives for the critical rate-independent buckling length, \( l = l_{k,0} \):

\[
l_{k,0} = \left( \frac{1.86635 \frac{D_0}{pg\hbar}}{9 D_0} \right)^{\frac{1}{8}} = 1.98635 \left( \frac{D_0}{pg\hbar} \right)^{\frac{1}{8}}.
\]

(111)

The analytical value \( l_{k,0} = 1.98635 \) can be compared against the rate-independent numerical values \( l_{k,0} = 2 \) (\( N = 5 \)), 1.98684 (\( N = 6 \)) and 1.98635 (\( N = 7 \)), which indicates a relative overestimation by the numerical approximations of 0.68%, 0.02% and 0.00%, respectively. From this comparison and the convergence behaviour of the numerical curves depicted in Fig. 12, it is concluded that the numerical result may be considered as highly accurate when the polygonal basis in Eq. (96) is constructed with \( N \geq 6 \). Accordingly, the forthcoming numerical results are computed with \( N = 6 \).

The critical buckling length of a free wall printed under a linear curing process can be closely approximated by the function

\[
l_{cr} = l_{cr,0} + 0.996 (\xi_k^5)^{0.793},
\]

(112)

where \( l_{cr,0} = 1.98635 \) is the rate-independent buckling length following from Eq. (111). Eq. (112) has been found from a least-squares fit on the (nearly) exact numerical solution plotted in Fig. 12 for \( N = 7 \), whereby \( R^2 = 0.999 \). The closed-form expression may serve as a convenient design formula, and will be used below for the construction of failure mechanism maps.

From the critical buckling length \( l_{k,0} \) and the 4 boundary conditions, the critical buckling mode can be determined via Eq. (96) by computing the coefficients \( C_n \) (up to an arbitrary value for the overall amplitude). The buckling mode is depicted in Fig. 13 by plotting the normalised deflection \( w^* \) against the dimensionless vertical coordinate \( X/k \) for \( \xi_k = 0 \) (rate-independent limit) and \( \xi_k = 4 \). Clearly, the critical buckling mode is not very sensitive to the value of the curing rate.

For a linear curing process the wall length \( l_{k,0} \) for plastic collapse can be plotted as a function of the curing rate \( \xi = \xi_k \) by using Eq. (92), see Fig. 14. Apparently, \( l_{k,0} \) grows substantially with increasing \( \xi_k \), whereby the increase in \( \xi_k \) may be caused by a larger curing rate \( \xi_k \), a higher initial yield strength \( |\sigma_p, o| \), and/or a lower printing rate \( l \), see Eq. (91).2. Notice that the vertical asymptote \( l_{k,0} \to \infty \) is reached when \( \xi_k \to 1 \). As explained in Section 3.2, for higher values of \( \xi_k \) plastic collapse can not occur. Further, the rate-independent limit \( l_{k,0} = 1 \) for \( \xi_k \to 0 \) is in correspondence with \( |\sigma_p,0| = \rho g E_{p,0} \), see Eq. (91). It is emphasised that the curve depicted in Fig. 14 is independent of the type of boundary condition in the \( y \)-direction, and therefore is applicable to any type of wall structure.

For a free wall the competition between elastic buckling and plastic collapse can be evaluated by reading off from Figs. 12 and 14 the buckling length \( l_{k,0} \) and the plastic collapse length \( l_{p} \) for specific values of
the curing rates $\tilde{\xi}_E$ and $\tilde{\xi}_p$, respectively, and substituting the result into the criterion given Eq. (95). Alternatively, from the closed-form expressions, Eqs. (92) and (112), a more general approach can be followed by constructing the length scale ratio $\tilde{L}_p/\tilde{L}_p^*$:

$$\tilde{L}_p = \left(1.986 + 0.996 \left(\tilde{\xi}_E\right)^{0.793}\right) \left(1 - \tilde{\xi}_E\right)$$

with $a = \frac{\tilde{\xi}_p}{\tilde{\xi}_E}$. (113)

and combining this function together with the criterion given by Eq. (95) into a failure mechanism map. Fig. 15 depicts Eq. (113) for selected values of the ratio of curing rates, $a = \tilde{\xi}_p/\tilde{\xi}_E$, in the range of 0.1 to 10. The depicted value of $\Lambda$ has been chosen arbitrarily, and indicates the transition from elastic buckling to plastic collapse. In specific, if a point on the curve $\tilde{L}_p/\tilde{L}_p^*$ lies above the value of $\Lambda$, plastic collapse may occur, whereas if the point lies below the value of $\Lambda$, elastic buckling may take place. The stiffness curing rate $\tilde{\xi}_E$ at which the transition from plastic collapse to elastic buckling happens (corresponding to the coincidence of $\tilde{\xi}_E$ with the curve $\tilde{L}_p/\tilde{L}_p^*$) obviously is dependent of the value of $a$. In the case of $a > 1$, for $\Lambda > 1.986$ plastic collapse can not occur, so that elastic buckling becomes the only possible failure mechanism. Conversely, in the range $0 < \Lambda < 1.986$, plastic collapse may take place for relatively low values of $\tilde{\xi}_E$, while elastic buckling becomes the potential failure mechanism for higher values of the curing rate. Note that the length scale ratio $\tilde{L}_p/\tilde{L}_p^*$ decreases towards zero with increasing curing rate; this is, because the plastic collapse length $\tilde{L}_p$ at this stage becomes infinitely large, see also Fig. 14. Consequently, at curing rates larger than the value at which $\tilde{L}_p/\tilde{L}_p^* = 0$, elastic buckling remains as the only potential failure mechanism. In the case of $a < 1$, for $\Lambda$ somewhat larger than 1.986 the potential failure mechanism under increasing $\tilde{\xi}_E$ initially is elastic buckling, then changes to plastic collapse, and subsequently turns back to elastic buckling. For example, for $a = 0.1$ and $\Lambda = 2.5$ this sequence of failure mechanisms is characterised by the ranges $0 < \tilde{\xi}_E < 0.62$ (elastic buckling), $0.62 < \tilde{\xi}_E < 5.85$ (plastic collapse), and $\tilde{\xi}_E \geq 5.85$ (elastic buckling), see Fig. 15. In addition, for $a = 0.1$ elastic buckling becomes the only possible failure mechanism when $\Lambda > \tilde{\xi}_p/\tilde{L}_p^*\max = 3.06$. Hence, in 3D printing applications characterised by a linear curing process, $\Lambda > 3.06$ can be used as a simple, though somewhat conservative design criterion to generally avoid plastic collapse of a free wall.

4.2. Free wall printed under an exponentially-decaying curing process

For a free wall printed under an exponentially-decaying curing process, Fig. 16 illustrates the critical buckling length $\tilde{L}_p$, as a function of the curing rate $\tilde{\xi}$ for selected values of $\gamma = \gamma_E = E_a/E_0$, ranging from 1 to 10. With $\gamma_E = 1$ the buckling length becomes independent of the curing rate $\tilde{\xi}_E$, and corresponds to the rate-independent value $\tilde{L}_p,0$ presented in Eq. (111). For $\gamma = 2.5$ and 10, under an increasing curing rate $\tilde{\xi}_E \rightarrow \infty$ the buckling length $\tilde{L}_p$ grows monotonically towards an asymptotic value, whereby the growth obviously is stronger for a higher stiffness ratio $\gamma = \gamma_E$. An expression for this limit can be deduced from Eqs. (45), (47), (52) and (75), as

$$\tilde{L}_p,\infty = \tilde{L}_p(\gamma_E) = \gamma_E^{1/4} \tilde{L}_p,0$$

whereby the rate-independent limit $\tilde{L}_p,0$ is presented by Eq. (111). The approximations plotted in Fig. 16 by the dashed lines are described by the following function:

$$\tilde{L}_p(\gamma_E) = \left(1.662 + 0.240\gamma_E\right) \tilde{L}_p,0$$

Fig. 14. Plastic collapse length $\tilde{L}_p$ versus curing rate $\tilde{\xi}$ for a wall printed under a linear curing process, see Eq. (92).

Fig. 15. Failure mechanism map for a free wall printed under a linear curing process. Length scale ratio $\tilde{L}_p/\tilde{L}_p^*$ versus curing rate $\tilde{\xi}$ (on a logarithmic scale) for selected values of the ratio of curing rates, $a = \tilde{\xi}_p/\tilde{\xi}_E$, in the range of 0.1 to 10. The curves presented are in accordance with Eq. (113), whereby the value depicted for $\Lambda$, see Eq. (95), has been chosen arbitrarily to indicate the transition from elastic buckling to plastic collapse.

Fig. 16. Critical buckling length $\tilde{L}_p$ versus curing rate $\tilde{\xi}$ for a free wall printed under an exponentially-decaying curing process, see Eq. (44). The stiffness ratios considered are $\gamma = \gamma_E = E_a/E_0 = 1, 2.5, 5$ and 10. For each stiffness ratio the asymptotic value of the buckling length $\tilde{L}_p,\infty$ at $\tilde{\xi} = \infty$ is indicated (in 2 decimals) at the right side of the figure. The closed-form approximations represented by the dashed lines are in accordance with Eq. (115).
\[ l = \frac{\alpha}{\gamma E}, \quad \gamma_E = \frac{E_0}{E} \quad \text{and} \quad \sigma_{p,\infty} = \frac{\sigma_{p,0}}{\sigma_{p,0}}. \] (118)

Combining Eq. (118) with the failure criterion, Eq. (95), leads to the failure mechanism map depicted in Fig. 18. As a basic choice, the stiffness and strength ratios characterising the exponential curing process are taken equal, \( \gamma = \gamma_E = \gamma_0 \), in correspondence with \( \gamma = 2 \) (Fig. 18a), \( \gamma = 5 \) (Fig. 18b), and \( \gamma = 10 \) (Fig. 18c). The general trend of \( \bar{\psi}_p/\bar{l}_p \) for the three different values of \( \gamma \) appears apparently similar. For \( \alpha < 1 \) the minimal value of \( \overline{\lambda} \) for which elastic buckling is the only possible failure mechanism clearly increases with increasing value of \( \gamma_E \); for example, for \( \alpha = 0.1 \) this happens when \( \overline{\lambda} > (\bar{\psi}_p/\bar{l}_p)_{\min} = 2.31 \) \((\gamma_E = 2), 2.80 \) \((\gamma_E = 5), \) and 3.25 \((\gamma_E = 10). \) Comparing the failure mechanism map with that for a printing process characterised by linear curing, see Fig. 15, shows that under exponential curing a maximal value of \( \overline{\lambda} \) can be identified below which plastic collapse unconditionally becomes the only possible failure mechanism. For example, for \( \alpha = 1.0 \) this corresponds to \( \overline{\lambda} < (\bar{\psi}_p/\bar{l}_p)_{\min} = 1.11 \) \((\gamma_E = 2), 0.47 \) \((\gamma_E = 5), \) and 0.24 \((\gamma_E = 10), \) indicating a decrease of this maximal value under an increasing value of \( \gamma_E \). For \( \overline{\lambda} \) falling in between \((\bar{\psi}_p/\bar{l}_p)_{\min} \) and \((\bar{\psi}_p/\bar{l}_p)_{\max} \) the potential failure mechanism either is elastic buckling or plastic collapse, depending on whether the condition \( \overline{\lambda} > (\bar{\psi}_p/\bar{l}_p)_{\min} \) or \( \overline{\lambda} < (\bar{\psi}_p/\bar{l}_p)_{\max} \) is met, respectively.

4.3. Simply-supported wall printed under linear and exponentially-decaying curing processes

For a simply-supported wall the equilibrium equation presented by Eqs. (49) and (50) is specified by the values \( n_{\psi} = n_{\psi} = 1 \) and \( \epsilon_{\psi} = \epsilon_{\psi} = 1 \) in Eq. (27), with the corresponding boundary conditions in accordance with Eqs. (54) and (55). In contrast to the free wall, the critical buckling length \( \bar{l}_p \) not only depends on the curing rate \( \bar{\psi}_p \), but also on the (dimensionless) width \( b \) of the wall, presented by Eq. (75b). For a linear curing process the critical bifurcation buckling length found after solving Eq. (73) has been depicted in Fig. 19 as a function of the curing rate \( \bar{\psi}_p = \bar{\psi}_p \), for selected values of the width \( b \), ranging from 3 to 20. The buckling curve for a free wall has been taken from Fig. 12 and sketched for comparison (dashed line); it indeed corresponds to the curve for the limit case of a simply-supported wall of infinite width, \( b \to \infty \). Further, as a reference the rate-independent buckling length \( \bar{l}_{cr,0} = \bar{l}_{cr,0} = 0 \) is presented (in 2 decimals) for a selection of wall widths.

Observe from Fig. 19 that the critical buckling length generally becomes larger for shorter wall widths \( b \). In addition, the buckling length increases with increasing curing rate \( \bar{\psi}_p \), whereby the growth turns out to be (significantly) larger at smaller wall width. As an example, for curing rates in the range of 0 \( \leq \bar{\psi}_p < 0.25 \), walls of \( b \leq 5 \) appear to be remarkably stable, resulting in a buckling length that may be more than 10 times larger than that of the free wall. The construction of a failure mechanism map for evaluating the competition between elastic buckling and plastic collapse is omitted here, since the dependency of the critical buckling length on both \( \bar{\psi}_p \) and \( b \) makes such a graphical representation relatively difficult to interpret. Instead, the determination of the potential failure mechanism follows from dividing the value of \( \bar{l}_{cr} \) obtained from Fig. 19 by the value of \( \bar{l}_p \) read off from Fig. 14, and comparing this ratio against the value of \( \overline{\lambda} \) computed via Eq. (95). Considering that the buckling length for a simply-supported wall typically is larger than for a free wall, see Fig. 19, failure by plastic collapse is more critical in the case of a simply-supported wall. Accordingly, the minimal value of \( \overline{\lambda} \) required for excluding plastic collapse for a simply-supported wall needs to be higher than for a free wall.

The critical buckling mode of a simply-supported wall has been illustrated in Fig. 20 for two different wall widths, \( b = 4 \) and 5. For the wall \( b = 5 \) the buckling mode is comparable to that of the free wall sketched in Fig. 13, and only slightly changes when increasing the curing rate from \( \bar{\psi}_p = 0 \) to \( \bar{\psi}_p = 0.4 \). In contrast, the buckling mode of the shorter
(a) Stiffness and strength ratio $\gamma = \gamma_E = \gamma_\sigma = 2$.

(b) Stiffness and strength ratio $\gamma = \gamma_E = \gamma_\sigma = 5$.

(c) Stiffness and strength ratio $\gamma = \gamma_E = \gamma_\sigma = 10$.

Fig. 18. Failure mechanism map for a free wall printed under an exponentially-decaying curing process. Length scale ratio $\xi_\sigma / \xi_\pi$ versus curing rate $\bar{\xi}$ (on a logarithmic scale) for selected values of the ratio of curing rates, $a = \xi_\pi / \xi_\sigma$, in the range of 0.1 to 10, thereby considering three different stiffness and strength ratios: $\gamma = \gamma_E = \gamma_\sigma = 2$ (a), $\gamma = \gamma_E = \gamma_\sigma = 5$ (b), and $\gamma = \gamma_E = \gamma_\sigma = 10$ (c). The curves are in accordance with Eq. (18), whereby the value depicted for $X$, see Eq. (95), has been chosen arbitrarily to indicate the transition from elastic buckling to plastic collapse.

Fig. 19. Critical buckling length $\tilde{I}_c$ versus curing rate $\bar{\xi}$ for a simply-supported wall printed under a linear curing process, see Eq. (43). The wall widths considered range from $\bar{b} = 3$ to 20. The buckling curve for a free wall (corresponding to $\bar{b} \to \infty$), has been taken from Fig. 12 and is plotted for comparison (dashed line). Values for the rate-independent buckling length $I_{cr,0}$ at $\bar{\xi} = 0$ are plotted for a selection of wall widths $\bar{b}$, and are presented as a reference.

Fig. 20. Critical buckling mode for a simply-supported wall printed under a linear curing process, see Eq. (43), for two different wall widths, $\bar{b} = 4$ (solid line) and $\bar{b} = 5$ (dashed line). The curing rates selected are $\bar{\xi} = \xi_E = 0, 0.2$ and 0.4.

wall $\bar{b} = 4$ shows a rather strong variation under the change in curing rate, and for all three curing rates selected appears to be rather different from that of the wall $\bar{b} = 5$. The change in critical buckling mode in the transition from $\bar{b} = 5$ to $\bar{b} = 4$ essentially is governed by an increasing aspect ratio $\xi_E / \xi_\pi$. In fact, the buckling response of relatively short walls with a high aspect ratio $I_{cr} / \bar{b}$ is characterized by a higher-order buckling mode. This behaviour, which is also known from plate structures of various aspect ratios subjected to in-plane loading, see for example [14], will be discussed in more detail below.

The critical buckling length for a simply-supported wall printed under an exponentially-decaying curing process is illustrated in Fig. 21 as a function of the curing rate $\bar{\xi} = \xi_E / \xi_\pi$ for different stiffness ratios $\gamma = \gamma_E = \gamma_\sigma = 2$ (Fig. 21a), $\gamma = \gamma_E = \gamma_\sigma = 5$ (Fig. 21b) and $\gamma = \gamma_E = \gamma_\sigma = 10$ (Fig. 21c). The range of wall widths considered, $3 \leq \bar{b} \leq 20$, is the same as for the linear curing process depicted in Fig. 19. As observed for the free wall, an important difference with the linear curing process is the appearance of an asymptotic buckling length, $I_{cr,\infty}$, when $\bar{\xi} \to \infty$. This limit value results from the asymptotic behaviour of the exponentially-decaying curing function, see Eq. (44) and Fig. 6. Values for $I_{cr,\infty}$ are depicted
(a) Curing stiffness ratio $\gamma = \gamma_E = E_\infty/E_0 = 2$.

(b) Curing stiffness ratio $\gamma = \gamma_E = E_\infty/E_0 = 5$.

(c) Curing stiffness ratio $\gamma = \gamma_E = E_\infty/E_0 = 10$.

Fig. 22. Limit buckling lengths $l_{c,r}$ (5 = 5) and $l_{c,\omega}$ (5 = 2, 5 and 10) versus wall width $b$ for a simply-supported wall printed under an exponentially-decaying curing process, see Eq. (44). The regime within which a change of the vertical buckling mode takes place corresponds to $0.65 < l_{c,r}/b < 1.5$, as designated by the dashed lines.

in Fig. 21 for a selection of wall widths $b$. It can be observed that in the case of large and small wall widths the corresponding value of $l_{c,\omega}$ is already approached closely at $\xi = 2$. For intermediate wall widths, $5 \leq b \leq 6 \mu E = 2$, $7 \leq b \leq 8 \mu E = 5$ and $9 \leq b \leq 10 \mu E = 10$, however, this limit value turns out to be relatively high, such that the curing rate required for closely reaching the limit falls outside the range considered in Fig. 21. The relatively high values of $l_{c,\omega}$ can be related to a change in the vertical buckling mode, which will be addressed below in more detail.

For a selection of wall widths Fig. 21 includes the rate-independent limit values $l_{c,\omega}$ obtained under $\xi = 0$. It can be easily confirmed that these values are identical to those found for a linear curing process, see Fig. 19. An additional graph that shows the limit buckling lengths $l_{c,\omega}$ and $l_{c,\omega}$ as a function of the wall width $b$ can be constructed by solving the weak form of the time-independent equilibrium condition, Eq. (25), together with the boundary conditions, Eqs. (28) and (29), using $D_0 = y_E D_0$ with $D_0$ given by Eq. (52). Accordingly, the value of $y_E = 1$ furnishes the curve for the rate-independent limit $l_{c,\omega} = l_{c,\omega}(\xi)$, while higher stiffness ratios $y_E > 1$ lead to the corresponding limits $l_{c,\omega} = l_{c,\omega}(\xi)$, see Fig. 22. Observe that the limit buckling lengths monotonically decrease with increasing width $b$. Also, $l_{c,\omega}$ generally becomes larger for a higher stiffness ratio $y_E$, whereby it can be confirmed that the curves incorporate the specific limit values depicted in Fig. 21. The relatively strong variation in slope observed between wall aspect ratios of $0.65 < l_{c,r}/b < 1.5$ (indicated by dashed lines) characterises a change of the vertical buckling mode; simulations not presented here have shown that for relatively stocky walls, $l_{c,r}/b < 0.65$, the buckling mode indeed appears to be different from that for longer walls, $l_{c,r}/b > 1.5$, comparable to what has been sketched in Fig. 20 for a simply-supported wall printed under a linear curing process. Due to this mode change, the values of $l_{c,\omega}$ at moderate wall width are relatively high, resulting in a slower asymptotic behaviour of $l_{c,\omega}$ towards $l_{c,\omega}$ under increasing $\xi$, see Fig. 21.

4.4. Fully-clamped wall printed under linear and exponentially-decaying curing processes

The buckling response for a fully-clamped wall is characterised by substituting the values $n_w = n_w = 2$ and $c_w = c_w = 0.5$ given by Eq. (27) into the equilibrium equation, Eqs. (49) and (50), and the boundary conditions, Eqs. (54) and (55). In Fig. 23 the critical buckling
length $\bar{l}_c$ for a fully-clamped wall subjected to a linear curing process has been depicted as a function of the curing rate $\bar{\tau} = \frac{\tau}{\tau_e}$ considering wall widths in the range $5 \leq \bar{b} \leq 30$. Again, the curve for the free wall, which is in agreement with the case $\bar{b} \rightarrow \infty$, is plotted for comparison. For a fully-clamped wall the boundary conditions in $y$-direction constrain the buckling response more than for a simply-supported wall, so that the wall width $\bar{b}$ at which the response closely approaches that of a free wall is larger, i.e., compare the curve for $\bar{b} = 30$ in Fig. 23 to the curve for $\bar{b} = 20$ in Fig. 21. For the same reason, the critical buckling length $\bar{l}_{cr}$ for a fully-clamped wall is (substantially) larger than for a simply-supported wall. Observe further that at moderate wall width $\bar{b}$ $\leq \bar{b} \leq 11$, under an increasing value of $\bar{l}_{cr}$ the curves at some stage slightly decrease in the value of $\bar{b}$. This is again caused by a change in the vertical buckling mode under an increasing wall aspect ratio $\bar{b}_w/\bar{b}$, which, as characterised by the typical kink in the curve for $\bar{b} = 8$, at some higher value of $\bar{l}_{cr}$ reaches completion, after which $\bar{b}$ again monotonically rises with increasing $\bar{l}_{cr}$. The transition in buckling modes for $\bar{b} = 8$ is visualised in Fig. 24, by selecting the curing rates as $\bar{\tau} = \frac{\tau}{\tau_e} = 0$, 0.2 and 0.4.

Fig. 23 illustrates that the lower curing rates $\bar{\tau} = \frac{\tau}{\tau_e} = 0$ and 0.2 relate to $\bar{l}_{cr}/\bar{b} = 0.35$ and 0.44, respectively, for which the buckling mode in Fig. 24 is indeed rather different than that for the higher curing rate, $\bar{\tau} = \frac{\tau}{\tau_e} = 0.4$, whereby $\bar{l}_{cr}/\bar{b} = 1.1$. Notice that the buckling modes sketched in Fig. 24 are rather similar to those observed for the simply-supported wall with $\bar{b} = 4$ and 5, see Fig. 20.

The buckling curves for a fully-clamped wall subjected to an exponentially-decaying curing process are sketched in Fig. 25 for stiffness ratios $\gamma = \frac{\tau}{\tau_e} = 2$ (Fig. 25a), 5 (Fig. 25b), and 10 (Fig. 25c). The trends are comparable to those for the simply-supported wall shown in Fig. 22, but the values of the critical buckling length are noticeably higher. For example, for relatively short walls ranging from $5 \leq \bar{b} \leq 8$, the critical buckling length $\bar{l}_{cr}$ for the fully-clamped wall appears to be 2 to 10 times larger than for the simply-supported wall, where the exact increase is dependent on the specific values of the stiffness ratio $\gamma_e$ and curing rate $\bar{\tau}_e$.

Obviously, such differences also become manifest when comparing the curves for the limit values $\bar{l}_{cr,m}$ and $\bar{l}_{cr,s}$, in Fig. 26 to those for the simply-supported wall in Fig. 22. Fig. 26 illustrates that for the fully-clamped wall the range within which the vertical buckling mode changes (indicated by the dashed line) corresponds to $0.45 < \bar{l}_{cr} < 0.9$, for which a large part lies below the range deduced for the simply-supported wall, see Fig. 22. Hence, the range of wall widths characterised by a relatively slow asymptotic behaviour of the critical buckling length $\bar{l}_{cr}$ towards the limit value $\bar{l}_{cr,m}$, i.e., $9 \leq \bar{b} \leq 10$ ($\gamma_e = 2$), $12 \leq \bar{b} \leq 13$ ($\gamma_e = 5$), and $15 \leq \bar{b} \leq 16$ ($\gamma_e = 10$), see Fig. 25, lies somewhat above that reported in Section 4.3 for the simply-supported wall.

For the linear and exponential curing processes, the differences in the ranges in which a change in the vertical buckling mode takes place are rather small; this generally will more or less happen at $0.65 < \bar{l}_{cr}/\bar{b} < 1.5$ for a simply-supported wall and at $0.45 < \bar{l}_{cr}/\bar{b} < 0.9$ for a fully-clamped wall. It is further noted that the buckling behaviour of very stocky walls with high aspect ratios $\bar{l}_{cr}/\bar{b} >> 1$ may be characterised by additional, higher-order buckling modes; however, these modes are left out of consideration here, due to their limited practical relevance.

4.5 Influence of imperfections

The effect of imperfections on the buckling behaviour is explored by adopting the displacement decomposition, Eq. (70), and computing the buckling response in accordance with Eqs. (73) and (74). The imperfection sensitivity of the buckling response is presented here for a free wall printed under a linear curing process; for other wall types and curing processes similar characteristics were found. In terms of the Lagrangian coordinate $x$, the following kinematically admissible imperfection profile is considered:

$$u^{E,0}(x) = u^{E,0}_m \left( -\sin \left( \frac{2\pi}{n_i t_1} x \right) + \frac{2\pi}{\omega m t_1} \left[ 1 - \exp(-\omega x) \right] \right). \quad (119)$$

The above idealisation basically reflects a harmonic imperfection, with the exponential term warranting that the essential boundary conditions at $x = 0$ are satisfied, see Eq. (28). Further, $u^{E,0}_m$ is the amplitude of the imperfection, $t_1$ is the height of an individual printed layer, $n_i$ is the number of printed layers defining the wavelength $L$ of the imperfection profile, i.e., $L = n_i t_1$, see also Fig. 27, and $\omega$ is a factor quantifying the influence length of the exponential term at the boundary $x = 0$. Applying the coordinate transformation, Eq. (48), together with Eqs. (41) and (53), the imperfection profile, Eq. (119), can be expressed in terms of the dimensionless Eulerian coordinate $\bar{X}$:

$$\bar{u}^{E,0}(\bar{X}) = \bar{u}^{E,0}_m \left( -\sin \left( \bar{k}_w \left( \bar{X} + k \right) \right) + \pi \left[ 1 - \exp\left( -\frac{\bar{k}_w}{\bar{X}} \left( \bar{X} + k \right) \right) \right] \right). \quad (120)$$

with the dimensionless imperfection amplitude given by $\bar{u}^{E,0}_m = u^{E,0}_m / h$, and the dimensionless wavenumber $\bar{k}_w$ and boundary factor $\bar{K}$ in accor-
Fig. 25. Critical buckling length \( \bar{l}_c \) versus curing rate \( \bar{\tau} = \tau \omega \) for a fully-clamped wall printed under an exponentially-decaying curing process, see Fig. (44). The wall widths considered range from \( \bar{\tau} = 5 \) to 20. The buckling curve for a free wall (corresponding to \( \bar{\tau} \to \infty \)), has been taken from Fig. 16 and is plotted for comparison. Values for the limit buckling lengths \( \bar{l}_{c,0} \) at \( \bar{\tau} = 0 \) and \( \bar{l}_{c,\infty} \) at \( \bar{\tau} \to \infty \) are plotted for a selection of wall widths \( \bar{\tau} \), and are presented as a reference. The curing stiffness ratios considered are \( \gamma = E_f/E_0 = 2 \) (a), 5 (b) and 10 (c).

(b) Curing stiffness ratio \( \gamma = \gamma_E = E_\infty/E_0 = 5 \).

(c) Curing stiffness ratio \( \gamma = \gamma_E = E_\infty/E_0 = 10 \).

(a) Curing stiffness ratio \( \gamma = \gamma_E = E_\infty/E_0 = 2 \).

Fig. 26. Limit buckling lengths \( \bar{l}_{c,0} (\gamma = \gamma_E = 1) \) and \( \bar{l}_{c,\infty} (\gamma = \gamma_E = 2, 5 \) and 10) versus wall width \( \bar{b} \) for a fully-clamped wall printed under an exponentially-decaying curing process, see Eq. (44). The regime within which a change of the vertical buckling mode takes place corresponds to \( 0.45 < \bar{\tau} < 0.9 \), as designated by the dashed lines.

\[
\bar{\kappa}_w = \frac{2\pi}{n_1 \bar{\tau}^*} \quad \bar{\tau} = \frac{2\pi}{\omega_n},
\]

From the wavenumber \( \bar{\kappa}_w \), the dimensionless wavelength \( \bar{\lambda}_w \) of the imperfect profile follows as

\[
\bar{\lambda}_w = \frac{2\pi}{\bar{\kappa}_w} = \frac{n_1 \bar{\tau}^*}{\bar{\tau}}.
\]

The influence of the exponential term in Eq. (120) is kept limited by taking a relatively small value for the boundary factor, \( \bar{\tau} = 0.5 \). So, the imperfection is essentially characterised by the two length scale parameters \( \bar{\tau} \) (or \( \bar{\kappa}_w \)) and \( \bar{\lambda}_w \).

Fig. 28 illustrates the deflection \( w_c \) at the top \( X = 0 \) of a free wall printed under a linear curing process for two different imperfection amplitudes, \( \bar{\omega}_m = 0.01 \) (Fig. 28a) and 0.05 (Fig. 28b). The curing rate has been selected as \( \gamma = 2 \), and the wavenumbers considered for the sinusoidal imperfection profile are \( \bar{\kappa}_w = 1, 2 \) and 20, in correspondence with the dimensionless wavelengths \( \bar{\lambda}_w = 6.28, 3.14 \) and 0.314, see Eq. (122). Both for the small and large imperfection amplitudes the curves for the three different wavenumbers asymptote towards the critical bifurcation buckling length \( \bar{l}_{c,0} \) at 3.72 indicated by the dashed line (taken from Fig. 12). However, for the large imperfection amplitude \( \bar{\omega}_m = 0.05 \) the convergence occurs at a larger wall top deflection than for the small imperfection amplitude \( \bar{\omega}_m = 0.01 \). The convergence behaviour further depends on the wavenumber \( \bar{\kappa}_w \) (or wavelength \( \bar{\lambda}_w \)) of the imperfection profile; for the intermediate wavenumber \( \bar{\kappa}_w = 2 (\bar{\lambda}_w = 3.14) \) the bifurcation buckling length is reached at a relatively large wall top deflection, for the smallest wavenumber \( \bar{\kappa}_w = 1 (\bar{\lambda}_w = 6.28) \) this occurs at a smaller deflection, while for the largest wavenumber \( \bar{\kappa}_w = 20 (\bar{\lambda}_w = 0.314) \) the wall top deflection only starts to develop when the wall length \( \bar{\tau} \) is close to the bifurcation buckling length \( \bar{l}_{c,0} \). It may be concluded that the stage at which lateral deflections start to grow during the printing process is determined by the typical interplay between the two length scales \( \bar{\lambda}_w \) and \( \bar{l}_{c,0} \); for the three cases considered in Fig. 28 the wall appears to be most sensitive to the imperfection of intermediate wavelength \( \bar{\lambda}_w = 3.14 (\bar{\kappa}_w = 2) \), followed by the imperfection with the largest wavelength \( \bar{\lambda}_w = 6.28 (\bar{\kappa}_w = 1) \), and finally the imperfection with the smallest wavelength \( \bar{\lambda}_w = 0.314 (\bar{\kappa}_w = 20) \). In conclusion, when the imperfections generated during 3D printing have a relatively small amplitude and short wavelength, for the wall geometries studied in this communi-
cation the bifurcation buckling length generally serves as an adequate design value.

### 4.6. Effect of coefficient of lateral stress

In the previous numerical analyses the coefficient of lateral stress appearing in the equilibrium equation, Eqs. (49) and (50), was taken as constant and set equal to the Poisson’s ratio, \( K_{yy} = \nu = 0.3 \). This is in correspondence with a wall that is fully constrained in the lateral direction, \( \epsilon_{yy} = 0 \), see Eq. (19). For a fully-clamped wall of width \( b = 10 \) printed under a linear curing process the effect of the coefficient of lateral stress on the buckling length is illustrated in Fig. 29 by showing \( \tilde{h}_{c} \) as a function of the curing rate \( \xi = \frac{\nu}{\nu} \) for the whole range of possible values of \( K_{yy} \), i.e., \( 0 \leq K_{yy} \leq 0.3 \). Here, the curve for \( K_{yy} = \nu = 0.3 \) has been taken from Fig. 23. For curing rates lower than \( \xi_{\nu} = 0.5 \) the differences between the individual curves appear to be small, such that the influence of \( K_{yy} \) on the value of the critical buckling length is only minor. At higher curing rates, \( 0.5 < \xi_{\nu} \leq 1.1 \), however, the slopes of the curves become steeper, whereby the critical buckling length \( \tilde{h}_{c} \) becomes more dependent of the specific value of \( K_{yy} \). Under a decreasing value of \( K_{yy} \) from \( K_{yy} = \nu = 0.3 \) to 0 the lateral constraint at the boundaries in y-direction decreases, see Eq. (19), by which the critical buckling length increases. Hence, the curve for \( K_{yy} = \nu \) provides a lower bound for the critical buckling length, so that the numerical results presented in this communication may be considered as conservative, and from a design viewpoint on the safe side. It is further expected that the level of lateral wall constraint in practical wall structures often is such that the coefficient of lateral stress is relatively close \( K_{yy} = \nu \). As an example, for the rectangular wall structure introduced in Section 3.1.5 a basic linear elastic 3D finite element analysis of the response under dead weight loading has demonstrated that the value of the coefficient of lateral stress for the case \( \nu = 0.3 \) typically falls within the range \( 0.2 < K_{yy} \leq 0.3 \) for most locations in the structure.

### 5. Experimental validation

The experimental validation of the modeling framework is carried out by considering two types of geometries constructed with 3D concrete printing, namely i) a free wall, and ii) a rectangular wall lay-out, see Fig. 30. The 3D concrete printing process is based on an extrusion-based technique similar to fused deposition modelling, whereby the viscous cementitious material is extruded from a nozzle to build the wall layer-by-layer along a calculated path. The curing process of the cementitious material occurs at room temperature, without the use of an external heat source. The experiments were performed using the 3D concrete printing facility at the Eindhoven University of Technology, see Fig. 31. The custom designed concrete applied in the 3D printing experiments is composed of Portland cement (CEM I 52.5 R), siliceous aggregate with a maximal particle size of 1 mm, limestone filler, rheology modifiers, ad- ditives and a small quantity of polypropylene fibres [8]. After the composition was mixed with water into a homogeneous viscous substance, the material was pumped through a hose towards the printer head, at which it was discharged from the printing nozzle to form a layer. The path followed by the printer head was governed by a motion-controlled gantry robot with 4 degrees of freedom, i.e., 3 mutually perpendicular translations and 1 rotation about the vertical axis. The determination of adequate printing process parameters, such as the concrete viscosity, the printing velocity, the pump pressure, the height of the printer head above a printed layer, the printing rotation angle, and the characteristics of the nozzle opening, was done by performing an extensive preliminary test program, see [8] for more details.

#### 5.1. Free wall

The process parameters applicable to the printing of a free wall of width \( b = 800 \) mm are listed in Table 1. With these parameters
the material volume discharged per unit time equals \( Q = \frac{\nu_c h t_f}{t_f} = 33340 \text{ mm}^3/\text{s} \), and the period for the printing of an individual material layer is \( T = b/\nu_c = 9.6 \text{ s} \). Inserting these values into Eq. (39), the wall growth velocity can be calculated as \( I = 0.958 \text{ mm/s} \).

The development of the strength and stiffness properties of the concrete under curing was measured by means of uniaxial compression tests on cylindrical concrete specimens with a diameter of 70 mm and a height of 140 mm, in accordance with the ASTM D2166 [20]. Correspondingly, the yield strength of the printing material is assumed to be determined by compressive failure, in accordance with Eq. (77). Specimens were prepared at 5 different curing times (0, 15, 30, 60 and 90 min), and were loaded displacement-controlled in an Instron test rig by applying a loading rate of 30 mm/min that mimics the experimental printing velocity. At each curing level 4 to 6 different specimens were tested to account for the statistical spread in material properties, see [21] for more details.

Fig. 28. Deflection \( \xi^* \) at the top (\( X = 0 \)) of a free wall printed under a linear curing process. The curing rate equals \( \xi^*_c = 2 \) and the wavenumbers of the sinusoidal imperfection profiles are \( k_w = 1, 2 \) and 20, in correspondence with the wavelengths \( L = 6.28, 3.14 \) and 0.314, respectively. The dashed line indicates the bifurcation buckling length for the ideally straight wall, in agreement with Fig. 12. The imperfection amplitudes selected are \( \bar{w}_{m}^{0.01} = 0.01 \) (a) and \( \bar{w}_{m}^{0.05} = 0.05 \) (b).

Fig. 29. Critical buckling length \( \tilde{\xi} \) versus curing rate \( \tilde{\xi} = \xi^*_c \) for a fully-clamped wall of width \( b = 10 \) printed under a linear curing process, see Eq. (43). The values of the coefficient of lateral stress have been selected in the range \( 0 \leq K_{n} \leq 0.3 \). The curve for \( K_{n} = 0.3 \) has been taken from Fig. 23.

Fig. 30. Geometrical characteristics of the free wall and rectangular wall-lay-out printed in the experiments.

Fig. 31. 3D concrete printing facility at the Eindhoven University of Technology, incorporating a 4-axis gantry robot, a control unit, and a concrete mixer and pump (inset).
The evolution of the elastic stiffness (in MPa) as a function of the curing time (in min.) has been determined by applying a least-squares procedure on the experimental data, see Fig. 32a. Adapting a best fit with $R^2 = 0.995$ on the average values of the measured stiffness modulus provides the linear relation:

$$\hat{E}_u(t) = 0.0781 + 0.0012t \quad \text{with} \quad E_u \text{ in MPa and } t \text{ in min.} \quad (123)$$

A similar procedure was followed for the determination of the compressive yield strength (in kPa) as a function of the curing time (in min.), see Fig. 32b. Using a best fit with $R^2 = 0.991$ on the average strength values results in the following linear relation for the average compressive yield strength:

$$\hat{\sigma}_{py}(t) = 5.984 + 0.147t \quad \text{with} \quad \sigma_{py} \text{ in kPa and } t \text{ in min.} \quad (124)$$

The value of the density $\rho$ presented in Table 1 appeared to be more or less insensitive to the curing time [21].

For the modelling of the experimental buckling behaviour it is reasonable to assume that the amplitudes of the imperfections in the printed wall structure are small (i.e., less than 5% of the wall thickness) and have a relatively short wavelength (i.e., smaller than 10% of the critical buckling length). Under these circumstances the buckling behaviour can be accurately determined from a bifurcation analysis, see Fig. 28. Accordingly, after combining Eq. (123) with Eqs. (42) and (43), the initial stiffness modulus of the printing material follows as $E_0 = 0.0781$ MPa, and the curing rate of the stiffness modulus becomes $\dot{E}_u = 0.0012/0.0781 = 0.0154 \text{ min}^{-1} = 2.6 \times 10^{-4} \text{ s}^{-1}$. With a Poisson’s ratio of $\nu = 0.3$ the initial bending stiffness can be calculated from Eq. (52) as $D_0 = 0.589 \text{ Nm}$. In a similar way as done for the stiffness, combining Eq. (124) with Eqs. (81) and (82) results in an initial yield strength $[\sigma_{py}] = 5.984 \text{ kPa}$ and a strength curing rate $\dot{\sigma}_{py} = 0.147/5.984 = 0.0246 \text{ min}^{-1} = 4.1 \times 10^{-4} \text{ s}^{-1}$.

With Eqs. (75), and (91), the dimensionless values of the two curing rates become $\tilde{\tau}_{Es} = 0.024$ and $\tilde{\tau}_{Ep} = 0.129$, respectively, leading to the ratio $\tilde{a} = \frac{\tilde{\tau}_{Ep}}{\tilde{\tau}_{Es}} = 5.4$. From these values, the length scale ratio $\tilde{l}\tilde{c}_{rs}/\tilde{l}\tilde{c}_{rp}$ can be read off from Fig. 15 as $\tilde{l}_{cr}/\tilde{l}_{pr} \approx 1.8$. This value is smaller than the value $N = 3.43$ calculated with Eq. (95), from which it is concluded that the free wall fails by elastic buckling. The dimensionless buckling length results from substituting the above value for $\tilde{\tau}_{Es}$ into Eq. (112), which renders $\tilde{l}_{cr} = 2.04$. The actual buckling length then follows from Eq. (75), as $\tilde{l}_{cr} = 0.179 \text{ m}$. This value lies 13% below the experimental buckling length of $l_{cr} = 0.202 \text{ m}$ (corresponding to 22 printed layers).

Note that the accuracy of the model prediction is established up to a deviation equal to the height $t_0$ of an individual layer, since in the experiments the critical buckling length is determined by the integer number of layers under which the wall starts to buckle, while in the model the printing process is considered as continuous, so that the critical buckling length generally does not correspond to an integer number of layers. This effect can be accounted for by rounding up the model prediction to an integer number of layers, which leads to $l_{cr} = 0.184 \text{ m}$ (corresponding to 20 printed layers), a value that lies 10% below the experimental value. This model prediction may be considered as accurate, considering the significant spread in the experimental values of the stiffness modulus and compressive yield strength, see Fig. 32. In specific, the relative standard deviation of both $E$ and $\sigma_{py}$ fluctuates between 13% and 21% for the different curing times tested. The test data has been reproduced from [21], with kind permission of the authors.

The rectangular wall lay-out has dimensions $b = 625 \text{ mm}$ and $d = 250 \text{ mm}$ and was printed with the same process parameters as applied for the free wall, see Table 1. In order to maintain an ideally constant printing velocity during the printing process, the corners of the rectangular wall lay-out were slightly rounded off with a radius $r = 50 \text{ mm}$, see Fig. 30. As a first step, the upper and lower bounds of the critical buckling length of the wall structure are determined from the design graphs presented in Section 4 for the simply-supported and fully-clamped walls. As explained in Section 3.1.5, the simply-supported wall is representative of global buckling of the complete rectangular structure, and thus furnishes a lower bound for the critical buckling length, while the fully-clamped wall reflects local buckling of the wall $b$, thereby resulting in an upper bound.

The period for the printing of an individual layer is larger than for the free wall, and equals $T_0 = 2(b+2d)/v = 210 \text{ s}$. Inserting this value together with the value $Q = 33340 \text{ mm}^2$/s and the values for $v_0$ and $h$ listed in Table 1 into Eq. (39) results in a wall growth velocity $\dot{t} = 0.438 \text{ mm/s}$. With $D_0 = 0.589 \text{ Nm}$ and $\tau_{Es} = 2.6 \times 10^{-4} \text{ s}^{-1}$, Eq. (75), then leads to $\tilde{\tau}_{Es} = 0.052$. Further, the wall width $b = 625 \text{ mm}$ via Eq. (75) provides the dimensionless value $\tilde{b} = 7.10$. Using the curve for $\tilde{b} = 7$ in
Figs. 19 (simply-supported wall) and 23 (fully-clamped wall), it follows that the corresponding dimensionless buckling lengths at the curing rate \( \frac{\nu}{\sigma_p} = 0.052 \) are equal to \( \bar{l}_{c.r} = 2.38 \) and 4.89, respectively. With Eq. (75)\(_1\), this results in the actual buckling lengths \( l_{c.r} = 0.210 \) m and 0.431 m, respectively. In addition, the length at which plastic collapse occurs can be determined from Eq. (92), which, with \( \frac{\nu}{\sigma_p} = 0.282 \) being calculated via Eq. (91)\(_2\), leads to \( \bar{l}_p = 1.39 \). With an initial strength of \( |\sigma_{p,0}| = 5.984 \) kPa, via Eq. (91)\(_1\), the actual plastic collapse length is determined as \( l_p = 0.421 \) m. Hence, the actual buckling length for the rectangular wall lay-out falls within the range 0.210 m \( \leq l_{c.r} \leq 0.421 \) m, with the upper bound being determined by plastic collapse (instead of elastic buckling of the fully-clamped wall) and the lower bound following from elastic buckling of the simply-supported wall.

Although for design purposes the above range for the critical buckling length may be acceptable, the model prediction can be further refined by applying the expressions presented in Section 3.1.5. From the aspect ratio \( h/f = 625/250 = 2.5 \) and the stiffness ratio \( D_f/D_p = 1 \) (i.e., \( h^* = h \)), the number of half-waves at infinite vertical distance from the clamped wall bottom is obtained from Eq. (57) as \( n_{a,m} = 1.429 \). Subsequently, the value for the dimensionless asymptotic rotational stiffness results from Eq. (58) as \( \bar{\kappa}_{c.r} = 3.55 \). Inserting this value into Eq. (66) provides the spatial variation of the rotational stiffness \( \hat{\kappa}(\bar{X}) \) at the connection between walls \( b \) and \( d \), with \( \hat{F}(\bar{X}) \) given by Eq. (69). This rotational stiffness is inserted into Eqs. (37) and (38), leading to the functions \( h^a_m(\bar{X}) \) and \( \hat{\kappa}_c(\bar{X}) \) that serve as input for the equilibrium equation, Eq. (49)\(^1\) and the natural boundary conditions, Eq. (55). Due to the complexity of the present problem, the solution of the equilibrium equation can only be obtained in a purely numerical fashion. This is done by computing the integral expression in Eq. (73) by means of Gaussian quadrature, whereby the number of Gauss points \( \bar{X} = X \) is set equal to 12, i.e., \( r \in \{ 1.2, ... , 12 \} \). For the simpler cases of a simply-supported wall and a fully-clamped wall, simulations not presented here have shown that with this number of Gauss points the numerical result of Eq. (73) is virtually identical to the analytical result. Accordingly, with the dimensionless wall width \( b = 7.10 \) the numerical integration of Eq. (73) together with the boundary conditions, Eq. (54) and (55), results for \( \bar{\kappa}_c = 0.052 \) in a dimensionless critical buckling length \( \bar{l}_{c.r} = 3.05 \). Substituting this value into Eq. (75)\(_1\) furnishes the actual critical buckling length as \( l_{c.r} = 0.269 \) m. This value indeed falls within the lower and upper bounds calculated above, and lies 13% below the experimental buckling length \( l_{c.r} = 0.304 \) m (corresponding to 33 printed layers). Rounding up the model prediction to an integer number of layers leads to \( l_{c.r} = 0.276 \) m (corresponding to 30 printed layers), which underestimates the experimental buckling length by 10%.

The experimental deformation profiles of wall \( b \), monitored at the onset of buckling and during buckling, are shown in Fig. 34a and b, respectively. The buckling mode in Fig. 34b can be compared to the critical buckling mode calculated by the model, which follows from Eq. (96). Because of the non-uniform boundary conditions at \( y = 0 \) and \( y = b \), the coefficients \( C_n \) in Eq. (96) are dependent of the \( \bar{X} \)-coordinate, \( C_n = C_n(\bar{X}) = C_n(\hat{\kappa}_c(\bar{X}), h^a_m(\bar{X})) \), and therefore need to be calculated for each Gauss point separately. The critical buckling mode computed with the model is depicted in Fig. 35 by plotting the normalised deflections \( \bar{w} \) at the 12 Gauss points, and connecting these data by straight lines. It can be concluded that the buckling mode is in good qualitative correspondence with the experimental buckling mode depicted in Fig. 34b.
showing limited deflections at the lower part of the wall and large deflections along an approximately constant inclination at the upper part. As an overview, Table 2 summarises the values for the critical buckling length $l_c$, measured experimentally and computed by the present model. The model predictions have been subjected to an additional verification procedure, by comparing these to 3D finite element analyses of the printing process of the free wall and the rectangular wall lay-out. The FEM results turned out to be in excellent agreement with the predicted buckling lengths in Table 2 and the corresponding buckling modes presented in Figs. 13 and 35, the details of which will be presented in a forthcoming communication. Despite the high accuracy of the modelling approach, both for the free wall and the rectangular wall lay-out the values computed for the critical buckling length lie somewhat below the experimental values. As already mentioned, this is likely to be due to the significant spread in the measured material properties, see Fig. 32. It has been validated that the material parameters needed for a highly accurate prediction of the experimental buckling lengths presented in Table 2 indeed fall within the spread of values illustrated in Fig. 32. Accordingly, it may be concluded that the accuracy of the model predictions will further increase when the spread in the measured material properties is reduced. A possible way to accomplish this is to investigate the application of alternative compositions of the printing material with more uniquely defined material properties. This experimental task, however, is considered as a topic for future research.

Table 2
Experimental value and model prediction of the critical buckling length $l_c$ of a free wall and a rectangular wall lay-out. The values between parentheses represent model predictions (and the corresponding relative differences with the experimental values) whereby the buckling length is rounded up to an integer number of layers. An upper-lower bound approximation of the critical buckling length of the rectangular wall lay-out can be established by using the design graphs presented in Section 4, which results in $0.210 \text{ m} \leq l_c \leq 0.421 \text{ m}$.

<table>
<thead>
<tr>
<th>Critical buckling length $l_c$</th>
<th>Experiment</th>
<th>Model prediction</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free wall</td>
<td>0.202 [m]</td>
<td>0.179 (0.184) [m]</td>
<td>12 (10) %</td>
</tr>
<tr>
<td>Rectangular wall lay-out</td>
<td>0.304 [m]</td>
<td>0.269 (0.276) [m]</td>
<td>13 (10) %</td>
</tr>
</tbody>
</table>

Fig. 35. Critical buckling mode computed for a rectangular wall lay-out of $b = 625 \text{ mm}$ and $d = 250 \text{ mm}$ printed under a linear curing process, see Eq. (43), with $\psi_c = 0.012$. The black dots indicate the normalised deflections of the 12 Gauss points at the centre line $y = b/2$ of wall $b$.

iii) a fully-clamped wall, printed under either linear curing or exponential curing, have been summarised in design graphs and failure mechanism maps. As demonstrated for a rectangular wall lay-out, the design graphs and failure mechanism maps furnish a practical tool for studying the performance of arbitrary wall structures under a broad range of possible printing process conditions. Here, the simply-supported wall provides a lower bound for the wall buckling length, corresponding to global buckling of the complete wall structure, while the fully-clamped wall gives an upper bound, reflecting local buckling of an individual wall. The range of critical buckling lengths defined by these bounds may be further narrowed by the critical wall length for plastic collapse. For arbitrary wall configurations an accurate model prediction for the critical buckling length and corresponding buckling mode can be obtained by deriving an expression for the non-uniform rotational stiffness provided by the support structure of a buckling wall. This has been demonstrated for a rectangular wall lay-out, which has proven to give a good agreement with the experimental buckling response of a wall structure manufactured with 3D printed concrete.

As future work, the optimisation of 3D printing processes needs to be studied further by means of additional comparisons between model predictions and experimental results performed at different printing velocities, curing characteristics, printing material properties and wall geometries. The present model can be applied to systematically and efficiently explore the influence by each of the 5 independent printing process parameters, Eqs. (75) and (91), on the failure resistance of wall structures, which should lead to clear directions for the optimisation on printing time and material usage. In addition, the model may be utilised as a validation tool for finite element models of wall structures printed under specific process conditions.

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Appendix A. Derivation of the equivalent translational stiffness for a cantilever wall segment

The derivation of the equivalent translational stiffness $\tilde{k}_e(\tilde{X})$, presented in Eq. (67), from the bending response of a cantilever wall segment $\bar{s}1$ is performed as follows. Segment $\bar{s}1$ with $\bar{X}$-coordinate $-k \leq \bar{X} \leq 0$ and unit width in the $y$-direction is printed under a linear curing process and subjected at $\bar{X} = \bar{X}_1$ to a dimensionless, uniform line load $\bar{q}_1$ acting in the out-of-plane direction $z$, see Fig. 10. For this bending problem the essential boundary conditions at $\bar{X} = -k$ are

$$\bar{w}_{x}^{1} = 0,$$
$$\bar{w}_{xx}^{1} = 0,$$
(A.1)

while the natural boundary conditions at $\bar{X} = \bar{X}_1$ are

$$\bar{w}_{xx}^{1} = 0,$$
$$\bar{q}_1 \bar{w}_{x}^{1} \bar{X}_1 = \bar{q}_1,$$
(A.2)
where $\tilde{w} = \tilde{w}(X)$ is the dimensionless deflection of the cantilever wall segment, i.e., $\tilde{w} = w / h$, with $h$ the wall thickness, and the abbreviation $s1$ indicating “segment 1”. By accounting for the curing process $\delta$, the equilibrium equation becomes relatively simple, such that the deflection $\tilde{w}$ of the wall segment can only be determined in an approximate fashion. This is done by solving the weak form of the equilibrium equation, which, based on Eq. (49), for the present bending problem turns into

$$\int_{\tilde{X}_{n-k}}^{\tilde{X}_k} \left( \tilde{R} \delta \tilde{w} \right) d\tilde{X} = 0, \quad (A.3)$$

with the residual $\tilde{R}$ as

$$\tilde{R} = \tilde{R}(\tilde{X}) = \left( \tilde{\Psi}_1 \frac{\partial \tilde{w}_n}{\partial \tilde{X}} \right)_X$$

and $\delta \tilde{w}_n$ representing the test function. The approximate solution adopted for solving Eq. (A.3) is of the polynomial form

$$\tilde{\Psi}(\tilde{X}) = \sum_{n=1}^{N} C_n \tilde{w}^{n-1} \quad \text{with} \quad n = 1, 2, \ldots, N, \quad (A.5)$$

whereby a polynomial function of the fifth degree ($N = 6$) has proven to describe the bending response with sufficiently high accuracy. The generalised coordinates $C_n$ in Eq. (A.5) are calculated by substituting Eq. (A.5) into the weak form of equilibrium, Eq. (A.3), and the boundary conditions, Eqs. (A.1) and (A.2), after which the translational stiffness $\bar{k}_i$ at $\tilde{X} = \tilde{X}_p$ is computed as

$$\bar{k}_i(\tilde{X}_p) = \frac{\bar{q}_i}{\tilde{w}^{(1)}(\tilde{X}_p)} = \frac{\bar{q}_i}{\tilde{w}^{(1)}(\tilde{X}_p)} = \left\{ \begin{array}{l}
30(\tilde{X}_p - 12\tilde{X}_p^2 + 15\tilde{X}_p^3 - 14\tilde{X}_p^4 + 42\tilde{X}_p^5 + 28) \\
[-73\tilde{X}_p^2 - 15\tilde{X}_p^3 + 900\tilde{X}_p^4 - 530\tilde{X}_p^5 + 195\tilde{X}_p^6 - 3\tilde{X}_p - 350\tilde{X}_p^2 - 480\tilde{X}_p^3 - 280\tilde{X}_p^4 - 210\tilde{X}_p^5 - 280\tilde{X}_p^6 - 840\tilde{X}_p^7 - 840\tilde{X}_p^8 - 280\tilde{X}_p^9 - 280\tilde{X}_p^{10})]^{-1}
\end{array} \right. \quad (A.6)$$

The above expression holds at any point $\tilde{X} = \tilde{X}_p$ within the wall dimension $-\kappa \leq \tilde{X} \leq 0$, so that in Eq. (A.6) the specific coordinate $\tilde{X}_p$ may be replaced by $\tilde{X}$, after which Eq. (67) is obtained.

References

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