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A quantitative assessment of the scale separation limits of classical and higher-order asymptotic homogenization

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Abstract

Classical homogenization techniques are known to be effective for materials with large scale separation between the size and spacing of their underlying heterogeneities on the one hand and the structural problem dimensions on the other. For low scale separation, however, they generally become inaccurate. This paper assesses the scale separation limit of classical asymptotic homogenization applied to periodic linear elastic composite materials and demonstrates the effectiveness of higher-order homogenization in stretching this limit. A quantitative assessment is performed on a two-dimensional elastic two-phase composite consisting of stiff circular particles in a softer matrix material and subjected to anti-plane shear, as introduced by Smyshlyaev and Cherednichenko (J. Mech. Phys. Solids 48:1325–1358, 2000). Reference solutions are created rigorously using full-scale numerical simulations in which a family of translated microstructures is considered and the ensemble average of their solutions is defined as the homogenized solution. This solution is used as a reference, which is compared with the periodic homogenization solution for a range of scale ratios. It is shown that the zeroth-order classical homogenization solution significantly deviates from the exact solution below a certain scale ratio for a given microstructure. Below this limit, the higher-order solutions provide a clear improvement of the match. Further, the performance of classical and higher-order asymptotic homogenization solution are evaluated for varying stiffness contrast ratio between the two phases of the microstructure and error contours are presented by comparison with full-scale numerical simulations.

Keywords: Homogenization, multiscale problems, scale separation, higher-order asymptotic homogenization

1. Introduction

All matter is heterogeneous at some scale, but for modeling at engineering scales it is convenient to replace it by an equivalent homogeneous material. Some of the well-known examples are metal alloys, concrete, foams, fibrous composites and filled polymers. The distinct phases (or orientations) of their microstructures may respond quite differently to mechanical loading and hence their deformation is heterogeneously distributed at the fine scale. However, it is the combination of the different microstructural phases which governs the overall response of the material to the loading.

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Mathematical homogenization is a technique for studying partial differential equations with rapidly oscillating coefficients, which are typical of the equations that govern the physics of heterogeneous materials. An important aspect in the analysis of multi-phase materials is to deduce their effective behavior (e.g. mechanical stiffness, thermal expansion properties, electrical and magnetic properties, etc.) from the corresponding phase behavior and the geometrical arrangement of the phases. This concept of rendering “homogeneous” a heterogeneous material is generally called homogenization.

Among the vast amount of work on homogenization available in the literature, one of the most rigorous approaches is asymptotic homogenization – also called as periodic or mathematical homogenization, or mathematical averaging. It was introduced by Bensoussan et al. [1] and Sanchez-Palencia [2]. Other classical contributions have been made in References [3, 4]. Comprehensive reviews of its application in the mechanics of periodic materials may be found in References [5–7]. The mathematical theory of asymptotic homogenization makes use of asymptotic expansions of the field variables of the full-scale problem to split it into separate microstructural and macrostructural problems. Solving the former allows one to compute the effective properties of the equivalent homogeneous medium, which appear in the macroscale problem. The approach has had implications in various applications. Some of the significant ones include the work of Ghosh et al. [8] which combined asymptotic homogenization theory with the Voronoi Cell finite element model to develop a multiple scale finite element model for elastic-plastic analysis of heterogeneous materials. Yu and Tang [9] developed the variational asymptotic method for unit cell homogenization to predict the effective properties of periodically heterogeneous materials and recover the local fields. Some of the other applications are given in References [10–14].

Conventional homogenization methods are based on a separation of scales, given by \( l \ll L \), where \( l \) represents the typical length scale characteristic of the microstructural heterogeneity and \( L \) the macroscopic length scale. If scale separation holds, i.e. the microstructure consists of relatively small heterogeneities, classical homogenization gives an adequate estimate of the average macroscopic properties. However, if the size of the heterogeneities is of the same order of magnitude as that of the macroscopic problem, most of the classical homogenization schemes are expected to break down. The requirement of scale separation constitutes one of the key limitations of classical homogenization, including asymptotic homogenization in its most common, “zeroth-order” form. A number of methods have been suggested in the literature to alleviate the above restriction on the separation of scales. They generally result in some form of nonlocality of the macroscopic, homogenized equations, either in integral form or as a result of higher-order gradients of the field variables. At the same time, and as a consequence, the microstructural length scale \( l \) appears as an intrinsic length scale in the macroscopic equations. The response of such theories is thus scale-dependent, i.e. predictions depend on the scale ratio \( L/l \). Gambin and Kröner [15] applied the method of two-scale asymptotic expansions to the displacement field in periodic elastic structures and explored the nonlocal contributions in the macroscopic stress–strain relation in the form of strain gradients up to infinite order. Boutin [16] studied microstructural effects of periodic elastic composites. It was thereby shown that the higher-order terms introduce successive gradients of macroscopic strain and tensors characteristic of the microstructure, which result in nonlocal effects. Fish and Chen [17] studied higher-order homogenization of one-dimensional initial and boundary value problems with oscillatory coefficients. Smyshlyaev and Cherednichenko [18] proposed an extension of the classical asymptotic homogenization to provide a rigorous link between the spatial scale of the heterogeneity and the intrinsic length scale of a higher-order continuum. Higher-order
homogenized equations were rigorously derived for an infinitely extended periodic elastic medium via a combination of variational and asymptotic techniques. The higher-order effective constitutive relations are in agreement with those proposed by phenomenological strain gradient theories. Peerlings and Fleck [19] extended this method to three-dimensional elasticity and computed the coefficients of the effective medium for an example problem. Drugan and Willis [20] studied the homogenization of random linear elastic composite materials and showed that for two-phase composites with a statistically uniform distribution of phases, the leading-order correction term to the macroscopically homogeneous constitutive equation is proportional to the second gradient of the ensemble average of strain. This corrected nonlocal constitutive equation was used to show that, for any volume fraction of the particles, the minimum RVE size is at most twice the particle diameter for a maximum error of 5% of the constant overall modulus tensor.

The present work critically investigates the scale separation limit of classical homogenization and the effectiveness of higher-order asymptotic homogenization in pushing this limit to smaller scale separations. The mathematical literature provides rigorous error bounds for classical homogenization, as well as for the higher-order correctors – see e.g. [5, 18] and references therein. Whereas the scaling of these bounds with the scale ratio is universal, the constants featuring in them generally can only be computed in closed form for simple microstructures such as laminates (e.g. [16]). This does not necessarily give engineers a good idea of the magnitude of the effect of scale ratio to be expected in a realistic setting, and of the ability of the higher-order terms to better capture this effect. The present contribution aims to address this question by applying the classical and higher-order method to a two-phase microstructure and making a comparison with a rigorous full-scale numerical solution. The authors are unaware of a similar comparison against such full-scale simulations.

The material considered in our study is an elastic two-phase composite consisting of stiff particles embedded in a softer matrix. The particle size and spacing is varied, maintaining the same volume fraction. The particle spacing defines the microstructural length scale \( l \). Following Smyshlyaev and Cherednichenko [18], an infinite volume of this composite is subjected to a periodic body force, with a constant wave length \( L \) which represents the macroscopic length scale. The accuracy of the homogenized equations is analyzed as \( L \) approaches \( l \) – or vice versa. For this purpose, reference solutions are rigorously generated using full-scale numerical simulations, by considering an entire family of translated microstructures and then applying an ensemble average to define the homogenized solution. Such reference homogenized solutions are created for scale ratios varying from 1 to 10 and their peak displacement is plotted against the scale ratio. On the other hand, classical and higher-order asymptotic homogenization are applied and their predictions of the peak displacement are compared against the numerical reference solutions. The limits of classical homogenization are analyzed and the contribution of each higher-order term in addressing this limitation is identified. The results provide a clear, quantitative understanding of when, for the elastic two-phase composite considered, the classical homogenization breaks down and how, and to which degree, the higher-order terms in the asymptotic homogenization framework can rectify this.

The paper is divided into five sections. The next section defines the problem to be studied, including the material properties, geometry and applied loading. It also describes the methodology adopted for this work. Section 3 details the reference solutions using full-scale numerical simulations. Section 4 deals with the asymptotic homogenization procedure and compares the predictions obtained using it with the numerically generated reference solutions. We close the paper with conclusions resulting from the present analysis in Section 5.
2. Problem description

The problem which we consider fits in the class of problems considered by Smyshlyaev and Cherednichenko [18]: an infinite, periodic heterogeneous linear elastic medium in two dimensions which is loaded by a periodic body force in anti-plane shear – see Figure 1 for a sketch of the problem. The period of the microstructure gives us the microstructural length scale \( l \) and the period of the body force, \( L \), is identified as the macroscopic length scale; we hence generally expect that \( l < L \). If in addition we assume \( L \) to be a multiple of \( l \), i.e. \( L/l \in \mathbb{N} \), our analysis can be limited to a single period of the macro-problem.

![Figure 1: Sketch of one period of the infinite two-dimensional problem considered: an out-of-plane periodic body force, with period \( L \), is applied to a two-dimensional periodic microstructure with period \( l \), resulting in anti-plane shear displacement. In the particular case shown, the scale ratio equals \( L/l = 10 \).](image)

The microstructure considered here is inspired by a two-phase composite consisting of isotropic stiff particles embedded in a softer, isotropic matrix. However, the stiffness distribution is regularized to be smooth in order to allow us to solve the problem on a regular computational grid. The data and results are non-dimensionalized.

2.1. Governing equation

The partial differential equation which governs equilibrium of the material can be written as

\[
\nabla \cdot \left( G(Lx/l + \zeta) \nabla u \right) + F(x) = 0
\]

(1)

In this equation, \( x = (x_1, x_2) \) denotes Cartesian coordinates in the plane of the problem; they have been normalized by the period \( L \) of the body force \( F \). This implies that we can limit our analysis to \( x \in (0, 1) \times (0, 1) \). The unknown \( u(x) \) is the out-of-plane displacement, again non-dimensionalized by a characteristic displacement which we will establish later, in Section 2.3. The function \( G(y) \) characterizes the shear modulus (i.e. stiffness) distribution in terms of normalized coordinates \( y = \)
The region \( Q = (0, 1) \times (0, 1) \) defines the periodic unit cell of the microstructure; beyond this unit cell the function \( G(y) \) is periodically extended according to \( G(y_1 + i, y_2 + j) = G(y_1, y_2) \) with \( i, j \in \mathbb{Z} \). The multiplication by \( L/l \) of its argument \( x \) in (1) turns it into an \( l/L \)-periodic stiffness distribution in terms of \( x \). The constant vector \( \zeta \in Q \) defines the position (or “phase”) of the microstructure relative to the applied body force. This vector for now is arbitrary, but it will be used below, in Section 2.4, to define the family of problems that needs to be considered to arrive at the homogenized solution. The gradient operator \( \nabla = (\partial/\partial x_1, \partial/\partial x_2) \) implies differentiation with respect to the normalized coordinates \( x; \nabla \cdot \) denotes the divergence. The solution of (1) is made unique by requiring it to satisfy the periodicity conditions \( u(x_1, 1) = u(x_1, 0) \) and \( u(1, x_2) = u(0, x_2) \) as well as to have mean zero in the unit cell, cf. Reference [18].

Equation (1) immediately highlights the two-scale character of the problem: the applied loading \( F \) varies at the macroscopic length scale unity in terms of the normalized coordinates \( x \) (or \( L \) in the corresponding dimensionful coordinates), whereas the stiffness \( G \) varies at the microstructural scale \( l/L \) (or \( l \)), which is generally expected to be smaller than one. The displacement solution \( u(x) \) is expected to reflect both influences and hence to have a “slow” variation induced by the body force and a superimposed “fast” variation due to the microstructure. We will refer to the ratio \( (L/l) \) as the scale ratio.

### 2.2. Microstructure considered

The microstructural stiffness distribution is inspired by a particle-reinforced matrix, where the matrix has shear modulus \( G_m \) and the particle \( G_p \). Unless indicated otherwise, the particle is circular, with radius \( r \) in terms of the normalized coordinates \( y \). The transition between the two moduli is regularized by a cubic interpolation on a thin interphase. Normalized by the matrix modulus \( G_m \), this renders a smooth stiffness distribution \( G(y) \), which is defined on the unit cell \( Q \) as

\[
G(y) = \begin{cases} 
\frac{G_p}{G_m} & \text{if } \rho < a \\
1 + \left( \frac{G_p}{G_m} - 1 \right) \left[ 1 - 3 \left( \frac{\rho - a}{b - a} \right)^2 + 2 \left( \frac{\rho - a}{b - a} \right)^3 \right] & \text{if } a \leq \rho \leq b \\
1 & \text{if } \rho > b 
\end{cases}
\]  
(2)

where

\[
\rho = \sqrt{(y_1 - \frac{1}{2})^2 + (y_2 - \frac{1}{2})^2}
\]  
(3)

is the radial distance of the material point from the centre of the particle and \( a \) and \( b \) represent the radial distances from the centre of the particle to the inner and outer interphase boundaries respectively, such that the thickness of the interphase is given by \( b - a \). Figure 2 shows the shear modulus distribution for a microstructure with a stiffness contrast of \( G_p/G_m = 20, r = 0.3, a = 0.25 \) and \( b = 0.35 \), in terms of a contour plot on the unit cell (Figure 2(a)) and as a function of the radial coordinate \( \rho \) (Figure 2(b)). The former clearly shows the regions of the stiff inclusion, matrix and the thin interphase. The above values are the reference values used in all results presented, unless stated otherwise.

### 2.3. Applied body force

The applied, non-dimensionalized, out-of-plane body force is taken to be

\[
F(x) = \sin(2\pi x_1) \sin(2\pi x_2)
\]  
(4)
It is instructive to observe that for a homogeneous medium with (normalized) shear modulus $G(y) \equiv 1$, the solution to the governing equation (1) reads, for any value of the translation $\zeta$,

$$u(x) = \frac{1}{8\pi^2} \sin(2\pi x_1) \sin(2\pi x_2)$$

(5)

Note that the normalizations adopted so far imply that $u$ is normalized by $F_0 L^2 / G_m$, where $F_0$ is the amplitude of the body force. We expect the solutions to the full-scale problem (1) to have a shape similar to (5), albeit with a superimposed fast fluctuation due to the microstructure.

2.4. Homogenization by ensemble averaging

It remains to define the homogenized solution of the full-scale problem given by Equation (1). For this purpose, we follow Smyshlyaev and Cherednicheko [18] in arguing that the precise position of the microstructure relative to the macroscopic problem generally cannot be controlled and that we should therefore consider an entire family of problems which covers all possible translations of that microstructure. Each of these realizations is assumed to have the same probability of occurrence. The homogenized solution is then defined as the ensemble average of the solutions to each of the translated problems.

In the case of our double-periodic problem given by (1), it is the position of the microstructure with respect to the period of the applied body force which is undeterminate. This relative position is characterized by the translation vector $\zeta$ in (1). An ensemble of full-scale problems may be constructed by considering all possible translations $\zeta \in Q$. The homogenized solution can then be written as

$$\bar{u}(x) = \int_Q u(\zeta)(x) \, d\zeta$$

(6)

where $u(\zeta)(x)$ denotes the solution of Equation (1) for a given translation vector $\zeta$. 

Figure 2: (a) Shear modulus distribution on the unit cell, representing a stiff particle embedded in a soft matrix. (b) Shear modulus $G(y)$ plotted against the radial distance $\rho$. 
Note that the averaging in (6) is done pointwise, i.e. for fixed $\mathbf{x}$. It gives us a rigourously defined, continuous homogenized solution, which does not depend on ad-hoc assumptions. In particular, it does not introduce any additional length scale, as for instance a moving volume averaging would do. This is a useful feature to have when the physical length scales of the problem, $l$ and $L$, are not well separated – the case which we are focussing on in this contribution.

2.5. Methodology

In what follows we first generate reference solutions for the problem detailed above by numerically solving the full-scale partial differential equation (1) for a finite, but large, number of translations of the microstructure and averaging these numerical solutions in accordance with (6). The process is repeated for a range of scale ratios $L/l$. This “brute force” computation of reference solutions is schematically shown in the left part of Figure 3; details of the approach and results are given in the next section.

![Figure 3: Sketch of the methodology](image)

Subsequently, in Section 4, periodic homogenization is used to compute the effective properties of the composite at the conventional, zeroth-order, but also including higher-order corrections. The corresponding homogenized macro-problems are subsequently solved analytically. This process is illustrated in the right part of Figure 3. It delivers predictions of the homogenized solutions, as a function of $L/l$, which are expected to be increasingly accurate as more higher-order terms are included – even at low $L/l$. These predictions are compared against the numerically computed reference solutions for the full range of scale ratios $L/l$ considered.

3. Reference solutions

Reference solutions are created by performing full-scale numerical simulations of a family of translated microstructures subjected to the periodic body force (4). For each of the translations,
the governing partial differential equation (1) of the anti-plane shear problem is solved numerically
using the finite difference method. The family of numerical solutions thus obtained is subsequently
averaged pointwise.

3.1. Numerical implementation

For the finite difference discretization we cover the problem domain \((0, 1) \times (0, 1)\) by a square
grid of spacing \(h = 1/n\) in the directions of \(x_1\) and \(x_2\). Discrete grid coordinates \((i, j)\), with
\(i, j = 1, 2, \ldots n\), refer to the grid points \(x_{ij} = (ih, jh)\). A second-order central difference scheme is
used to discretize the governing partial differential equation (1), resulting in

\[
\frac{1}{h} \left( G_{i+\frac{1}{2},j} \left( \frac{u_{(i+1)j} - u_{ij}}{h} - G_{i-\frac{1}{2},j} \frac{u_{ij} - u_{(i-1)j}}{h} \right) \right) + \frac{1}{h} \left( G_{i,j+\frac{1}{2}} \left( \frac{u_{i(j+1)} - u_{ij}}{h} - G_{i,j-\frac{1}{2}} \frac{u_{ij} - u_{i(j-1)}}{h} \right) \right) + F_{ij} = 0
\]

for \(i, j = 1, 2, \ldots n\). In this equation \(u_{ij}\) denotes the numerical approximation of \(u(x)\) in grid
point \(x_{ij} = (ih, jh)\), the shear moduli \(G_{ij}\) are computed taking into account the translation vector
\(\zeta\) according to \(G_{ij} = G(Lx_{ij}/l + \zeta) = G(iLh/l + \zeta_1, jLh/l + \zeta_2)\), and \(F_{ij} = F(x_{ij}) = F(ih, jh)\).

Periodicity is enforced by replacing \(i = 0\) by \(i = n\), \(i = n+1\) by \(i = 1\) and making the corresponding
substitutions for \(j\). To enforce the average to be zero, the solution is first computed fixing an
arbitrary degree of freedom to be zero, and subsequently subtracting the average of the solution
thus obtained.

The numerical solutions reported below are all based on a grid spacing of \(1/80\) of the period of
the microstructure, i.e. \(n = 80L/l\). Note that this corresponds with at least 5 grid points across
the thickness \(b-a=0.1\) of the interphase between particle and matrix. A convergence study,
not shown here, has established that this discretization gives an accuracy which is an order of
magnitude better than the trends which we observe.

The number of translated microstructures considered at each scale ratio equals the number of
grid points per microstructural unit cell, i.e. \(80 \times 80\). The reference homogenized solutions are thus
based on 6400 numerical solutions for each value of the scale ratio.

3.2. Illustration of full-scale numerical solutions obtained

Figure 4 shows a few of the full-scale solutions computed, for a scale ratio \(L/l = 10\). As
expected, each of the displacement solutions exhibits a slow variation which follows that of the
applied body force. The influence of the microstructure is visible as a fluctuation with a wave
length of \((L/l)^{-1} = 0.1\). The positions of the circular particles can clearly be recognized as a
relatively flat region in the displacement solution, due to the higher stiffness of the particles. Most
of the deformation is generated in the bands of matrix material in between the circular particles.
For the case \(\zeta = 0\), in Figure 4(a), the boundaries of the problem domain coincide with such bands
and exactly \(10 \times 10\) complete unit cells of the microstructure fit in the problem domain. In the
other solutions shown, for \(\zeta = (0, 0.25)\) and \(\zeta = (0, 0.5)\) (Figure 4(b) and (c), respectively), the
circular flat regions associated with the particles are translated by the vector \(-l\zeta/L\).
Figure 4: Full scale solutions computed for a scale ratio of $L/l = 10$ and different positions $\zeta$ of the microstructure relative to the body force: (a) $\zeta = (0, 0)$, (b) $\zeta = (0, 0.25)$, and (c) $\zeta = (0, 0.5)$. 
3.3. Influence of the scale ratio $L/l$

Figure 5 shows cross-sections of all numerical solutions along the line $x_2 = \frac{1}{4}$, for the scale ratio $L/l = 10$ considered above, in Figure 5(a), as well as for smaller ratios of $L/l = 3$, and 1, in Figure 5(b) and (c). The reference homogenized solutions, obtained by ensemble averaging, are also included for each scale ratio, as dark solid curves. Note that the case $L/l = 1$ represents a complete breakdown of scale separation, as the entire macrostructural periodic cell (i.e. one wavelength of the body force) contains only a single, large particle.

We observe in Figure 5 that the ensemble averaged, homogenized solutions are similar between the different scale ratios – a more quantitative comparison follows below. The individual solutions for different realizations of the microstructure fluctuate around these averaged solutions. Clearly, and not surprisingly, the bandwidth of these fluctuations is larger as $L/l$ is smaller, i.e. as the size of the microstructure increases with respect to the macroscopic length scale. Whereas for a scale ratio of $L/l = 10$ it is on the order of 10% of the peak homogenized solution, it increases to approximately 100% for the case $L/l = 1$.

3.4. Quantitative comparison of the reference solutions

For a more careful comparison of the homogenized reference solutions obtained for the different scale ratios, the amplitudes of the ensemble averaged displacement solutions have been plotted versus the length scale ratio $L/l$ in Figure 6. The amplitude is defined here as the infinity norm of

![Figure 5: Cross-sections of all full-scale numerical solutions along the line $x_2 = \frac{1}{4}$ for scale ratios equal to (a) $L/l = 10$, (b) $L/l = 3$ and (c) $L/l = 1$. The ensemble averaged solutions are also shown as dark curves.](image-url)
the ensemble averaged solution, $\|\bar{u}\|_\infty$, i.e. the maximum absolute value of $u(x)$ on the domain. A range of scale ratios $1 \leq L/l \leq 10$ has been explored. Data is shown for the circular particles with stiffness contrast $G_p/G_m = 20$ considered in Figures 4 and 5, as well as for circular particles with a lower stiffness contrast $G_p/G_m = 5$ and for rectangular particles as indicated in the diagram. The latter also have a stiffness contrast of 5; no regularization of the stiffness distribution was used for them (see Section 2.2).

![Figure 6: Amplitude (infinity norm) of the numerically determined homogenized solutions as a function of scale ratio $L/l$, for the reference case of a circular particle with $G_p/G_m = 20$, the same microstructure but with stiffness contrast $G_p/G_m = 5$ and a rectangular particle with $G_p/G_m = 5$.](image)

To make the trends more visible in Figure 6, some data points have been included at non-integer scale ratios. For these ratios the macroscopic problem size has been extended to more than one period of the body force to accommodate an integer number of microstructural periodic cells in each direction. For example, the computations for $L/l = 1.5$ were done on a domain $(0, 2) \times (0, 2)$ containing $3 \times 3$ particles.

Let us first consider the result for the reference case of a circular particle with stiffness contrast $G_p/G_m = 20$, given by the black markers in Figure 6. For large scale ratios, $L/l \geq 8$ say, the peak homogenized displacement becomes constant at approximately half of that one would find for the homogeneous matrix material only, $1/(8\pi^2) \approx 0.013$ – see Equation (5). One indeed expects that in the limit $L/l \to \infty$ the microstructure becomes so fine relative to the macroscopic problem, that the material effectively behaves as a homogeneous medium. This is the limit which conventional homogenization methods are generally based on. As a result, they deliver homogenized material models (and properties) which are scale-independent, i.e. which are uninformed about the scale of the microstructure, because the role of that very scale is neglected in setting up the homogenization. The diagram shows that, for the linear elastic system studied here, this may be quite an acceptable assumption for scale ratios as low as $L/l = 8$ – depending, of course, on the desired accuracy.

As the scale ratio is reduced below 8, the homogenized response starts to become scale-
dependent. The microstructure interferes with the macroscopic loading in such a way that even the homogenized solution (and its amplitude $||\bar{u}||_\infty$) is affected. The peak homogenized displacement drops to a minimum of approximately 90% of that for large $L/l$ at $L/l = 2$. Upon further reduction of $L/l$, towards unity, the effect is reduced again and the peak displacement increases. This fluctuation of the homogenized response as a function of the scale ratio would not be captured by conventional, scale-independent, homogenization methods.

The data for circular, but softer particles, with $G_p/G_m = 5$, confirms the trend, but shows that its magnitude depends on the stiffness contrast. For this reduced contrast, the peak displacement at $L/l = 2$ is only 3.5% lower than that at $L/l = 10$. At the same time the curve has shifted upward, closer to the level which would be obtained for the homogeneous matrix, because the softer particles result in an overall softer material and hence in a larger displacement amplitude.

The microstructure with rectangular particles shows a similar trend too. The smallest peak displacement is in this case found at $L/l = 2$.5. It is less than 2% smaller than the value at $L/l = 10$. Because of the smaller volume fraction of particles, the amplitude $||\bar{u}||_\infty$ of the homogenized displacements for this case is even larger than that for the same contrast but circular particles.

The magnitude of the trends observed here and the scale ratios at which they occur, match well with earlier observations in the literature. In particular, Drugan and Willis [20], based on the leading order correction term of a higher-order asymptotic expansion (see the next section), conclude that conventional homogenization, based on only the leading-order term, is accurate within 5% for scale ratios larger than 2. Based on the full, numerically computed reference solutions employed here we find a deviation which is about twice as large (10%) at $L/l = 2$ for the larger stiffness contrast (of 20), whereas at a contrast of 5 the error is indeed within 5%.

We wish to emphasize that, although these effects may be quite small – perhaps too small to be practically significant – in the elastic regime, one expects them to be more prominent for non-linear material behaviour of one or both of the phases. This has for instance been shown recently by Biswas and Poh [21], who observe a deviation of 16% at a scale ratio $L/l = 5$ for a composite of soft elastic–ideally plastic particles in a harder elasto-plastic matrix.

4. Asymptotic homogenization

Asymptotic homogenization, or periodic homogenization, is a rigorous method used to compute effective, homogenized properties for heterogeneous media, thereby yielding simplified macroscopic models based on geometrically complex, but periodic, microstructural models [1, 2, 5]. It employs a series expansion of the unknown field to systematically separate the full-scale problem into a series of microscale problems and a parallel series of macroscale problems. The microscale problems are formulated on the periodic unit cell of the microstructure. They use only the geometry and properties of the microstructure, and are thus independent of the loading and boundary conditions. Once solved, they allow one to compute effective properties of the microstructure. The macroscale problems are defined on the problem domain of the underlying full-scale problem and use its loading and boundary conditions. However, the microstructure of the full-scale problem is replaced by a homogenous medium which has the effective properties obtained from the microscale problems. Solving the macroscale problems results in an estimate of the homogenized solution of the full-scale problem, i.e. of the “average”, slowly varying part of its solution.

Conventionally, the asymptotic series is truncated after the leading terms. This results in a classical macroscopic continuum description at the macroscale, which is scale-independent. The consequences of taking into account additional terms of the expansion have been explored in the
Asymptotic homogenization is founded on the observation that for large scale separation the inverse of the scale ratio \( L/l \) appearing in (1), \( \eta = l/L \), is a small parameter. Together with the expectation that the solution of this equation has a fast variation, at scale \( \eta \), superimposed on a slow variation, at scale unity, this suggests searching for a solution of (1) in the form of a series

\[
\eta \frac{\partial u}{\partial x} = \eta v + \eta^2 v_2(x, x/\eta) + \eta^3 v_3(x, x/\eta) + \ldots
\]

where the functions \( u_k(x, y) \) are periodic with period 1 in \( x_1 \) and \( x_2 \), as well as in \( y_1 \) and \( y_2 \).

The substitution of \( u_k(x, y) \) for the second argument of each \( u_k \) in (8) introduces fast displacement fluctuations, at the microscale \( \eta \), while the dependence on \( x \) via the first argument is slow, at scale unity.

The method proceeds by substituting (8) in (1) while taking account of the fact that the gradient operator (as well as the divergence) acts on both arguments of \( u \) and \( \eta \) fluctuations, at the microscale \( \eta \), and of the corresponding homogenized macroscopic equations for the double-periodic anti-plane shear problem considered here.
The leading-order term in (10) is a slowly varying function \( v(x) \), which itself is given in terms of a series by (11). The remaining terms in (10) consist of products of \( p^{th}\)-order tensor-valued microfluctuation functions \( N_p(y) \) and (higher-order) gradients of \( v \). They characterize the fast displacement fluctuations due to interaction of the the applied loading (body force) and the microstructure. Note that the slow function \( v \) in (10)–(11), as well as the slow functions \( v_q \), are independent of the translation of the microstructure as characterized by \( \zeta \). The fast variations, however, are translated along with that microstructure by the appearance of \( \zeta \) in the argument of the microfluctuation functions in (10).

The microfluctuation functions \( N_p(y) \) are defined as the \( Q \)-periodic, zero-average functions which satisfy the following set of partial differential equations defined on the microstructural unit cell \( Q \):

\[
\nabla_y \cdot \left[ G(y) \left( \nabla_y N_1 + I \right) \right] = 0 \tag{12}
\]
\[
\nabla_y \cdot \left[ G(y) \left( \nabla_y N_2 + I N_1 \right) \right] + G(y) \left( \nabla_y N_1 + I \right) = C_0 \tag{13}
\]
\[
\nabla_y \cdot \left[ G(y) \left( \nabla_y N_p + I N_{p-1} \right) \right] + G(y) \left( \nabla_y N_{p-1} + I N_{p-2} \right) = C_{p-2} \quad \text{for } p \geq 3 \tag{14}
\]

where \( I \) denotes the second-order identity tensor and the constant, \( (p + 2)^{th}\)-order tensors \( C_p \) are defined as

\[
C_0 = \int_Q G(y) \left( \nabla_y N_1 + I \right) dy \tag{15}
\]
\[
C_p = \int_Q G(y) \left( \nabla_y N_{p+1} + I N_p \right) dy \quad \text{for } p \geq 1 \tag{16}
\]

Note that each of the problems (12)–(14) requires the solutions of the two previous problems. These problems must thus be solved sequentially, for increasing \( p \). The slow functions \( v_q(x) \) which contribute to \( v(x) \) are the solutions to the macroscopic partial differential equations

\[
\nabla_x \cdot \left( C_0 \cdot \nabla_x v_0 \right) + F(x) = 0 \tag{17}
\]
\[
\nabla_x \cdot \left( C_0 \cdot \nabla_x v_1 + C_1 : \nabla_x \nabla_x v_1^T \right) = 0 \tag{18}
\]
\[
\nabla_x \cdot \left( C_0 \cdot \nabla_x v_2 + C_1 : \nabla_x \nabla_x v_1^T + C_2 : \nabla_x \nabla_x \nabla_x v_0^T \right) = 0 \tag{19}
\]

etc. For the double-periodic composite problem considered throughout this paper, they must furthermore be periodic, at the slow scale of unity, and have average zero. These problems, too, must be solved for increasing order, as the governing equation for \( v_q \) features gradients of all \( v_r \) for which \( r < q \). Note that the constant tensors \( C_p \) according to (15)–(16) appear in these equations as well – they carry the relevant information on the microstructure to the macro-problems. This implies that in order to solve for the macroscopic field \( v_q \), all \( C_p \) with \( p \leq q \) must be known, which in turn implies that \( q + 1 \) micro-problems need to have been solved first.

The asymptotic expression (10)–(11) still contains the complete, full-scale detail of the displacement field. In order to obtain only the coarse-scale, overall response, we call upon the very same reasoning introduced in Section 2.4 and ensemble-average it by employing (6). As a result, due to the fact that the microfluctuation functions have average zero, all microfluctuation terms vanish
and only the leading term remains in (10). This allows us to identify the slow function \( v(x) \), given by (11), as the exact homogenized solution: \( \bar{u}(x) = v(x) \).

In order to be of any practical use, the series (11) defining the homogenized solution, or, equivalently, the double series (10)–(11) defining the underlying full-scale solution, must be truncated. Conventional periodic homogenization truncates (10) after the second and (11) after the first term. It thus requires solving only the leading microstructural problem (12) and the leading macroscopic problem (17). The latter may be interpreted as a simple, homogenous elastic problem, in which the tensor \( C_0 \) characterizes the homogenized elastic (shear) stiffness. We refer to its solution, \( v_0 \), as the zeroth-order homogenized solution. It can be shown to converge to \( \bar{u} \) as \( \eta \to 0 \). But even for finite, but small \( \eta \), i.e. \( \eta \ll 1 \), the contribution of the higher-order terms in the series is negligible and the conventional zeroth-order approximation hence is accurate. More precisely, it can be stated (see e.g. [5] for details) that for a fixed microstructure and body force there exists a constant \( C^{(0)} \), which is independent of \( \eta \), such that

\[
\int (v_0(x) - \bar{u}(x))^2 \, dx \leq C^{(0)} \eta^2 \tag{20}
\]

To determine this higher-order approximation of \( \bar{u} \), one needs to solve \( K + 1 \) macro-problems akin to (17)–(19), each of which we refer to as homogenized equation of order \( q \). To determine the higher-order homogenized stiffness tensors in them, \( K + 1 \) micro-problems need to be solved, according to (12)–(14). Doing so allows one to reduce the error to (cf. (20))

\[
\int (v^{(K)}(x) - \bar{u}(x))^2 \, dx \leq C^{(K)} \eta^{2K+2}. \tag{22}
\]

Note, finally, that the higher-order homogenized solutions \( v^{(K)} \) thus computed (for \( K > 0 \)) are scale-dependent: they depend on the scale ratio via \( \eta = l/L \) appearing in (21). This is not true for the conventional, zeroth-order homogenized solution \( v^{(0)} = v_0 \), which is independent of \( \eta \).

4.2. Microscale problem solutions and effective properties

We now put the theory summarized above to work on the two-scale problem defined in Section 2. The first requirement is to obtain the microfluctuation functions \( N_p(y) \) by solving the unit-cell problems (12)–(14). For this we employ a finite difference discretization which is fully analogous to that for the full-scale numerical solutions of Section 3, cf. Equation (7). A grid spacing of \( h = 1/80 \) is used unless otherwise indicated.

Figure 7 shows one component of each of the first three (tensor-valued) microfluctuation functions for the reference microstructure of circular particles with stiffness contrast \( G_p/G_m = 20 \). These functions may be thought of as corrections to the smooth field \( v \) due to gradients of \( v \) in particular \( \partial v / \partial x_1 \), \( \partial^2 v / \partial x_1^2 \) and \( \partial^3 v / \partial x_1^3 \), respectively.
Figure 7: Examples of microfluctuation functions of different orders for the microstructure with circular particles and $G_p/G_m = 20$: (a) $(N_1)_1$, (b) $(N_2)_{11}$ and (c) $(N_3)_{111}$. 
Once the microfluctuation functions have been computed, the effective constants of each order are obtained by numerically evaluating the integrals in (15)–(16). The symmetric stiffness distribution in the unit cell, as shown by Smyshlyaev and Cherednichenko [18], results in two simplifications: (i) symmetry of the effective constants and (ii) all odd-ordered tensors of effective constants vanish. Hence the effective material tensors of each order are characterized by only a limited number of independent constants, as shown in Table 1 up to the fourth order for the reference case of circular particles with \( G_p/G_m = 20 \).

Table 1: Table of homogenized coefficients of different orders for the composite with circular particles and \( G_p/G_m = 20 \). Components which follow by symmetry or are equal to zero are not listed.

<table>
<thead>
<tr>
<th>Zeroth order</th>
<th>Second order</th>
<th>Fourth order</th>
<th>Fourth order (cont’d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{11} )</td>
<td>( C_{1111} )</td>
<td>( C_{1111} )</td>
<td>( C_{12112} )</td>
</tr>
<tr>
<td>1.9206</td>
<td>2.1015 \times 10^{-4}</td>
<td>(-7.4913 \times 10^{-5})</td>
<td>(-2.3508 \times 10^{-3})</td>
</tr>
<tr>
<td>( C_{1112} )</td>
<td>(-3.2913 \times 10^{-3})</td>
<td>(9.8569 \times 10^{-4})</td>
<td>(4.2325 \times 10^{-3})</td>
</tr>
<tr>
<td>( C_{1121} )</td>
<td>(-5.0744 \times 10^{-4})</td>
<td>(-2.4780 \times 10^{-4})</td>
<td>(-8.0200 \times 10^{-3})</td>
</tr>
<tr>
<td>( C_{1221} )</td>
<td>(8.9220 \times 10^{-4})</td>
<td>(9.3843 \times 10^{-4})</td>
<td>(6.7974 \times 10^{-3})</td>
</tr>
</tbody>
</table>

4.3. Macroscale problem solutions

Once the homogenized moduli \( C_p \) have been established, the macroscopic problems (17), (18), (19), etc. may be solved. Given the fact that these equations have constant coefficients and all odd-order terms in them vanish (see above), they may easily be solved analytically for the problem considered here. Their solutions all turn out to be of the form

\[
v_q(\mathbf{x}) = A_q \sin(2\pi x_1) \sin(2\pi x_2)
\]

where in addition \( A_q = 0 \) for odd \( q \). The computed amplitudes \( A_q \) for even orders, up to an order of four, are tabulated in Table 2.

Table 2: Non-vanishing amplitudes \( A_q \) of the slow variation functions \( v_q \) for the reference microstructure with circular particles and \( G_p/G_m = 20 \).

| \( A_0 \) | \( 6.5944 \times 10^{-3} \) |
| \( A_2 \) | \( -4.4667 \times 10^{-3} \) |
| \( A_4 \) | \( 5.9853 \times 10^{-3} \) |

The homogenized solution of order \( K \) is now obtained by adding up the individual \( v_q \) according to (21), resulting in an expression of the form

\[
v^{(K)}(\mathbf{x}) = A^{(K)} \sin(2\pi x_1) \sin(2\pi x_2)
\]
with the amplitude $A^{(K)}$ given by

$$A^{(K)} = \sum_{q=0}^{K} \eta^q A_q$$

(25)

Note that this amplitude is scale-dependent, as a result of the fact that the individual functions $v_q$ are multiplied by different orders of $\eta$ in (21). For $K = 0$, however, we recover the conventional, scale-independent homogenized solution $v_0$.

4.4. Comparison with the reference solutions

The periodic homogenization solutions at different orders as obtained above are now compared to the reference solution which was computed numerically in Section 3. Figure 8 shows the maximum displacements determined for each as a function of the scale ratio $L/l$. The black data points marked “Reference solution” are the values computed by brute force for the microstructure with circular particles and a contrast ratio of 20. This data was earlier reported in Figure 6, but the displacement has now been normalized by that predicted by conventional, zeroth-order homogenization, $||v_0||_\infty = A_0$, which is independent of $L/l$. The fact that the reference solution approaches 1 as $L/l$ becomes large shows that in this limit the conventional homogenization is indeed accurate: for $L/l = 10$ the difference is on the order of 1%. As was already observed in Figure 6, the reference solution drops significantly below this level for scale ratios $L/l < 8$, reaching a minimum of approximately 0.9 at $L/l = 2$.

Figure 8: Comparison of the normalized peak homogenized displacement predicted by higher-order asymptotic homogenization and the numerically determined reference solution as a function of the scale ratio $L/l$, for the reference case of a circular particle with $G_p/G_m = 20$. The displacements have been normalized by that of the classical, zeroth-order homogenized solution, $||v_0||_\infty$. 

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The higher-order asymptotic homogenization solutions capture this trend with increasing accuracy as the order of the approximation is increased. The second-order solution, in green, clearly predicts the decrease of $||\bar{u}||$ as $L/l$ is reduced. This scale-dependent trend is the result of the negative value of $A_2$ (Table 2) combined with the quadratic dependence of the quadratic corrector term on $\eta = l/L$. It however continues to decrease well beyond the minimum exhibited by the reference solution as $L/l \to 1$. The fourth-order solution initially follows the same trend, until at $L/l \approx 4$ the fourth-order corrector term in (25) starts to become dominant. As a result, this trend is reversed and a minimum occurs. This prediction, shown in red in the diagram, is accurate to approximately 1% to a scale ratio as low as $L/l = 2$. It is anticipated that the trend for such low scale ratios would be followed even more closely if more corrector terms were included in the asymptotic expansion.

4.5. Influence of stiffness contrast

The effect of the stiffness contrast ratio on the classical and higher-order asymptotic homogenization solutions is now explored, again by comparison with numerically computed ensemble averaged solutions. Figure 9 shows the peak homogenized displacements obtained as a function of the contrast ratio $G_p/G_m$. The data corresponds to the reference problem consisting of a microstructure with circular particles at a scale ratio $L/l = 2$. On the horizontal axis, the contrast ratio is increased from the homogeneous case ($G_p/G_m = 1$) to $G_p/G_m = 1000$. Note that the latter value is unrealistically high for most practical materials – it has been selected here merely to illustrate the trends over a wide range of contrasts. The reference solution, indicated with a black dashed line, is constructed out of full-scale simulations followed by ensemble averaging; the computed data points are indicated with black circles. The data has been normalized by the peak displacement of the respective classical (zeroth-order) homogenized solution; note that this normalization depends on $G_p/G_m$. The predictions made by the higher-order homogenization of order two and four, also normalized by the classical homogenized solution, are shown in green and red, respectively. To compute these predictions, the numerical discretization of the unit cell had to be substantially refined, down to $h = 1/640$, in order to ensure converged higher-order microscale solutions for the highest contrast values.

The normalized reference solution equals 1 at $G_p/G_m = 1$, as expected. As the contrast ratio increases, the reference solution decreases significantly below the classical homogenization solution, reaching a constant value of approximately 0.84 beyond $G_p/G_m = 100$. Beyond this value the particle essentially behaves as rigid. The size effect observed in this limit is 50% larger than at the reference stiffness contrast $G_p/G_m = 20$ considered above. The higher-order homogenization solutions pick up the trend. Quantitatively, however, they show an increasing error with respect to the reference solution as the level of contrast is increased. In the rigid-particle limit, the fourth-order homogenized solution nevertheless still has an error of less than 5%. It is expected that this error would be further reduced if additional higher-order terms were taken into account in the homogenization.

4.6. Influence of inverse stiffness contrast

The case of an inverse stiffness contrast ratio is investigated next, where the matrix is stiffer than the circular inclusions. The effect of the classical and higher-order asymptotic homogenization solutions for this case is again explored, by comparison with the numerically computed ensemble averaged solution. Similar to the previous case, a finer discretization is adopted in this case in order
Figure 9: Comparison of the peak homogenized displacement predicted by higher-order asymptotic homogenization and the numerically determined reference solution as a function of the contrast ratio $G_p/G_m$, for the reference case of a circular particle at a scale ratio $L/l = 2$. The displacement amplitudes have been normalized by that of the classical, zeroth-order homogenized solution, $||v_0||_\infty$.

to accommodate the higher stiffness contrast between the matrix and the inclusions. Figure 10 shows the maximum displacements obtained for each of these homogenized solutions as a function of the contrast ratio $G_p/G_m$ for the reference problem consisting of a microstructure with circular particles at a scale ratio $L/l = 2$. On the horizontal axis, the contrast ratio is decreased from the homogeneous case ($G_p/G_m = 1$) to a very low contrast ratio of $G_p/G_m = 0.001$. On the vertical axis, the maximum displacements normalized by the classical homogenization solution is shown. The black dashed curve denotes the reference solution, i.e. full-scale simulation followed by ensemble averaging, and the computed data points are indicated with black circles. The normalized reference solution equals 1 at $G_p/G_m = 1$, as in the case of Figure 9. As the contrast ratio decreases, the reference solution increases rather slowly till a contrast ratio $G_p/G_m = 0.1$, after which it starts to increase significantly above the classical homogenization solution, without a horizontal asymptote as for the case of stiffer inclusions. This, however, is due to the applied body force which induces very large deformations inside the soft inclusions. Note that the stiffness of the matrix is kept a constant (in all cases). The amplitude of the body force is always kept constant. Hence, as the stiffness of the particles reduces, the deformation induced by the body force inside these particles increases. This is reflected in the trend shown by the reference solution. Even in the limit of an inclusion becoming a void (i.e. at a very low stiffness), the effective shear modulus of the homogenized material does not become a constant. This effect can be potentially avoided by making use of a configuration-dependent body force (cf. e.g. [22]). For each realization of the microstructure considered, a body force with a very low magnitude inside the inclusions needs to
be applied. For example, it could be made a function of the shear modulus distribution for each realization $\zeta$. In general, the configuration-dependent body force $f^{(\zeta)}(\mathbf{x})$ for any given $\zeta$ can be given by

$$f^{(\zeta)}(\mathbf{x}) = F(\mathbf{x}) \ast \rho(\mathbf{x}/\eta + \zeta)$$  \hspace{1cm} (26)$$

where $F(\mathbf{x})$ can be the standard bisinusoidal function and $\rho(\mathbf{x}/\eta + \zeta)$ denotes the realization of the microstructure. $F(\mathbf{x})$ is fixed in space and varies slowly, while $\rho(\mathbf{x}/\eta + \zeta)$ varies at the scale of the microstructure and shifts with it, bringing in the configuration-dependence. This, however, deviates from the main goal of this work and is not further investigated here.

Figure 10: Comparison of the maximum homogenized displacement predicted by higher-order asymptotic homogenization and the numerically determined reference solution as a function of the contrast ratio $G_p/G_m$, for the reference case of circular particles at a scale ratio $L/l = 2$. Note that the shear modulus of the matrix $G_m$ is kept constant, while that of the circular particles $G_p$ is reduced gradually. The displacement amplitudes have been normalized by the classical, zeroth-order homogenized solution, $||\mathbf{v}_0||_\infty$.

From Figure 10, it can be observed that the classical homogenization has an error as high as 1500% for $G_p/G_m = 0.001$. The higher-order homogenization solutions captures the trend of the reference solution with increasing accuracy. As the stiffness contrast ratio decreases the second order solution, in green, predicts the increase of $||\mathbf{u}||_\infty$, yet starts to deviate from the solution beyond $G_p/G_m = 0.01$ reaching an error of about 15% at $G_p/G_m = 0.001$. The fourth-order solution, in red, predicts the solution trend rather accurately. Up to a contrast ratio of $G_p/G_m = 0.01$, the error is less than 0.1%. Even at a stiffness contrast ratio of $G_p/G_m = 0.001$, the error is still less than 2%.
Figure 11: Relative error, $\epsilon$, as a function of the scale ratio $L/l$ and the stiffness contrast ratio $G_p/G_m$ for the three cases: (a) classical, zeroth order homogenization solution ($\epsilon^{(0)}$), (b) second order homogenization solution ($\epsilon^{(2)}$) and (c) fourth order homogenization solution ($\epsilon^{(4)}$).
4.7. Error contours

Figure 11 shows the contour plots of the error in the classical homogenization and the higher-order asymptotic homogenization for a wide range of scale ratios $1.5 \leq L/l \leq 10$ and stiffness contrast ratios $1 \leq G_p/G_m \leq 1000$. Note that the limit case $L/l = 1$ is not included in these plots as it represents a case when homogenization is no longer valid. The relative error, $\epsilon$, defined as the relative difference between the corresponding homogenization solution and the reference solution, is given by

$$\epsilon^{(K)} = \frac{||v^{(K)}||_\infty - ||\bar{u}||_\infty}{||\bar{u}||_\infty}$$  \hspace{1cm} (27)

Note that a fine discretization is used for the solutions here to accommodate the high contrast in geometric and material properties considered. A finite grid spacing of $1/640$ of the period of the microstructure is employed for all the three plots.

From Figure 11(a), it can be observed that the error in the classical homogenization solution ($\epsilon^{(0)}$) is negligible for scale ratios larger than $L/l = 8$ and for stiffness contrast ratios smaller than $G_p/G_m = 2$. $\epsilon^{(0)}$ starts to increase for increasing values of stiffness contrast ratio and decreasing values of scale ratio. The highest error of about 20% is observed for the range of scale ratios $1.5 \leq L/l \leq 2.1$ and stiffness contrast ratios $100 \leq G_p/G_m \leq 1000$. The second order homogenization solution predicts the reference solution very accurately for all stiffness contrast ratios up to scale ratio $L/l = 2$, as can be observed in Figure 11(b). For scale ratios less than $L/l = 2$, the error $\epsilon^{(2)}$ grows rather rapidly and reaches a very high value of about 25% close to $L/l = 1.5$ and stiffness contrast ratio in the range of $100 \leq G_p/G_m \leq 1000$. Note that this high error at low scale ratios has previously been observed in Figure 8. The fourth order asymptotic homogenization solution further improves the solution as can be observed in Figure 11(c). The corresponding error $\epsilon^{(4)}$ is negligibly small below scale ratio $L/l = 2$ for any stiffness contrast ratio. Below this scale ratio, $\epsilon^{(4)}$ increases gradually reaching a maximum of about 10% at $L/l = 1.5$ and $100 \leq G_p/G_m \leq 1000$.

5. Conclusions

The aim of this work was to investigate the scale separation limit for homogenizing periodic linear elastic composite materials and demonstrate the effectiveness of the higher-order asymptotic homogenization method. A quantitative assessment was performed on a two-dimensional elastic two-phase composite loaded in anti-plane shear. Reference solutions were constructed using full-scale numerical simulations where the effective solution was defined as the ensemble average of the solutions of the entire family of translated microstructures. These reference solutions show a clear size effect as the scale of the microstructure approaches the macroscopic scale. Using the method of asymptotic homogenization, we showed that the zeroth-order (classical) homogenization is independent of the scale ratio, as expected. Higher-order solutions have been computed, which capture the scale effect and which predict increasingly accurate solutions for smaller scale ratios as more higher-order terms are included. The performance of the classical and higher-order asymptotic homogenization method were evaluated for varying stiffness contrast ratio between the two phases of the microstructure. The results were again compared with the full-scale numerical reference solution and relative error contours were extracted. The asymptotic homogenization method predicts the solutions with increasing accuracy as more higher-order terms are added, for a wide range of stiffness contrast, even at small scale ratios.
The results provide clear insights into the low scale separation regime of an elastic periodic two-phase composite. The data presented shows the limitation of classical homogenization methods and the effectiveness of higher-order asymptotic homogenization in addressing this limitation. Some key results from this work are:

- The classical homogenization solution is accurate (with less than 1% error for the reference stiffness contrast of 20) only for cases where the spacing between the particles \( l \) is at least one order of magnitude smaller than the macroscale length \( L \).

- Higher-order periodic homogenization is an accurate method for tackling problems in linear elasticity with low scale separation. The scale separation can actually become as small as \( L/l = 2 \) before a significant difference emerges. At such low scale ratios, higher-order periodic homogenization solutions can be used to correct the classical solution with an adequate accuracy.

- The accuracy of classical and higher-order homogenized solutions depends mildly on the amount of stiffness contrast. For the particle-reinforced system considered here, the error increases with stiffness contrast, until it becomes constant in the rigid-particle limit. For the fourth-order solution, it remains within 5%, even at an unrealistic stiffness contrast of 1000. For the inverse stiffness contrast case studied here, the error increases rather rapidly with inverse stiffness contrast, without a horizontal asymptote. However, the second-order and fourth-order solutions still captures this trend accurately. The error in the fourth-order homogenization solution remains within 2%, even at \( G_p/G_m = 0.001 \).

- Higher-order periodic homogenization is not constrained by the conventional separation of scales. It enables a relaxed scale separation constraint \( l \leq L \).

Let us finally note that the quantitative results discussed here for a specific microstructure and loading cannot be generalized rigorously. For analyzing more complex microstructures, the methodology needs to be extended for non-periodic microstructures [23]. The qualitative results, however, give a clear indication that higher order periodic homogenization can be made use of in problems with a clear lack of scale separation.

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References


